

# A PROOF OF VAUGHT'S CONJECTURE FOR $\omega$ -STABLE THEORIES

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*Dedicated to the memory of Abraham Robinson  
on the tenth anniversary of his death*

ABSTRACT

In this paper it is proved that if  $T$  is a countable complete  $\omega$ -stable theory in ordinary logic, then  $T$  has either continuum many, or at most countably many, non-isomorphic countable models.

## Introduction

Vaught's conjecture is the statement that for any countable theory  $T$  in  $L_{\omega_1, \omega}$ ,  $T$  has either  $2^{\aleph_0}$ , or at most countably many isomorphism types of countable models. Vaught's conjecture is not known to hold even for theories in finitary logic. In this paper, a proof is given for Vaught's conjecture for  $T$  a countable  $\omega$ -stable (totally transcendental) theory.

The results of this paper are due to Shelah. Makkai has written down the proof in the way Harrington patiently explained it to him. Makkai has no part in the work other than checking it and writing it up.

Related earlier results can be found in [1], [2], [3], [5], [6].

The expository paper [M] is used as preliminaries to the present paper. References such as A.2, B.3, etc., refer to [M]. Besides [M], Section 2 of the paper [H-M] is also essential for the present paper; however, if the reader is willing to accept two (crucial) propositions without proof, he can read the present paper without reference to [H-M].

<sup>†</sup> The author thanks the United States-Israel Binational Science Foundation for supporting his research.

<sup>††</sup> Supported by the Natural Sciences and Engineering Research Council of Canada, and FCAC Quebec.

Received January 10, 1982 and in revised form April 28, 1983

Makkai would like to express his thanks to Bradd Hart, Anand Pillay, Gabriel Srouf and the referee for their careful reading of the paper and for their valuable remarks.

### §1. ENI types, dimension, ENI-NDOP

Throughout the paper,  $T$  is a countable complete  $\omega$ -stable theory.

DEFINITION 1.1. A type (or ideal type)  $p$  is an *eventually non-isolated, strongly regular* type (briefly:  $p$  is an ENI type) if it is strongly regular (SR; see D.13), and for some finite set  $B$ ,  $p \upharpoonright B$  is defined (see Section B) and is non-isolated.  $p$  is NENI if it is SR, and it is not ENI.

Note that by the “open mapping theorem” (A.8), if  $p$  does not fork over  $B$ , and  $p \upharpoonright B$  is non-isolated, then  $p$  is non-isolated. Hence, in the above definition, with  $B$ , any finite superset of  $B$  will work as well.

To emphasize the elementary character of the following proposition, let us temporarily call a type  $p \in S(A)$  eni if  $A$  is finite, and there is a finite  $B \supset A$  such that all nf extensions of  $p$  to  $B$  are non-isolated. In other words,  $p$  is ENI iff it is eni and SR. Note that in the following proposition, if, in addition,  $p$  is SR, then  $|I| = \dim(p, M)$ .

PROPOSITION 1.2. *Let  $A$  be a finite set,  $p \in S(A)$ ,  $A \subset M$ ,  $I$  a maximal  $A$ -independent set of realizations of  $p$  in  $M$ .*

- (i) *If  $I$  is finite, then  $p$  is eni.*
- (ii) *If  $M$  is prime over some finite set, and  $p$  is eni, then  $I$  is finite.*
- (iii) *If  $p$  is eni, then there is finite  $B \supset A$  such that all nf extensions of  $p$  to  $B$  are non-isolated, and in addition,  $t(\vec{B}/A)$  is isolated.*

PROOF. *ad(i).* By the defining property of  $I$ , no nf extension of  $p$  to  $A \cup I$  is realized in  $M$ ; thus, all nf extensions of  $p$  to  $A \cup I$  are non-isolated. Thus, if  $I$  is finite, then (with  $B = A \cup I$ )  $p$  is eni.

*ad(ii).* Suppose  $M$  is prime over the finite set  $A' \supset A$ , and that  $I$  is infinite. Let  $B$  be any finite set containing  $A'$ . We will show that  $p$  has an isolated nf extension to  $B$ ; by the open mapping theorem, this will imply that  $p$  is not eni as desired. Let  $N$  be a model prime over  $B$ . Since  $M$  is prime over  $A'$ , there is an elementary  $A'$ -isomorphism  $f: N \xrightarrow{\sim} N'$  such that  $M \subset N'$ ; let  $B' = f(B)$ . Let  $I^0$  be a finite subset of  $I$  such that  $B' \downarrow_{A \cap I^0} I$ ; let  $c \in I - I^0$ ; it follows that  $c \downarrow_A B'$ , in other words  $t(c/B')$  is a nf extension of  $p$ . Since  $c \in M \subset N'$ , and  $N'$  is prime over  $B'$ ,  $t(c/B')$  is isolated. We have found an isolated nf extension of  $p$  to  $B'$ ; since  $\vec{B} \equiv \vec{B}'(A)$ , there is one for  $B$  as well.

*ad (iii).* Suppose that  $p$  is eni. Let  $M$  be a model prime over  $A$ , and let  $I$  be a maximal independent  $/A$  set of realizations of  $p$  in  $M$ . By (ii),  $I$  is finite. Since no nf extension of  $p$  to  $A \cup I$  is realized in  $M$ , all such extensions must be non-isolated. Of course, for  $B = A \cup I$ ,  $\vec{B}$  has an isolated type over  $A$  since  $B \subset M$ .  $\square$

PROPOSITION 1.3. *For SR types, the property of being ENI is invariant under the equivalence relation  $\mathcal{L}$ . If  $p_1, p_2$  are SR,  $p_1 \not\mathcal{L} p_2$ , and  $p_1$  is ENI, then  $p_2$  is ENI.*

PROOF. Equivalently, we show that if  $p_1$  is ENI,  $p_2$  is NENI, then  $p_1 \perp p_2$ . Let  $A$  be a large enough finite set such that  $p_1 \upharpoonright A$  is non-isolated, and  $p_2 \upharpoonright A$  is defined. Without loss of generality,  $p_1 = p_1 \upharpoonright A$  and  $p_2 = p_2 \upharpoonright A$ . Let  $M$  be a model prime over  $A$ . By 1.2(i),  $\dim(p_2, M) = \omega$ , and of course,  $\dim(p_1, M) = 0$ . The latter fact and D.21(ii) give us  $p_1^\circ \perp M/A$  (now  $I = \emptyset$ ). Since  $\dim(p_2, M) = \omega$ ,  $p_2^\circ$  is realized in  $M$ ; therefore  $p_1^\circ \perp p_2^\circ$ . By C.7(i), we conclude that  $p_1 \perp p_2$ , as promised.  $\square$

The following proposition is due to Bouscaren and Lascar [2]. For the sake of completeness, we include the proof of it.

PROPOSITION 1.4. *Suppose that  $p \in S(B)$  is SR,  $A \subset B$ ,  $B$  finite, and  $p \not\mathcal{L} A$ . Then for any  $B'$  with  $\vec{B}' \equiv \vec{B}(A)$ , and any  $M$  containing  $A$ ,  $B$  and  $B'$ ,  $\dim(p_{\vec{B}'}, M) = \dim(p, M)$ .*

PROOF. We start by two easy remarks. If  $p, q$  are SR types over a model  $N$ ,  $M \supset N$ , and  $p \not\mathcal{L} q$ , then  $\dim(p, M) = \dim(q, M)$ . Namely, if  $\langle a_i \rangle_{i \in I}$  is a  $p$ -basis for  $M$ , for each  $i \in I$  we can find  $b_i \in M$  realizing  $q$  such that  $a_i \not\downarrow_N b_i$  (see D.18); since the  $a_i, b_i$  have weight 1 over  $N$ , it is easy to see that  $\langle b_i \rangle_{i \in I}$  is independent over  $N$ ; this shows  $\dim(p, M) \leq \dim(q, M)$ , and the other inequality is symmetric. Secondly, if  $p \in S(M)$  is of weight 1, and  $N = M(a)$  with a finite tuple  $a$ , then  $\dim(p, N)$  is finite; if we had an infinite  $p$ -basis  $I$  for  $N$ , there would be  $b \in I$  such that  $b \not\downarrow_M a$  (any  $b \in I - I_0$  with a finite  $I_0 \subset I$  such that  $a \not\downarrow_{M_0} I$ ), contradicting  $b \in M(a) - M$  and the open mapping theorem A.8 (the last two facts imply that  $b \not\downarrow_M a$ ).

Turning to the proof of the proposition, let us note that without loss of generality,  $A = \emptyset$  (pass to a new theory  $T' = \text{Th}(\mathcal{G}, a)_{a \in A}$ ). First assume in addition to all the hypotheses of the proposition, that  $B' \not\downarrow B$ . Let  $M_0 \subset M$  be a model prime over  $B \cup B'$ . Notice that  $\vec{B} \wedge \vec{B}' \equiv \vec{B}' \wedge \vec{B}$ ; this follows from the finite equivalence relation theorem (B.3),  $B' \not\downarrow B$ , and  $\vec{B}' \equiv \vec{B}$ . It follows that there is an automorphism of  $M_0$  that maps  $B$  into  $B'$  and  $B'$  into  $B$ . This clearly shows that, for  $p' = p_{\vec{B}'}$ ,  $\dim(p, M_0) = \dim(p', M_0)$ . Since  $p \not\mathcal{L} \emptyset$  and  $B' \not\downarrow B$ , we

have  $p \not\perp p'$  by C.6(i). It follows by the first of the above two remarks that  $\dim(p \upharpoonright M_0, M) = \dim(p' \upharpoonright M_0, M)$ . Hence, by the addition formula, D.21(iii),  $\dim(p, M) = \dim(p, M_0) + \dim(p \upharpoonright M_0, M) = \dim(p', M_0) + \dim(p' \upharpoonright M_0, M) = \dim(p', M)$ , as required.

