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THE COVERING NUMBERS OF MYCIELSKI IDEALS ARE ALL EQUAL

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Abstract. The Mycielski ideal $\mathfrak{M}_k$ is defined to consist of all sets $A \subseteq {}^{\mathbb{N}}k$ such that $\{f \upharpoonright X : f \in A\} \neq {}^{X}k$ for all $X \in [\mathbb{N}]^{\aleph_0}$. It will be shown that the covering numbers for these ideals are all equal. However, the covering numbers of the closely associated Rosłanowski ideals will be shown to be consistently different.

§1. **Introduction.** In [6] J. Mycielski defined a class of ideals which have been studied in various contexts by several authors [7, 11, 8, 10, 5, 1, 9, 2, 4, 3]. This paper is devoted to examining the covering numbers of these ideals as well as those of a closely related class of ideals. It will be shown that, while the covering number of the Mycielski ideals is independent of their dimension, the covering numbers of the related ideals are very closely connected to their dimension.

DEFINITION 1.1. The Mycielski ideal $\mathfrak{M}_k$ is defined to consist of all sets $A \subseteq {}^{\mathbb{N}}k$ such that for all $X \in [\mathbb{N}]^{\aleph_0}$

$$\{f \upharpoonright X : f \in A\} \neq {}^{X}k. \tag{1.1}$$

A function Φ on $[\mathbb{N}]^{\aleph_0}$ will be said to witness that $A \in \mathfrak{M}_k$ if $\Phi(X) \in {}^{X}k \setminus \{f \upharpoonright X : f \in A\}$ for each $X \in [\mathbb{N}]^{\aleph_0}$.

Notice that if $A \in \mathfrak{M}_k$ and X is an infinite subset of $\mathbb{N}$ then not only is there some $g \in {}^{X}k$ such that for all $f \in A$ there is some $x \in X$ such that $f(x) \neq g(x)$ but, by partitioning X into infinitely many infinite sets, one sees that there is actually some $g \in {}^{X}k$ such that for all $f \in A$ there are infinitely many $x \in X$ such that $f(x) \neq g(x)$. The next definition will generalize this version of the Mycielski ideals.

DEFINITION 1.2. Let $\mathbb{S}_k$ denote the set of all functions $f : X \to k$ where X is a co-infinite subset of $\mathbb{N}$. This can be thought of as k dimensional Silver forcing. The Rosłonowski ideal $\mathfrak{R}_k$ is defined to consist of all sets $A \subseteq {}^{\mathbb{N}}k$ such that for all $g \in \mathbb{S}_k$ there is an extension $g' \supseteq g$ such that $g' \in \mathbb{S}_k$ and $g' \not\subseteq^* f$ for all $f \in A$. A function Φ on $\mathbb{S}_k$ will be said to witness that $A \in \mathfrak{R}_k$ if $g \subseteq \Phi(g) \in \mathbb{S}_k$ for each $g \in \mathbb{S}_k$ and $\Phi(g) \not\subseteq^* f$ for all $f \in A$.

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It is worth noting that neither of these ideals has a simple definition. Indeed, since the definition given is Π^1_2, many of the usual arguments which apply to Borel ideals must be applied with great care, if at all, in this context. For an alternate approach to finding a nice base for the Mycielski ideals see [10].

The covering numbers of the ideals $\mathfrak{N}_k$ have a connection to gaps in $\mathscr{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ since the assertion that $\mathrm{cov}(\mathfrak{N}_2) = \aleph_1$ can be interpreted as saying there are many Hausdorff gaps. To see this, suppose that $\{A_\xi\}_{\xi\in\omega_1}$ is a cover of $2^{\mathbb{N}}$ by sets in $\mathfrak{N}_2$ witnessed by $\{\Phi_\xi\}_{\xi\in\omega_1}$. If $\{f_\xi\}_{\xi\in\omega_1}$ is any $\subseteq^*$-increasing sequence in $\mathbb{S}_2$ such that $f_{\xi+1} = \Phi_\xi(f_\xi)$ then $\{(f_\xi^{-1}\{0\}, f_\xi^{-1}\{1\})\}_{\xi\in\omega_1}$ is a Hausdorff gap[1]. Hence a large tree all of whose branches are Hausdorff gaps can be constructed using $\mathrm{cov}(\mathfrak{N}_2) = \aleph_1$. It will be shown that similar assertions for $\mathrm{cov}(\mathfrak{N}_n) = \aleph_1$ are not equivalent to $\mathrm{cov}(\mathfrak{N}_2) = \aleph_1$ for $n > 2$.

§2. Equality and inequality.

THEOREM 2.1. *If k and n are integers greater than* 1 *then* $\mathrm{cov}(\mathfrak{M}_k) = \mathrm{cov}(\mathfrak{M}_n)$.

PROOF. To begin, notice that if Φ witnesses that $A \in \mathfrak{M}_k$ then

$$\{f \in {}^{\mathbb{N}}(k+1) : (\forall X \in [\mathbb{N}]^{\aleph_0}) f \restriction X \neq \Phi(X)\}$$

belongs to $\mathfrak{M}_{k+1}$. It follows that $\mathrm{cov}(\mathfrak{M}_k) \geq \mathrm{cov}(\mathfrak{M}_{k+1})$. It therefore suffices to show that $\mathrm{cov}(\mathfrak{M}_{k^2}) \geq \mathrm{cov}(\mathfrak{M}_k)$ for each $k \geq 2$.

To this end, let $\beta : \mathbb{N} \to [\mathbb{N}]^2$ be a bijection and let $\beta_s(n)$ be the smallest member of $\beta(n)$ and $\beta_g(n)$ be the greatest member of $\beta(n)$. Define a relation $\equiv_\beta$ on $\mathbb{PF}_k \times \mathbb{PF}_{k^2}$ by $f \equiv_\beta g$ if and only if the following conditions (2.1) and (2.2) hold:

$$(\forall \{n,m\} \in [\mathrm{domain}(g)]^2) \beta(n) \cap \beta(m) = \emptyset \tag{2.1}$$

$$(\forall n \in \mathrm{domain}(g)) g(n) = kf(\beta_s(n)) + f(\beta_g(n)). \tag{2.2}$$

Now suppose that $\mathscr{A}$ is a cover of ${}^{\mathbb{N}}(k^2)$ by sets in $\mathfrak{M}_{k^2}$ and that Φ_A witnesses that $A \in \mathfrak{M}_{k^2}$ for each $A \in \mathscr{A}$. Now, for $A \in \mathscr{A}$ define

$$A^* = \{f \in {}^{\mathbb{N}}k : (\forall X \in [\mathbb{N}]^{\aleph_0})(\forall Z \in [\mathbb{N}]^{\aleph_0}) f \restriction X \not\equiv_\beta \Phi(Z)\}. \tag{2.3}$$

It will be shown that $\{A^* : A \in \mathscr{A}\}$ is a cover of ${}^{\mathbb{N}}k$ by sets in the ideal $\mathfrak{M}_k$.

