

How much sweetness is there in the universe?

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We continue investigations of forcing notions with strong ccc properties introducing new methods of building sweet forcing notions. We also show that quotients of topologically sweet forcing notions over Cohen reals are topologically sweet while the quotients over random reals do not have to be such.

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0 Introduction

One of the main ingredients of the construction of the model for “all projective sets of reals have the Baire property” presented in Shelah [7, §7] was a strong ccc property of forcing notions called *sweetness*. This property is preserved in amalgamations and also in compositions with the Hechler forcing notion \mathbb{D} and the Universal Meager forcing UMI (see [7, §7]; a full explanation of how this is applied can be found in [3]). Stern [10] considered a slightly weaker property, *topological sweetness*, which is also preserved in amalgamations and compositions with \mathbb{D} and UMI . We further investigated the sweet properties of forcing notions in [6, §4], where we introduced a new property called *iterable sweetness* and we showed how one can build sweet forcing notions. New examples of iterably sweet forcing notions can be used in constructions like [7, §7] and [9], but it could be that there is no need for this – the old forcing notions could be adding generic objects for all of them. In [4] we proved that this is exactly what happens with the natural examples of sweet forcing notions determined by the universality parameters as in [6, §2.3]: a sequence Cohen real – dominating real – Cohen real produces generic filters for many of them.

In the present paper we show that sweetness is not so rare after all and we give more constructions of sweet forcing notions. In the first section we present a new method of building sweet forcing notions and we give our first example: a forcing notion $\mathbb{Q}^{\mathcal{T}^{\text{sc}}}$ associated with scattered subtrees of $2^{<\omega}$. We do not know if the iterations of “old” forcing notions add generic objects for $\mathbb{Q}^{\mathcal{T}^{\text{sc}}}$, but in Proposition 1.8 we present an indication that this does not happen. In the second section we use our method to introduce two large families of sweet forcing notions, in some sense generalizing the known examples from [6]. This time we manage to show that *some* of our forcing notions are really new by showing that we have too many different examples (in Theorems 2.10 and 2.15).

In the last section of the paper we investigate the preservation of topological sweetness under some operations. We note that a complete subforcing of a topologically sweet separable partial order is equivalent to a topologically sweet forcing (in Proposition 3.5). We also show that the quotient of a topologically sweet forcing notion by a Cohen subforcing is topologically sweet (Theorem 3.7), but quotients by random real do not have to be topologically sweet (Corollary 3.10).

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0.1 Notation

Our notation is rather standard and compatible with that of classical textbooks (like Jech [2] or Bartoszyński and Judah [1]). In forcing we keep the older convention that a stronger condition is the larger one. Our main conventions are listed below:

1. For a forcing notion \mathbb{P} , $\Gamma_{\mathbb{P}}$ stands for the canonical \mathbb{P} -name for the generic filter in \mathbb{P} . Other \mathbb{P} -names for objects in the extension via \mathbb{P} will be denoted with a tilde below (e. g. $\tilde{\tau}$, \tilde{X}). The weakest element of \mathbb{P} will be denoted by $\emptyset_{\mathbb{P}}$ (and we will always assume that there is one and that there is no other condition equivalent to it).

2. The complete Boolean algebra determined by a forcing notion \mathbb{P} is denoted by $\mathbf{BA}(\mathbb{P})$. For a complete Boolean algebra \mathbb{B} , \mathbb{B}^+ is $\mathbb{B} \setminus \{0_{\mathbb{B}}\}$ treated as a forcing notion (so the order is the reverse Boolean order). Also, for a formula φ , the Boolean value (with respect to \mathbb{B}) of φ will be denoted by $\llbracket \varphi \rrbracket_{\mathbb{B}}$.

3. Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet ($\alpha, \beta, \gamma, \delta, \dots$) and also by i, j (with possible sub- and superscripts). Cardinal numbers will be called κ, λ, μ .

4. For two sequences η, ν we write $\nu \triangleleft \eta$ whenever ν is a proper initial segment of η , and $\nu \leq \eta$ when either $\nu \triangleleft \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\text{lh}(\eta)$.

5. The quantifier $(\exists^{\infty} n)$ is an abbreviation for $(\forall m \in \omega) (\exists n > m)$.

6. The Cantor space 2^{ω} and the Baire space ω^{ω} are the spaces of all functions from ω to 2 and from ω to ω , equipped with the natural (Polish) topology.

0.2 Background on sweetness

Let us recall basic definitions related to sweet forcing notions.

Definition 0.1 (Shelah [7, Definition 7.2]) A pair (\mathbb{P}, \bar{E}) is *model of sweetness* whenever

- (i) \mathbb{P} is a forcing notion;
- (ii) $\bar{E} = \langle E_n : n < \omega \rangle$, each E_n is an equivalence relation on \mathbb{P} such that \mathbb{P}/E_n is countable;
- (iii) equivalence classes of each E_n are $\leq_{\mathbb{P}}$ -directed, $E_{n+1} \subseteq E_n$;
- (iv) if $\{p_i : i \leq \omega\} \subseteq \mathbb{P}$ and $p_i E_i p_{\omega}$ (for $i \in \omega$), then

$$(\forall n \in \omega) (\exists q \geq p_{\omega}) (q E_n p_{\omega} \ \& \ (\forall i \geq n) (p_i \leq q));$$

- (v) if $p, q \in \mathbb{P}$, $p \leq q$ and $n \in \omega$, then there is $k \in \omega$ such that

$$(\forall p' \in [p]_{E_k}) (\exists q' \in [q]_{E_n}) (p' \leq q').$$

If there is a model of sweetness based on \mathbb{P} , then we say that \mathbb{P} is *sweet*.

Definition 0.2 (Stern [10, Definition 1.2]) A *model of topological sweetness* is a pair $M = (\mathbb{P}, \mathcal{B})$ such that $\mathbb{P} = (\mathbb{P}, \leq)$ is a forcing notion, \mathcal{B} is a countable basis of a topology τ on \mathbb{P} and

- (i) $\emptyset_{\mathbb{P}}$ is an isolated point in τ ;
- (ii) if a sequence $\langle p_n : n < \omega \rangle \subseteq \mathbb{P}$ is τ -converging to $p \in \mathbb{P}$, $q \geq p$ and W is a τ -neighbourhood of q , then there is a condition $r \in \mathbb{P}$ such that

$$(a) \ r \in W, \ r \geq q,$$

$$(b) \ \text{the set } \{n \in \omega : p_n \leq r\} \text{ is infinite.}$$

If there is a model of topological sweetness $(\mathbb{P}, \mathcal{B})$, then the forcing notion \mathbb{P} is *topologically sweet*.

Lemma 0.3 (See [6, Lemma 4.2.3]) Assume that $(\mathbb{P}, \mathcal{B})$ is a model of topological sweetness.

1. If $p, q \in \mathbb{P}$, $p \leq q$ and $q \in U \in \mathcal{B}$, then there is an open neighbourhood V of p such that

$$(\forall r \in V) (\exists r' \in U) (r \leq r').$$

2. If $m \in \omega$, $p \in U \in \mathcal{B}$, then there is an open neighbourhood V of p such that any $p_0, \dots, p_m \in V$ have a common upper bound in U .

Definition 0.4 (See [6, Definition 4.2.1]) Let \mathcal{B} be a countable basis of a topology on a forcing notion \mathbb{Q} . We say that $(\mathbb{Q}, \mathcal{B})$ is a model of iterable sweetness if

- (i) \mathcal{B} is closed under finite intersections;
- (ii) each $U \in \mathcal{B}$ is directed and if $p \leq q \in U$, then $p \in U$;
- (iii) if $\langle p_n : n \leq \omega \rangle \subseteq U$ and the sequence $\langle p_n : n < \omega \rangle$ converges to p_ω (in the topology generated by \mathcal{B}), then there is a condition $p \in U$ such that $(\forall n \leq \omega) (p_n \leq p)$.

Proposition 0.5 (See [6, Proposition 4.2.2]) If \mathbb{P} is a sweet forcing notion in which any two compatible conditions have a least upper bound, then \mathbb{P} is iterably sweet.

1 sw-closed families and scattered trees

In this section we present a new method of building sweet forcing notions. This method is, essentially, a generalization of that determined by the universality parameters of [6, §2.3].

Definition 1.1

1. A tree is a family T of finite sequences such that for some $\text{root}(T) \in T$ we have

$$(\forall \nu \in T) (\text{root}(T) \trianglelefteq \nu) \quad \text{and} \quad \text{root}(T) \trianglelefteq \nu \trianglelefteq \eta \in T \Rightarrow \nu \in T.$$

Elements of a tree T may be called *nodes* of T .

2. If η is a node in the tree T , then

$$\begin{aligned} \text{succ}_T(\eta) &= \{\nu \in T : \eta \triangleleft \nu \ \& \ \text{lh}(\nu) = \text{lh}(\eta) + 1\}, \\ T^{[\eta]} &= \{\nu \in T : \eta \trianglelefteq \nu\}. \end{aligned}$$

3. For a tree T , the family of all ω -branches through T is denoted by $[T]$, and we let

$$\max(T) \stackrel{\text{def}}{=} \{\nu \in T : \text{there is no } \rho \in T \text{ such that } \nu \triangleleft \rho\}$$

and

$$\text{split}(T) \stackrel{\text{def}}{=} \{\nu \in T : |\text{succ}_T(\nu)| \geq 2\}.$$

4. A tree T is *normal* if $\max(T) = \emptyset$ and $\text{root}(T) = \langle \rangle$.

Definition 1.2 Suppose that \mathcal{T} is a family of normal subtrees of $\omega^{<\omega}$. We say that \mathcal{T} is *sw-closed*¹⁾ whenever

1. if $T_1 \in \mathcal{T}$, $T_2 \subseteq T_1$ and T_2 is a normal tree, then $T_2 \in \mathcal{T}$;
2. if $T_1, T_2 \in \mathcal{T}$, then $T_1 \cup T_2 \in \mathcal{T}$;
3. if $\langle T_n : n \leq \omega \rangle \subseteq \mathcal{T}$ is such that $(\forall n < \omega) (T_n \cap \omega^{\leq n} = T_{n+1} \cap \omega^{\leq n})$, then $\bigcup_{n < \omega} T_n \in \mathcal{T}$.