Finally, we turn to the general case. Let  $\bar{B}''$  be such that  $\bar{B}'' \stackrel{s}{\equiv} \bar{B}' \stackrel{s}{\equiv} \bar{B}$ , and  $BB' \downarrow B''$ ; let  $p'' = p_{\bar{B}''}$ , and let  $N = M(B'')$ . By the case handled above,  $\dim(p, N) = \dim(p'', N) = \dim(p', N)$ . Also, since  $p \not\perp p''$ ,  $p' \not\perp p''$  (and thus  $p \not\perp p'$ ), we have  $\dim(p \upharpoonright M, N) = \dim(p' \upharpoonright M, N)$  by our first remark; moreover, this dimension is finite by our second remark. Since  $\dim(p, N) = \dim(p, M) + \dim(p \upharpoonright M, N) = \dim(p', M) + \dim(p' \upharpoonright M, N) (= \dim(p', N))$ , it follows that  $\dim(p, M) = \dim(p', M)$ , as desired.  $\square$

LEMMA 1.5. *Suppose  $p$  is stationary and has weight 1,  $p \not\perp B$ ,  $p \perp A$  and  $B \downarrow_A C$ . Then  $p \perp C$ .*

PROOF. Find an  $a$ -model  $M$  containing  $B$  such that  $M \downarrow_A C$ . Since  $p \not\perp M$ , there is a weight 1 type  $q \in S(M)$  such that  $p \not\perp q$  (see D.11(v)). Since  $p \perp A$ , we have  $q \perp A$  (see D.5'). By C.8,  $q \perp C$ ; by D.5' again,  $p \perp C$ .  $\square$

LEMMA 1.6. *Suppose  $B$  finite,  $q \in S(B)$  a stationary type of weight 1, and  $q \perp A$ . Then there are: an  $a$ -model  $M$  containing  $A$ , an element  $c$ , and an  $a$ -model  $M[c]$   $a$ -prime over  $Mc$ , such that  $c/M$  is SR,  $B \subset M[c]$ , and  $q \perp M$ .*

PROOF. Choose an  $a$ -model  $M_0$  containing  $A$ , such that  $B \downarrow_A M_0$ . By C.8,  $q \perp M_0$ . Let  $N = M_0[B]$ . Let  $\bar{c} = \langle c_i \rangle_{i < n}$  be a SR basis for  $N$  over  $M_0$ , i.e.,  $\bar{c}$  is a maximal  $M_0$ -independent system of elements in  $N$ , each having a SR type over  $M_0$ ; among others  $N = M_0[\bar{c}]$  (see especially D.15). If we put  $M_{k+1} = M_k[c_k]$  ( $k < n$ ), then it is easily seen that  $M_n$  is  $a$ -prime over  $M_0\bar{c}$ , and more generally, over  $M_k \langle c_i \rangle_{k \leq i < n}$ . Therefore, without loss of generality,  $M_0 \subset M_1 \subset \dots \subset M_n = N$ . Since  $q \perp M_0$ , and  $q \not\perp M_n$ , there is  $k < n$  such that, for  $M' = M_k$ ,  $M'' = M_{k+1}$ ,

$$(1) \quad q \perp M', \quad q \not\perp M''.$$

Let  $c = c_k$ . We claim that there is an  $a$ -model  $M$   $a$ -prime over  $M' \langle c_i \rangle_{k < i < n}$  contained in  $N$  such that  $N = M[c]$ . In fact, let  $\hat{M}$  be an  $a$ -model  $a$ -prime over  $D = M' \langle c_i \rangle_{k < i < n}$ . Then  $\hat{M}[c]$  is  $a$ -prime over  $Ac$ . Note that also,  $N$  is  $a$ -prime over  $Dc$ . Hence, there is an elementary isomorphism over  $Dc$  mapping  $\hat{M}[c]$  onto  $N$ ; this maps  $\hat{M}$  onto the desired model  $M$ . With  $M'$ ,  $M''$ ,  $M$  so defined, we have  $M'' \downarrow_{M'} M$ . Since  $w(q) = 1$ , (1) implies that  $q \perp M$  (see 1.5). Since  $B \subset N = M[c]$ , the lemma is proved.  $\square$

We consider  $a$ -models  $M_0, M_1, M_2$  and  $M$  satisfying the following conditions

$$(2) \quad \left\{ \begin{array}{l} M_0 \subset M_1, \quad M_0 \subset M_2 \\ M_1 \downarrow_{M_0} M_2 \\ M \text{ } a\text{-prime over } M_1 \cup M_2 \end{array} \right.$$

DEFINITION 1.7.  $T$  has ENI-NDOP if for all  $a$ -models  $M_0, M_1, M_2$  and  $M$  satisfying (2), whenever  $p \in S(M)$  is an ENI type, then either  $p \not\perp M_1$ , or  $p \not\perp M_2$ .

REMARK. Compare Section 1 in [H-M].

PROPOSITION 1.8. (i)  $T$  having ENI-NDOP is equivalent to saying that whenever we have (2), and  $p$  is an ENI type, then  $p \not\perp M$  iff  $p \not\perp M_1$  or  $p \not\perp M_2$ .

(ii) If  $T$  has ENI-NDOP, then whenever  $M_0, M_1, \dots, M_n$  are  $a$ -models such that

$$(3) \quad \left\{ \begin{array}{l} M_0 \subset M_i \quad (1 \leq i \leq n), \\ \langle M_i \rangle_{1 \leq i \leq n} \text{ is independent over } M_0, \\ M \text{ is } a\text{-prime over } \bigcup_{1 \leq i \leq n} M_i, \end{array} \right.$$

and  $p$  is an ENI type, then  $p \not\perp M$  implies  $p \not\perp M_i$  for some  $i, 1 \leq i \leq n$ .

(iii) If  $T$  has ENI-DOP (the negation of ENI-NDOP), then there are: a finite tuple  $a$ , finite tuples  $d_1, d_2$  extending  $a$ , a finite tuple  $b$  extending  $d_1$  and  $d_2$ , and an ENI type  $p$  over  $b$  such that  $d_1/a, d_2/a$  are of weight 1,  $d_1 \downarrow_a d_2$ ,  $b$  is dominated by  $d_1 d_2/a$ , by  $d_2/d_1$ , and by  $d_1/d_2$ , moreover,  $p \perp d_1$  and  $p \perp d_2$ . Furthermore, we can arrange that  $d_1 \equiv d_2(a)$ , or  $d_1/a \perp d_2/a$ .

PROOF. The proofs are identical to corresponding proofs in [H-M], with the important addition of Proposition 1.3. In particular, (i) is proved like 1.2 in [H-M], by using 1.3 (and also D.19). The proof of (ii) is done by arguments used in the proof of 1.4 in [H-M], and (iii) is proved just as 1.5 in [H-M].  $\square$

The last two propositions in this section are somewhat of an afterthought to [M].

PROPOSITION 1.9. Let  $M_0$  be a countable model,  $\mathcal{P}$  a set of SR types over  $M_0$ . Then there is a countable model  $M$  extending  $M_0$  such that  $\dim(p, M) = 0$  for all  $p \in \mathcal{P}$ , and  $\dim(q, M)$  is infinite for all stationary  $q$  with  $\text{dom}(q)$  a finite subset of  $M$  such that  $q \perp p$  for all  $p \in \mathcal{P}$ .

PROOF. The proof is identical to that of D.12'.  $\square$

Let us call  $p \in S(A)$  *nearly SR via  $q$* , if  $p$  is stationary,  $q \in S(A)$  is SR, and for some  $a^0$  realizing  $p$ , and some  $b^0$  realizing  $q$ , we have that  $a$  is dominated by  $b^0/A$ ,  $b$  is dominated by  $a^0/A$ , and the types  $t(a^0/Ab^0)$  and  $t(b^0/Aa^0)$  are isolated. Clearly, in this case  $p$  has weight 1 in particular.

Suppose  $p \in S(A)$  is nearly SR via  $q$ , and  $J$  is a  $q$ -basis for  $M$ . Then, since  $t(a^0/Ab^0)$  (with  $a^0$  and  $b^0$  as above) is isolated, we can find, for each  $b \in J$ , an  $a$  in  $M$  with  $ab \equiv a^0b^0(A)$ . Taking one such  $a$  for each  $b$  in  $J$ , we form a set  $I$ ; by C.11(iv),  $I$  is an independent set over  $A$ . Moreover, given any independent set  $I'$  of realizations of  $p$  in  $M$ , the reverse process always gives rise to an independent set  $J'$  of realizations of  $q$ . It follows that, for  $I$  obtained from  $J$  as described, if  $J$  is a  $q$ -basis for  $M$ , then  $I$  is a  $p$ -basis for  $M$ , and conversely. Now we can easily prove the following versions of D.21(ii).

**PROPOSITION 1.10.** *Suppose  $p \in S(A)$  is nearly SR,  $A \subset M$  and  $I$  is a  $p$ -basis for  $M$ . Then*

$$p \upharpoonright AI \vdash p \upharpoonright M.$$

**PROOF.** Let  $p$  be nearly SR via  $q$ . Let  $a$  realize  $p \upharpoonright AI$ ; in other words,  $I \cup \{a\}$  is an independent set of realizations of  $p$ . Let us form  $J$ , a  $q$ -basis for  $M$ , out of  $I$  as described above, and in fact, let  $b$  be a realization of  $q$  such that  $ab \equiv a^0b^0(A)$  (with  $a^0, b^0$  as above). Thus  $b$  realizes  $q \upharpoonright AJ$ . Since, by D.21(i),  $q \upharpoonright AJ \vdash q \upharpoonright M$ ,  $b$  realizes  $q \upharpoonright M$ . Since  $a$  is dominated by  $b/A$ ,  $a$  realizes  $p \upharpoonright M$ , completing the proof.  $\square$

## §2. Consequences of having few countable models

From now on, for the rest of the paper we assume that  $T$  has less than  $2^{\aleph_0}$  isomorphism types of countable models.

At the referee's suggestion, we give a few introductory words trying to illuminate the rather technical contents of this section. Since at some critical points (notably, the proofs of 2.2 and 2.3), we will inevitably have to rely on the paper [H-M], we feel free to make that paper the point of departure for the purposes of this introduction as well.

The analysis of the models of the theory  $T$  in [H-M] relies on the concept of a *concrete chain*: a finite sequence  $N_0 \subset \cdots \subset N_n$  of models such that  $N_0$  is  $a$ -prime over  $\emptyset$ , and  $N_{k+1} = N_k[a_k]$  for some  $a_k$  SR over  $N_k$  such that  $a_k/N_k \perp N_{k-1}$  for  $0 < k < n$ . (We deliberately use here the  $a$ -variant of the notion described in detail at the beginning of section 5 of [H-M]; this variant is the more basic one, and it is the one that goes into the proof of theorem 5.4 there.) The isomorphism

types, called simply chains, of concrete chains, in an appropriate sense, form a tree in a natural way; the (foundation-)rank of this tree is an important invariant, called the depth, of the theory.