To see that each $A^* \in \mathfrak{M}_k$ let $A \in \mathfrak{M}_{k^2}$ and $X \in [\mathbb{N}]^{\aleph_0}$. Let $\{\{e_i, d_i\}\}_{i\in\omega}$ be disjoint pairs from X such that $e_i < d_i$ for all i. Let $Z = \{\beta^{-1}(\{e_i, d_i\})\}_{i\in\omega}$ and define $h : \bigcup_{i\in\omega}\{e_i, d_i\} \to k$ such that $\Phi_A(Z)(i) = kh(e_i) + h(d_i)$ for all i. It follows that no member of A^* extends h.

To see that $\{A^* : A \in \mathscr{A}\}$ is a cover of ${}^{\mathbb{N}}k$ let $f \in {}^{\mathbb{N}}k$. Let $g : \mathbb{N} \to k^2$ be defined such that $g(n) = kf(\beta_s(n)) + f(\beta_g(n))$. Then there is some $A \in \mathscr{A}$ such that $g \in A$. It is easy to check that $f \in A^*$. ⊣

It is worth observing that $\mathrm{cov}(\mathfrak{M}_j) = \mathrm{add}(\mathfrak{M}_j)$ for all values of j. It suffices to note that $\mathrm{cov}(\mathfrak{M}_j) \leq \mathrm{add}(\mathfrak{M}_j)$ since the covering number of any ideal is bounded

[1]The term "Hausdorff gap" here is used to denote a pair of towers, increasing with respect to $\subseteq^*$ of length ω_1, such that any proper initial segment can be separated but the towers themselves can not be separated.

by its additivity. If $\{A_\xi\}_{\xi\in\kappa} \subseteq \mathfrak{M}_j$ is such that $\bigcup_{\xi\in\kappa} A_\xi \notin \mathfrak{M}_j$ then there is some infinite $X \subseteq \mathbb{N}$ such that

$$\left\{ f \restriction X : f \in \bigcup_{\xi\in\kappa} A_\xi \right\} = {}^X j$$

and, hence, $\bigcup_{\xi\in\kappa}\{f \restriction X : f \in A_\xi\} = {}^X j$ is a cover of ${}^X j$ by sets in $\mathfrak{M}_j$ under the obvious bijection of ${}^X j$ and ${}^{\mathbb{N}} j$.

PROPOSITION 1. *If* $i \geq j$ *then* $\operatorname{cov}(\mathfrak{R}_i) \leq \operatorname{cov}(\mathfrak{R}_j)$.

PROOF. Let $\bigcup_{\zeta\in\kappa} A_\zeta$ be a cover of ${}^{\mathbb{N}}j$ by sets in $\mathfrak{R}_j$. Let $\Phi_\zeta : \mathbb{S}_j \to \mathbb{S}_j$ witness that A_ζ belongs to $\mathfrak{R}_j$. Define $S : \mathbb{PF}_i \to \mathbb{PF}_j$ by

$$S(f)(m) = \begin{cases} f(m) & \text{if } f(m) \in j \\ j-1 & \text{if } f(m) \notin j \end{cases}$$

and then let $\Psi_\zeta : \mathbb{S}_i \to \mathbb{S}_i$ be defined by

$$\Psi_\zeta(f)(m) = \begin{cases} \Phi_\zeta(S(f))(m) & \text{if } m \notin \operatorname{domain}(f) \\ f(m) & \text{if } m \in \operatorname{domain}(f) \end{cases}$$

Let $B_\zeta = \{f \in {}^{\mathbb{N}}i : (\forall g \in \mathbb{S}_i)(\Psi_\zeta(g) \not\subseteq^* f)\}$ and note that if $f \in {}^{\mathbb{N}}i \setminus \bigcup_{\zeta\in\kappa} B_\zeta$ then $S(f) \in {}^{\mathbb{N}}j \setminus \bigcup_{\zeta\in\kappa} A_\zeta$. ⊣

§3. Covering numbers of many Rosłonowski ideals may be different. In this section it will be shown that any combination of values for the cardinal invariants $\operatorname{cov}(\mathfrak{R}_k)$ is consistent so long as it does not violate the basic monotonicity result of Proposition 1. Intervals of integers will be denoted by the usual notation; so, for example, $n \setminus m$ will be denoted by $[m, n)$

THEOREM 3.1. *Let* κ *be a nowhere increasing function from* $[1, \infty)$ *to the uncountable regular cardinals. It is consistent, relative to the consistency of set theory itself, that* $\operatorname{cov}(\mathfrak{R}_i) = \kappa(i)$ *for each* $i \geq 2$ *and* $2^{\aleph_0} = \kappa(1)$.

Denote $\kappa(1)$ by κ. The basic construction will be a finite support iteration of length κ of countable chain condition partial orders. Simultaneously with this construction, a sequence of trees $\{T_i\}_{i=2}^{\infty}$ will be constructed such that the height of T_i is $\kappa(i)$ and the width of each T_i is κ. The tree T_i will be thought of as a subset of $\mathbb{S}_i$ and the tree ordering will agree with $\subseteq^*$. The construction will guarantee that each level of the tree T_i corresponds to a subset of $\mathbb{S}_i$ which belongs to $\mathfrak{R}_i$. The fact that the tree has no cofinal branches will be used to show that the union of these sets covers $\mathbb{S}_i$ thus providing an upper bound on $\operatorname{cov}(\mathfrak{R}_i)$. On the other hand, if $\lambda < \kappa(i)$ and $\bigcup_{\alpha\in\lambda} X_\alpha$ is a cover of $\mathbb{S}_i$ by sets from $\mathfrak{R}_i$ then, at the typical limit stage an approximation to a function Φ_α witnessing that $X_\alpha \in \mathfrak{R}_i$ will have been trapped. A tower of partial functions $\{f_\alpha\}_{\alpha\in\lambda}$ with respect to $\subseteq^*$ will be constructed so that $f_{\alpha+1} \supseteq^* \Phi_\alpha(f_\alpha)$ and a new function will be added to the top of this tower. This new function will prevent the approximations from witnessing that $\operatorname{cov}(\mathfrak{R}_i)$ is smaller than $\kappa(i)$. The countable chain condition of the forcing which adds a top to this tower is not an obstacle since this will follow from the genericity of the construction. However, more care will have to be taken to preserve the key property of the trees which guarantees that there are no cofinal branches. The remainder of this section will supply the details of this outline.

Let V be a model where $2^{\aleph_0} = \aleph_1$ holds and the following version of $\diamondsuit_\kappa$ holds:

HYPOTHESIS 1. There is a sequence $\{D_\alpha\}_{\alpha\in\kappa}$ such that for each cardinal $\lambda < \kappa$ and each family $\{X_\zeta\}_{\zeta\in\lambda}$ of subsets of κ and for each closed unbounded set $C \subseteq \kappa$ and each $\mu \in \kappa$ there is some $\gamma \in \kappa$ such that

- the cofinality of γ is λ
- $D_\gamma = (c, \{X_\zeta \cap \gamma\}_{\zeta\in\lambda}, J)$ for some integer J
- $c : \lambda \to \gamma \cap (C \setminus \mu)$ is a continuous, increasing mapping whose range is cofinal in γ
- $D_{c(\xi)} = (c \restriction \xi, \{X_\zeta \cap c(\xi)\}_{\zeta\in\xi}, J)$ for each limit $\xi \in \lambda$.

