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ON ENDO-RIGID, STRONGLY $\aleph_1$-FREE ABELIAN GROUPS IN $\aleph_1$

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ABSTRACT

Assuming $2^{\aleph_0} < 2^{\aleph_1}$ we prove that there is an endo-rigid strongly $\aleph_1$-free group of power $\aleph_1$.

Here group will mean an abelian group.

1. Definition. A group G is endo-rigid if every endomorphism $h : G \to G$ has the form $h(x) = nx$ ($n \in \mathbf{Z}$ fixed).

2. History. Fuchs [5] with the help of Coroner proved the existence of such groups up to very large cardinalities, Shelah [7] in all cardinals ($> \aleph_0$), Eklof and Mekler [4] prove the existence of strong κ-free, indecomposable groups of power κ, κ regular, under the hypothesis $V = L$, and Dugas [3] replaces indecomposable by endo-rigid.

3. Theorem. ($2^{\aleph_0} < 2^{\aleph_1}$) *There is an endo-rigid, strongly $\aleph_1$-free group of power $\aleph_1$.*

Remark. We can get $2^{\aleph_1}$ such groups with no non-zero homomorphism from one to another (see [1]).

4. Claim. Let G be a countable free abelian group, $c, b \in G$, $c \neq 0$, $b \neq 0$, b, c have no common multiple (by integers).

Let $G = \bigcup_{n<\omega} G_n$, $G_n \subseteq G_{n+1}$, G_{n+1}/G_n free (hence G/G_n is free). Let $a_n \in G_{n+1}$ be such that $a_n + G_n \in G_{n+1}/G_n$ is not divisible by any natural number, and for $l = 0, 1$ and $i < \omega$, $k_i^l \in \mathbf{Z}$ such that for infinitely many i's, $k_i^0 = k_i^1 = 0$, and for infinitely many i's $k_i^0 - k_i^1 = 1$.

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Let G^l $(l = 0, 1)$ be the group freely generated by G, x^l, y_i^l $(i < \omega)$ except the relations $p_i y_i^l = x^l - a_i - k_i^l b$ where $\langle p_i : i < \omega \rangle$ is a list of the primes.

Then

(1) G^l is countable and free; moreover, it is a pure extension of G, i.e., $nx \in G \wedge x \in G^l \wedge n \neq 0$ implies $x \in G$.

(2) There are no homomorphisms $h_l : G^l \to G^l$, $h_0 \restriction G = h_1 \restriction G$, $h_0(b) = c$.

(3) If $G^l \subseteq G'$, G'/G^l is $\aleph_1$-free, f an endomorphism of G' mapping G into itself, then f maps G^l into itself.

5. Proof of 4. Part (1) of the claim is trivial, and so is part (3) (as in G'/G, G^l/G is the set of elements of G'/G divisible by infinitely many primes and f induces an endomorphism of G'/G).

So we concentrate on (2), and let $h = h_l \restriction G$. For some m, $k_l \in \mathbf{Z}$, $d_l \in G$, $m \neq 0$, $mh_l(x^l) = k_l x^l + d_l$ (there are such m, k_l, d_l as $h_l(x^l) \in G^l$). (Why m and not m_l? Use the least common multiple.)

So for every $i < \omega$, $l \in \{0, 1\}$, as $p_i y_i^l = x^l - a_i - k_i^l b$, clearly (remember that $h_l \restriction G = h$)

$$mp_i h_l(y_i^l) = m(h_l(x^l) - h(a_i) - k_i^l h(b))$$

$$= k_l x^l + d_l - mh(a_i) - m_l k_i^l c.$$

So in G^l, $(k_l x^l + d_l - mh(a_i) - mk_i^l c)$ is divisible by p_i. But also $(x^l - a_i - k_i^l b)$ is divisible by p_i in G^l. Hence in G^l

$$z_i^l = (k_l x^l + d_l - mh(a_i) - mk_i^l c) - k_l(x^l - a_i - k_i^l b)$$

$$= d_l - mh(a_i) + k_l a_i + k_i^l(k_l b - mc) \in G$$

is divisible by p_i in G^l, but G is a pure subgroup of G^l, hence z_i^l is divisible by p_i in G. Hence

$$z_i^0 - z_i^1 = (d_0 - d_1) + k_i^0(k_0 b - mc) - k_i^1(k_1 b - mc) + (k_0 - k_1)a_i$$

$$= (d_0 - d_1) + (k_0 - k_1)a_i + (k_i^0 k_0 - k_i^1 k_1)b - m(k_i^0 - k_i^1)c$$

is divisible by p_i in G.