Now, the first main point is that here we are interested, essentially, only in chains for which, using the above notation for chains, the top type,  $t(a_{n-1}/N_{n-1})$ , is ENI. The reason ultimately is that the dimensions of NENI types over finite sets in a countable model cannot matter, since they must necessarily be equal to  $\aleph_0$ . Call such a chain an ENI chain (being ENI is invariant under isomorphism of concrete chains).

It is a crucial fact that, under the standing hypothesis of this section of  $T$  having less than  $2^{\aleph_n}$  non-isomorphic countable models, we can show that all ENI chains are of length at most 2 (i.e.,  $n$ , in the above notation, is at most 2). This is an equivalent way of stating what Proposition 2.4 below says.

We may define a notion of ENI-depth of any t.t.  $T$  (or, even a more general  $T$ ), although it is unclear how useful this notion is. We take the subset of the tree of chains consisting of all ENI chains; closing this set down under taking initial segments results in a subtree; the rank of this subtree might be called the ENI-depth. Of course, the above statement is equivalent to saying that, under the hypothesis of this section, the ENI depth is at most 2.

Another fact is that, for an ENI chain,  $a_0/N_0$  is necessarily trivial, analogously to a stronger fact shown in [H-M] under a different hypothesis; this fact is equivalent to 2.2 below. This fact is used among others to prove 2.3, ENI-NDOP (defined in Section 1); but also, the triviality of such types goes in a direct way into the "final analysis" of models given in Section 3 in a crucial manner.

The reader will notice that the terminology used here does not appear in the body of this section. Rather, we have to make a finer analysis of types over sets, mostly finite sets. In particular, we have to make a more general statement of the kind of relationship that the type  $t(a_{k+1}/M_{k+1})$  bears to the element  $a_k$  and the set  $M_k$  (here again, we used the above notation for chains;  $M_{k+1} = M_k[a_k]$ ). The resulting notion is that of type  $p$  *needing* an element  $c$  over a set  $A$  ( $p$  *needs*  $c/A$ ); in particular, it will be true that  $p = t(a_{k+1}/M_{k+1})$  needs  $a_k$  over  $M_k$  in the above situation. The notion is such that it is invariant under replacing  $p$  by a parallel, in fact, by a non-orthogonal, SR type. The most reasonable general formulation of this notion seems to be this:  $p$ , a stationary weight 1 type, *needs*  $c/A$  iff  $p \perp A$ , and for some  $B$  dominated by  $c/A$ , we have  $p \not\perp B$ . It turns out that, under the hypothesis of few countable models, and for  $p$  an ENI type, 'needing' can be formulated in a simpler way; this simpler formulation will be adopted after Proposition 2.7. In fact, we feel that talking about needing does

not really help much in understanding the technicalities of the first (larger) half of this section; that is why we do not use the terminology up to 2.8.

The last two results of this section are finiteness results whose significance for a structural description will likely be accepted easily. It should, however, be born in mind that in the “final analysis” not only these results, but others in this section as well, play important roles.

We first repeat a definition from [H–M].

**DEFINITION 2.1.** A stationary type  $q$  is called *trivial* if every nf extension  $q'$  of  $q$  satisfies the following: whenever  $I$  is a set of elements realizing  $q'$  such that any two-element subset of  $I$  is independent over  $\text{dom } q'$ , then  $I$  is independent over  $\text{dom } q'$ .

**PROPOSITION 2.2.** *Let  $A$  be a set,  $c$  a tuple of weight 1 over  $A$ ,  $c/A$  stationary,  $p$  an ENI type such that  $p \perp A$  and  $\text{dom}(p)$  is dominated by  $c/A$ . Then  $c/A$  is trivial.*

**PROOF.** The proof is essentially identical to that of 2.2 in [H–M]. The proof remains valid upon replacing  $\lambda$  by  $\aleph_0$ , and changing all references to  $a$ -models to ordinary models, in particular,  $a$ -prime models to prime models. The reason is that  $p$ , the type ‘supported by’  $c/A$ , is not only of weight 1, but it is ENI. This latter fact is used to show at the appropriate place that  $\dim(p_d, M_0) < \aleph_0$ . In fact, for  $M_1$ , the prime model over  $D$ , we have  $\dim(p_d, M_1) < \aleph_0$  by 1.2(ii), since  $p$  is ENI. The equality  $\dim(p, M_0) = \dim(p, M_1)$  follows by the same (easy) argument as the one used in the proof of D.12”(iii), since we have  $p_d \perp D'$ .  $\square$

**PROPOSITION 2.3.**  *$T$  has ENI-NDOP.*

**PROOF.** Again, the proof is essentially the same as that of 2.3 in [H–M]; the changes to be made are the same ones as in the case of 2.2. Of course, the “reduction” 1.5 in [H–M] is replaced by 1.8(iii).  $\square$

**PROPOSITION 2.4.** *Suppose  $p$  is an ENI type,  $C/A$  is stationary, and has weight 1,  $p \not\perp C$  and  $p \perp A$ . Then  $C/A \not\perp \emptyset$ .*

**PROOF.** Without loss of generality,  $C$  is finite,  $C = c_0$ . Suppose, on the contrary, that there are  $p_0$ ,  $c_0$  and  $A$  such that  $p_0$  is ENI,  $c_0/A$  has weight 1,  $p_0 \not\perp c_0$ ,  $p_0 \perp A$  and  $c_0/A \perp \emptyset$ . We first *claim* that then there are finite tuples  $a \subset b \subset c$  and an ENI type  $p$  with  $\text{dom}(p) = c$  such that  $c/b$ ,  $b/a$  are stationary, and of weight 1,  $p \perp b$ , and  $c/b \perp a$ . By applying 1.6 to  $q = t(c_0/A)$ , we can find

an  $a$ -model  $M$ ,  $b_0$  SR over  $M$ , such that  $A \subset M[b_0]$ ,  $c_0 \upharpoonright_A M[b_0]$ , and  $c_0/M[b_0] \perp M$ . By 1.5,  $p_0 \perp M[b_0]$ . Since  $p_0 \not\leq c_0$ , by D.19 and 1.3 there is an ENI type  $p$  over  $(M[b_0])[c_0]$  such that  $p \not\leq p_0$ . Let  $c_1$  be a finite tuple in  $M[b_0][c_0]$  containing  $c_0$  such that  $p$  is based on  $c_1$ , let  $b_1$  be a finite tuple in  $M[b_0]$  containing  $b_0$  such that  $c_1/M[b_0]$  is based on  $b_1$ , and let  $a$  be a finite tuple in  $M$  such that  $c_1 b_1/M$  is based on  $a$ . Let  $b = ab_1$ ,  $c = ab_1 c_1$ . Then  $c/M[b_0]$  is based on  $b$ ; by C.12(ii),  $c$  is dominated by  $c_0/M[b_0]$ , hence by D.2(ix),  $c/M[b_0]$  is of weight 1, and so by D.2(iv),  $c/b$  is of weight 1. Similarly,  $b/a$  is of weight 1. Since  $p_0 \perp M[b_0]$ , and  $p \not\leq p_0$ , we have  $p \perp b$ ; since  $c/M[b_0] \perp M$ , we have  $c/b \perp a$ . We have proved the *claim*.

Now, using the data of the claim, we proceed to construct  $2^{\aleph_0}$  pairwise non-isomorphic countable models, in contradiction to our assumption on  $T$ . Without loss of generality,  $a$  is the empty tuple (namely, having proved the existence of many models under the above conditions with  $a = \emptyset$ , we can apply the result to  $T' = \text{Th}(\mathcal{U}, a)$ ; but the conclusion for  $T'$  implies that for the original  $T$ ).

Let  $X$  be an arbitrary subset of  $\omega - \{0\}$ . For  $n \in X$ , let  $b^n$  be a tuple such that  $b^n \equiv b$ , and  $\langle c_i^n \rangle_{i < n}$  be a system independent over  $b^n$  of elements  $c_i^n$  such that  $c_i^n b^n \equiv cb$ . Let  $C_n = \{c_i^n : i < n\}$ . We furthermore make sure that  $\langle C_n \rangle_{n \in X}$  is independent. Let  $p_i^n = p_{c_i^n}$ . We *claim* that there is a model  $M_X$  such that  $\dim(p_i^n, M_X)$  is finite for all  $n \in X$ ,  $i < n$  and  $\dim(q, M_X)$  is infinite for all stationary  $q$  with a finite  $\text{dom}(q)$  included in  $M_X$  such that  $q \perp p_i^n$  for all  $n \in X$ ,  $i < n$ .

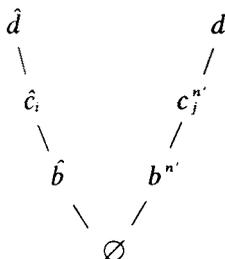
Let  $n \in X$  and  $i < n$  be fixed, and let  $M$  be a model prime over  $C_n$  such that  $M \upharpoonright_{b^n} C_{X-(n)}$ . Since  $p_i^n \perp b^n$ , we have  $p_i^n \upharpoonright M \perp C_{X-(n)}/M$ , from which it follows that in  $M(C_{X-(n)})$ ,  $p_i^n \upharpoonright M$  is not realized (any element  $d$  of  $M(C_{X-(n)}) - M$  satisfies  $d \not\perp_M C_{X-(n)}$ ). On the other hand, by 1.2(ii),  $\dim(p_i^n, M)$  is finite. It follows (by D.21(iii)) that  $\dim(p_i^n, M(C_{X-(n)}))$  is finite, hence for  $M_0$ , the prime model over  $C_X$ ,  $\dim(p_i^n, M_0)$  is finite (since  $M_0$  is embedded in  $M(C_{X-(n)})$ ). Since this is true for all  $n \in X$  and  $i < n$ , the existence of a suitable  $M_X$  follows from 1.9 applied to  $\mathcal{P} = \{p_i^n \upharpoonright M_0 : n \in X, i < n\}$ .