Definition 1.3 For a family \mathcal{T} of normal subtrees of $\omega^{<\omega}$ we define a forcing notion $\mathbb{Q}^{\mathcal{T}}$ as follows. A *condition* in $\mathbb{Q}^{\mathcal{T}}$ is a pair $p = (N^p, T^p)$ such that $N^p < \omega$ and $T^p \in \mathcal{T}$. The order $\leq_{\mathbb{Q}^{\mathcal{T}}}$ of $\mathbb{Q}^{\mathcal{T}}$ is given by $p \leq_{\mathbb{Q}^{\mathcal{T}}} q$ if and only if $N^p \leq N^q$, $T^p \subseteq T^q$ and $T^q \cap \omega^{N^p} = T^p \cap \omega^{N^p}$.

The relation between the forcing $\mathbb{Q}^{\mathcal{T}}$ and the family \mathcal{T} is similar to that in the case of the Universal Meager forcing notion \mathbb{UM} and nowhere dense subtrees of $2^{<\omega}$. Note that $\mathbb{Q}^{\mathcal{T}}$ does not have to be ccc in general, however in many natural cases it is.

Proposition 1.4 Assume that \mathcal{T} is an sw-closed family of normal subtrees of $\omega^{<\omega}$ such that every $T \in \mathcal{T}$ is finitely branching. Then $\mathbb{Q}^{\mathcal{T}}$ is a sweet forcing notion in which any two compatible conditions have a least upper bound (and consequently $\mathbb{Q}^{\mathcal{T}}$ is iterably sweet).

¹⁾ sw comes from sweet. The closure property introduced here is instrumental for the sweetness of the associated forcing notion $\mathbb{Q}^{\mathcal{T}}$.

Proof. One easily verifies that \mathbb{Q}^T is indeed a forcing notion and that any two compatible conditions in \mathbb{Q}^T have a least upper bound. For an integer $n < \omega$ let E_n be a binary relation on \mathbb{Q}^T defined by $qE_n p$ if and only if $N^q = N^p$ and $T^q \cap \omega^{\leq N^q+n} = T^p \cap \omega^{\leq N^q+n}$, and let $\bar{E} = \langle E_n : n < \omega \rangle$. We claim that (\mathbb{Q}^T, \bar{E}) is a model of sweetness. Conditions (i) – (iii) of Definition 0.1 should be clear. To verify Definition 0.1(iv) suppose that $p_i \in \mathbb{Q}^T$ for $n \leq i \leq \omega$ are such that $p_i E_i p_\omega$ (for $i < \omega$). Thus, for $n \leq i < \omega$, $N^{p_i} = N^{p_\omega}$ and

$$T^{p_i} \cap \omega^{\leq N^{p_i}+i} = T^{p_\omega} \cap \omega^{\leq N^{p_\omega}+i}.$$

Put $N = N^{p_\omega}$ and $T = \bigcup \{T^{p_i} : n \leq i \leq \omega\}$. Then it follows from Definition 1.2, 3. that $T \in \mathcal{T}$, and plainly $q = (N, T) \in \mathbb{Q}^T$, $qE_n p_\omega$ and $(\forall i \geq n) (p_i \leq q)$, finishing justification of Definition 0.1(iv).

Finally, to check Definition 0.1(v) suppose that $p, q \in \mathbb{Q}^T$, $p \leq q$ and $n < \omega$. Let $k = N^q + n$. It should be clear that $(\forall p' \in [p]_{E_k}) (\exists q' \in [q]_{E_n}) (p' \leq q')$. \square

Now we are going to present our first example of an sw-closed family: the family of scattered subtrees of $2^{<\omega}$.

Definition 1.5

1. For a closed set $A \subseteq 2^\omega$, let $\text{rk}(A)$ be the Cantor-Bendixson rank of A , that is

$$\text{rk}(A) = \min\{\alpha < \omega_1 : A^\alpha = A^{\alpha+1}\},$$

where A^α denotes the α^{th} Cantor-Bendixson derivative of A .

2. We say that a tree $T \subseteq 2^{<\omega}$ is *scattered* if it is normal and $[T]$ is countable. The family of all scattered subtrees of $2^{<\omega}$ will be denoted by \mathcal{T}^{sc} .

3. For a scattered tree $T \subseteq 2^{<\omega}$, let $g^T : [T] \rightarrow \text{rk}([T])$ and $h^T : [T] \rightarrow \omega$ be such that for each $\eta \in [T]$ we have

$$g^T(\eta) = \min\{\alpha < \text{rk}(T) : \eta \notin [T]^{\alpha+1}\}$$

and

$$h^T(\eta) = \min\{m < \omega : (\forall \nu \in [T]) (\nu \upharpoonright m = \eta \upharpoonright m \Rightarrow (\eta = \nu \vee g^T(\nu) < g^T(\eta)))\}.$$

Proposition 1.6 *Assume that $T \subseteq 2^{<\omega}$ is a normal tree. Then T is scattered if and only if there is a mapping $\varphi : T \rightarrow \omega_1$ such that*

- (\otimes) $^0_{\varphi, T}$ $(\forall \eta, \nu \in T) (\nu \triangleleft \eta \Rightarrow \varphi(\nu) \geq \varphi(\eta))$,
- (\otimes) $^1_{\varphi, T}$ $(\forall \eta \in \text{split}(T)) (\varphi(\eta \frown \langle 0 \rangle) < \varphi(\eta) \vee \varphi(\eta \frown \langle 1 \rangle) < \varphi(\eta))$.

Proof. It should be clear that if there is a function $\varphi : T \rightarrow \omega_1$ such that (\otimes) $^0_{\varphi, T}$ + (\otimes) $^1_{\varphi, T}$ holds true, then the tree T contains no perfect subtree and hence T is scattered.

We will show the converse implication by induction on $\text{rk}([T])$.

Suppose that T is a scattered tree. Let $S = \{\eta \upharpoonright h^T(\eta) : \eta \in [T]\}$. Clearly, for every $\eta \in [T]$ there is $n < \omega$ such that $\eta \upharpoonright n \in S$, so the set

$$F \stackrel{\text{def}}{=} \{\nu \in S : \forall \varrho (\varrho \triangleleft \nu \Rightarrow \varrho \notin S)\}$$

is a front of T . Choose $\{\eta_\ell : \ell < n\} \subseteq [T]$, $n < \omega$, such that $F = \{\eta_\ell \upharpoonright h^T(\eta_\ell) : \ell < n\}$ and let

$$A \stackrel{\text{def}}{=} \{\nu \in T : (\exists \ell < n) (h^T(\eta_\ell) < \text{lh}(\nu) \ \& \ \nu \upharpoonright (\text{lh}(\nu) - 1) \triangleleft \eta_\ell \ \& \ \nu \not\triangleleft \eta_\ell)\}.$$

Note that if $\ell < n$, $\nu \in [T] \setminus \{\eta_\ell\}$ and $\nu \upharpoonright h^T(\eta_\ell) = \eta_\ell \upharpoonright h^T(\eta_\ell)$, then $g^T(\nu) < g^T(\eta_\ell)$. Hence

$$(\forall \nu \in A) (\text{rk}([T^{\upharpoonright \nu}]) < \text{rk}([T])),$$

so by the induction hypothesis for each $\nu \in A$ we may choose $\varphi_\nu : T^{[\nu]} \rightarrow \omega_1$ such that $(\otimes)_{\varphi_\nu, T^{[\nu]}}^0 + (\otimes)_{\varphi_\nu, T^{[\nu]}}^1$ holds true. Put $\alpha^* = \sup\{\varphi_\nu(\nu) : \nu \in A\} < \omega_1$, $k^* = \max\{h^T(\eta_\ell) : \ell < n\} + 1$ and let $\varphi : T \rightarrow \omega_1$ be defined by

$$\varphi(\eta) = \begin{cases} \alpha^* + k^* - \text{lh}(\eta) & \text{if no initial segment of } \eta \text{ belongs to } F, \\ \alpha^* + 1 & \text{if an initial segment of } \eta \text{ belongs to } F \text{ but no initial segment of } \\ & \text{belongs to } A, \\ \varphi_\nu(\eta) & \text{if } \nu \in A \text{ and } \nu \leq \eta. \end{cases}$$

One easily verifies that the function φ (is well defined and) satisfies $(\otimes)_{\varphi, T}^0 + (\otimes)_{\varphi, T}^1$. \square

Proposition 1.7 \mathcal{T}^{sc} is an sw-closed family and consequently $\mathbb{Q}^{\mathcal{T}^{\text{sc}}}$ is iterably sweet.

Proof. Plainly \mathcal{T}^{sc} satisfies the conditions 1. and 2. of Definition 1.2.

To verify Definition 1.2, 3. suppose that $\langle T_n : n \leq \omega \rangle \subseteq \mathcal{T}^{\text{sc}}$ is a sequence of scattered trees such that

$$(\forall n < \omega) (T_\omega \cap 2^{\leq n} = T_n \cap 2^{\leq n}).$$

Let $T = \bigcup_{n \leq \omega} T_n$. We are going to show that T is a scattered tree, and for this we have to show that $[T]$ is countable.

Note that if $n < \omega$, $\nu \in 2^{<\omega} \setminus T_\omega$ and $\text{lh}(\nu) \leq n$, then $\nu \notin T_n$. Therefore, if $\nu \in 2^{<\omega} \setminus T_\omega$, then

$$[\nu] \cap [T] \subseteq \bigcup \{[T_n] : n < \text{lh}(\nu)\},$$

so $[\nu] \cap [T]$ is countable. Hence $[T] \setminus [T_\omega]$ is countable and thus (since $[T_\omega]$ is countable) so is $[T]$.

