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On the Structure of $\mathrm{Ext}_p(G, \mathbf{Z})$

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We will prove a theorem on the cardinality of inverse limits of systems of groups. The following is an instance of the theorem:

THEOREM. *Let λ be a strong limit cardinal of cofinality $\aleph_0$. For every torsion free abelian group G of cardinality λ and a prime p $|\mathrm{Ext}_p(G, \mathbf{Z})| < \lambda$ or $|\mathrm{Ext}_p(G, \mathbf{Z})| = 2^\lambda$.*

We made an effort to make this paper readable also by non-logicians.

0. INTRODUCTION

History and motivation. For a discussion of the importance and the history of problems about the structure of the group $\mathrm{Ext}(G, \mathbf{Z})$ see Fuchs [6, 7] and Nunke [13]. The major question in the area was Whitehead's problem: "Does there exist a nonfree group G such that $\mathrm{Ext}(G, \mathbf{Z}) = \{0\}$?" In an early stage it was clear that without loss of generality we may assume:

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Assumption 0.1. From now on (unless explicitly stated) the groups mentioned always will be uncountable torsion free abelian groups.

Shelah proved in [17] that the problem is independent of the usual axioms of set theory (an alternative presentation can be found in Eklof's book [2]). The positive result is:

THEOREM 0.2. *Assume* $\mathbf{V}=\mathbf{L}$. *Let* G *be an abelian group.* G *is free* $\Leftrightarrow \operatorname{Ext}(G, \mathbf{Z})=\{0\}$.

After the solution of Whitehead's problem, the next natural question is the investigation of the structure of the group $\operatorname{Ext}(G, \mathbf{Z})$ (see [13]). A basic question to ask is: If $\operatorname{Ext}(G, \mathbf{Z}) \neq \{0\}$, what can the cardinality of $\operatorname{Ext}(G, \mathbf{Z})$ be?

Since $\operatorname{Ext}(G, \mathbf{Z})$ is a divisible group (as we are assuming that G is torsion free; see [6]), $\operatorname{Ext}(G, \mathbf{Z})$ is determined by the invariants $\nu_p(\operatorname{Ext}(G, \mathbf{Z}))$, where for p prime ν_p is the p-rank, and $\nu_0(\operatorname{Ext}(G, \mathbf{Z}))$ is the torsion free rank of the group. Let $\operatorname{Ext}_p(G, \mathbf{Z})$ be the p part of the group $\operatorname{Ext}(G, \mathbf{Z})$. A more concrete (and harder) question is the following:

Question 0.3. Given that $\operatorname{Ext}_p(G, \mathbf{Z}) \neq \{0\}$, what can the cardinality of $\operatorname{Ext}_p(G, \mathbf{Z})$ be?

Because of the independence results of [17, 21] and Theorem 0.2 it was natural to deal with the last question assuming the axiom $\mathbf{V}=\mathbf{L}$ or at least *GCH*. (For groups of cardinality $\aleph_1$ there is a positive result assuming $2^{\aleph_0}>\aleph_1+MA_{\aleph_1}$, see [3]). Since the early sixties everything was known about the structure of $\operatorname{Ext}(G, \mathbf{Z})$ for G countable (see S. Chase [1] and the book of Hilton and Stambach [10]). The first instance of the question was answered by Hiller and Shelah in [9]:

THEOREM 0.4. *Assume* $\mathbf{V}=\mathbf{L}$. *If* $\operatorname{Ext}(G, \mathbf{Z}) \neq \{0\}$ *then there are at least* $\aleph_1$ *many elements in* $\operatorname{Ext}(G, \mathbf{Z})$.

Additional results on the rank of $\operatorname{Ext}(G, \mathbf{Z})$ assuming only $2^{\aleph_0}<2^{\aleph_1}$ can be found in [20, Chap. 14].

Change of notation. From now on, given a group G, by $\nu_p(G)$ we denote $\nu_p(\operatorname{Ext}(G, \mathbf{Z}))$.