By a *class* we now mean an equivalence class of elements  $\hat{b}$  in  $M_X$  such that  $\hat{b} \equiv b$ , under the equivalence relation  $( \ ) \not\perp ( \ )$ . The class containing  $\hat{b}$  is denoted  $[\hat{b}]$ . Given a class  $\mathcal{B}$ , and  $\hat{b} \in \mathcal{B}$ , let  $m(\hat{b})$  be the supremum of all cardinals  $m \leq \omega$  such that there is a system  $\langle \hat{c}_i \rangle_{i < m}$ , independent over  $\hat{b}$ , of elements  $\hat{c}_i$  in  $M_X$  such that  $\hat{c}_i \hat{b} \equiv cb$  and such that for all  $i < m$ ,  $\dim(p_{\hat{c}_i}, M_X)$  is finite, and let  $m(\mathcal{B}) = \sup\{m(\hat{b}) : \hat{b} \in \mathcal{B}\}$ . We *claim* that  $m([\hat{b}^n]) = n$ , and that  $m(\mathcal{B}) = 0$  for a class  $\mathcal{B}$  distinct from any  $[\hat{b}^n]$ ,  $n \in X$ . This claim will clearly

imply that for distinct  $X, X' \subset \omega - \{0\}$ ,  $M_X \neq M_{X'}$ , since  $X$  is recovered from  $M_X$  as the set  $\{m(\mathcal{B}) : \mathcal{B} \text{ a class in } M_X, m(\mathcal{B}) \neq 0\}$ .

To show the first assertion of the claim, we note that  $m([b^n]) \cong n$  holds directly by the construction. On the other hand, let  $\hat{b} \in [b^n]$ ,  $\langle \hat{c}_i \rangle_{i < m}$  independent over  $\hat{b}$ ,  $\hat{c}_i \hat{b} \equiv cb$ , let  $\hat{p}_i = p_{\varepsilon_i}$ , and assume that  $\dim(\hat{p}_i, M)$  is finite for all  $i < m$ .

Note that  $\hat{p}_i \perp p_j^{n'}$  for all  $n' \in X - \{n\}$ ,  $j < n'$ ; this follows from 3.3 in [H-M], since the "tree"



with  $\hat{d}$  realizing  $\hat{p}_i$ ,  $d$  realizing  $p_j^{n'}$  is a normal tree; also, the assertion can be shown by an elementary computation using given (non-)orthogonalities and  $\hat{b} \perp b_j^{n'}$ . Therefore, it follows by the defining property of  $M_X$ , and the fact that  $\dim(\hat{p}_i, M)$  is finite, that for every  $i < m$  there is  $i' < n$  such that  $\hat{p}_i \not\perp p_{i'}^{n'}$ . Now, we notice that  $\hat{p}_i \perp \hat{p}_{i'}$  for  $i \neq i'$ ,  $i, i' < m$ ; this is because  $\hat{c}_i \perp_{\hat{b}} \hat{c}_{i'}$ , and  $\hat{p}_i \perp \hat{b}$ . Therefore,  $i \neq i'$  implies  $i \neq i'$ ; this proves that  $m \leq n$ , as promised.

If  $\mathcal{B} = [\hat{b}] \neq [b^n]$  for all  $n \in X$ , i.e.,  $\hat{b} \perp b^n$  for all  $n \in X$ , then if  $\hat{c} \hat{b} \equiv cb$ , and  $\hat{p} = p_\varepsilon$ , we have  $\hat{p} \perp p_j^{n'}$  for all  $n \in X$ ,  $j < n$ , just as in the previous paragraph. By the defining property of  $M_X$ , we conclude that  $m(\mathcal{B}) = 0$ , as required for the second assertion of the claim.

The proof is complete.  $\square$

**PROPOSITION 2.5.** *Suppose  $p$  is an ENI type,  $p \not\perp B$ , and  $B$  is dominated by  $C/A$ . Then  $p \not\perp AC$ .*

**PROOF.** Without loss of generality,  $B$  is a finite set. Assume, contrary to the assertion, that  $p \perp AC$ . Choose an  $a$ -model  $M \supset A$  such that  $BC \perp_A M$ , and an  $a$ -model  $M' \supset M$  such that  $B \perp_{MC} M'$ . It follows that we have  $b \perp_A M'$ , hence by 1.5 and the assumptions,  $p \perp M'$ . Let  $\vec{b} = \langle b_i \rangle_{i < n}$  be a sequence of elements realizing SR types over  $M'$  such that  $M'[B] = M'[\vec{b}]$ . By a suitable choice of the models  $M'[b_i]$ , we have that  $M'[B]$  is  $a$ -prime over  $\bigcup_{i < n} M'[b_i]$ . Hence, by ENI-NDOP (1.8(ii)), and  $p \not\perp B$ , there is  $i < n$  such that  $p \not\perp N$  for  $N = M'[b_i]$ . Since  $p \perp M'$ , by 2.4 it follows that  $b_i/M' \not\perp \emptyset$ . Hence, there is  $q \in S(M)$ , an SR type, such that  $b_i/M' \not\perp q$ . By D.18, there is  $b \in N$  realizing  $q \upharpoonright M'$  such that

$N = M'[b]$ . We claim that there is a copy of  $M[b]$  such that  $N$  is  $a$ -prime over  $M' \cup M[b]$ . Indeed, let  $M[b]$  be any copy of the model  $a$ -prime over  $Mb$ , and let  $N'$  be the model  $a$ -prime over  $M' \cup M[b]$ . Then,  $N'$  is  $a$ -prime over  $M'b$ ; to see this, we use  $M' \downarrow_M b$ , and  $M' \downarrow_M M[b]$  as a consequence. Therefore, there is an isomorphism mapping  $N'$  onto  $N$  over the set  $M'b$ . The image of  $M[b]$  under this isomorphism is the desired copy of  $M[b]$ . Having  $M[b]$  in the desired way, by ENI-NDOP again, and by  $p \perp M'$ , we conclude that  $p \not\perp M[b]$ . Now, note again that  $c \downarrow_M b$ ; since  $B$  is dominated by  $c/M$ ,  $B \downarrow_M b$ , and hence  $B \downarrow_M M[b]$ . Since  $p \not\perp B$ , and  $p \not\perp M[b]$ , by 1.5 we obtain  $p \not\perp M$ , a contradiction to  $p \perp M'$ .  $\square$

**PROPOSITION 2.6.** *Let  $p$  be an ENI type, suppose  $c/A$ ,  $c'/A$  are stationary and weight 1,  $c \not\downarrow_A c'$ ,  $p \not\perp Ac$ , and  $p \perp A$ . Then  $p \not\perp Ac'$ .*

**PROOF.** With  $B$  a finite set such that  $p$  is based on  $B$ , let  $M$  be an  $a$ -model containing  $A$  such that  $Bcc' \downarrow_A M$ . Then  $p \perp M$ . Note that we have  $c \not\downarrow_M c'$ . Let  $\vec{c} = \langle c_i \rangle_{i < n}$  be a sequence independent over  $M$  of elements  $c_i$  of weight 1 over  $M$  such that  $N = M[Bcc'] \stackrel{\text{df}}{=} M[\vec{c}]$  and such that  $c_0 = c$ . Since  $c \not\downarrow_M c'$ ,  $\vec{c}' = c' \wedge \langle c_i \rangle_{1 \leq i < n}$  also is a maximal independent system of elements having weight 1 over  $M$ , hence  $N = M[\vec{c}']$ . Now, since  $p \not\perp M[c]$ ,  $p \perp M$ , and  $M[c] \downarrow_M c_i$  ( $1 \leq i \leq n$ ), by 1.5 we have  $p \perp Mc_i$  for  $1 \leq i < n$ . Since  $N = M[\vec{c}']$ , there are copies  $M[c']$ ,  $M[c_i]$  ( $1 \leq i < n$ ) of the models  $a$ -prime over  $Mc'$  and  $Mc_i$ , respectively, such that  $N$  is  $a$ -prime over  $M[c'] \cup \bigcup_{1 \leq i < n} M[c_i]$ . Since  $p \perp Mc_i$ , and  $M[c_i]$  is dominated by  $c_i/M$ , by 2.5,  $p \perp M[c_i]$ . Therefore, by ENI-NDOP (1.8(ii)), necessarily  $p \not\perp M[c']$ , and by 2.5 again,  $p \not\perp Mc'$ . Since we have  $B \downarrow_{Ac'} Mc'$ , and  $p \not\perp B$ , we have  $p \not\perp Ac'$  as required.  $\square$

**PROPOSITION 2.7.** *Suppose  $p$  is an ENI type,  $A \subset A'$ ,  $c \downarrow_A A'$ ,  $p \not\perp A'c$ ,  $p \perp A'$ . Then  $p \not\perp Ac$ .*

**PROOF.** Without loss of generality,  $A$  is an  $a$ -model  $M$ . (To see this, choose an  $a$ -model  $M$  such that  $A \subset M$  and  $M \downarrow_A A'c$ ; in particular,  $c \downarrow_M A'$  and  $c \downarrow_A M$ . From  $p \not\perp A'c$  and  $p \perp A'$ , it follows that  $p \perp MA'$  by 1.5. Thus, we have the hypotheses with  $M$  for  $A$  and  $MA'$  for  $A'$ . From  $p \not\perp Mc$ ,  $p \not\perp A'c$  and  $Mc \downarrow_{Ac} A'c$ ,  $p \not\perp Ac$  follows by 1.5.) Let  $N = [M[A'], M[c]]$ , the  $a$ -prime model over the union of  $M[A']$  and  $M[c]$ . Since  $p \not\perp N$ , and  $M[A'] \downarrow_M M[c]$ , by ENI-NDOP we have either  $p \not\perp M[A']$  or  $p \not\perp M[c]$ . But the first possibility is ruled out, since by 2.5 it would imply  $p \not\perp MA'$ , contrary to  $p \perp A'$  ( $M \subset A'$ ). Thus,  $p \not\perp M[c]$ , and by 2.5 again,  $p \not\perp Mc$  as desired.  $\square$

We now introduce some terminology. We say that the ENI type  $p$  needs  $c/A$  if

$c/A$  is stationary and of weight 1,  $p \not\perp Ac$  and  $p \perp A$ . Notice that 1.6 together with 2.5 says that  $p \perp A$  implies that  $p$  needs  $c/A'$  with some  $A' \supset A$  and some  $c$ , with  $c/A'$  SR. Also, 2.6 says that if  $p$  needs  $c/A$ , and  $c \downarrow_A c'$ ,  $c'/A$  is stationary and of weight 1, then  $p$  needs  $c'/A$ . 2.7 says that if  $c/A$  is stationary,  $A \subset A'$ ,  $c \downarrow_A A'$ , and  $p$  needs  $c/A'$ , then  $c$  needs  $c/A$ . In all the above,  $p$  is assumed to be ENI. We call a type  $q \in S(A)$  *supportive* if for some (any)  $c$  realizing  $q$ , there is an ENI type  $p$  needing  $c/A$ . Note that 2.4 says that every supportive type is  $\not\perp \emptyset$ . Also, it is clear that every supportive type is  $\not\perp$  to a SR supportive type (*exercise*).