This is easily seen to be a consequence of $\diamondsuit^+_\kappa$, but is, in fact, considerably weaker.

The first step is to define a finite support iteration of countable chain condition partial orders $\{\mathbb{Q}_\alpha\}_{\alpha\in\kappa}$. The iteration of $\{\mathbb{Q}_\alpha\}_{\alpha\in\eta}$ will be denoted by $\mathbb{P}_\eta$. Before proceeding, using the cardinal arithmetic implied by Hypothesis 1, let all sets of hereditary cardinality less than κ be enumerated by $\{F_\eta\}_{\eta\in\kappa}$.

If $\alpha = \beta + 2$ then $\mathbb{Q}_\alpha$ is simply defined to be Cohen forcing for adding a generic function $c_\alpha : \mathbb{N} \to \mathbb{N}$. Defined simultaneously with $\mathbb{P}_\alpha$ will be $\mathbb{P}_\alpha$-names for subtrees $T^\alpha_j \subseteq \Omega_j = {}^{<\kappa(j)}\kappa$ and functions Θ^α_j with domain T^α_j such that, for each $j \geq 2$

- if $\beta \in \alpha$ then $T^\beta_j \subseteq T^\alpha_j$
- if $\beta \in \alpha$ then $\Theta^\beta_j \subseteq \Theta^\alpha_j$
- $1 \Vdash_{\mathbb{P}_\alpha}$ "if $\xi \in T^\alpha_j$ then $\Theta^\alpha_j(\xi) \in \mathbb{S}_j$"
- $1 \Vdash_{\mathbb{P}_\alpha}$ "if ξ and ξ' belong to T^α_j and $\xi \subseteq \xi'$ then $\Theta^\alpha_j(\xi) \subseteq^* \Theta^\alpha_j(\xi')$"
- $1 \Vdash_{\mathbb{P}_\alpha}$ "if ξ and ξ' are distinct elements of T^α_j of the same height then $|\{n \in \mathbb{N} : \Theta^\alpha_j(\xi)(n) \neq \Theta^\alpha_j(\xi')(n)\}| = \aleph_0$"
- if α is a limit then $T^\alpha_j = \bigcup_{\beta\in\alpha} T^\beta_j$ and $\Theta^\alpha_j = \bigcup_{\beta\in\alpha}\Theta^\beta_j$
- if $\alpha = \beta + i$ where $i \in 3$ and β is a limit then $T^\alpha_j = T^\beta_j$ and $\Theta^\alpha_j = \Theta^\beta_j$.

Notice that by the induction hypothesis, if $F \in \mathbb{PF}_j$ and $B^\alpha_j(F)$ is defined to be $\{\xi \in T^\alpha_j : \Theta^\alpha_j(\xi) \subseteq^* F\}$ then $B^\alpha_j(F)$ forms a chain in T^α_j. The following additional induction hypothesis will play a crucial role in the construction:

INDUCTION HYPOTHESIS.

$$(\forall j \geq 2)(\forall F \in \mathbb{PF}_j)(|B^\alpha_j(F)| < \kappa(j)) \tag{3.1}$$

If there is some β such that $\alpha = \beta + 3$ then let $\varphi(j, \alpha)$ be the least ordinal such that $F_{\varphi(j,\alpha)}$ is a $\mathbb{P}_{\beta+2}$-name for an element of $\mathbb{S}_j$ which does not equal, modulo a finite set, an element of the range of $\Theta^{\beta+2}_j$. (Such an ordinal must exist because α is a successor and, hence, many new Cohen reals have been added at the previous stage.) Given a generic $G \subseteq \mathbb{P}_\alpha$ let $\bar{\xi}$ be a name for the lexicographically least member of $\Omega_j[G] \setminus T^{\beta+2}_j[G]$ which extends each member of $B^{\beta+2}_j(F_{\varphi(j,\alpha)})$ and let T^α_j be a name for $T^{\beta+2}_j[G] \cup \{\bar{\xi}\}$. Note that by Hypothesis 3.1 the sequence $\bar{\xi}$

belongs to $\Omega_j[G]$. Define $\Theta^\alpha_j(\bar{\xi})$ by

$$\Theta^\alpha_j(\bar{\xi})(i) = \begin{cases} F_{\varphi(j,\alpha)}(i) & \text{if } i \in \text{domain}(F_{\varphi(j,\alpha)}) \\ c_\alpha(i) & \text{if } i \in \mathbb{N} \setminus \text{domain}(F_{\varphi(j,\alpha)}) \text{ and } c_\alpha(i) < j \\ \text{undefined} & \text{if } i \in \mathbb{N} \setminus \text{domain}(F_{\varphi(j,\alpha)}) \text{ and } c_\alpha(i) \geq j \end{cases}$$

and notice that this definition will satisfy the induction hypotheses because of the genericity of c_α. Observe also, that adding a Cohen real does no harm to the Induction Hypothesis 3.1.

The next step is to define $\mathbb{Q}_\alpha$ when α is a limit or the successor of a limit ordinal.

DEFINITION 3.1. If $\mathscr{H}$ is an $\subseteq^*$-increasing chain in $\mathbb{S}_k$ then the partial order $\mathbb{Q}(\mathscr{H})$ is defined to be the set of all functions $f \in \mathbb{S}_k$ such that there is some $h \in \mathscr{H}$ such that $f \subseteq^* h$, and the ordering on $\mathbb{Q}(\mathscr{H})$ is inclusion. If G is a filter on $\mathbb{Q}(\mathscr{H})$ then define $f_G = \cup G$ and note that if G is a sufficiently generic filter then f_G is a total function from $\mathbb{N}$ to k.

Observe that if $\mathscr{Z} \subseteq \mathscr{H}$ is a $\subseteq^*$-cofinal set then $\mathbb{Q}(\mathscr{Z})$ is equal to $\mathbb{Q}(\mathscr{H})$. This fact will be used in the sequel without further mention. The function f_G is intended to be used to extend the given chain and obtain a new partial order extending the given one. However, since f_G is a total function, it will be necessary to cut it down to obtain a member of $\mathbb{S}_k$. The following partial order is designed to do this.