The “consequently” part follows from Proposition 1.4 (remember that members of \mathcal{T}^{sc} are subtrees of $2^{<\omega}$ so finitely branching). \square

Recall that a forcing notion \mathbb{P} has \aleph_1 -caliber if for every uncountable family $\mathcal{F} \subseteq \mathbb{P}$ there is a condition $p \in \mathbb{P}$ such that $|\{q \in \mathcal{F} : q \leq p\}| = \aleph_1$ (see Truss [11]).

Proposition 1.8

1. If a forcing notion \mathbb{P} has \aleph_1 -caliber, then in $\mathbf{V}^{\mathbb{P}}$ there is no tree $T \subseteq 2^{<\omega}$ such that

(a) for every $\alpha < \omega_1$ there is a countable closed set $A \subseteq 2^\omega$ coded in \mathbf{V} such that $\text{rk}(A) = \alpha$ and $A \subseteq [T]$;

(b) T includes no perfect subtree from \mathbf{V} .

Consequently, \mathbb{P} does not add generic objects for $\mathbb{Q}^{\mathcal{T}^{\text{sc}}}$.

2. If $\mathfrak{b} > \aleph_1$, then neither the Hechler forcing notion \mathbb{D} nor its composition $\mathbb{D} * \mathbb{C}$ with the Cohen real forcing add generic objects for $\mathbb{Q}^{\mathcal{T}^{\text{sc}}}$.

Proof.

1. Suppose towards contradiction that \mathbb{P} has an \aleph_1 -caliber, $p \in \mathbb{P}$ and \underline{T} is a \mathbb{P} -name for a subtree of $2^{<\omega}$ such that the condition p forces that both (a) and (b) of Proposition 1.8, 1. hold true for \underline{T} . Then for each $\alpha < \omega_1$ we may choose a scattered tree $T_\alpha \subseteq 2^{<\omega}$ and a condition $p_\alpha \in \mathbb{P}$ such that $(T_\alpha \in \mathbf{V}$ and)

$$\text{rk}([T_\alpha]) = \alpha \quad \text{and} \quad p \leq p_\alpha \quad \text{and} \quad p_\alpha \Vdash_{\mathbb{P}} T_\alpha \subseteq \underline{T}.$$

Since \mathbb{P} has an \aleph_1 -caliber we find a condition $p^* \in \mathbb{P}$ such that the set

$$Y \stackrel{\text{def}}{=} \{\alpha < \omega_1 : p_\alpha \leq p^*\}$$

is uncountable. Put $T^* = \bigcup_{\alpha \in Y} T_\alpha$. Clearly T^* is a non-scattered tree and $(T^* \in \mathbf{V}$ and) $p^* \Vdash T^* \subseteq \underline{T}$, contradicting (b).

Concerning the “consequently” part it is enough to note that if $\underline{T}^{\text{sc}}$ is the canonical $\mathbb{Q}^{\mathcal{T}^{\text{sc}}}$ -name for a subset of $2^{<\omega}$ such that

$$\Vdash_{\mathbb{Q}^{\mathcal{T}^{\text{sc}}}} \underline{T}^{\text{sc}} = \bigcup \{T^p : p \in \Gamma_{\mathbb{Q}^{\mathcal{T}^{\text{sc}}}}\},$$

then $\Vdash \underline{T}^{\text{sc}}$ is a tree satisfying Proposition 1.8, 1.(a), (b)”.

2. If the unbounded number \mathfrak{b} is greater than \aleph_1 , then both \mathbb{D} and $\mathbb{D} * \mathbb{C}$ have the \aleph_1 -caliber, so 1. applies. \square

Remark 1.9 The forcing notion $\mathbb{Q}^{T^{sc}}$ is somewhat similar to the universal forcing notions discussed in [6, §2.3] and [4]. However it follows from Proposition 1.8, 2. that if MA holds true, then the composition $\mathbb{C} * \mathbb{D} * \mathbb{C}$ does not add generic real for $\mathbb{Q}^{T^{sc}}$. This is somewhat opposite to the result presented in [4, Theorem 2.1] and it may indicate that the answer to the following question is negative.

Problem 1.10 Can a finite composition (or, in general, an FS iteration) of the Hechler forcing notions add a generic object for $\mathbb{Q}^{T^{sc}}$?

2 More sweet examples

In this section we will present two classes of sw-closed families of trees, producing many new examples of sweet forcing notions. Let us start with extending the framework of universality parameters to that of sw-closed families.

The sweet forcing notions determined by the universality parameters were introduced in [6, §2.3]. In [4] we showed that, unfortunately, the use of them may be somewhat limited because the composition of, say, the Universal Meager forcing notions adds generic reals for many examples of the forcing notions determined by universality parameters. However, as we will show here, *families* of universality parameters may determine forcing notions which cannot be embedded into the known examples of sweet forcing notions.

Let us start with recalling definitions concerning universality parameters and the related forcing notions. We will cut down the generality of [6, §2.3] and we will quote here the somewhat simpler setting of [4]. Let \mathbf{H} be a function from ω to $\omega \setminus 2$.

Definition 2.1

1. We let $\mathbf{T} = \bigcup_{i < \omega} \prod_{j < i} \mathbf{H}(j)$, $\mathbf{T} \upharpoonright n = \bigcup_{i \leq n} \prod_{j < i} \mathbf{H}(j)$.
2. A *finite \mathbf{H} -tree* is a tree $S \subseteq \mathbf{T} \upharpoonright N$ with $N < \omega$, $\text{root}(S) = \langle \rangle$ and $\max(S) \subseteq \prod_{i < N} \mathbf{H}(i)$. The integer N will be denoted by $\text{lev}(S)$.
3. An *infinite \mathbf{H} -tree* is a normal tree $T \subseteq \mathbf{T}$.

Definition 2.2 A *simplified universality parameter* \mathfrak{p} for \mathbf{H} is a pair $(\mathcal{G}^{\mathfrak{p}}, F^{\mathfrak{p}}) = (\mathcal{G}, F)$ such that

- (α) elements of \mathcal{G} are triples $(S, n_{\text{dn}}, n_{\text{up}})$, where S is a finite \mathbf{H} -tree, $n_{\text{dn}} \leq n_{\text{up}} \leq \text{lev}(S)$, $(\langle \rangle, 0, 0) \in \mathcal{G}$;
- (β) if $(S^0, n_{\text{dn}}^0, n_{\text{up}}^0) \in \mathcal{G}$, S^1 is a finite \mathbf{H} -tree, $\text{lev}(S^0) \leq \text{lev}(S^1)$, $S^1 \cap (\mathbf{T} \upharpoonright \text{lev}(S^0)) \subseteq S^0$, $n_{\text{dn}}^1 \leq n_{\text{dn}}^0$, and $n_{\text{up}}^0 \leq n_{\text{up}}^1 \leq \text{lev}(S^1)$, then $(S^1, n_{\text{dn}}^1, n_{\text{up}}^1) \in \mathcal{G}$;
- (γ) $F \in \omega^\omega$ is increasing;
- (δ) if

1. $(S^\ell, n_{\text{dn}}^\ell, n_{\text{up}}^\ell) \in \mathcal{G}$ (for $\ell < 2$), $\text{lev}(S^0) = \text{lev}(S^1)$,
2. S is a finite \mathbf{H} -tree, $\text{lev}(S) < \text{lev}(S^\ell)$, and $S^\ell \cap (\mathbf{T} \upharpoonright \text{lev}(S)) \subseteq S$ (for $\ell < 2$),
3. $\text{lev}(S) < n_{\text{dn}}^0, n_{\text{up}}^0 < n_{\text{dn}}^1, F(n_{\text{up}}^1) < \text{lev}(S^1)$,

then there is $(S^*, n_{\text{dn}}^*, n_{\text{up}}^*) \in \mathcal{G}$ such that

1. $n_{\text{dn}}^* = n_{\text{dn}}^0, n_{\text{up}}^* = F(n_{\text{up}}^1)$, $\text{lev}(S^*) = \text{lev}(S^0) = \text{lev}(S^1)$;
2. $S^0 \cup S^1 \subseteq S^*$ and $S^* \cap (\mathbf{T} \upharpoonright \text{lev}(S)) = S$.

Remark 2.3

1. The function F in Definition 2.2 is to allow more examples of universal parameters. In condition (δ) of Definition 2.2 it “creates room” for taking care of some “extra” nodes which we may be forced to take to S^* because S can be a *proper* superset of $(S^0 \cap (\mathbf{T} \upharpoonright \text{lev}(S))) \cup (S^1 \cap (\mathbf{T} \upharpoonright \text{lev}(S)))$.

2. Note that Definition 2.2(β) implies that \mathcal{G} is closed under “end extensions”.

Definition 2.4 Let $\mathfrak{p} = (\mathcal{G}, F)$ be a simplified universality parameter for \mathbf{H} . We say that an infinite \mathbf{H} -tree T is \mathfrak{p} -*narrow* if for infinitely many $n < \omega$, for some $n = n_{\text{dn}} < n_{\text{up}}$ we have $(T \cap (\mathbf{T} \upharpoonright (n_{\text{up}} + 1)), n_{\text{dn}}, n_{\text{up}}) \in \mathcal{G}$. The family of all \mathfrak{p} -narrow infinite \mathbf{H} -trees will be denoted by $\mathcal{T}^*(\mathfrak{p}, \mathbf{H})$.

Proposition 2.5 If \mathfrak{p} is a simplified universality parameter, then $\mathcal{T}^*(\mathfrak{p}, \mathbf{H})$ is an sw-closed family (of finitely branching normal trees). Consequently, $\mathbb{Q}^{T^*(\mathfrak{p}, \mathbf{H})}$ is an iterably sweet forcing notion.

Proof. It should be clear that $\mathcal{T}^*(\mathbf{p}, \mathbf{H})$ satisfies Definition 1.2, 1. and 2. The proof of Definition 1.2, 3. is, basically, included in the proof of [6, Proposition 4.2.5(3)]. \square

The examples of simplified universality parameters include the following.