**PROPOSITION 2.8.** *Suppose  $p, p'$  are ENI types,  $p$  needs  $c/b$ ,  $p'$  needs  $c'/b'$ . Suppose furthermore that  $c/b \perp c'/b'$ . Then  $p \perp p'$ .*

**PROOF.** Start by choosing tuples  $\hat{b}$  and  $\hat{b}'$  such that  $\hat{b} \stackrel{s}{=} b$ ,  $\hat{b}' \stackrel{s}{=} b'$ , and

$$(3) \quad bcb'c', \hat{b}, \hat{b}' \text{ are independent (over } \emptyset \text{)}.$$

Since  $q = t(c/b) \not\perp \emptyset$  ( $q$  is a supportive type), we have  $q_b \not\perp q$  (see C.6(i)). Let  $M$  be an  $a$ -model containing  $b\hat{b}\hat{b}'$  such that  $c \downarrow_{b\hat{b}\hat{b}'} M$ ; then  $c \downarrow_b M$  by (3). Let  $\hat{c}$  realizing  $q_b \upharpoonright M$  be such that  $c \downarrow_M \hat{c}$ . Since  $p$  needs  $c/b$ , we have (by  $c \downarrow_b M$ ) that  $p$  needs  $c/M$ . Hence, by  $c \downarrow_M \hat{c}$ ,  $p$  needs  $\hat{c}/M$ . Since  $\hat{c} \downarrow_b M$  ( $\hat{c}$  realizes  $q_b \upharpoonright M$ ),  $p$  needs  $\hat{c}/\hat{b}$ . Note also that we have  $\hat{c} \downarrow_b \hat{b}'$  (since  $\hat{b}'$  belongs to  $M$ ).

By a symmetric argument, we can find  $\hat{c}'$  realizing  $q'_{b'}$  (for  $q' = t(c'/b')$ ) such that  $p'$  needs  $\hat{c}'/\hat{b}'$ , and  $\hat{c}' \downarrow_b \hat{b}$ . Now, we apply the fact that  $q \perp q'$ . Since  $q_b \not\perp q$ ,  $q'_{b'} \not\perp q'$ , and all the types involved are stationary and of weight 1, it follows that  $q_b \perp q'_{b'}$ , hence  $q_b \upharpoonright b\hat{b}' \perp q'_{b'} \upharpoonright b\hat{b}'$ . Since  $\hat{c}$  realizes the first,  $\hat{c}'$  the second, of the last two types, we conclude that we have

$$(4) \quad \hat{c} \downarrow_{b\hat{b}'} \hat{c}'.$$

Since  $p$  needs  $\hat{c}/\hat{b}$ , and  $\hat{c} \downarrow_b \hat{b}'$ ,  $p$  needs  $\hat{c}/b\hat{b}'$ . Similarly,  $p'$  needs  $\hat{c}'/b\hat{b}'$ . By (4) and 1.5, we have  $p \perp b\hat{b}'\hat{c}'$ . Since  $p' \not\perp b\hat{b}'\hat{c}'$ , it follows that  $p \perp p'$ .  $\square$

**PROPOSITION 2.9.** *There are only finitely many equivalence classes of supportive types under the equivalence relation  $\not\perp$ .*

**PROOF.** Suppose, on the contrary, that there are infinitely many pairwise  $\perp$  supportive types. We'll construct  $2^{\aleph_0}$  non-isomorphic countable models.

Suppose  $q \in S(b)$  is a supportive type. Notice that there are finitely many strong types over  $\emptyset$  extending  $r = t(b)$ , and for any  $b', b''$  realizing  $r$ , if  $b' \stackrel{s}{=} b''$ ,

then  $q_b \not\perp q_{b'}$  (since  $q \not\perp \emptyset$ ). Hence there are altogether finitely many  $\perp$ -classes that contain an isomorphic copy  $q_{b'}$  of  $q$  ( $b' \equiv b$ ).

Suppose further that  $q'$  is another supportive type. If  $q' \not\perp q$ , and  $\hat{q}$  is an isomorphic copy of  $q$ , then for  $\hat{q}'$ , the isomorphic copy of  $q'$  under the automorphism of  $\mathcal{C}$  that maps  $q$  into  $\hat{q}$ , we clearly have  $\hat{q}' \not\perp \hat{q}$ . In other words, if  $q, q'$  are two supportive types (in fact, if  $q, q'$  are stationary weight-1 types), then the set of equivalence classes under  $\perp$  of isomorphic copies of  $q$  is either identical to, or disjoint from, the set of equivalence classes under  $\perp$  of isomorphic copies of  $q'$ .

It now follows that there is a sequence  $\langle q_n \rangle_{n < \omega}$  of supportive types such that for  $n, n' < \omega, n \neq n'$ , every isomorphic copy of  $q_n$  is  $\perp$  to every isomorphic copy of  $q_{n'}$ .

Let  $b_n = \text{dom}(q_n)$ , let  $c_n$  be a realization of  $q_n$ ; we may assume that  $b_n$  is a subtuple of  $c_n$ ; let  $p_n$  be an ENI type needing  $c_n/b_n$ ; let  $d_n = \text{dom}(p_n)$ ; we may assume that  $c_n$  is a subtuple of  $d_n$ , and also that  $p_n$  is not isolated.

Let  $X$  be an arbitrary subset of  $\omega$ . For  $n \in X$ , let  $\hat{b}_n, \hat{c}_n, \hat{d}_n$  be such that  $\hat{b}_n \equiv b_n, \hat{b}_n \hat{c}_n \hat{d}_n \equiv b_n c_n d_n$ , and let us ensure that the system pictured by

$$\begin{array}{ccccccc}
 \dots & \hat{d}_n & \dots & \hat{d}_{n'} & \dots & & \\
 & | & & | & & & \\
 \dots & \hat{c}_n & \dots & \hat{c}_{n'} & \dots & & (n, n' < \omega) \\
 & | & & | & & & \\
 \dots & \hat{b}_n & \dots & \hat{b}_{n'} & \dots & & 
 \end{array}$$

is independent relative to the obvious "height-3" partial ordering (see A.11, A.12). Let  $\hat{p}_n = (p_n)_{\hat{d}_n}$ . With  $M_0$  the prime model over  $\hat{d}_X (= \{\hat{d}_n : n \in X\})$ ,  $\text{dim}(\hat{p}_n, M_0) = 0$ . The reason is that since  $\hat{p}_n \perp \emptyset$ , and  $\hat{d}_n \downarrow \hat{d}_{X-n}$ , we have  $\hat{p}_n \upharpoonright \hat{d}_X \perp \hat{p}_n$ , hence  $\text{dim}(\hat{p}_n, M_0) = \text{dim}(\hat{p}_n \upharpoonright \hat{d}_X, M_0)$ ; the latter is 0 since  $\hat{p}_n \upharpoonright \hat{d}_X$  is non-isolated (since  $\hat{p}_n$  is non-isolated), and  $M_0$  is prime over  $\hat{d}_X$ . By 1.9, there is a countable model  $M = M_X$  extending  $M_0$  such that  $\text{dim}(p_n, M) = 0$  for all  $n \in X$ , and  $\text{dim}(p, M)$  is infinite for any SR type  $p$  with  $\text{dom}(p)$  a finite subset of  $M$ , and  $p \perp \hat{p}_n$  for all  $n \in X$ .

We claim that  $n \in X$  iff there are  $\tilde{b}, \tilde{c}, \tilde{d}$  in  $M$  such that  $\tilde{b}\tilde{c}\tilde{d} \equiv b_n c_n d_n$  and  $\text{dim}((p_n)_{\tilde{d}}, M)$  is finite. Indeed, the 'only if' direction is true by construction (take  $\tilde{b} = \hat{b}_n, \tilde{c} = \hat{c}_n, \tilde{d} = \hat{d}_n$ ). Conversely, assume  $n \in \omega, \tilde{b}\tilde{c}\tilde{d}$  is in  $M$ , and has the same type as  $b_n c_n d_n$ . For  $n' \neq n, \tilde{c}/\tilde{b} \perp \hat{c}_{n'}/\hat{b}_{n'}$ , by the choice of the types  $q_n$ . Since  $\tilde{p} = (p_n)_{\tilde{d}}$  needs  $\tilde{c}/\tilde{b}$ , and  $\hat{p}_{n'}$  needs  $\hat{c}_{n'}/\hat{b}_{n'}$ , by 2.8 we have that  $\tilde{p} \perp \hat{p}_{n'}$ . Thus, if  $n \notin X$ , then  $\tilde{p}$  is  $\perp$  to all  $\hat{p}_{n'}$  for  $n' \in X$ , hence by the defining property of  $M$ ,  $\text{dim}(\tilde{p}, M)$  is infinite; this shows the claim.

The claim clearly shows that  $X \neq X'$  implies  $M_X \neq M_{X'}$ , which proves the proposition.  $\square$

PROPOSITION 2.10. *Let  $A$  be any finite set. There are only finitely many equivalence classes of ENI types  $\mathcal{L}$  to  $A$  under the equivalence relation  $\mathcal{L}$ .*

PROOF. Without loss of generality,  $A = \emptyset$ . Let  $M_0$  be a fixed copy of the prime model of  $T$ . By D.19, and 1.3, every ENI type  $\mathcal{L}$  to  $\emptyset$  is  $\mathcal{L}$  to one over  $M_0$ , hence one over a finite tuple in  $M_0$ . Suppose the assertion of the proposition fails. Using also an argument used in the proof of 2.9, we find that there is a sequence  $\langle p_n \rangle_{n < \omega}$  of ENI types such that  $b_n \stackrel{\text{df}}{=} \text{dom}(p_n)$  is in  $M_0$ , and every isomorphic copy of  $p_n$  is  $\perp$  to every isomorphic copy of  $p_{n'}$ , for  $n \neq n'$ . Let  $X$  be an arbitrary subset of  $\omega$ . Since  $\dim(p_n, M_0)$  is finite (see 1.2(ii)), for all  $n < \omega$ , by 1.9 there is a model  $M_X$  extending  $M_0$  such that  $\dim(p_n, M_X)$  is finite for all  $n \in X$ , and  $\dim(p, M_X)$  is infinite for every stationary weight-1 type  $p$  with  $\text{dom}(p)$  a finite subset of  $M_X$  such that  $p \perp p_n$  for all  $n \in X$ .