Hiller, Huber, and Shelah [8] improved Theorem 0.4 and supplied a partial answer to Question 0.3:

THEOREM 0.5. *Assume* $\mathbf{V}=\mathbf{L}$. *Suppose that* G *is not free, and let* $\mu(G)=^{\text{def}} \operatorname{Min}\{|B|+\aleph_0$: *$B$ is a direct summand of the group G, and G/B is free*$\}$. *Then* $\nu_0(G)=2^{\mu(G)}$, *and for every prime* $\nu_p(G) \leqslant \nu_0(G)$.

Note that these results imply that (in **L**) there is no G such that $\mathrm{Ext}(G, \mathbf{Z})$ is the group of the rationals (which answers one of Nunke's questions [13]). In light of Theorem 0.5 it was natural to conjecture that $\nu_0(G) = \nu_p(G)$. But by any of [4, 16, 18] this is not true in general. The impression that there is no other restriction was contradicted by [16]: if the cardinality of G is weakly compact then $\nu_p(G) \neq |G|$. On the other hand, recently Mekler and Shelah [12] proved the following:

THEOREM 0.6. *Assume* $\mathbf{V} = \mathbf{L}$, *and suppose* λ *is a regular uncountable cardinal smaller than the first weakly compact. Then for every sequence* $\{\lambda_p \leqslant \lambda^+ : p \text{ a prime}\}$ *there exists a* λ*-free group* G *of cardinality* λ *such that for every prime* p, $\nu_p(G) = \lambda_p$, *but* $\nu_0(G) = \lambda^+$.

However, this does not provide any information about the case when $|G| = \lambda$ is singular, for example, whether $\lambda = |G| \leqslant \nu_p(G) < \nu_0(G)$ is possible. The aim of this paper is to show that this is impossible when λ is a strong limit cardinal of cofinality $\aleph_0$.

The structure of the paper. An answer will be presented to Question 0.3. In the next section we present a proof of the following theorem:

THEOREM 1.0. *Let* λ *be a strong limit cardinal of cofinality* $\aleph_0$. *For every abelian group* G *of cardinality* λ *and prime* p, *either* $|\mathrm{Ext}_p(G, \mathbf{Z})| < \lambda$ *or* $|\mathrm{Ext}_p(G, \mathbf{Z})| = 2^\lambda$.

Combining Theorems 1.0 and 0.5 we obtain:

COROLLARY 0.7. (Assume $\mathbf{V} = \mathbf{L}$.) *Let* λ *be a singular cardinal of cofinality* $\aleph_0$. *For every group* G *of cardinality* λ *we have that* $\nu_p(G) \geqslant \lambda \Rightarrow \nu_0(G) = \nu_p(G) = 2^\lambda$.

It is natural to ask: Is the corollary the best possible? Since $\mathrm{Ext}(\cdot, \mathbf{Z})$ is a multiplicative functor, starting with the above example (from Theorem 0.6), there exists a group G such that $\nu_p(G) < \lambda$ but $\nu_0(G) = 2^\lambda$. So it is clear that the assumption of Corollary 0.7 cannot be weakened to $\nu_p(G) < \lambda$.

In the last section some generalizations will be discussed. The theorem mentioned in the abstract is presented. It is a generalization of Theorem 1.4, the main theorem of Section 1, which implies Theorem 1.0.

Explanation of the proof of Theorem 1.0. First we show that it is enough to prove $[G^p : (G^*/p)] \geqslant \lambda \Rightarrow [G^p : (G^*/p)] \geqslant \lambda^{\aleph_0}$, which is the statement of Theorem 1.4 (for the terminology see Notation 1.2 in Sect. 1).