DEFINITION 3.2. If $\mathbb{Q}(\mathscr{H})$ is as in Definition 3.1 and G is a filter on $\mathbb{Q}(\mathscr{H})$ then define $\mathbb{A}(G)$ to consist of all quadruples $(a, p, \mathscr{F}, \nu)$ ordered under coordinatewise inclusion such that:

- $a \in [\mathbb{N}]^{<\aleph_0}$
- $p \in G$
- $a \cup \text{domain}(p) \supseteq [0, max(a)]$
- $a \cap \text{domain}(p) = \emptyset$
- $\mathscr{F}$ is a finite set of nice $\mathbb{Q}(\mathscr{H})$-names[2] for elements of ${}^{\mathbb{N}}\mathbb{N}$
- $\nu : \mathscr{F} \to \mathbb{N}$
- for each $\theta : \nu(f) \cap a \to k$, for each $i \in a \cap [\nu(f), \infty)$, for each $\tau : [\nu(f), i] \cap a \to k$ there is some integer $m_{\theta,\tau,f}$ such that
$$p \cup \theta \cup \tau \Vdash_{\mathbb{Q}(\mathscr{H})} \text{“} f(K) = m_{\theta,\tau,f} \text{”}$$
where $K = |a \cap [\nu(f), i]|$.

If $H \subseteq \mathbb{A}(G)$ is a filter then define $A_H = \bigcup_{(a,p,\mathscr{F},\nu) \in H} a$ and define $f_{G,H} = f_G \restriction (\mathbb{N} \setminus A_H)$.

Observe that $\mathbb{A}(G)$ has $\aleph_1$ as a precalibre regardless of the cofinality of $\mathscr{H}$ — indeed, each regular uncountable cardinal is a precalibre of $\mathbb{A}(G)$. Ameoba forcing is usually not this nice. Hence $\mathbb{Q}(\mathscr{H}) * \mathbb{A}(G)$ has the countable chain condition so long as $\mathbb{Q}(\mathscr{H})$ does. Furthermore, $\mathbb{Q}(\mathscr{H}) \subseteq \mathbb{Q}(\{f_{G,H}\})$. The main question to be addressed is: Do dense sets in $\mathbb{Q}(\mathscr{H})$ remain dense in $\mathbb{Q}(\{f_{G,H}\})$? The next pair of lemmas provide some information on this.

[2]Nice names are not crucial here. All that is required is that the rank of the elements of $\mathscr{F}$ is bounded so that $\mathbb{A}(G)$ is a set rather than a class.

LEMMA 1. *If* $\mathscr{H}$ *is an increasing tower in* $\mathbb{S}_k$, $p \in \mathbb{Q}(\mathscr{H})$, $g : l \to k$, $a \in [\mathbb{N} \setminus \text{domain}(p)]^{<\aleph_0}$ *and* D *is a dense subset of* $\mathbb{Q}(\mathscr{H})$ *then there is* $p' \supseteq p$ *such that* $a \cap \text{domain}(p') = \emptyset$ *and* $(p' \restriction [l, \infty) \cup \theta \cup g \in D$ *for each* $\theta : a \cap [l, \infty) \to k$.

PROOF. This is the standard argument which is used, among other things, to prove that Silver forcing is proper. ⊣

LEMMA 2. *Let* $\mathscr{H}$ *be an increasing tower in* $\mathbb{S}_k$ *and let* G *be* $\mathbb{Q}(\mathscr{H})$*-generic over the model* V. *Suppose also that* H *is* $\mathbb{A}(G)$ *generic over* $V[G]$. *If* D *belongs to* $V[G]$ *and is predense in* $\mathbb{Q}(\mathscr{H})$ *then it remains so in* $\mathbb{Q}(\mathscr{F})$ *for any increasing tower* $\mathscr{F} \subseteq \mathbb{S}_k$ *such that* $f_{G,H} \in \mathscr{F}$.

PROOF. Given $f \in \mathbb{Q}(\mathscr{F})$ it may, without loss of generality, be assumed that $f \supseteq^* f_{G,H}$. Therefore, it is possible to choose $l \in \mathbb{N}$ such that $f \restriction [l, \infty) \supseteq f_{G,H} \restriction [l, \infty)$. Also without loss of generality, it may be assumed that $l \subseteq \text{domain}(f)$. Now, let $g = f \restriction l$. From Lemma 1 it follows that the set

$$D_g = \{(a, p, \mathscr{F}, v) \in \mathbb{A}(G) : (\forall \theta : a \cap [l, \infty) \to k)(p \restriction [l, \infty) \cup \theta \cup g \in D\}$$

is dense in $\mathbb{A}(G)$. Now, choose $(a, p, \mathscr{F}, v) \in D_g \cap H$. Let

$$\theta = f \restriction a \cap [l, \infty)$$

and, using the definition of D_g, conclude that $p \restriction [l, \infty) \cup \theta \cup g \in D$. Since $p \restriction [l, \infty) \subseteq f_{G,H} \restriction [l, \infty) \subseteq f$ it follows that $p \restriction [l, \infty) \cup \theta \cup g \subseteq f$ and hence, f extends an element of D. ⊣

Whenever α is a limit ordinal such that $\text{cof}(\alpha) < \kappa$, the partial order $\mathbb{Q}_\alpha$ will be defined to be of the form $\mathbb{Q}(\mathscr{H}_\alpha)$ where $\mathscr{H}_\alpha \subseteq \mathbb{S}_J$ is an increasing tower with respect to $\subseteq^*$ which has the same cofinality as α and $J \geq 2$ is such that $\kappa(J) > \text{cof}(\alpha)$. Moreover, in this case, $\mathbb{Q}_{\alpha+1}$ will always be of the form $\mathbb{A}(G)$ where G is the generic filter on $\mathbb{Q}(\mathscr{H}_\alpha)$. Keeping this in mind, let H be the generic filter on $\mathbb{A}(G)$ and define $H_\alpha = f_{G,H} \in \mathbb{S}_J$. The only point which requires elaboration is how to choose $\mathscr{H}_\alpha$. There are three cases to consider.

CASE ONE. There exist c, a limit ordinal $\rho \in \kappa$, $\{\Phi_\zeta\}_{\zeta \in \rho}$ and integer J such that:

- $\text{cof}(\alpha) < \kappa(J)$
- $D_\alpha = (c, \{\Phi_\zeta\}_{\zeta \in \rho}, J)$
- $c : \rho \to \alpha$ is a continuous, increasing mapping whose range is cofinal in α
- ξ and $c(\xi)$ have the same cofinality for each $\xi \in \rho$
- $1 \Vdash_{\mathbb{P}_\alpha}$ "$(\forall \zeta \in \rho)(\Phi_\zeta : \mathbb{S}_J \to \mathbb{S}_J$ is such that $(\forall f)(f \subseteq \Phi_\zeta(f)))$"
- $D_{c(\xi)} = (c \restriction \xi, \{X_\zeta \cap c(\xi)\}_{\zeta \in \xi}, J)$ for each limit $\xi \in \rho$
- there is some $\rho' \in \rho$ such that $\rho = \rho' + \omega$

CASE TWO. The hypothesis is the same as in Case 1 except that the last requirement fails; in other words, there is no $\rho' \in \rho$ such that $\rho = \rho' + \omega$.

CASE THREE. Both Case 1 and Case 2 fail.