Definition 2.6 (Compare [4, Definition 1.7, Example 1.9(2)]) Suppose that the function \mathbf{H} is increasing and $g \in \omega^\omega$ is such that $(\forall i \in \omega) (0 < g(i) < \mathbf{H}(i))$. Let $A \in [\omega]^\omega$. We define $\mathcal{G}_{\mathbf{H}}^{g,A}$ as the family consisting of $(\{\langle \rangle\}, 0, 0)$ and of all triples $(S, n_{\text{dn}}, n_{\text{up}})$ such that

- (α) S is a finite \mathbf{H} -tree, $n_{\text{dn}} \leq n_{\text{up}} \leq \text{lev}(S)$, $A \cap [n_{\text{dn}}, n_{\text{up}}] \neq \emptyset$;
- (β) for some sequence $\langle w_i : i \in A \cap [n_{\text{dn}}, n_{\text{up}}] \rangle$ such that $w_i \in [\mathbf{H}(i)]^{\leq g(i)}$ (for $i \in A \cap [n_{\text{dn}}, n_{\text{up}}]$) we have

$$(\forall \eta \in \max(S)) (\exists i \in A \cap [n_{\text{dn}}, n_{\text{up}}]) (\eta(i) \in w_i).$$

Proposition 2.7 Assume that \mathbf{H}, g, A are as in Definition 2.6, and $F \in \omega^\omega$ is an increasing function such that

$$(\forall n < \omega) (\prod_{i \leq n} \mathbf{H}(i) \leq |n, F(n) \cap A|).$$

Then $\mathbf{p}_{\mathbf{H}}^{g,A} \stackrel{\text{def}}{=} (\mathcal{G}_{\mathbf{H}}^{g,A}, F)$ is a simplified universality parameter (and it even is a regular universality parameter in the sense of [4, Definition 1.14]).

The universality parameters $\mathbf{p}_{\mathbf{H}}^{g,A}$ above are related to the strong PP-property (see [8, Chapter VI, 2.12*], compare also with [5, § 7.2]). Note that an infinite \mathbf{H} -tree T is $\mathbf{p}_{\mathbf{H}}^{g,A}$ -narrow if and only if there exist sequences $\bar{w} = \langle w_i : i \in A \rangle$ and $\bar{n} = \langle n_k : k < \omega \rangle$ such that

- 1. $(\forall i \in A) (w_i \subseteq \mathbf{H}(i) \ \& \ |w_i| \leq g(i))$;
- 2. $n_k < n_{k+1} < \omega$ for each $k < \omega$;
- 3. $(\forall \eta \in [T]) (\forall k < \omega) (\exists i \in A \cap [n_k, n_{k+1})) (\eta(i) \in w_i)$.

It should be clear that the intersection of a family of sw-closed sets of normal trees is sw-closed. So now we are going to look at the intersections of the families of $\mathbf{p}_{\mathbf{H}}^{g,A}$ -narrow trees.

Definition 2.8 Let \mathbf{H}, g be as in Definition 2.6 and let $\emptyset \neq \mathcal{B} \subseteq [\omega]^\omega$.

- 1. Put $\mathcal{T}(\mathcal{B}) = \mathcal{T}_{\mathbf{H}}^g(\mathcal{B}) \stackrel{\text{def}}{=} \bigcap \{ \mathcal{T}^*(\mathbf{p}_{\mathbf{H}}^{g,B}, \mathbf{H}) : B \in \mathcal{B} \}$ and $\mathbb{P}_{\mathcal{B}} = \mathbb{Q}^{\mathcal{T}(\mathcal{B})}$.
- 2. Let $\underline{T}_{\mathcal{B}}$ be a $\mathbb{P}_{\mathcal{B}}$ -name such that

$$\Vdash_{\mathbb{P}_{\mathcal{B}}} \underline{T}_{\mathcal{B}} = \bigcup \{ T^p : p \in \Gamma_{\mathbb{P}_{\mathcal{B}}} \}.$$

- 3. For a set $A \in [\omega]^\omega$ put

$$S_A = \{ \eta \in \mathbf{T} : (\forall i \in \text{lh}(\eta) \cap A) (\eta(i) = 0) \}.$$

Lemma 2.9 Suppose that \mathbf{H}, g are as in Definition 2.6.

- 1. Let $A, C \in [\omega]^\omega$. Then the tree S_A is $\mathbf{p}_{\mathbf{H}}^{g,C}$ -narrow if and only if $A \cap C$ is infinite.
- 2. Let $\emptyset \neq \mathcal{B} \subseteq [\omega]^\omega$. Then, in $\mathbf{V}^{\mathbb{P}_{\mathcal{B}}}$, $\underline{T}_{\mathcal{B}}$ is an infinite \mathbf{H} -tree which is $\mathbf{p}_{\mathbf{H}}^{g,B}$ -narrow for all $B \in \mathcal{B}$, and
 - (a) if $T \in \mathbf{V}$ is an infinite \mathbf{H} -tree which is $\mathbf{p}_{\mathbf{H}}^{g,B}$ -narrow for all $B \in \mathcal{B}$, then there is $n < \omega$ such that

$$(\forall \nu \in \underline{T}_{\mathcal{B}}) (\forall \eta \in T) (n = \text{lh}(\nu) < \text{lh}(\eta) \Rightarrow \nu \frown \eta \upharpoonright [n, \text{lh}(\eta)) \in \underline{T}_{\mathcal{B}});$$

- (b) if an infinite \mathbf{H} -tree $T \in \mathbf{V}$ is not $\mathbf{p}_{\mathbf{H}}^{g,B}$ -narrow for some $B \in \mathcal{B}$, then

$$(\forall n < \omega) (\exists \eta \in T) (\text{lh}(\eta) > n \ \& \ (\forall \nu \in \prod_{i < n} \mathbf{H}(i)) (\nu \frown \eta \upharpoonright [n, \text{lh}(\eta)) \notin \underline{T}_{\mathcal{B}}).$$

Theorem 2.10 Suppose that \mathbb{P} is a ccc forcing notion, $\Vdash_{\mathbb{P}} 2^{\aleph_0} = \kappa$, $\kappa < 2^{2^{\aleph_0}}$. Then there is a family $\mathcal{B} \subseteq [\omega]^\omega$ such that \mathbb{P} does not add a generic object for the (iterably sweet) forcing notion $\mathbb{P}_{\mathcal{B}}$.

Proof. Note that if \mathcal{U} is a uniform ultrafilter on ω , then $(\forall A, B \in \mathcal{U}) (|A \cap B| = \omega)$ and hence, by Lemma 2.9, 1., for every $A \in \mathcal{U}$ and every $B \in \mathcal{U}$, the tree S_A is $\mathfrak{p}_H^{g, B}$ -narrow. Also by Lemma 2.9, 1., for every $A \in [\omega]^\omega$ the tree S_A is not $\mathfrak{p}_H^{g, \omega \setminus A}$ -narrow.

Now, if $\mathcal{U}', \mathcal{U}'' \subseteq [\omega]^\omega$ are two distinct uniform ultrafilters on ω , then we may pick $A \in [\omega]^\omega$ such that $A \in \mathcal{U}'$ and $\omega \setminus A \in \mathcal{U}''$. Then the tree S_A is $\mathfrak{p}_H^{g, B}$ -narrow for every $B \in \mathcal{U}'$, but is not $\mathfrak{p}_H^{g, \omega \setminus A}$ -narrow, $\omega \setminus A \in \mathcal{U}''$. Therefore, by Lemma 2.9, 2., the interpretations of the names $\underline{T}_{\mathcal{U}'}, \underline{T}_{\mathcal{U}''}$ by the corresponding generic filters must be different. Since there are $2^{2^{\aleph_0}}$ ultrafilters on ω we easily get the conclusion. \square

Corollary 2.11 *There exists an iterably sweet forcing notion \mathbb{Q} which cannot be embedded into the forcing notion constructed in [7, § 7].*

Let us present now a different class of sw-closed families of normal trees and corresponding forcing notions.

Definition 2.12 The sw-closure $\text{cl}^{\text{sw}}(\mathcal{T})$ of a family \mathcal{T} of normal subtrees of $\omega^{<\omega}$ is the smallest family \mathcal{T}^* of subtrees of $\omega^{<\omega}$ which includes \mathcal{T} and is sw-closed.

Clearly, $\text{cl}^{\text{sw}}(\mathcal{T})$ is well defined for any family \mathcal{T} of normal subtrees of $\omega^{<\omega}$.

Lemma 2.13

1. Suppose that T^* is a normal subtree of $\omega^{<\omega}$ and let \mathcal{T}^* be the family of all normal subtrees of T^* . Then \mathcal{T}^* is sw-closed. Consequently, if $\mathcal{T} \subseteq \mathcal{T}^*$, then $\text{cl}^{\text{sw}}(\mathcal{T}) \subseteq \mathcal{T}^*$.

2. Assume that \mathcal{T} is an sw-closed family of normal subtrees of $\omega^{<\omega}$ and $A \subseteq \omega^\omega$ is a non-empty closed set. Let

$$\mathcal{T}^-(A) = \{T \in \mathcal{T} : [T] \cap A \text{ is nowhere dense in } A\}.$$

Then $\mathcal{T}^-(A)$ is sw-closed.

3. If \mathcal{T} is a family of normal subtrees of $\omega^{<\omega}$, $T \subseteq \omega^{<\omega}$ is a normal tree and

$$(\forall T' \in \mathcal{T}) ([T] \cap [T'] \text{ is nowhere dense in } [T]),$$

then $T \notin \text{cl}^{\text{sw}}(\mathcal{T})$.

Proof.