For large enough i, $d_0, d_1, b, c \in G_i$, hence $(k_0 - k_1)a_i + G_i$ is divisible by p_i (in G/G_i), but by the choice of a_i this implies $k_0 - k_1 = 0$, i.e., $k_0 = k_1$ (as otherwise we can choose i such that p_i does not divide $k_0 - k_1$).

So for every i, $(d_0 - d_1) + k_0(k_i^0 - k_i^1)b - m(k_i^0 - k_i^1)c$ is divisible by p_i. As for infinitely many i's, $k_i^0 = k_i^1$, for infinitely many primes p, $d_0 - d_1$ is divisible by p.

As G is free, $d_0 - d_1 = 0$. Similarly as for infinitely many i's, $k_i^0 - k_i^1 = 1$, $k_0 b - mc = 0$ hence $k_0 b = mc$. Clearly $m \neq 0$, thus we contradict a hypothesis.

6. Fact. If h is an endomorphism of an $\aleph_1$-free (abelian) group G, and there is no $n \in \mathbf{Z}$ such that for every x, $h(x) = nx$, then there are $b, c \in G$, $h(b) = c$, and $nb \neq mc$ for $n, m \neq 0$, and $b \neq 0$, $c \neq 0$.

Proof. Suppose h is a counterexample. Then for every $x \in G$ not divisible by any $n \in \mathbf{Z}$, $n \neq 0, 1$, there is n_x such that $h(x) = n_x x$, hence for every $x \in G$ there is such n_x. Clearly if $k_0 x = k_1 y$ $(k_0 k_1 \neq 0)$ then $n_x = n_y$. If the rank of G is 1, h is nx for some n, so there are $x, y \in G$ which are a basis of a free pure subgroup of G, and $n_x \neq n_y$. Trivially $b = x + y$, $c = h(b) = n_x x + n_y y$ are as required.

7. Proof of Theorem 3. Let $\langle S_\alpha : \alpha < \omega_1 \rangle$ be a list of $\aleph_1$ pairwise disjoint non-small stationary subsets of $\aleph_1$ (see [2], or e.g. [1]) such that $y \in S_\alpha \Rightarrow \alpha < y$.

Let $\{\langle b_\alpha, c_\alpha \rangle : \alpha < \omega_1\}$ be a list of the pairs of ordinals smaller than ω_1, such that $b_\alpha, c_\alpha \leqq 1 + \alpha$.

Now we define by induction on $\alpha < \omega_1$, for every $\eta \in {}^\alpha 2$, a group G_η such that:

(1) G_η is a free (abelian) group with universe $\omega(1+\alpha) = \omega(1 + l(\eta))$,

(2) if $\nu = \eta \restriction \beta$ then G_ν is a pure subgroup of G_η,

(3) if $\nu = \eta \restriction (\beta + 1)$ then G_η / G_ν is free, also $G_\eta / G_{\eta|0}$ is free,

(4) if $\alpha \in S_i$, α limit, $\eta \in {}^\alpha 2$,

then there are no homomorphisms $h_l : G_{\eta\hat{}\langle l \rangle} \to G_{\eta\hat{}\langle l \rangle}$, $h_0 \restriction G_\eta = h_1 \restriction G_\eta$, $h_l(b_i) = c_i$, except when $mc_i = nb_i$ for some $m, n \in \mathbf{Z} - \{0\}$, also if $G_{\eta\hat{}\langle l \rangle} \subseteq G'$, $G'/G_{\eta\hat{}\langle l \rangle}$ $\aleph_1$-free, h an endomorphism of G', h maps G_η to G_η then h maps $G_{\eta\hat{}\langle l \rangle}$ into $G_{\eta\hat{}\langle l \rangle}$.

There is no problem in the definition; for (4) use the claim.

For each α, let F_α be the following function: If $\delta < \omega_1$, $\omega\delta = \delta$, $\eta \in {}^\delta 2$, $h : \delta \to \delta$, h an endomorphism of G_η into G_η, $h(b_\alpha) = c_\alpha$ and h can be extended to an endomorphism of $G_{\eta\hat{}\langle 0 \rangle}$, then $F_\alpha(\eta, h) = 1$, otherwise $F_\alpha(\eta, h) = 0$.

