

Erratum

Maximal chains in ${}^\omega\omega$ and ultrapowers of the integers

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In the above article the proof of Theorem 3.3 of [1] was flawed in that the definition of *somewhat determined* was inaccurate. This should be replaced by the following.

A condition $p \in \mathbb{P}_{\omega_2}$ will be said to be determined if there is some $\Sigma_p \in [\omega_2]^{<\aleph_0}$ such that Σ_p is the support of p and for each $\sigma \in \Sigma_p$ there is a quadruple

$(a_p^\sigma, f_p^\sigma, \Delta_p^\sigma, g_p^\sigma)$ such that:

- $p \restriction \sigma \Vdash_{\mathbb{P}_\sigma} "p(\sigma) = a_p^\sigma * (f_p^\sigma, \Delta_p^\sigma)"$ for each $\sigma \in \Sigma_p$
- $\Delta_p^\sigma \subseteq \Sigma_p \cap \sigma$ for each $\sigma \in \Sigma_p$
- $p \restriction \sigma \Vdash_{\mathbb{P}_\sigma} "g_{\sigma(\sigma)} \restriction \text{dom}(a_p^\sigma) = g_p^\sigma"$ for each $\sigma \in \Sigma_p$
- for each $\{\sigma, \tau\} \in [\Sigma_p]^2$ such that $\sigma \prec \tau$ there is some $k_p(\sigma, \tau) \in 2$ such that $p \restriction \tau \Vdash_{\mathbb{P}_\tau} "A_\sigma^{-1}\{k_p(\sigma, \tau)\} \in D_\tau"$
- $\text{dom}(f_p^\sigma) \supseteq \text{dom}(a_p^\sigma)$ for each $\sigma \in \Sigma_p$
- $\text{dom}(f_p^\tau) \subseteq \text{dom}(f_p^\sigma)$ for each $\{\sigma, \tau\} \in [\Sigma_p]^2$ such that $\sigma \prec \tau$.

This definition of determined differs in a substantial way from the definition of *somewhat determined* in [1]. The next lemma shows that every condition can be extended to a determined condition; this is problematic for the somewhat determined conditions.

Lemma 0.1. *The set of determined conditions is dense in \mathbb{P}_{ω_2} .*

Proof. Induction on $\alpha \in \omega_2 + 1$ will be used to prove the following stronger statement: For each $m \in \omega$ and each $p \in \mathbb{P}_\alpha$ there is a determined condition $q \geq p$ such that if σ is the maximal element of Σ_q then $m \subseteq a_q^\sigma$ and σ is the maximal element of the support of p . Note that a_q^σ has the smallest domain of any function appearing in q so the requirement that $m \subseteq a_q^\sigma$ implies that m is in the domain of any function appearing in q .

To prove this, suppose the statement is true for all $\alpha \in \beta$. If β is a limit ordinal the result follows from the finite support of the iteration; therefore suppose that $\beta = \gamma + 1$. Then extend p so that $p \Vdash_{\mathbb{P}_\gamma} "p(\gamma) = a * (f, \Delta)"$. By extending, it may be assumed that

$m \subseteq \text{dom}(a) \subseteq \text{dom}(f)$. Let \bar{m} be the maximal element of $\text{dom}(f)$. Let $p' \geq p \upharpoonright \gamma$ be such that Δ is contained in the support of p' .

There are now two cases to consider: Either β is a successor in \prec or it is a limit. If it is a successor then let β^* be the predecessor of β in \prec . Otherwise, let β^* be such that β^* is greater than the support of p' and $\beta^* \prec \beta$ and β^* is the successor of β^{**} in the ordering \prec . In the first case, let $p'' \geq p'$ be such that $p'' \Vdash_{\mathbb{P}_\gamma} "A_{\beta^*}^{-1}k \in D_\beta"$. In the second case, choose p'' such that $p'' \Vdash_{\mathbb{P}_{\beta^*}} "A_{\beta^{**}}^{-1}k \in D_{\beta^*}"$ and such that β^{**} belongs to the support of p'' .

Now use the induction hypothesis to find a determined condition q such that if σ is the maximal element of Σ_q then $\bar{m} \in \text{dom}(a_q^\sigma)$. Moreover, in the case that β is a limit of \prec , then the induction hypothesis can be used to ensure that $\sigma < \beta^*$. It will be shown that the transitivity of \prec guarantees that $q * p(\gamma) = r$ is a determined condition satisfying the extra induction requirements. Let $\Sigma_r = \Sigma_q \cup \{\beta\}$ and let f_r^σ , a_r^σ and Δ_r^σ have the values inherited from q and $p(\beta)$. Furthermore, $k_r(a, \tau)$ can be defined to be $k_q(\alpha, \tau)$ unless $\beta = \tau$. Here the choice of p'' helps.

In the case that β is the successor of β^* , then p'' decides that $A_{\beta^*}^{-1}k \in D_{\beta^*}$ so $k_r(\beta^*, \beta)$ can be defined to be k and, moreover $k_r(\mu, \beta)$ can be defined to be k for each $\mu \in \Sigma_q$ such that $\mu \prec \beta^*$. Since β is the successor of β^* in \prec there are no new instances with which to deal. In the case that β is a limit in the partial order \prec , it is possible to define $k_r(\beta^{**}, \beta) = k$ because of the transitivity of \prec . For the same reason it is possible to define $k_r(\mu, \beta)$ to be k for each $\mu \in \Sigma_q$ such that $\mu \prec \beta^{**}$. Since the support of q is contained in β^* and β^* is the successor of β^{**} in the partial order \prec , it follows that there are no new instances to consider in this case as well. \square

The reader who wishes to see all the details of the proof is advised to download the file *chainserrata* by anonymous ftp from the directory `\pub\logic\shelahsteprans` at the site `ftp.math.ufl.edu`.

References

1. Shelah, S., Steprāns, J.: Maximal Chains in ω_ω and Ultrapowers of the Integers. Arch. Math. Log. 32, 305–319 (1993)