$X$  can be recovered from  $M_X$  as follows:  $n \in X$  iff there is an isomorphic copy  $p$  of  $p_n$ , with  $\text{dom}(p)$  in  $M_X$ , such that  $\dim(p, M_X)$  is finite. Certainly, the ‘only if’ direction is clear. Supposing that  $p$  is an isomorphic copy of  $p_n$ , we have that  $p \perp p_{n'}$  for all  $n' \neq n$ . Hence, if  $n \notin X$ , then  $\dim(p, M_X)$  is infinite, by the defining property of  $M_X$ . This shows what we want.  $\square$

### §3. The final analysis

In this section, we state and prove a theorem that amounts to a characterisation up to isomorphism of an arbitrary countable model in terms of certain invariants. The statement of the characterisation takes some preparation. Once the characterisation is stated, however, the Vaught conjecture for  $T$  is seen to be an essentially trivial consequence of it. Namely, using the characterisation, one can write down a Scott sentence  $\sigma_M$  of any countable model of  $T$  ( $\sigma_M$  is a sentence of  $L_{\omega_1, \omega}$  such that the countable models of  $\sigma_M$  are exactly those isomorphic to  $M$ ), and one sees that the quantifier rank of  $\sigma_M$  is low, in particular,  $\text{qr}(\sigma_M) < \omega \cdot \omega$ . It is well-known that from this the Vaught conjecture for  $T$  follows (see [4]). The writing out of  $\sigma_M$  will be left to the reader as a trivial but tedious exercise.

First, we set up the “reference-points” of the invariants.

Let  $M$  be a fixed copy of the prime model.

Let  $Q$  be a finite set of pairwise  $\perp$  SR supportive types over  $M$  such that every supportive type is  $\mathcal{L}$  to a member of  $Q$ . Since every supportive type  $q$  is  $\mathcal{L}$  to the

empty set (2.4), and since  $q$  is supportive,  $q'$  SR and  $q \not\mathcal{L} q'$  imply that  $q'$  is supportive (2.6), every supportive type is  $\mathcal{L}$  to a supportive type over  $M$  (by using D.19). Also, there are only finitely many  $\mathcal{L}$ -classes of supportive types (2.9). It follows that  $Q$  with the stated properties exists.

Choose and fix a realization  $c_q$  of each  $q \in Q$ ; put  $C = \{c_q : q \in Q\}$ .

Let  $A$  be a finite subset of  $M$  such that every  $q \in Q$  is based on  $A$ .

For each  $q \in Q$ , let  $M_q \supset M$  be a fixed copy of  $M(c_q)$ ; we write  $M_c$  for  $M_q$  if  $c = c_q$ .

For each  $q \in Q$ , let  $P_q$  be a finite set of pairwise  $\perp$  ENI types over  $M_q$  such that every ENI type which is  $\mathcal{L}$  to  $Ac_q$  is  $\mathcal{L}$  to a member of  $P_q$ .  $P_q$  exists by 2.10. We also write  $P_c$  for  $P_q$  with  $c = c_q$ .

For each  $c \in C$ , let  $B_c$  be a finite subset of  $M_c$  such that  $Ac \subset B_c$ , every  $p \in P_c$  is based on  $B_c$ , and  $p \upharpoonright B_c$  is non-isolated (since  $p$  is ENI, by 1.2(iii) such  $B_c$  can be chosen).

Let  $A'$  be a finite subset of  $M$  such that  $A \subset A'$ , each  $t(\vec{B}_c/M)$  ( $c \in C$ ) is based on  $A'$ , and each  $t(\vec{B}_c/A'c)$  is isolated.

For each  $q \in Q$ , let  $P'_q (= P'_c)$  be the subset of  $P_q$  consisting of those members of  $P_q$  that are  $\perp$  to  $A'$ . Thus, every member of  $P'_c$  needs  $c/A'$ . Moreover, if  $p$  is any ENI type that needs  $c/A'$ , we have that  $p$  needs  $c/A$  by 2.7, since  $c \upharpoonright_A A'$ , hence  $p$  is  $\mathcal{L}$  to some member  $p'$  of  $P_c$ , and  $p'$  must belong to  $P'_c$ .

Let  $P$  be a finite set of pairwise  $\perp$  ENI types, each  $\mathcal{L}$  to  $A'$ , each over  $M$ , each  $\perp$  to every member of  $Q$ , such that every ENI type which is  $\mathcal{L}$  to  $A$ , is  $\mathcal{L}$  to either a member of  $Q$ , or a member of  $P$ .  $P$  exists by 2.10.

Let  $B$  be a finite subset of  $M$  such that  $A' \subset B$ , every member of  $P$  is based on  $B$ , and such that for  $p \in P$ ,  $p \upharpoonright B$  is non-isolated.

Let  $A''$  be a finite set such that  $A' \subset A'' \subset \text{acl}(A')$  (with the algebraic closure meant in  $\mathcal{C}^{\text{eq}}$ ) such that  $\vec{B}/A''$  is stationary, and  $\vec{B}_c/A''$  is stationary for every  $c \in C$ . By B.4 and by the fact that the theory is t.t., it is easy to see that such  $A''$  exists.

Let us put:

$$\left. \begin{aligned} Q' &= \{q \mid A'' : q \in Q\}, \\ B'_c &= B_c \cup A'', \\ P''_c &= \{p \mid B'_c : p \in P'_c\}, \end{aligned} \right\} c \in C$$

$$B' = B \cup A'',$$

$$P' = \{p \mid B' : p \in P\}.$$

Primed (or double-primed) items are updated versions of the corresponding previous items; these latter can be discarded. Also, let  $B^0$  be the set of all the finite tuples  $\vec{B}'_c$  for  $c \in C$ ,  $B^0 = \{\vec{B}'_c : c \in C\}$ . We can discard the  $c$ 's in  $C$ , and work with the  $b$ 's in  $B^0$ . Also, let  $R^0$  be the set of all types  $t(b/A'')$  for  $b \in B^0$ ; thus,  $B^0 = \{b : r \in R^0\}$  for a unique assignment  $r \mapsto b$ , such that  $b$ , realizes  $r$ . Let us write  $a^0$  for  $\vec{A}''$ ,  $d^0$  for  $\vec{B}'$ ,  $P^0$  for  $P'$ , and  $P^0_b$  for  $P'_c$  if  $b = \vec{B}'_c$ ; also,  $P^0_r = P^0_b$  for  $r \in R^0$ .

Besides the items marked with superscript 0, we do not need the others above, and we may use the notation used for them for other things. Here is the list of the properties of the items we'll need.

$a^0$  is a subtuple of  $d^0$ ,  $t(d^0/\emptyset)$  is isolated, and  $t(d^0/a^0)$  is stationary.

$R^0$  is a set of pairwise  $\perp$ , stationary weight-1 types over  $a^0$  such that every supportive type is  $\not\perp$  to a member of  $R^0$ .

(The fact that  $t(b_c/a)$  is of weight 1 follows from the fact that  $B'_c$  is dominated by  $c$  over  $A''$ , and that  $c/A''$  is of weight 1, see D.2(ix). The same reasons, together with the fact that the elements of  $Q$  are pairwise  $\perp$ , are sufficient for the elements of  $R^0$  to be pairwise  $\perp$ , as is easily seen.)

If  $r \in R^0$ ,  $M$  is any model containing  $a^0$ ,  $I$  is an  $r$ -basis in  $M$ , then  $r \upharpoonright a^0 I \vdash r \upharpoonright M$ . (This follows from 1.10 since clearly  $r$  is nearly SR.)

Every type in  $R^0$  is trivial (this follows from 2.2).

$B^0$  is a set of representative realizations of the types in  $R^0$ ;  $B^0 = \{b_r : r \in R^0\}$ ,  $b_r$  realizes  $r$ .  $a^0$  is a subtuple of each  $b$  in  $B^0$ .

For each  $b \in B^0$ ,  $P^0_b$  is a finite set of pairwise  $\perp$  non-isolated SR types over  $b$ , each  $\perp$  to  $a^0$ , such that any ENI type that needs  $b/a^0$  is  $\not\perp$  to a member of  $P^0_b$ . (Indeed, if  $P$  is an ENI type that needs  $b/a^0$ , then, for  $c$ , the ' $c$ -part' of  $b$ ,  $p$  needs  $c/a^0$  (this follows from 2.5, and the fact that  $b$  is dominated by  $c/a^0$ ); hence, by the above,  $p$  is  $\not\perp$  to a member of  $P^0_b$ .)

$P^0$  is a finite set of pairwise  $\perp$  non-isolated SR types over  $d^0$ , each  $\not\perp$  to  $a^0$ , each  $\perp$  to every member of  $R^0$ , such that every ENI type  $\not\perp$  to  $a^0$  is  $\not\perp$  either to a member of  $R^0$ , or to a member of  $P^0$ . (Note that passing from  $A'$  to  $A'' (= a^0)$  did not affect the essential properties of  $A'$ , since  $A' \subset A'' \subset \text{acl}(A')$ .)

This completes the list of "reference-points" for the invariants.

Now, let  $M$  be an arbitrary countable model. We choose  $a$  in  $M$  such that  $a \equiv a^0$ , and  $d$  in  $M$  such that  $ad \equiv a^0 d^0$ . We consider the types  $p_a$  for all  $p \in P^0$ ;  $p_a$  is the copy of  $p$  under the change of  $d^0$  to  $d$ . Let  $\vec{m} = \langle \dim(p_a, M) : p \in P^0 \rangle$  be the vector of dimensions of the types  $p_a$  in  $M$ . Note that by 1.4, and by the fact that  $t(d/a)$  is stationary,  $d \equiv d'(a)$  implies that  $\vec{m}$  computed with  $d$  and that computed with  $d'$  are equal; in other words,  $\vec{m}$  depends only on  $(M$  and)  $a$ .

Next, we consider the types  $r_a$  for  $r \in R^0$ ; in other words, if  $b$  realizes  $r_a$ , we have  $ba \equiv b_a a^0$ . For  $r \in R^0$ , we let  $X_r$  be the family of all  $\downarrow_a$ -classes of realizations of  $r_a$ ;  $X_r$  is the set of all classes of the form  $\{b' \in M : b' \downarrow_a b, b' \models r_a\}$  for all  $b$  in  $M$  realizing  $r_a$ . Put  $X = \bigcup_{r \in R^0} X_r$ .