1. Should be clear.

2. Clearly $\mathcal{T}^-(A)$ is closed under finite unions. Assume now that $T_n, T_\omega \in \mathcal{T}^-(A)$ are such that

$$(\forall n < \omega) (T_\omega \cap \omega^{\leq n} = T_n \cap \omega^{\leq n})$$

and let $T = \bigcup_{n < \omega} T_n$. We want to show that $T \in \mathcal{T}^-(A)$. Since \mathcal{T} is sw-closed we see that $T \in \mathcal{T}$, so we need to show that $[T] \cap A$ is nowhere dense in A . To this end let $S \subseteq \omega^{<\omega}$ be a normal tree such that $A = [S]$ and suppose that $\nu \in S$. Since $T_\omega \in \mathcal{T}^-(A)$, we may find $\eta_0 \in S$ such that $\nu \triangleleft \eta_0$ and $\eta_0 \notin T_\omega$. Then, by our assumptions on $\langle T_n : n \leq \omega \rangle$, also for each $k \geq \text{lh}(\eta_0)$ we have $\eta_0 \notin T_k$. Since $T_n \in \mathcal{T}^-(A)$ (for $n < \text{lh}(\eta_0)$), the set $\bigcup_{n < \text{lh}(\eta_0)} [T_n] \cap A$ is nowhere dense in A and hence we may find $\eta \in S$ such that $\eta_0 \triangleleft \eta$ and $\eta \notin \bigcup_{n < \text{lh}(\eta_0)} T_n$. Then we also have $\nu \triangleleft \eta \in S$ and $\eta \notin T$.

3. Follows from 2. \square

Definition 2.14

1. For a set $A \in [\omega]^\omega$ let \mathcal{T}^A be the collection of all normal subtrees T of $2^{<\omega}$ such that

$$(\forall \nu \in \text{split}(T)) (\text{lh}(\nu) \in A).$$

2. For a family $\mathcal{A} \subseteq [\omega]^\omega$ let $\mathcal{T}_{\mathcal{A}} = \text{cl}^{\text{sw}}(\bigcup \{\mathcal{T}^A : A \in \mathcal{A}\})$.

Theorem 2.15 *Suppose that \mathbb{P} is a ccc forcing notion, $\Vdash_{\mathbb{P}} 2^{\aleph_0} = \kappa$, $\kappa < 2^{2^{\aleph_0}}$. Then there is a family $\mathcal{A} \subseteq [\omega]^\omega$ such that \mathbb{P} does not add the generic object for the (iterably sweet) forcing notion $\mathbb{Q}^{\mathcal{T}_{\mathcal{A}}}$ (see Definition 1.3).*

Proof. Let us start with some observations of a more general character.

Claim 2.15.1 Assume that \mathcal{T} is an sw-closed family of subtrees of $2^{<\omega}$ such that

(*) for every $\eta \in 2^\omega$ and $T \in \mathcal{T}$ we have $T \dashv_2 \eta \stackrel{\text{def}}{=} \{\nu \in 2^{<\omega} : \nu \dashv_2 (\eta \upharpoonright \text{lh}(\nu)) \in T\} \in \mathcal{T}$.

Let $\mathbb{T}^{\mathcal{T}}$ be a $\mathbb{Q}^{\mathcal{T}}$ -name such that

$$\Vdash_{\mathbb{Q}^{\mathcal{T}}} \mathbb{T}^{\mathcal{T}} = \bigcup \{T^p : p \in \Gamma_{\mathbb{Q}^{\mathcal{T}}}\}.$$

Then, in $\mathbf{V}^{\mathbb{Q}^{\mathcal{T}}}$, $\mathbb{T}^{\mathcal{T}}$ is a subtree of $2^{<\omega}$ such that

1. for every $T \in \mathcal{T}$ there is $n < \omega$ such that if $\nu_1 \in \mathbb{T}^{\mathcal{T}} \cap 2^n$ and $\eta \in T$, $\text{lh}(\eta) > n$, then

$$\nu_1 \frown \eta \upharpoonright [n, \text{lh}(\eta)) \in \mathbb{T}^{\mathcal{T}};$$

2. for every normal tree $T \subseteq 2^{<\omega}$ such that $T \notin \mathcal{T}$, $T \in \mathbf{V}$, we have

$$(\forall n < \omega) (\exists \eta \in T) (\text{lh}(\eta) > n \ \& \ (\forall \nu \in 2^n) (\nu \frown \eta \upharpoonright [n, \text{lh}(\eta)) \notin \mathbb{T}^{\mathcal{T}}).$$

Proof.

1. Suppose that $p \in \mathbb{Q}^{\mathcal{T}}$ and $T \in \mathcal{T}$. Let $\langle \eta_\ell : \ell < 2^{N^p} \rangle$ list all elements of 2^ω which are constantly zero on $[N^p, \omega)$. It follows from our assumption (*) that

$$(\forall \ell < 2^{N^p}) (T \dashv_2 \eta_\ell \in \mathcal{T} \ \& \ T^p \dashv_2 \eta_\ell \in \mathcal{T}).$$

Since \mathcal{T} is sw-closed we may now conclude that (by Definition 1.2, 2.)

$$T_0 \stackrel{\text{def}}{=} \bigcup_{\ell < 2^{N^p}} (T \dashv_2 \eta_\ell) \cup \bigcup_{\ell < 2^{N^p}} (T^p \dashv_2 \eta_\ell) \in \mathcal{T},$$

and hence also (by Definition 1.2, 1.)

$$T_1 \stackrel{\text{def}}{=} \{\eta \in T_0 : (\text{lh}(\eta) \leq N^p \ \& \ \eta \in T^p) \vee (\text{lh}(\eta) > N^p \ \& \ \eta \upharpoonright N^p \in T^p)\} \in \mathcal{T}.$$

Now, letting $N^q = N^p$ and $T^q = T_1$ we get a condition $q \in \mathbb{Q}^{\mathcal{T}}$ stronger than p and such that

$$q \Vdash (\forall \nu_1 \in \mathbb{T}^{\mathcal{T}} \cap 2^{N^q}) (\forall \eta \in T) (\text{lh}(\eta) > N^q \Rightarrow \nu_1 \frown \eta \upharpoonright [N^q, \text{lh}(\eta)) \in \mathbb{T}^{\mathcal{T}}).$$

2. Now suppose that $p \in \mathbb{Q}^{\mathcal{T}}$, $n < \omega$ and $T \subseteq 2^{<\omega}$ is a normal tree which does not belong to \mathcal{T} . Let $N = N^p + n$ and let $\langle \eta_\ell : \ell < 2^N \rangle$ list all elements of 2^ω which are constantly zero on $[N, \omega)$. From (*) follows that $T_0 \stackrel{\text{def}}{=} \bigcup_{\ell < 2^N} (T^p \dashv_2 \eta_\ell) \in \mathcal{T}$ and since $T \notin \mathcal{T}$ we may conclude by Definition 1.2, 1. that $T \setminus T_0 \neq \emptyset$. Pick $\eta \in T \setminus T_0$ and note that necessarily $\text{lh}(\eta) > N \geq n$. Letting $N^q = \text{lh}(\eta)$ and $T^q = T_0$ we get a condition $q \in \mathbb{Q}^{\mathcal{T}}$ stronger than p and such that

$$q \Vdash (\forall \nu \in 2^n) (\nu \frown \eta \upharpoonright [n, \text{lh}(\eta)) \notin \mathbb{T}^{\mathcal{T}}). \quad \square \text{ (Claim 2.15.1)}$$

Claim 2.15.2 If \mathcal{T} is a collection of normal subtrees of $2^{<\omega}$ such that the demand in Claim 2.15.1(*) holds for \mathcal{T} , then also $\text{cl}^{\text{sw}}(\mathcal{T})$ satisfies this condition. Consequently, for each $\mathcal{A} \subseteq [\omega]^\omega$, (*) of Claim 2.15.1 holds true for $\mathcal{T}_{\mathcal{A}}$.

Proof. Should be clear. □ (Claim 2.15.2)

Claim 2.15.3 Suppose that $A \in [\omega]^\omega$ and $\mathcal{A} \subseteq [\omega]^\omega$ are such that

$$(\forall B \in \mathcal{A}) (|A \setminus B| = \omega).$$

Then $\mathcal{T}^A \not\subseteq \mathcal{T}_{\mathcal{A}}$.

Proof. Let $T = \{\nu \in 2^{<\omega} : (\forall n < \text{lh}(\nu)) (\nu(n) = 1 \Rightarrow n \in A)\}$. Plainly $T \in \mathcal{T}^A$. Also, for each $B \in \mathcal{A}$ and $T' \in \mathcal{T}^B$ the set $[T] \cap [T']$ is nowhere dense in $[T]$, so by Lemma 2.13, 3.,

$$T \notin \text{cl}^{\text{sw}}(\bigcup\{\mathcal{T}^B : B \in \mathcal{A}\}) = \mathcal{T}_A. \quad \square \text{ (Claim 2.15.3)}$$

Now choose a family $\mathcal{I} \subseteq [\omega]^\omega$ of almost disjoint sets, $|\mathcal{I}| = 2^{\aleph_0}$.

Suppose that $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}$, $\mathcal{A} \neq \mathcal{B}$, say $A \in \mathcal{A} \setminus \mathcal{B}$. Then $(\forall B \in \mathcal{B}) (|A \setminus B| = \omega)$ and hence (by Claim 2.15.3) we get $\mathcal{T}^A \not\subseteq \mathcal{T}_B$, so we have a normal tree $T \in \mathcal{T}_A \setminus \mathcal{T}_B$. Now look at Claim 2.15.1 – by Claim 2.15.2 it is applicable to $\mathbb{Q}^{\mathcal{T}_A}$, $\mathbb{Q}^{\mathcal{T}_B}$ and we get from it that if $T_A, T_B \subseteq 2^{<\omega}$ are trees generic over \mathbf{V} for $\mathbb{Q}^{\mathcal{T}_A}$, $\mathbb{Q}^{\mathcal{T}_B}$, respectively, then

1. $(\exists n < \omega) (\forall \nu \in T_A \cap 2^n) (\forall \eta \in T) (\text{lh}(\eta) > n \Rightarrow \nu \frown \eta \upharpoonright [n, \text{lh}(\eta)) \in T_A)$,
2. $(\forall n < \omega) (\exists \eta \in T) (\text{lh}(\eta) > n \ \& \ (\forall \nu \in 2^n) (\nu \frown \eta \upharpoonright [n, \text{lh}(\eta)) \notin T_B)$.