By [2] there are $\nu_\alpha \in {}^{\omega_1} 2$, such that for every $h : \omega_1 \to \omega_1$, $\eta \in {}^{\omega_1} 2$, the set $\{\delta \in S_\alpha : F_\alpha(\eta \restriction \delta, h \restriction \delta) = \nu_\alpha(\delta)\}$ is stationary (because S_α is not small).

Let $\nu \in {}^{\omega_1} 2$ be defined such that $i \in S_\alpha \Rightarrow \nu(i) = \nu_\alpha(i)$. Suppose h is an endomorphism of G_ν, such that for no n is $h(x) = nx$ for every $x \in G_\nu$. By Fact 6, $h(b) = c$, b, c with no common multiple $\neq 0$, for some $b, c \in G_\nu$. For some α, $\langle b_\alpha, c_\alpha \rangle = \langle b, c \rangle$ (as $\{\langle b_\alpha, c_\alpha \rangle : \alpha < \omega_1\}$ list all pairs of ordinals $< \omega_1$). Also $S^* =$

$\{\delta : h$ maps δ into $\delta, \omega\delta = \delta\}$ is a closed unbounded set of ω_1. On the other hand, $S^*_\alpha = \{\delta \in S_\alpha : F_\alpha(\nu \restriction \delta, h \restriction \delta) = \nu_\alpha(\delta)\}$ is stationary. So there is $\delta \in S^* \cap S^*_\alpha$. Now $h \restriction \delta$ is an endomorphism of $G_{\nu \restriction \delta}$; it can be extended to an endomorphism of $G_{\nu \restriction (\delta+1)} = G_{\nu \restriction \delta \hat{} \langle \nu(\delta) \rangle}$. What is $\nu(\delta)$? If it is zero, then $F_\alpha(\nu \restriction \delta, h \restriction \delta) = 1$ (by its definition) hence $\nu_\alpha(\delta) = 1$ (as $\delta \in S^*_\alpha \cap S^*$), but $\nu(\delta) = \nu_\alpha(\delta)$ as $\delta \in S_\alpha$, contradiction. If, on the other hand, $\nu(\delta) = 1$ then $h \restriction G_{\nu \restriction \delta}$ can be extended to an endomorphism of some $G' \supseteq G_{\nu \restriction (\delta+1)} = G_{\nu \restriction \delta \hat{} \langle 1 \rangle}$ (use G_ν), and also of some $G' \supseteq G_{\nu \restriction \delta \hat{} \langle 0 \rangle}$ (as $1 = \nu(\delta) = \nu_\alpha(\delta) = F_\alpha(\nu \restriction \delta, h \restriction \delta)$ and the definition of F_α). This contradicts (4) in the requirements on the G_η's.

We can now ask: when does this proof generalize to cardinals $\lambda > \aleph_1$? For example:

8. THEOREM. *Suppose*
(i) λ *is a regular cardinal* $> \aleph_0$,
(ii) $S \subseteq \{\delta < \lambda : \mathrm{cf}\, \delta = \aleph_0\}$,
(iii) S *is not small* (*hence stationary, see* [2]),
(iv) S *has no initial segment stationary* (*but is stationary*).
Then there is a strongly λ*-free abelian group of power* λ *which is endo-rigid.*

9. REMARK. (A) So in the proof $G = \bigcup_{i<\lambda} G_i$, G_i increasing continuous, each G_i free and $i < j \wedge i \notin S \Rightarrow G_j / G_i$ is free.

(B) In the proof of 8 we need λ disjoint non-small subsets of S. Let $\delta = \bigcup_n \alpha(\delta, n)$, $\alpha(\delta, n) < \alpha(\delta, n+1)$ for $\delta \in S$, then for some n for λ α_0's, $\{\delta : \alpha(\delta, n) = \alpha_0\}$ is not small; otherwise use the normality of the ideal of non-small subsets of λ. (This proof is well known and appears in Solovay [9].)

(C) If G.C.H., $\lambda = \mu^+$, $\mathrm{cf}\, \mu \neq \aleph_0$, we can omit (iii) (= non-smallness) as by Gregory [6] and Shelah [8] $\diamondsuit^*\{\delta < \lambda : \mathrm{cf}\, \delta \neq \mathrm{cf}\, \mu\}$ holds, hence for every stationary $S \subseteq \lambda$, $(\forall \delta \in S)\, \mathrm{cf}\, \delta \neq \mathrm{cf}\, \mu$, $\diamondsuit_S$ holds, hence S is not small (see [2]).

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