Now let  $\mathcal{B}$  be a class in  $X_r$  ( $r \in R^0$ ). Pick  $b \in \mathcal{B}$ ; in particular,  $b$  realizes  $r$ . Consider the types  $p_b$  for  $p \in P_r^0$  and let  $\vec{n} = \langle n_p \rangle_{p \in P_r^0}$  be the vector of dimensions  $n_p \stackrel{\text{def}}{=} \dim((p)_b, M)$ . Of course,  $\vec{n}$  depends on  $b$ ;  $\vec{n} = \vec{n}(b)$ . With a fixed  $\mathcal{B}$  in  $X_r$ , we consider the set of all possible dimension-vectors  $D(\mathcal{B}) = \{\vec{n}(b) : b \in \mathcal{B}\}$ . For  $r \in R^0$ , let us denote by  $\mathcal{D}_r^0$  the set of all countable sets of vectors  $\vec{n} = \langle n_p \rangle_{p \in P_r^0}$ , with arbitrary  $n_p \leq \omega$ ;  $\mathcal{D}_r^0$  is a set defined independently of  $M$ . Clearly,  $D(\mathcal{B}) \in \mathcal{D}_r^0$  for  $\mathcal{B} \in X_r$ . We count how many times each  $D \in \mathcal{D}_r^0$  occurs as  $D(\mathcal{B})$  in  $M$  for  $\mathcal{B} \in X_r$ ; this number is denoted by  $s_D$ . In other words,  $s_D$  is the cardinality of the set of all  $\mathcal{B} \in X_r$  such that  $D(\mathcal{B}) = D$ . We have that  $s_D \leq \aleph_0$ , and  $s_D = 0$  except for countably many  $D \in \mathcal{D}_r^0$ .

The system of invariants for  $(M, a)$  consists, by definition, of the vector  $\vec{m}$ , and the vectors

$$\langle s_D \rangle_{D \in \mathcal{D}_r^0}$$

for all  $r \in R^0$ . (Again, notice that once the copy  $a$  of  $a^0$  has been chosen in  $M$ , the above data are all uniquely defined.)

**THEOREM 3.1.** *Suppose  $T$  is a countable  $\omega$ -stable theory with less than  $2^{\aleph_0}$  many non-isomorphic countable models. Let  $a \equiv a' \equiv a^0$ ,  $a \in M$ ,  $a' \in M'$ ,  $M$  and  $M'$  are countable models. Then, if  $(M, a)$  and  $(M', a')$  have the same invariants, they are isomorphic.*

The rest of the section is devoted to the proof of the theorem. As we indicated above, the Vaught conjecture for  $T$  is an essentially trivial consequence of the theorem.

Let  $M$  be a countable model. Let us make all the specifications and choices as above relative to  $M$ . Let  $R$  be the set of all types  $(r)_a$ , for  $r \in R^0$ . For  $r \in R$ , let  $B_r$  be a maximal independent set of elements of  $M$  each realizing  $r$ , and let  $B = \bigcup_{r \in R} B_r$ . Note that  $B$  is a maximal independent set of elements of  $M$  each realizing a type in  $R$ .

Let  $N$  be a *maximal atomic* model over  $aB$  in  $M$ :  $aB \subset N \subset M$ ,  $N$  atomic over  $aB$ , and if  $N \subset D \subset M$ ,  $D$  atomic over  $aB$ , then  $N = D$ . Since the union of an increasing chain of sets atomic over  $aB$  is again atomic over  $aB$ , and for every  $D \subset M$  atomic over  $aB$  there is  $N'$ ,  $D \subset N' \subset M$ ,  $N'$  atomic over  $aB$ , such  $N'$  clearly exists

For  $b \in B_r$ ,  $r \in R$ , let  $P_b$  be the set of all the types  $p_b \upharpoonright N$ , for  $p \in P_r^0$ . Let  $P_1$  be the set of all the types  $p_a \upharpoonright N$  for  $p \in P^0$ . Put  $P = \bigcup_{b \in B} P_b \cup P_1$ . Finally, let  $I$  be a  $P$ -basis for  $M$ , i.e. a maximal independent set of realizations of types in the set  $P$ .

CLAIM 3.2.  $M$  is atomic over  $N \cup I$ .

PROOF. Let  $M'$  be maximal atomic over  $N \cup I$  in  $M$ . We are going to show that  $M' = M$ .

First we show (*Subclaim 1*) that no NENI type  $p \in S(N)$  is realized in  $M$ . Indeed, suppose  $p \in S(N)$  is NENI, and  $p$  is realized in  $M$  by  $c$ , say. Then  $p \perp r$  for every  $r \in R$ ; otherwise  $r \upharpoonright N$  is realized in  $M$ , contradicting the maximality of  $B$ . We now show that for arbitrarily large finite subsets  $D$  of  $N$  (for every finite subset  $D_0$  of  $N$  there is a finite  $D$ ,  $D_0 \subset D \subset N$ ) such that  $t(c/aBD)$  is isolated; this will show that the set  $N \cup \{c\}$  is atomic over  $aB$ , contradicting the maximality of  $N$ .

Let  $D$  be any finite subset of  $N$  such that  $c \downarrow_D N$ ,  $a \subset D$ . Find finite  $B' \subset B$  such that  $D \downarrow_{B'} B$ . Since  $B$  is an independent set over  $a$ , and we have  $B - B' \downarrow_a DB'$ , we have that  $B - B'$  is an independent set over  $DB'$ , and for each  $b \in B - B'$ ,  $t(b/DB')$  is parallel to  $t(b/a)$ , an element of  $R$ . Since  $p$  is orthogonal to every member of  $R$ ,  $t(c/DB')$  (a type parallel to  $p$ ) is orthogonal to  $B - B'/DB'$ . It follows that

$$(1) \quad p \upharpoonright DB' \vdash p \upharpoonright DB' \cup (B - B') = p \upharpoonright DB.$$

Since  $p$  is NENI, and  $D \cup B'$  is finite,  $p \upharpoonright DB'$  is isolated. By (1),  $p \upharpoonright DB$  is isolated as well. Since this holds for arbitrarily large finite  $D \subset N$ , we obtain a contradiction as said. *Subclaim 1* is established.

Next, we show that (*Subclaim 2*), in fact, no NENI type over  $M'$  is realized in  $M$ . The proof is similar to that of *Subclaim 1*. Suppose  $p \in S(M')$  is NENI, and is realized by  $c$  in  $M$ . Let  $D \subset M'$  be any finite set such that  $p$  is based on  $D$ , let  $I'$  be a finite subset of  $I$  such that  $D \downarrow_{N'} I'$ , and let  $D'$  be a finite subset of  $N$  such that  $a \in D'$  and  $DI' \downarrow_{D'} N$ . By the maximality of  $I$ ,  $p$  is orthogonal to every type in the set  $P$ . Therefore, it follows as (1) does above that

$$(2) \quad p \upharpoonright NDI' \vdash p \upharpoonright NDI.$$

We have that  $p \perp N$ ; otherwise there is a SR type  $p' \in S(N)$  such that  $p \not\perp p'$ . (see D.19); by 1.3,  $p'$  is NENI, and by D.18,  $p'$  (in fact, even  $p' \upharpoonright M'$ ) is realized in  $M$ , contradicting *Subclaim 1*. Hence, we have  $p \perp D'$ . It follows that  $p \perp N/D'$ , and by  $DI' \downarrow_{D'} N$  that  $p \perp N/D'DI'$ , hence

$$(3) \quad p \upharpoonright D'DI' \vdash p \upharpoonright NDI'.$$

(2) and (3) give  $p \mid D'DI' \vdash p \mid NDI$ . Since  $p$  is NENI,  $p \mid D'DI'$  is isolated; therefore, so is  $p \mid NDI$ , and we obtain a contradiction to the maximality of  $M'$  as above, completing the proof of Subclaim 2.

*Subclaim 3* is the statement that every ENI type over  $M$  is  $\not\perp$  to an element of  $P \cup R$ .

Let  $p \in S(M)$  be an ENI type. If  $p \not\perp a$ , then, by choice of the set  $P_1 \subset P$ ,  $p$  is  $\not\perp$  to a member of  $P_1 \cup R$ . So, we may assume that  $p \perp a$ .

By 1.6, there is a set  $A$  containing  $a$ , and an element  $\hat{d}$  such that  $\hat{d}/A$  has weight 1, and  $p$  needs  $\hat{d}/A$ . Also,  $A$  can be chosen "sufficiently large", e.g. we may assume  $A$  is an  $a$ -model. The type  $\hat{d}/A$  thus shown to be a supportive type, we have that  $\hat{d}/A$  is  $\not\perp$  to a member  $r$  of  $R$  (this is because of the choice of  $R^0$ , and the fact that the property of  $R^0$  in question is invariant under automorphisms of  $\mathcal{U}$ , hence it is shared by  $R$ ). Find  $\hat{b}$  realizing  $r \mid A$  such that  $\hat{b} \not\perp_A \hat{d}$  (recall that  $A$  is sufficiently large). By 2.6, and since  $p$  is ENI, we have that  $p$  needs  $\hat{b}/A$ . By 2.7, it follows that  $p$  needs  $\hat{b}/a$ . We have  $p \not\perp a\hat{b}$  and  $p \perp a$ , hence necessarily,  $M \not\perp_a \hat{b}$ . We claim that  $B_r \not\perp_a \hat{b}$ . Indeed, if we had  $B_r \perp_a \hat{b}$ , then  $\hat{b}$  would realize  $r \mid aB_r$ ; hence, by the version of D.21(ii) for  $r$ , stated above,  $\hat{b}$  realizes  $r \mid M$ , i.e.  $M \perp_a \hat{b}$ , which is false, showing the claim. Since  $r$  is trivial (see above), and  $B_r \not\perp_a \hat{b}$ , there is  $b \in B_r$  such that  $b \not\perp_a \hat{b}$ . Since  $p$  needs  $\hat{b}/a$ , it follows that  $p$  needs  $b/a$ . By choice of the set  $P_b \subset P$ , we have that  $p \not\perp$  to some member of  $P_b$ , completing the proof of Subclaim 3.