As a further induction hypothesis it will be assumed that:

INDUCTION HYPOTHESIS. If $\eta' \in \eta$ and either Case 1 or Case 2 holds at η and $D_\eta = (c, \{\Phi_\zeta\}_{\zeta \in \rho}, J)$ and $\eta' = c(\xi)$ for some $\xi \in \rho$ such that $\text{cof}(\xi) \leq \text{cof}(\rho)$ (note that this implies that either Case 1 or Case 2 holds at η' as well) then $H_{\eta'} \subseteq^* H_\eta$.

Now consider Case 1. Note that $H_{c(\rho')}$ is already defined since $c(\rho') < \alpha$. It is therefore possible to choose inductively $H_{c(\rho'+n+1)}$ to be a $\mathbb{P}_\alpha$-name such that

$$1 \Vdash_{\mathbb{P}_\alpha} \text{“}H_{c(\rho'+n+1)} = \Phi_{\rho'+n}(H_{c(\rho'+n)})\text{”}$$

for each $n \in \mathbb{N}$. Let $\mathscr{H}_\alpha = \{H_{c(\xi)}\}_{\xi\in\rho}$ and note that this is increasing with respect to $\subseteq^*$ since $\{H_{\rho'+n}\}_{n\in\omega}$ is.

Observe that in Case 2 it follows from the construction and the induction hypothesis that the set $\{H_{c(\xi)}\}_{\xi\in\rho}$ is an increasing tower in $\mathbb{S}_J$ for some J such that the cofinality of α is less than $\kappa(J)$. To see this, proceed by induction on η and note that if $\eta' < \eta$ then either $\eta = \eta' + k$ for some integer k or there is some ζ of cofinality not exceeding that of η' such that $\eta' < \zeta < \eta$. The first possibility is handled using the construction in Case 1 and the second is dealt with using the induction hypothesis. It is therefore possible to let $\mathscr{H}_\alpha = \{H_{c(\xi)}\}_{\xi\in\rho}$.

In Case 3 let $\mathscr{H}_\alpha$ be any increasing countable family; in other words, $\mathbb{Q}_\alpha$ will be Cohen forcing. It will become apparent that $\mathbb{Q}_{\alpha+1}$ is irrelevant in this case.

LEMMA 3. *The partial order $\mathbb{P}_\kappa$ has the countable chain condition.*

PROOF. Proceed by induction to show that

$$1 \Vdash_{\mathbb{P}_\alpha} \text{“}\mathbb{Q}_\alpha \text{ has the countable chain condition”}$$

for each α. The countable chain condition for $\mathbb{Q}(\mathscr{H})$ is problematic only when the cofinality of α is uncountable. Indeed, if $\operatorname{cof}(\alpha) = \omega$ or $\operatorname{cof}(\alpha) = 1$ then $\mathbb{Q}(\mathscr{H})$ is σ-centred. The same is also true if either Case 1 or Case 3 in the inductive construction of $\mathbb{P}_\alpha$ holds. So assume that Case 2 holds and that $D_\alpha = (c, \{\Phi_\zeta\}_{\zeta\in\rho}, J)$ and, hence, $\mathbb{Q}_\alpha = \mathbb{Q}(\{H_{c(\zeta)}\}_{\zeta\in\rho})$. Now, if $A \subseteq \mathbb{Q}_\alpha$ is a maximal antichain then, using the fact that c is continuous and its range is cofinal in α, it is possible to find some $\xi \in \rho$ such that $A \cap \mathbb{Q}(\{H_{c(\zeta)}\}_{\zeta\in\xi})$ is a maximal antichain. By the induction hypothesis, it follows that $A \cap \mathbb{Q}(\{H_{c(\zeta)}\}_{\zeta\in\xi})$ is countable. By Lemma 2 and the definition of $H_{c(\xi)}$, it follows that $A \cap \mathbb{Q}(\{H_{c(\zeta)}\}_{\zeta\in\xi})$ is also maximal in $\mathbb{Q}_\alpha$. ⊣

Notice that it is immediate that if G is $\mathbb{P}_\kappa$ generic over a model V where $2^{\aleph_0} = \aleph_1$ then $2^{\aleph_0} = \kappa$ in $V[G]$. Before proceeding some notation will be introduced.

DEFINITION 3.3. Suppose that $\mathbb{P} \subseteq \mathbb{P}'$ and that X is $\mathbb{P}'$-name. The $\mathbb{P}$-name $X \restriction \mathbb{P}$ is defined by induction on the rank of the inductive definition of names. If X is of the form $X \subseteq \mathbb{P}' \times Z$ where Z is a ground model set then $X \restriction \mathbb{P} = X \cap (\mathbb{P} \times Z)$. In general, $X \restriction \mathbb{P} = \{(p, A \restriction \mathbb{P}) : (p, A) \in X\}$.

LEMMA 4. *If G is $\mathbb{P}_\kappa$ generic over V then* $\operatorname{cov}(\mathfrak{R}_j) \geq \kappa(j)$ *in $V[G]$ for $j \geq 2$.*

PROOF. If $\operatorname{cov}(\mathfrak{R}_j) < \kappa(j)$ then let $\Phi_\xi : \mathbb{S}_j \to \mathbb{S}_j$ be such that $\{\Phi_\xi\}_{\xi\in\lambda}$ witness this fact for some $\lambda < \kappa(j)$. Let $\tilde{\Phi}_\xi$ be a name for Φ_ξ and suppose that

$$p \Vdash_{\mathbb{P}_\kappa} \text{“}\{\tilde{\Phi}_\xi\}_{\xi\in\lambda} \text{ witnesses that } \operatorname{cov}(\mathfrak{R}_j) \leq \lambda\text{”}.$$

Using the regularity of κ, the fact that $|\mathbb{P}_\alpha| < \kappa$ for each $\alpha \in \kappa$ and that V is a model of $2^{\aleph_0} = \aleph_1$, let C be a closed unbounded set in κ such that for each $\alpha \in C$ the restricted names $\tilde{\Phi}_\xi \restriction \mathbb{P}_\alpha$ satisfy that

$$p \Vdash_{\mathbb{P}_\alpha} \text{“}\{\tilde{\Phi}_\xi \restriction \mathbb{P}_\alpha\}_{\xi\in\lambda} \text{ witnesses that } \operatorname{cov}(\mathfrak{R}_j) \leq \lambda\text{”}.$$

Find some η such that $\operatorname{cof}(\eta) = \lambda$,

$D_\eta = (c, \{\tilde{\Phi}_\xi \restriction \mathbb{P}_\eta\}_{\xi\in\lambda}, j)$ for some $c : \lambda \to C \setminus \sup(\text{domain}(p))$. It follows directly from the construction of $\mathbb{P}_\kappa$ that $\{H_{c(\xi)}\}_{\xi\in\lambda}$ is an increasing sequence in $\mathbb{S}_j$ and $\text{cof}(\eta) = \lambda < \kappa(j)$. Moreover, the construction at isolated limit ordinals guarantees that $H_{c(\xi+1)} \supseteq^* \Phi_\xi(H_{c(\xi)})$ for $\xi \in \lambda$. This yields that $f = f_{G\cap\mathbb{Q}(\mathscr{H}_\eta)}$ extends each $\Phi_\xi(H_{c(\xi)})$ for $\xi \in \lambda$. Hence f does not belong to any of the members of the ideal $\mathfrak{R}_j$ defined by the witnesses Φ_ξ. ⊣