Hence $T_A \neq T_B$. Since \mathbb{P} satisfies the ccc and $\Vdash_{\mathbb{P}} 2^{\aleph_0} = \kappa$ and $\kappa < 2^{2^{\aleph_0}}$, we may find a family \mathcal{F} of subsets of \mathcal{I} such that $|\mathcal{F}| = \kappa$ and

$$\Vdash_{\mathbb{P}} \text{“for no } \mathcal{A} \subseteq \mathcal{I} \text{ with } \mathcal{A} \notin \mathcal{F}, \text{ there is a } \mathbb{Q}^{\mathcal{T}_A}\text{-generic filter over } \mathbf{V}\text{”}. \quad \square$$

One should note that the examples of sweet forcing notions which cannot be embedded into the one constructed in [7, §7] which we gave in this section are not very nice – it may well be that the parameters \mathcal{A}, \mathcal{B} needed to define them are not definable from a real. Even the candidate for a somewhat definable example from the previous section, the forcing notion $\mathbb{Q}^{\mathcal{T}^{\text{sc}}}$, is not Souslin. Thus the following variant of [6, Problem 5.5] may be of interest.

Problem 2.16 *Is there a Souslin ccc iterably sweet forcing notion \mathbb{Q} such that no finite composition of the Universal Meager forcing notion adds a \mathbb{Q} -generic real? Such that the forcing of [7, § 7] does not add \mathbb{Q} -generic real?*

3 Subforcings, quotients and likes

Topological sweetness, as defined in Definition 0.2, is the property of a particular representation of a forcing notion. It is only natural to ask if a forcing notion having a topologically sweet dense subforcing is topologically sweet, or, in general, if a forcing notion equivalent to a topologically sweet one is topologically sweet. We start this section with some results in these directions.

Definition 3.1 We say that a forcing notion \mathbb{P} has a GLB-property provided that for every $q_0, q_1 \in \mathbb{P}$, there is $q \in \mathbb{P}$ such that

- (α) $q \leq q_i$ for $i = 0, 1$;
- (β) if $q^* \in \mathbb{P}$ satisfies $(\forall i < 2) (q^* \leq q_i)$, then $q^* \leq q$.

Remark 3.2

1. GLB stands for “Greatest Lower Bound”. Note the direction of the inequality on the definition of GLB and remember that $p \leq q$ means “ q is stronger than p ”.

2. If \mathbb{B} is a Boolean algebra, then \mathbb{B}^+ is a forcing notion with the GLB-property. Also the forcing notions \mathbb{R} and \mathbb{A} defined in Definition 3.8 later have this property.

Proposition 3.3 *Suppose that a forcing notion \mathbb{P} has the GLB-property and $\mathbb{Q} \subseteq \mathbb{P}$ is its dense subforcing. If \mathbb{Q} is topologically sweet, then so is \mathbb{P} .*

Proof. Let $(\mathbb{Q}, \mathcal{B})$ be a model of topological sweetness and let τ be the topology on \mathbb{Q} generated by \mathcal{B} . For sets $U_0, \dots, U_k \in \mathcal{B}$, $k < \omega$, define

$$W(U_0, \dots, U_k) = \{p \in \mathbb{P} : (\forall i \leq k) (\exists q \in U_i) (p \leq q)\},$$

and let

$$\mathcal{B}^* = \{W(U_0, \dots, U_k) : k < \omega \ \& \ U_0, \dots, U_k \in \mathcal{B}\} \cup \{\{\emptyset_{\mathbb{P}}\}\}.$$

It should be clear that

1. \mathcal{B}^* is closed under finite intersections;
2. it is a countable basis of a topology τ^* on \mathbb{P} ;
3. $\emptyset_{\mathbb{P}}$ is an isolated point in τ^* .

We are going to show that the topology τ^* satisfies the demand of Definition 0.2(ii). So suppose that a sequence $\bar{p} = \langle p_n : n < \omega \rangle \subseteq \mathbb{P}$ is τ^* -converging to $p \in \mathbb{P}$ and $q \geq p$ and W is a τ^* -neighbourhood of q . Pick $U_0, \dots, U_k \in \mathcal{B}$ such that $q \in W(U_0, \dots, U_k) \subseteq W$ and let $q_i \in U_i$ (for $i \leq k$) be such that $q \leq q_i$. Furthermore, for $i \leq k$, let $\{V_n^i : n < \omega\}$ be a basis of τ -neighbourhoods of $q_i \in \mathbb{Q}$ such that

$$(\forall n_0 < n_1 < \omega) (q_i \in V_{n_1}^i \subseteq V_{n_0}^i \subseteq U_i).$$

Since $p \in W(V_n^0, V_n^1, \dots, V_n^k) \in \mathcal{B}^*$ (for each $n < \omega$) and the sequence \bar{p} τ^* -converges to p , we may choose an increasing sequence $\langle m_n : n < \omega \rangle \subseteq \omega$ such that $(\forall n < \omega) (p_{m_n} \in W(V_n^0, V_n^1, \dots, V_n^k))$. Then we may also pick $p_{n,i}^*$ (for $n < \omega, i \leq k$) such that $p_{m_n} \leq p_{n,i}^* \in V_n^i$. Fix $i \leq k$ and look at the sequence $\bar{p}_i^* = \langle p_{n,i}^* : n < \omega \rangle$: clearly it τ -converges to q_i . Consequently, we may easily choose (by repeated application of Definition 0.2(ii) for τ) conditions $q_i^* \in \mathbb{Q}$ such that

1. $q_i \leq q_i^* \in U_i$ for $i \leq k$;
2. $(\exists^\infty n < \omega) (\forall i \leq k) (p_{n,i}^* \leq q_i^*)$.

Since \mathbb{P} has the GLB-property we may pick $q^* \in \mathbb{P}$ such that

- (α) $q^* \leq q_i$ for $i \leq k$;
- (β) if $r \in \mathbb{P}$ is weaker than q_0^*, \dots, q_k^* , then $r \leq q^*$.

Then, plainly, $q^* \in W(U_0, \dots, U_k)$ and $q \leq q^*$ and $(\exists^\infty n < \omega) (p_{m_n} \leq q^*)$. \square

Proposition 3.4 *Assume that \mathbb{P} is a topologically sweet forcing notion. Then there is a model $(\mathbb{P}, \mathcal{B}^*)$ of topological sweetness such that all members of \mathcal{B}^* are downward closed.*

Proof. Let $(\mathbb{P}, \mathcal{B})$ be a model of topological sweetness. For $U \in \mathcal{B}$ put

$$W(U) = \{p \in \mathbb{P} : (\exists q \in U) (p \leq q)\},$$

and let $\mathcal{B}^* = \{W(U) : U \in \mathcal{B}\}$. Note that if $p \in W(U_0) \cap W(U_1)$ and $p \leq p_0 \in U_0, p \leq p_1 \in U_1$, then there is $V \in \mathcal{B}$ such that $p \in V$ and $V \subseteq W(U_0) \cap W(U_1)$ (remember Lemma 0.3, 1.). Hence we easily conclude that \mathcal{B}^* is a base of a topology τ^* on \mathbb{P} . Similarly as in Proposition 3.3 one shows that $(\mathbb{P}, \mathcal{B}^*)$ is a model of topological sweetness. \square

Proposition 3.5 *Assume that \mathbb{P} is a topologically sweet and separative partial order. Suppose also that \mathbb{B}_0 is a complete subalgebra of the complete Boolean algebra $\mathbf{BA}(\mathbb{P})$ determined by \mathbb{P} . Then \mathbb{B}_0^+ (as a forcing notion) is topologically sweet.*

Proof. It follows from our assumptions on \mathbb{P} that it is (isomorphic to) a dense subset of $\mathbf{BA}(\mathbb{P})^+$ and hence, by Propositions 3.3 and 3.4, there is a model $(\mathbf{BA}(\mathbb{P})^+, \mathcal{B})$ of topological sweetness such that all members of \mathcal{B} are downward closed. By our assumptions $\mathbb{B}_0 \triangleleft \mathbf{BA}(\mathbb{P})$, so let $\pi : \mathbf{BA}(\mathbb{P}) \rightarrow \mathbb{B}_0$ be the projection. We will use the symbol \leq to refer to the forcing order of \mathbb{B}_0^+ (which is the reverse Boolean order). Put

$$\mathcal{B}' = \{U \cap \mathbb{B}_0^+ : U \in \mathcal{B}\}.$$

We claim that $(\mathbb{B}_0^+, \mathcal{B}')$ is a model of topological sweetness. It is easy to verify Definition 0.2(i), so let us only argue that Definition 0.2(ii) holds true. To this end suppose that a sequence $\bar{p} = \langle p_n : n < \omega \rangle \subseteq \mathbb{B}_0^+$ converges to $p \in \mathbb{B}_0^+$ (in the topology generated by \mathcal{B}') and let $p \leq q \in U \cap \mathbb{B}_0^+, U \in \mathcal{B}$. Then also \bar{p} converges to p in the topology generated by \mathcal{B} on $\mathbf{BA}(\mathbb{P})^+$, so we may find $r \in \mathbf{BA}(\mathbb{P})^+$ such that $q \leq r \in U$ and

$$(\exists^\infty n < \omega) (p_n \leq r).$$

Let $r^* = \pi(r) \in \mathbb{B}_0$. Then we have

1. $q \leq r^*$ (as π is the projection and $q \in \mathbb{B}_0^+$, $q \leq r$);
2. $(\exists^\infty n < \omega) (p_n \leq r^*)$ (as π is the projection and $p_n \in \mathbb{B}_0^+$);
3. $r^* \in U$ (as U is downward closed, $r^* \leq r \in U$). □

Sweetness and topological sweetness are important properties because they are preserved in amalgamations of forcing notions. Since the amalgamation can be represented as the composition with the product of two quotients (see, e. g., [3] on that), one may ask if sweetness is also preserved in quotients.