To complete the proof of 3.2, let us assume that  $M - M' \neq \emptyset$ . Then there is  $c \in M - M'$  such that  $c/M'$  is SR. By Subclaim 2,  $p = t(c/M')$  is ENI. By Subclaim 3,  $p \mid M$ , hence  $p$  itself, is  $\not\perp$  either to a member of  $R$ , or to a member of  $P$ . The first possibility contradicts the maximality of  $B$ , the second the maximality of  $I$ .  $\square$

Let  $\hat{p} = (p^0)_b$  for some  $r \in R^0$ ,  $b \in B_r$ ,  $p^0 \in P^0$ .

CLAIM 3.3.  $\dim(\hat{p}, N) = 0$ .

PROOF.  $b = \text{dom}(\hat{p})$  is an element of  $B$ . We have  $b \perp_a B - \{b\}$ . By C.8, and since  $\hat{p} \perp a$ , we have that  $\hat{p} \perp a(B - \{b\})$ , hence  $\hat{p} \vdash \hat{p} \mid ab(B - \{b\}) = \hat{p} \mid aB$ .  $\hat{p} \mid aB$  is non-isolated (since  $\hat{p}$  is), and since  $N$  is atomic over  $aB$ ,  $\hat{p} \mid aB$  is not realized in  $N$ . It follows that  $\hat{p}$  is not realized in  $N$  either.  $\square$

Next, consider a type  $\hat{p} = (p^0)_a$ ,  $p^0 \in P^0$ .

CLAIM 3.4.  $\dim(\hat{p}, N)$  is finite.

PROOF. Choose a finite subset  $B'$  of  $B$  such that  $B \perp_{ab'} d$ . Let  $N'$  be a model

contained in  $N$ , and prime over  $dB'$ . Since  $p$  is a non-isolated SR type,  $\dim(\hat{p}, N')$  is finite (see 1.2(ii)). On the other hand,

$$\dim(\hat{p}, N) = \dim(\hat{p}, N') + \dim(\hat{p} \upharpoonright N', N);$$

we'll show that the second term of the sum is 0, which will complete the proof.

Recall that  $\hat{p}$  is  $\perp$  to every member of  $R$ . We have  $B \downarrow_{aB} d$  and  $B - B' \downarrow_a B'$ , hence  $B - B' \downarrow_a dB'$ . It follows that  $q \stackrel{\text{df}}{=} t(\overline{B - B'} / dB')$  is a product of types that are nf extensions of members of  $R$ . Hence,  $q$  is orthogonal to  $\hat{p}$ , and to  $\hat{p} \upharpoonright dB'$  as well. It follows that

$$(4) \quad \hat{p} \upharpoonright dB' \vdash \hat{p} \upharpoonright dB' \cup (B - B') = \hat{p} \upharpoonright dB.$$

$N$  is atomic over  $aB$ , hence over  $dB$  as well. Since  $\hat{p} \upharpoonright dB$  is non-isolated,  $\dim(\hat{p} \upharpoonright dB, N) = 0$ . From the relation (4), it follows that  $\dim(\hat{p} \upharpoonright dB', N) = 0$ . A fortiori,  $\dim(\hat{p} \upharpoonright N', N) = 0$ , as desired.  $\square$

Now, we can prove Theorem 3.1.

Let  $M$  and  $M'$  be two countable models,  $a \in M$ ,  $a' \in M'$ ,  $a \equiv a' \equiv a^0$ , and assume that  $(M, a)$  and  $(M', a')$  have the same invariants. Introduce the items referred to in 3.2–3.4 in the model  $M$  (but not yet in  $M'$ ). Next, consider a fixed  $r \in R^0$ , and the set  $X'_r$  of all the equivalence classes  $\mathcal{B}'$  of realizations of  $r_a$  in  $M'$  (just as  $X_r$  was defined in  $M$ ). Since  $s_D^{(M)} = s_D^{(M')}$  for every  $D \in \mathcal{D}_r^0$ , clearly, there is a 1-1 correspondence between the classes  $\mathcal{B}$  in  $X_r$ , and the classes  $\mathcal{B}'$  in  $X'_r$ , such that if  $\mathcal{B}'$  corresponds to  $\mathcal{B}$  under this correspondence, then  $D^{(M)}(\mathcal{B}) = D^{(M')}(\mathcal{B}')$ . Consider any  $b \in B$ , and  $\mathcal{B}$ , the class in  $X_r$  containing  $b$ . Let  $\vec{n} = \vec{n}(b)$  be the vector of the  $p_b$ -dimensions of  $M$ , for  $p \in P_r^0$ . The equality  $D^{(M)}(\mathcal{B}) = D^{(M')}(\mathcal{B}')$  (for  $\mathcal{B}$  and  $\mathcal{B}'$  corresponding classes) implies that there is  $b' \in \mathcal{B}'$  such that  $\vec{n}^{(M)}(b') = \vec{n}(b)$ . Let us choose one such  $b' \in \mathcal{B}'$ , and let us define a map,

$$b \mapsto b'$$

defined for  $b \in B = \bigcup_{r \in R^0} B_r$ , with  $b'$  determined as described for each  $b \in B$ . Since the  $b'$  are pairwise in distinct classes, they are pairwise independent over  $a'$ . Since the types in  $R^0$  are trivial, it follows that the family of all  $b'$ 's (for  $b \in B$ ) is independent over  $a'$ . Since  $B$  was an  $R$ -basis for  $M$  (for  $R = \{(r)_a : r \in R^0\}$ ), and hence  $B$  has (exactly) one element in each class  $\mathcal{B} \in X$ , moreover since the correspondence  $\mathcal{B} \mapsto \mathcal{B}'$  is bijective on classes, it follows that  $B' = \{b' : b \in B\}$  is an  $R'$ -basis for  $M'$  ( $R' = \{(r)_{a'} : r \in R^0\}$ ). We also have that the map that assigns  $a'$  to  $a$ , and  $b'$  to  $b$  for each  $b \in B$  is an elementary isomorphism between the sets  $\{a\} \cup B$  and  $\{a'\} \cup B'$ .

Now, let  $N'$  be an (elementary) submodel of  $M'$  maximal atomic over  $a'B'$ . The above elementary isomorphism can be extended to one from  $N$  onto  $N'$ . Let  $d'$  be the tuple in  $N'$  which is the image of  $d$  under the said isomorphism. The types  $p_b \upharpoonright N$  in  $P_b$  ( $p \in \bigcup_{r \in R^0} P_r^0$ ) correspond to  $p_{b'} \upharpoonright N'$  under the isomorphism; let  $P'_b = \{(p)_{b'} \upharpoonright N'; p \in \bigcup_{r \in R^0} P_r^0\}$ . The types  $p_a \upharpoonright N$  in  $P_1$  ( $p \in P^0$ ) correspond to  $p_{a'} \upharpoonright N'$ ; let  $P'_1 = \{(p)_{a'} \upharpoonright N'; p \in P^0\}$ . Put  $P' = \bigcup_{b' \in B'} P'_b \cup P'_1$ .

Notice that each type in  $P_{b_1}$  is  $\perp$  to every type in  $P_{b_2}$ , for  $b_1 \neq b_2$ , both in  $B$ , as well as to every type in  $P_1$ . This follows easily from the facts that every type in  $P_{b_1}$  is  $\perp$  to  $a$ , and that every type in  $P_1$  is  $\not\perp$  to  $a$ . It follows that if we decompose  $I$  into the union of the  $I_b$  ( $b \in B$ ) and  $I_1$ , with  $I_b = \{c \in I : c \text{ realizes some type in } P_b\}$ ,  $I_1 = \{c \in I : c \text{ realizes some type in } P_1\}$ , then  $I_b$  is a  $P_b$ -basis for  $M$ , and  $I_1$  is a  $P_1$ -basis for  $M$ . Furthermore, the types in a fixed  $P_b$ , or in  $P_1$ , are pairwise  $\perp$  by choice, hence

$$I_p = \{c \in I : c \text{ realizes } p\} \quad (p \in P_b)$$

is a  $p$ -basis for  $M$ . Let  $p \in P_b$ ,  $\hat{p} = p \upharpoonright b$ ;  $\hat{p} = (p^0)_b$ . Then, since  $\dim(\hat{p}, N) = 0$ , by the addition formula D.21(iii) it follows that  $\dim(p, M) = \dim(\hat{p}, M)$ . Of course, the latter is what we denoted by  $n_p^0(b)$ , and it is the same as the cardinality of  $I_p$ . If  $b'$  corresponds to  $b$  under the specific isomorphism, then  $n_p^{(M')}(b') = n_p^{(M)}(b)$ , by the choice of the correspondence  $b \mapsto b'$ . Thus, if we let  $\hat{p}' = (p^0)_{b'}$ , and  $p' = \hat{p}' \upharpoonright N'$ , the type in  $P'$  corresponding to  $p$ , and  $I'_p$  any  $p'$ -basis for  $M'$ , then (because of  $\dim(\hat{p}', M') = \dim(p', M')$ ) we have that  $|I'_p| = |I_p|$ .

For any  $p \in P_1$ ,  $\hat{p} = p \upharpoonright d$ ,  $k = \dim(\hat{p}, N)$  is finite by 3.4. If  $p'$  corresponds to  $p$ ,  $\hat{p}' = p \upharpoonright d'$ , then of course  $\dim(\hat{p}', N') = \dim(\hat{p}, N) = k$ . Also, by assumption,  $\dim(\hat{p}, M) = \dim(\hat{p}', M')$  (since  $\tilde{m}^{(M)} = \tilde{m}^{(M')}$ ). Since

$$\dim(\hat{p}, M) = \dim(\hat{p}, N) + \dim(p, M) = k + \dim(p, M),$$

and similarly for  $M'$ , it follows that  $\dim(p, M) = \dim(p', M')$ . So, if  $I'_p$  is a  $p'$ -basis for  $M'$ , then  $|I'_p| = |I_p|$ .

Let  $I' = \bigcup_{p' \in P'} I'_p$ . Then, clearly,  $I'$  is a  $P'$ -basis for  $M'$ . If we let  $c \mapsto c'$  be any bijection of  $I_p$  and  $I'_p$ , for each  $p \in P$  and the corresponding  $p'$  in  $P'$ , then these bijections together with the given isomorphism of  $N$  onto  $N'$  is an isomorphism from  $N \cup I$  onto  $N' \cup I'$ . Applying 3.2 to  $M'$  now,  $M'$  is prime over  $N' \cup I'$ , as well as  $M$  was prime over  $N \cup I$ . It follows that the isomorphism of  $N \cup I$  onto  $N' \cup I'$  can be extended to an isomorphism of  $M$  onto  $M'$ .

This completes the proof of the theorem.

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