Lemma 5. *If G is $\mathbb{P}_\kappa$ generic over V, $j \geq 2$ and $g \in {}^{\mathbb{N}}j$ in $V[G]$ then there is some $\alpha \in \kappa(j)$ such that $g \not\subseteq^* \Theta^\kappa_j(\sigma)$ for all $\sigma \in T^\kappa_j$ of length greater than α.*

Proof. Let $g \in {}^{\mathbb{N}}j$ in $V[G]$. By Induction Hypothesis 3.1 it follows that the branch $B^\kappa_j(g)$ has length α for some α less than $\kappa(j)$. Hence, if σ has length greater than α then $\sigma \notin B^\kappa_j(g)$. By definition $g \not\subseteq^* \Theta^\kappa_j(\sigma)$. ⊣

Lemma 6. *If G is $\mathbb{P}_\kappa$ generic over V then* $\text{cov}(\mathfrak{R}_j) \leq \kappa_j$ *in* $V[G]$.

Proof. In $V[G]$, for each $\alpha \in \kappa(j)$, let E_α be the set of all $g : \mathbb{N} \to j$ such that $\Theta^\kappa_j(\sigma) \not\subseteq^* g$ for all $\sigma \in T^\kappa_j$ of length greater than α. To see that $E_\alpha \in \mathfrak{R}_j$ let $f \in \mathbb{S}_j$. Then f has a name of hereditary cardinality less than κ and so, there is some $\mu \in \kappa$ be such that $F_{\varphi(j,\mu)}$ is interpreted as f in $V[G]$. It follows from the construction of Θ that there is some sequence $\bar{\xi}$ such that $1 \Vdash_{\mathbb{P}_\mu}$ "$\Theta^\mu_j(\bar{\xi}) \supseteq F_{\varphi(j,\mu)}$". Now let $\xi \in T^\kappa_j$ be an extension of $\bar{\xi}$ of length greater than α and note that $f' \equiv^* \Theta^\kappa_j(\xi)$ has the property that $g \not\supseteq^* f'$ for all $g \in E_\alpha$. From Lemma 5 it follows that $\bigcup_{\alpha\in\kappa(j)} E_\alpha = {}^{\mathbb{N}}j$. ⊣

Hence, in order to finish the proof of Theorem 3.1, it suffices to show that Hypothesis 3.1 holds. The basic idea here is that it suffices to show that the induction hypothesis holds at a single stage for any particular name for a function; at later stages Cohen genericity can be used. The next three lemmas provide the details to this sketch.

Lemma 7. *Let G be $\mathbb{P}_\kappa$ generic over V and $J < j$. If $\alpha \in \beta \in \kappa$ and T is a J-branching subtree of ${}^{\mathbb{N}\smile}\mathbb{N}$ which belongs to $V[G \cap \mathbb{P}_\alpha]$ then for any $\xi \in T^\beta_j \setminus T^\alpha_j$ there are infinitely many integers i such that there is some $i' > i$ so that*

$$\Theta^\beta_j(\xi) \restriction [i, i') \not\subseteq b \restriction [i, i')$$

for any $b \in T$.

Proof. Recall that a tree T is said to be J-branching of height n if $T \subseteq \bigcup_{k\leq n} {}^k\mathbb{N}$ and no node has more than J successors. The following fact is easily proven by induction on n: If $\{T_i\}_{i\in n}$ is a family of K-branching trees of height n then $\bigcup_{i\in n} T_i \not\supseteq {}^n(K+1)$. A direct corollary of this fact is that if $T \subseteq {}^{\mathbb{N}\smile}\mathbb{N}$ is a J-branching tree and $i \in \mathbb{N}$ then there is a function $f : [i, i + J^i) \to J + 1$ such that $f \neq b \restriction [i, i + J^i)$ for any $b \in T$. Due to technical details, this simple fact can not be used directly; but, a slight variation of it will be combined with Cohen genericity to obtain the desired conclusion.

Before this can be done however, let T and G be given and let $t \in T$ be a sequence whose domain is i. Let A denote the domain of the interpretation of $F_{\varphi(j,\beta)}$ in $V[G \cap \mathbb{P}_\beta]$. Define a tree $T(t)$ in $V[G \cap \mathbb{P}_\beta]$ by

$$T(t) = \{s \in T : s \restriction A \cap [i, \infty) \subseteq F_{\varphi(j,\beta)}\}$$

and let ψ_i be the order preserving bijection from $\mathbb{N}$ to $[i, \infty) \setminus A$. Define $T^*(t) = \{s \circ \psi : s \in T(t)\}$ and notice that $T^*(t)$ is a J-branching tree. Using this and the Cohen genericity of c_β it is possible to apply the first observation of the previous paragraph to conclude that there are infinitely many integers i such that $c_\beta \circ \psi_i(k) \in J \setminus \{b(k)\}$ for each $k \in J^i$, $t \in T$ of length i and for any $b \in T^*(t)$. Given any such i let $i' = i + J^i + |A \cap [i, \psi_i(J^i))|$. It follows that $c_\beta \restriction [i, i') \not\subseteq b \restriction [i, i')$ for any $b \in T$. ⊣

DEFINITION 3.4. If for some model of set theory V and integer k

- $\mathscr{H} \subseteq \mathbb{S}_k$
- f is a $\mathbb{Q}(\mathscr{H})$-name such that $1 \Vdash_{\mathbb{Q}(\mathscr{H})}$ "$f \in {}^{\mathbb{N}}\mathbb{N}$"
- G is a $\mathbb{Q}(\mathscr{H})$ generic filter over V
- H is an $\mathbb{A}(G)$ generic filter over $V[G]$
- $(a, p, \mathscr{F}, v) \in H$
- $f \in \mathscr{F}$
- $l > v(f)$

then, for any $g : [0, l] \to k$ let $\tau(G, H, f, g)$ denote the k-branching tree defined by $t \in \tau(G, H, f, g)$ if and only if there exists some $(\bar{a}, \bar{p}, \bar{\mathscr{F}}, \bar{v}) \in H$ and $\tau : \bar{a} \cap [l, |\bar{a}|) \to k$ such that $|t| = |\tau|$ and

$$p \cup g \cup \tau \Vdash_{\mathbb{Q}(\mathscr{H})} \text{“}f(i) = m_{g,\tau,f} = t(i)\text{”} \tag{3.2}$$

for each $i < |t|$. Note that this is well defined.