Definition 3.6 Let \mathbb{P}, \mathbb{Q} be forcing notions and suppose that $\mathbb{Q} \triangleleft \mathbb{P}$. The quotient $(\mathbb{P} : \mathbb{Q})$ is the \mathbb{Q} -name for the subforcing of \mathbb{P} consisting of all $p \in \mathbb{P}$ such that p is compatible (in \mathbb{P}) with all members of $\Gamma_{\mathbb{Q}}$. Thus for $p \in \mathbb{P}$ and $q \in \mathbb{Q}$,

$$q \Vdash_{\mathbb{Q}} p \in (\mathbb{P} : \mathbb{Q}) \quad \text{if and only if} \quad (\forall r \in \mathbb{Q}) (q \leq r \Rightarrow r, p \text{ are compatible in } \mathbb{P}).$$

Theorem 3.7 Let \mathbb{C} be the standard Cohen forcing notion (so it is a countable atomless partial order). Suppose that $(\mathbb{P}, \mathcal{B})$ is a model of topological sweetness and $\mathbb{C} \triangleleft \mathbb{P}$. Let $\mathcal{B}^{\mathbb{C}}$ be the \mathbb{C} -name for the family $\{U \cap (\mathbb{P} : \mathbb{C}) : U \in \mathcal{B}\}$. Then

$$\Vdash_{\mathbb{C}} \text{“}((\mathbb{P} : \mathbb{C}), \mathcal{B}^{\mathbb{C}}) \text{ is a model of topological sweetness”}.$$

Proof. First note that, in $\mathbb{V}^{\mathbb{C}}$, $\mathcal{B}^{\mathbb{C}}$ is a countable basis of a topology on $(\mathbb{P} : \mathbb{C})$, and $\emptyset_{(\mathbb{P} : \mathbb{C})} = \emptyset_{\mathbb{P}}$ is an isolated point in this topology. Thus the only thing that we should verify is the demand in Definition 0.2(ii).

Suppose that $\eta \in \mathbb{C}$ and \mathbb{C} -names $\langle p_i : i < \omega \rangle$, p, q and W are such that

$$\eta \Vdash_{\mathbb{C}} \text{“} \langle p_i, p, q \in (\mathbb{P} : \mathbb{C}), W \in \mathcal{B}^{\mathbb{C}}, p \leq q \in W \text{ and the sequence } \langle p_i : i < \omega \rangle \text{ converges to } p \text{ in the topology generated by } \mathcal{B}^{\mathbb{C}} \text{”}.$$

Replacing η by a stronger condition in \mathbb{C} (if necessary), we may assume that for some $p, q \in \mathbb{P}$ and $W \in \mathcal{B}$ we have

$$\eta \Vdash_{\mathbb{C}} p = p \& q = q \& W = W \cap (\mathbb{P} : \mathbb{C}).$$

Then also $\eta \Vdash_{\mathbb{C}} p, q \in (\mathbb{P} : \mathbb{C})$ and $p \leq q \in W$. Let us choose a condition $q^+ \in \mathbb{P}$ which is stronger than both q and η , and let $U \in \mathcal{B}$ be a neighborhood of q^+ such that any two members of U are compatible in \mathbb{P} (remember Lemma 0.3, 2.). Next, choose $W^+ \in \mathcal{B}$ such that $q \in W^+ \subseteq W$ and every member of W^+ has an upper bound in U (possible by Lemma 0.3, 1.).

Pick $V_i \in \mathcal{B}$ (for $i < \omega$) such that $\{V_i : i < \omega\}$ forms a neighbourhood basis at p (for the topology generated by \mathcal{B}) such that for each $i < \omega$:

$$(\alpha) \quad p \in V_{i+1} \subseteq V_i;$$

$$(\beta) \quad \text{any } i + 1 \text{ elements of } V_{i+1} \text{ have a common upper bound in } V_i.$$

(The choice is clearly possible; remember Lemma 0.3.)

Clearly

$$\eta \Vdash_{\mathbb{C}} \text{“}\{V_i \cap (\mathbb{P} : \mathbb{C}) : i < \omega\} \text{ forms a neighbourhood basis at } p \text{ (for the topology generated by } \mathcal{B}^{\mathbb{C}}\text{)”}.$$

Hence, without loss of generality, we may assume that $\eta \Vdash_{\mathbb{C}} p_i \in V_i$ (as we may change the names p_i reflecting a passage to a subsequence). Let us fix a list $\{\nu_\ell : \ell < \omega\}$ of all conditions in \mathbb{C} stronger than η , and for every $i, \ell < \omega$ let us pick $p_{i,\ell} \in \mathbb{P}$ such that $\nu_\ell \Vdash_{\mathbb{C}} p_i \neq p_{i,\ell}$. Note that then $p_{i,\ell} \in V_i$, so by clause (β) above we may choose $p_i^* \in V_i$ such that for each $i > 0$ we have

$$(\forall \ell \leq i) (p_{i+1,\ell} \leq p_i^*).$$

The sequence $\langle p_i^* : i < \omega \rangle$ converges to p so (by Definition 0.2(ii) for $(\mathbb{P}, \mathcal{B})$) there are a condition $r \in \mathbb{P}$ and an infinite set $A \subseteq \omega$ such that

$$r \in W^+ \quad \text{and} \quad q \leq r \quad \text{and} \quad (\forall i \in A) (p_i^* \leq r).$$

By the choice of W^+ , the condition r has an upper bound in U and hence (by the choice of U), r, q^+ are compatible in \mathbb{P} . Therefore, as q^+ is stronger than η (in \mathbb{P}), there is $\nu \in \mathbb{C}$ stronger than η such that $\nu \Vdash_{\mathbb{C}} r \in (\mathbb{P} : \mathbb{C})$.

Now the proof follows from the following claim.

Claim 3.7.1 $\nu \Vdash_{\mathbb{C}} (\exists^\infty i < \omega) (p_i \leq r)$.

Proof. If not, then we may find $\nu' \in \mathbb{C}$ stronger than ν and $i' < \omega$ such that $\nu' \Vdash_{\mathbb{C}} (\forall i \geq i') (p_i \not\leq r)$. Let $\ell < \omega$ be such that $\nu' = \nu_\ell$ and let $i \in A$ be larger than $\ell + i' + 1$. Look at our choices before – we know that:

- (i) $p_i^* \leq r$,
- (ii) $p_{i+1, \ell} \leq p_i^*$,
- (iii) $\nu_\ell \not\Vdash_{\mathbb{C}} p_{i+1} \neq p_{i+1, \ell}$.

Therefore some condition $\nu^* \in \mathbb{C}$ stronger than ν_ℓ forces that $p_{i+1} \leq r$, contradicting the choice of $\nu' = \nu_\ell$ (as $i + 1 > i'$). □ (Claim 3.7.1)

□

In the rest of this section we are going to show that the result of Theorem 3.7 cannot be very much improved: when taking a quotient over a random real forcing we may lose topological sweetness. Let us start with recalling some notation and definitions, which we will need later.

Definition 3.8

1. The Lebesgue (product) measure on 2^ω is denoted by μ^{Leb} , $\mathbf{Borel}(2^\omega)$ is the σ -field of Borel subsets of 2^ω and \mathbb{L} is the σ -ideal of Lebesgue null subsets of 2^ω . The quotient Boolean algebra $\mathbb{B} = \mathbf{Borel}(2^\omega)/\mathbb{L}$ is called *the random algebra* (it is a complete ccc Boolean algebra).

2. The random forcing notion \mathbb{R} is defined as follows: a *condition in* \mathbb{R} is a closed subset of 2^ω of positive Lebesgue measure, *the order of* \mathbb{R} is the reverse inclusion.

3. The amoeba for measure forcing notion \mathbb{A} is defined as follows: a *condition in* \mathbb{A} is a closed subset F of 2^ω such that $\mu^{\text{Leb}}(F) > \frac{1}{2}$, *the order of* \mathbb{A} is the reverse inclusion.

Of course, $\mathbb{B} = \mathbf{BA}(\mathbb{R})$. Let us also recall that both \mathbb{R} and \mathbb{A} are topologically sweet (see [10, 1.3.3]).

Proposition 3.9

- 1. $\Vdash_{\mathbb{B}} \text{“}\mathbb{A}^{\mathbf{V}} \text{ is not topologically sweet”}$.
- 2. $\Vdash_{\mathbb{B}} \text{“}\mathbb{R}^{\mathbf{V}} \text{ is not topologically sweet”}$.

Proof.

- 1. Suppose towards contradiction that

$$\llbracket \text{there is a model of topological sweetness based on } \mathbb{A}^{\mathbf{V}} \rrbracket_{\mathbb{B}} \neq \mathbf{0}_{\mathbb{B}}.$$

Since the random algebra is homogeneous, we may assume that we have \mathbb{B} -names \underline{U}_n for subsets of $\mathbb{A}^{\mathbf{V}}$ such that

$$(*)_0 \quad \Vdash_{\mathbb{B}} \text{“}(\mathbb{A}^{\mathbf{V}}, \{U_n : n < \omega\}) \text{ is a model of topological sweetness”}.$$

For $i < \omega$ let $m_i = \lfloor -\frac{i \cdot 2^i}{\log_2(1 - 2^{-2^{i+1}})} \rfloor + 2$, so $\frac{m_i}{2^i} > \frac{-i}{\log_2(1 - 2^{-2^{i+1}})}$ and thus

$$(*)_1^i \quad (1 - 2^{-2^{i+1}})^{m_i/2^i} < 2^{-i}.$$

Let μ be the product Lebesgue measure on the space $\prod_{i < \omega} m_i$, and let μ^* be the corresponding outer measure.