LEMMA 8. *If $\mathscr{H} \subseteq \mathbb{S}_k$, then:*

1. *If f is a nice name such that $1 \Vdash_{\mathbb{Q}(\mathscr{H})}$ "$f \in {}^{\mathbb{N}}\mathbb{N}$" then the set of all $(a, p, \mathscr{F}, v) \in \mathbb{A}(G)$ such that $f \in \mathscr{F}$ is dense in $\mathbb{A}(G)$*
2. *If $j \in \mathbb{N}$ then the set of all $(a, p, \mathscr{F}, v) \in \mathbb{A}(G)$ such that $a \cap [j, \infty) \neq \emptyset$ is dense in $\mathbb{A}(G)$.*

PROOF. To prove (1), given $q = (a, p, \mathscr{F}, v) \in \mathbb{A}(G)$ define

$$q' = (a, p, \mathscr{F} \cup \{f\}, v \cup \{(f, \max(a) + 1)\}) \in \mathbb{A}(G)$$

and note that q' is stronger than q.

For (2), let j and $q = (a, p, \mathscr{F}, v) \in \mathbb{A}(G)$ be given and choose an integer $u \in [j, \infty)$ which does not belong to the domain of p. Using Lemma 1 it follows that the set of $p' \in \mathbb{Q}(\mathscr{H})$ such that $(a \cup \{u\}, p', \mathscr{F}, v)$ satisfies the conclusion of (2) is dense. Hence, there is $p' \in G$ such that $(a \cup \{u\}, p', \mathscr{F}, v)$ is stronger than q and satisfies the requirements of (2). ⊣

LEMMA 9. *If it is given that*

- $\mathrm{cof}(\alpha) = \kappa(j)$
- *G is $\mathbb{P}_{\alpha+1}$ generic over V*
- *$f \in {}^{\mathbb{N}}\mathbb{N}$ in $V[G]$*

then there is $J < j$ and a J-branching tree $T \subseteq {}^{\underset{\smile}{\mathbb{N}}}\mathbb{N}$ in $V[G \cap \mathbb{P}_\alpha]$ such that $f \in \overline{T}$.

PROOF. Using the same notation as in the construction of $\mathbb{P}_\kappa$, let $\mathbb{Q}_\alpha = \mathbb{Q}(\{H_{c(\eta)}\}_{\eta \in \rho})$. Using the countable chain condition of $\mathbb{Q}_\alpha$, the uncountable cofinality of α, and hence $c(\alpha)$, it is possible to find a limit ordinal $\beta \in \rho$ such that

f is a $\mathbb{Q}(\{H_{\bar{c}(\eta)}\}_{\eta\in\beta})$-name and the name f belongs to $V[G\cap\mathbb{P}_\beta]$. Notice that $\mathrm{cof}(\alpha)=\kappa(j)$ implies that $\{H_{c(\eta)}\}_{\eta\in\rho}\subseteq\mathbb{S}_J$ for some $J<j$.

Let H be $\mathbb{A}(G\cap\mathbb{Q}_\alpha)$ generic over $V[G\cap(\mathbb{P}_\alpha*\mathbb{Q}(\mathscr{H}_\alpha))]$ and recall that $H_\alpha=f_{G\cap\mathbb{Q}_\alpha,H}$. If $h\in\mathbb{S}_J$ and $h\supseteq^* H_\alpha$ let $l\in\mathbb{N}$ be such that $h\supseteq H_\alpha\restriction[l,\infty)$ and let $g=h\restriction l$. Now use Lemma 8 to conclude that $\tau(G,H,g,f)$ is a J-branching tree whose closure contains f. The desired result now follows directly from Lemma 2. ⊣

The next task is to check that the Induction Hypothesis 3.1 holds at each stage of the construction. The countable chain condition and Cohen genericity guarantee this at limit stages of large cofinality but the argument at limit stages of countable cofinality requires a bit more care. The next three lemmas provide the details and finish the proof of Theorem 3.1.

Lemma 10. *If the Induction Hypothesis* 3.1 *holds at all previous stages and* $\mathrm{cof}(\alpha)$ *is uncountable then*

$$(\forall F\in\mathbb{PF}_j)(|B_j^\alpha(F)|<\kappa(j)).$$

Proof. Given $F\in\mathbb{PF}_j$, it follows by the countable chain condition of the partial order that there is some $\beta\in\alpha$ such that $F\in V[G\cap\mathbb{P}_\beta]$ for any generic $G\subseteq\mathbb{P}_\alpha$. By the induction hypothesis it follows that $|B_j^\beta(F)|<\kappa(j)$ in $V[G\cap\mathbb{P}_\beta]$. However, the genericity of c_γ over $V[G\cap\mathbb{P}_\beta]$ for $\gamma>\beta$ guarantees that $\Theta_j^\alpha(\xi)\not\subseteq^* F$ for every $\gamma\in\alpha\setminus\beta$ and $\xi\in T_j^\alpha\setminus T_j^\beta$.. Hence $|B_j^\alpha(F)|<\kappa(j)$ in $V[G\cap\mathbb{P}_\alpha]$ as well. ⊣

Lemma 11. *If the Induction Hypothesis* 3.1 *holds at all previous stages and* $\mathrm{cof}(\alpha)<\kappa(j)$ *then*

$$(\forall F\in\mathbb{PF}_j)(|B_j^\alpha(F)|<\kappa(j)).$$

Proof. Let $\mathrm{cof}(\alpha)=\gamma<\kappa(j)$ and suppose that G is $\mathbb{P}_\alpha$ generic over V. If F is a function from $\mathbb{N}$ to j in $V[G]$ then notice that, if $B_j^\alpha(F)$ has length $\kappa(j)$ then, by the cofinality of α, there is some $\beta\in\alpha$ such that there is a cofinal subset $B\subseteq B_j^\beta(F)$. This determines the branch through T_j^β in $V[G\cap\mathbb{P}_\beta]$. Hence, it suffices to show that if $B\subseteq T_j^\beta$ is a branch of length $\kappa(j)$ in $V[G\cap\mathbb{P}_\beta]$ and F is in $V[G\cap\mathbb{P}_\alpha]$ then $B\not\subseteq B_j^\beta(F)$.

To this end, let B be a $\mathbb{P}_\beta$-name for a long branch through T_j^β and F a $\mathbb{P}_\alpha$-name. Let $\{\beta_\xi\}_{\xi\in\gamma}$ be an increasing sequence of ordinals cofinal in α such that $\beta_0>\beta$. For any $p\in\mathbb{P}_\alpha$ define $F_p=\{(i,j):p\Vdash_{\mathbb{P}_\alpha}\text{“}F(i)=j\text{”}\}$. It will first be shown that for each $\xi\in\gamma$ the set

$$D(\xi)=\{q\in\mathbb{P}_{\beta_\xi}:(\exists\sigma\in B)(\forall r\leq q)(F_r\not\supseteq^*\Theta_j^\beta(\sigma))\}$$

is dense in $\mathbb{P}_{\beta_\xi}$. To see that this is so, suppose that $q\in\mathbb{P}_{\beta_\xi}$ is such that for each $\sigma\in B$ and $\bar{q}\leq q$ there is some $r\leq\bar{q}$ such that $F_r\supseteq^*\Theta_j^\beta(\sigma)$. Then let $\bar{F}$ to be the $\mathbb{P}_{\beta_\xi}$-name defined by $p\Vdash_{\mathbb{P}_{\beta_\xi}}$ “$\bar{F}(i)=j$” if and only if $p\Vdash_{\mathbb{P}_\alpha}$ “$F(i)=j$”. It follows that $q\Vdash_{\mathbb{P}_{\beta_\xi}}$ “$\bar{F}\supseteq^*\Theta_j^\beta(\sigma)$” for each $\sigma\in B$ contradicting the induction hypothesis.

Using the density of each $D(\xi)$, let $\mathscr{A}_\xi\subseteq D(\xi)$ be a maximal antichain and, for each $q\in\mathscr{A}_\xi$, let $\sigma_q^\xi\in B$ witness that $q\in D(\xi)$. Since B has length $\kappa(j)$, it is possible to find $\sigma\in B$ be such that $\sigma\supseteq\sigma_q^\xi$ for each $\xi\in\gamma$ and $q\in\mathscr{A}_\xi$. Now suppose that

$p \in \mathbb{P}_\alpha$ is such that $p \Vdash_{\mathbb{P}_\alpha}$ "$F \cup (\Theta^\beta_j(\sigma) \restriction m) \supseteq \Theta^\beta_j(\sigma)$" for some integer m. Let ξ be such that $p \in \mathbb{P}_{\beta_\xi}$ and choose $q \in \mathscr{A}_\xi$ such that there is some $r \in \mathbb{P}_{\beta_\xi}$ such that $r \leq q$ and $r \leq p$. Since $q \in D(\xi)$ it follows that $F_r \not\supseteq^* \Theta^\beta_j(\sigma^\xi_q) \subseteq^* \Theta^\beta_j(\sigma)$. Hence, there is some $i > m$ in the domain of $\Theta^\beta_j(\sigma)$ such that either $r \Vdash_{\mathbb{P}_\alpha}$ "$F(i) \neq \Theta^\beta_j(\sigma)(i)$" or r does not decide a value for $F(i)$. The first case directly contradicts that $r \leq p$ and, in the second case, it is possible to extend r to r' such that $r' \Vdash_{\mathbb{P}_\alpha}$ "$F(i) \neq \Theta^\beta_j(\sigma)(i)$". This again yields a contradiction. ⊣

It remains to consider successor ordinals. If $\alpha = \beta + 1$ and β itself is a successor or has cofinality less than $\kappa(j)$ then $\kappa(j)$ is a precalibre of $\mathbb{Q}_\alpha$. Hence, a standard argument shows that it preserves the induction hypothesis. So the only problem may arise when β is a limit of large cofinality.

Lemma 12. *Suppose that α is a limit ordinal of cofinality greater than or equal to $\kappa(j)$. Given that each preceding stage satisfies the Induction Hypothesis* 3.1, *the partial order $\mathbb{P}_{\alpha+1}$ will also satisfy the induction hypothesis.*

Proof. Let G be $\mathbb{P}_\alpha$ generic over V and argue in $V[G]$. There are two types of branches which might provide difficulties. To begin, consider branches which occur at some stage β before α. Let B be a branch through T^β_j of length $\kappa(j)$ in $V[G \cap \mathbb{P}_\beta]$ and let F be a $\mathbb{Q}_\alpha$-name for a function from $\mathbb{N}$ to j such that

$$1 \Vdash_{\mathbb{Q}_\alpha} \text{"}(\forall \sigma \in B)(F \supseteq^* \Theta^\beta_j(\sigma))\text{"}$$

If $\mathbb{Q}_\alpha$ has a dense subset of size less than $\kappa(j)$ then a simple pigeonhole argument shows that there is some $m \in \mathbb{N}$ and a single condition $q \in \mathbb{Q}_\alpha$ such that the set of $\sigma \in B$ such that $q \Vdash_{\mathbb{Q}_\alpha}$ "$F \cup (\Theta^\beta_j(\sigma) \restriction m) \supseteq \Theta^\beta_j(\sigma)$" is cofinal in B. This means that

$$\bigcup\{\Theta^\beta_j(\sigma) : q \Vdash_{\mathbb{Q}_\alpha} \text{"}F \cup (\Theta^\beta_j(\sigma) \restriction m) \supseteq \Theta^\beta_j(\sigma)\text{"}\}$$

is cofinal in B contradicting the induction hypothesis. Use the notation of the construction of $\mathbb{Q}_\alpha$. If Case 1 or Case 3 holds at α then $\mathbb{Q}_\alpha$ has a countable dense subset and so this possibility has already been considered. Hence, assume that Case 2 holds at α. Using the countable chain condition of $\mathbb{Q}_\alpha = \mathbb{Q}(\{h_{c(\eta)}\}_{\eta \in \rho}$, let $\xi \in \rho$ be such that $F \restriction \mathbb{Q}(\{h_{c(\eta)}\}_{\eta \in \xi})$ is a $\mathbb{Q}(\{h_{c(\eta)}\}_{\eta \in \rho})$-name. From the induction hypothesis it follows that

$$1 \Vdash_{\mathbb{Q}_{c(\xi)}} \text{"}(\exists \sigma \in B)(F \not\supseteq^* \Theta^\beta_j(\sigma))\text{"}$$

Using the uncountable cofinality of B and the countable chain condition of $\mathbb{Q}_{c(\xi)}$ it follows that there is some fixed $\sigma \in B$ such that

$$1 \Vdash_{\mathbb{Q}_{c(\xi)}} \text{"}F \not\supseteq^* \Theta^\beta_j(\sigma))\text{"}$$

so it is possible to use Lemma 2 to conclude that the dense subsets of $\mathbb{Q}_{c(\xi)}$ witnessing that $F \not\supseteq^* \Theta^\beta_j(\sigma)$ remain dense in $\mathbb{Q}_\alpha$.

The second possibility is that a cofinal branch is added to T^α_j which is not cofinal in any previous T^β_j. To see that this can not happen, suppose that $1 \Vdash_{\mathbb{Q}_\alpha}$ "$F : \mathbb{N} \to j$". Then, by Lemma 9, there is some J-branching tree T such that $J < j$ and

$$1 \Vdash_{\mathbb{Q}_\alpha} \text{"}F \text{ is in the closure of } T\text{"}.$$

Since α has uncountable cofinality and the iterands all have the countable chain condition, it follows that if G is a generic set for $\mathbb{P}_\kappa$ then there is some $\beta \in \alpha$ such that T belongs to $V[G \cap \mathbb{P}_\beta]$. Choose $\sigma \in B_j^\alpha(F) \setminus T_j^\beta$. Now use Lemma 7 to conclude that there are infinitely many integers i for which there is some $i' > i$ so that

$$\Theta_j^\beta(\sigma) \restriction [i, i') \not\subseteq b \restriction [i, i')$$

for any $b \in T$. In particular, there are infinitely many i such that there is some $i' > i$ so that $\Theta_j^\beta(\sigma) \restriction [i, i') \not\subseteq F \restriction [i, i')$. In other words, $\Theta_j^\beta(\sigma) \not\subseteq^* F$. ⊣

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