Define $\langle n_i : i < \omega \rangle$ by $n_0 = 0$, $n_{i+1} = n_i + m_i \cdot 2^{i+1}$, and for $i < \omega$, $j < m_i$ put

$$t_j^i \stackrel{\text{def}}{=} \{\sigma \in 2^{[n_i, n_{i+1})} : (\exists \ell < 2^{i+1}) (\sigma(n_i + j \cdot 2^{i+1} + \ell) = 1)\}.$$

Note that

$$(*)_2^i \quad \text{if } j_0 < j_1 < \dots < j_k < m_i, \text{ then } |t_{j_0}^i \cap t_{j_1}^i \cap \dots \cap t_{j_k}^i| = (1 - 2^{-2^{i+1}})^{k+1} \cdot 2^{m_i \cdot 2^{i+1}}.$$

For $x \in \prod_{i < \omega} m_i$ let

$$Z_x \stackrel{\text{def}}{=} \{\eta \in 2^\omega : (\forall i < \omega) (\eta \upharpoonright [n_i, n_{i+1}) \in t_{x(i)}^i)\}$$

and note that Z_x is a closed set and $\mu^{\text{Leb}}(Z_x) > \frac{1}{2}$, so $Z_x \in \mathbb{A}$. For each $x \in \prod_{i < \omega} m_i$ and $n < \omega$ we may pick a Borel set $B(x, n) \subseteq 2^\omega$ such that $\llbracket Z_x \in \mathcal{U}_n \rrbracket_{\mathbb{B}} = \llbracket B(x, n) \rrbracket_{\mathbb{L}}$. Next, for each $k < \omega$ (and $x \in \prod_{i < \omega} m_i$, $n < \omega$) choose a clopen set $C(x, n, k) \subseteq 2^\omega$ such that $\mu^{\text{Leb}}(B(x, n) \Delta C(x, n, k)) < 2^{-k}$. Now, for $n < \omega$, consider the binary relation \sim_n on $\prod_{i < \omega} m_i$ given by

$$x \sim_n y \quad \text{if and only if} \quad (\forall k, \ell \leq n) (C(x, \ell, k) = C(y, \ell, k)).$$

It should be clear that (for each $n < \omega$) \sim_n is an equivalence relation on $\prod_{i < \omega} m_i$ such that

$$(*)_3^n \quad x \sim_{n+1} y \Rightarrow x \sim_n y \quad (\text{for each } x, y \in \prod_{i < \omega} m_i),$$

$$(*)_4^n \quad \prod_{i < \omega} m_i / \sim_n \text{ is countable.}$$

Consequently we may pick $x^* \in \prod_{i < \omega} m_i$ such that for each $n < \omega$ we have

$$\lim_{\ell \rightarrow \infty} \frac{\mu^*(\{x \in \prod_{i < \omega} m_i : x \upharpoonright \ell = x^* \upharpoonright \ell \ \& \ x \sim_n x^*\})}{\mu(\{x \in \prod_{i < \omega} m_i : x \upharpoonright \ell = x^* \upharpoonright \ell\})} = 1.$$

So now we may choose an increasing sequence $\langle \ell_i : i < \omega \rangle \subseteq \omega$ such that for $i < \omega$ we have

$$\mu^*(\{x \in \prod_{j < \omega} m_j : x \upharpoonright \ell_i = x^* \upharpoonright \ell_i \ \& \ x \sim_i x^*\}) > \frac{1}{2} \mu(\{x \in \prod_{j < \omega} m_j : x \upharpoonright \ell_i = x^* \upharpoonright \ell_i\}),$$

and then for each $i < \omega$ we may choose $v_i \subseteq m_{\ell_i}$ and $\langle y_k^i : k \in v_i \rangle \subseteq \prod_{j < \omega} m_j$ such that

$$(*)_5^i \quad |v_i| > \frac{1}{2} m_{\ell_i},$$

$$(*)_6^i \quad y_k^i \upharpoonright \ell_i = x^* \upharpoonright \ell_i, y_k^i(\ell_i) = k \text{ and } y_k^i \sim_i x^* \text{ for } k \in v_i.$$

It follows from the definition of the relations \sim_n and from $(*)_6^i$ that for each $k \in v_i$ and all $\ell \leq i$ we have

$$\mu^{\text{Leb}}(B(x^*, \ell) \Delta B(y_k^i, \ell)) < 2^{1-i}.$$

Thus, for each $i < \omega$, we may pick a partition $\langle B_k^i : k \in v_i \rangle$ of 2^ω into disjoint Borel sets such that for all $k \in v_i$ we have

$$(*)_7^{i,k} \quad \mu^{\text{Leb}}(B_k^i) = \frac{1}{|v_i|},$$

$$(*)_8^{i,k} \quad \mu^{\text{Leb}}(B_k^i \cap (B(x^*, \ell) \Delta B(y_k^i, \ell))) < \frac{2^{1-i}}{|v_i|} \quad \text{for all } \ell \leq i.$$

Let x_i be a \mathbb{B} -name for a member of $\mathbf{V} \cap \prod_{j < \omega} m_j$ such that

$$(\forall k \in v_i) (\llbracket x_i = y_k^i \rrbracket_{\mathbb{B}} = \llbracket B_k^i \rrbracket_{\mathbb{L}}).$$

Claim 3.9.1 $\Vdash_{\mathbb{B}} (\forall n < \omega) (\forall^\infty i < \omega) (Z_{x^*} \in \mathcal{U}_n \Rightarrow Z_{x_i} \in \mathcal{U}_n)$.

Proof. Note that for $n, i < \omega$ we have

$$\llbracket Z_{x_i} \notin \mathcal{U}_n \rrbracket_{\mathbb{B}} = \llbracket \bigcup_{k \in v_i} B_k^i \setminus B(y_k^i, n) \rrbracket_{\mathbb{L}},$$

and thus $\llbracket Z_{x^*} \in \mathcal{U}_n \ \& \ Z_{x_i} \notin \mathcal{U}_n \rrbracket_{\mathbb{B}} = \llbracket \bigcup_{k \in v_i} (B(x^*, n) \setminus B(y_k^i, n)) \cap B_k^i \rrbracket_{\mathbb{L}}$. It follows from $(*)_8^{i,k}$ that we have (for $n \leq i < \omega$)

$$\mu^{\text{Leb}}(\bigcup_{k \in v_i} (B(x^*, n) \setminus B(y_k^i, n)) \cap B_k^i) < 2^{1-i}.$$

Hence for, each $n < \omega$,

$$\mu^{\text{Leb}}(\bigcap_{m < \omega} \bigcup_{i > m} (\bigcup_{k \in v_i} (B(x^*, n) \setminus B(y_k^i, n)) \cap B_k^i)) = 0,$$

so

$$\llbracket (\exists^\infty i < \omega) (Z_{x^*} \in \underline{U}_n \ \& \ Z_{x_i} \notin \underline{U}_n) \rrbracket_{\mathbb{B}} = \mathbf{0}_{\mathbb{B}},$$

and the claim follows. □ (Claim 3.9.1)

It follows from $(*)_0$ and Claim 3.9.1 that

$$\llbracket (\exists F \in \mathbb{A}^{\mathbf{V}}) (F \subseteq Z_{x^*} \ \& \ (\exists^\infty i < \omega) (F \subseteq Z_{x_i})) \rrbracket_{\mathbb{B}} = \mathbf{1}_{\mathbb{B}},$$

and therefore we may find $F \in \mathbb{A} \cap \mathbf{V}$ such that $F \subseteq Z_{x^*}$ and $a \stackrel{\text{def}}{=} \llbracket (\exists^\infty i < \omega) (F \subseteq Z_{x_i}) \rrbracket_{\mathbb{B}} \neq \mathbf{0}_{\mathbb{B}}$. For $i < \omega$ put

$$w_i = \{k \in v_i : F \subseteq Z_{y_k^i}\} \quad \text{and} \quad C_i = \bigcup_{k \in w_i} B_k^i.$$

Plainly, $a = \llbracket \bigcap_{m < \omega} \bigcup_{i > m} C_i \rrbracket_{\mathbb{B}}$ so (as $a \neq \mathbf{0}_{\mathbb{B}}$) $\sum_{i=1}^{\infty} \mu^{\text{Leb}}(C_i) = \infty$, and hence the set

$$I \stackrel{\text{def}}{=} \{i < \omega : \mu^{\text{Leb}}(C_i) > 2^{1-i}\}$$

is infinite.

Fix $i \in I$ for a moment. Then

$$2^{1-i} < \mu^{\text{Leb}}(C_i) = \sum_{k \in w_i} \mu^{\text{Leb}}(B_k^i) = \frac{|w_i|}{|v_i|},$$

and thus (by $(*)_5^i$)

$$|w_i| > |v_i| \cdot 2^{1-i} > \frac{1}{2} \cdot m_{\ell_i} \cdot 2^{1-i} \geq \frac{m_{\ell_i}}{2^{\ell_i}}.$$

Hence, by $(*)_1^{\ell_i}$, we get $(1 - 2^{-2^{\ell_i+1}})^{|w_i|} < 2^{-\ell_i} \leq 2^{-i}$. Now (for our $i \in I$) consider the closed set

$$Y_i \stackrel{\text{def}}{=} \bigcap_{k \in w_i} Z_{y_k^i}$$

and note that

$$Y_i \subseteq \{\eta \in 2^\omega : (\forall k \in w_i) (\eta \upharpoonright [n_{\ell_i}, n_{\ell_i+1}) \in t_k^{\ell_i})\}.$$

Thus, by $(*)_2^{\ell_i}$, we may conclude that (for our $i \in I$)

$$\mu^{\text{Leb}}(Y_i) \leq \frac{|\bigcap_{k \in w_i} t_k^{\ell_i}|}{2^{m_{\ell_i} \cdot 2^{\ell_i+1}}} = (1 - 2^{-2^{\ell_i+1}})^{|w_i|} < 2^{-i}.$$

Since I is infinite and for every $i \in I$ we have $F \subseteq \bigcap_{k \in w_i} Z_{y_k^i} = Y_i$ we may now conclude that $\mu^{\text{Leb}}(F) = 0$, contradicting $F \in \mathbb{A}$.

2. The same proof as for 1. works here too. □

Putting together Proposition 3.3 and Proposition 3.9 (and remembering that products of sweet forcings are sweet) we may easily conclude the following.

Corollary 3.10 *Both $\mathbb{R} \times \mathbb{R}$ and $\mathbb{A} \times \mathbb{A}$ are topologically sweet, but*

$\Vdash_{\mathbb{R}}$ “no dense subforcing of $(\mathbb{R} \times \mathbb{R} : \mathbb{R})$ ($(\mathbb{R} \times \mathbb{A} : \mathbb{R})$, respectively) is topologically sweet”.

(Above, in $(\mathbb{R} \times \mathbb{R} : \mathbb{R})$ we consider the quotient over the first coordinate of the product. Similarly in $(\mathbb{R} \times \mathbb{A} : \mathbb{R})$.)

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