We show that $[G^p : (G^*/p)] \geqslant \lambda^{\aleph_0}$ by constructing a family of $\lambda^{\aleph_0}$ elements of G^p such that the difference of any two does not belong to the subgroup G^*/p. How do we carry out the construction? Since $cf\lambda = \aleph_0$, fix $\{G_n : n < \omega\}$ an increasing sequence of subgroups of G such that

$|G_n| = \lambda_n < \lambda$ and $G = \bigcup_{n<\omega} G_n$. For $f \in G^p$ and $g \in G_n^*$ in Definition 1.5 we introduce a notion of rank such that $\mathrm{rk}(g, f) = \infty \Leftrightarrow$ there exists an element $g' \in G^*$ extending g such that $f = g'/p$. In order to show that $[G^p : (G^*/p)] \geqslant \lambda^{\aleph_0}$ it is enough to find $\{f_\eta \in G^p;\ \eta \in B(T)\}$ such that for every $\eta, \nu \in B(T)$ we have $f_\eta - f_\nu \notin G^*/p$, where $B(T)$ is the set of infinite branches of the tree T, which is defined as

$$T \overset{\text{def}}{=} \{\eta \in {}^{\omega>}\lambda : (\forall n < \omega)\eta[n] < \lambda_n\}.$$

Hence by the above property of $\mathrm{rk}(\cdot, \cdot)$ we can ensure this by requiring the existence of a natural number n such that for every $g \in G_n^*$ and every $\eta, \nu \in T$, $\eta \mid n \neq \nu \mid n \Rightarrow \mathrm{rk}(g, f_\eta - f_\nu) < \infty$.

The family $\{f_\eta \in G^p : \eta \in B(T)\}$ is constructed by finite approximations by constructing $\{f_\eta \in G^p : \eta \in T\}$ satisfying a strong induction hypothesis on explicit bounds on $\mathrm{rk}(g, f_\eta - f_\nu)$, which is why the rank is bounded.

We have some lemmas which investigate the notion of rank and simplify the computation of bounds for it. *The fact that we are working with groups is used quite heavily in many places*: Lemma 1.8, Lemma 1.10, and in the last stage of the proof of Theorem 1.4.

Additional remarks. The statement of Corollary 0.7 $[\nu_p(G) \geqslant \lambda \Rightarrow \nu_p(G) = 2^\lambda]$ for $\lambda = \aleph_0$ is reminiscent of results on the number of classes of Σ_1^1, $\prod_1^1$ equivalence relations of reals of Burgess, Silver, and Harrington and Shelah.

Shelah has a theorem which has a conclusion similar to Corollary 0.7: Let X be a "nice" topological space. If the fundamental group of X is not finitely generated then it is generated by $2^{\aleph_0}$ many elements (see [21]).

Open problem. Is the statement of Theorem 1.0 true for singular strong limit cardinals with uncountable cofinality?

Notation. α, β, γ, δ, i, ξ, ζ stand for ordinals; λ, κ, μ, χ are cardinal numbers. l, n, m, k are integers, p is a prime number, ω is the first infinite ordinal, and also stands for the set of natural numbers ${}^\alpha\lambda$ is the set of sequences of length α whose elements are ordinals less than λ, ${}^{\alpha\geqslant}\lambda =^{\text{def}} \bigcup_{\beta\leqslant\alpha} {}^\beta\lambda$. We say that $T \subseteq {}^{\omega\geqslant}\lambda$ is a tree if for every $\eta \in T$ all its initial segments are also elements of T. We denote by η, ν sequences, for $\eta \in {}^{\alpha\geqslant}\lambda$, $\beta < \alpha$ $\eta \mid \beta$ is the restriction of η to β, i.e., $\eta \mid \beta$ is the sequence ν of length β such that for every $\gamma < \beta$ we have $\eta \mid \beta[\gamma] = \nu[\gamma]$. $B(T)$ is the set of limit points of T, i.e., sequences of length ω, $h(\eta, \nu) = \mathrm{Max}\{k : \eta \mid k = \nu \mid k\}$-the length of the maximal common initial sequence, and $l(\eta)$ is the length of the sequence η. For a linearly ordered set S, $[S]^2$ is the set of increasing pairs from S.

Models will be denoted by the letters M, N. $L(M)$ is the similarity type (language, or signature) of the model M.

In this paper λ will stand always for a strong limit cardinal (i.e., λ satisfies $(\forall \mu < \lambda) 2^\mu < \lambda$) of cofinality $\aleph_0$ (= there exists an increasing sequence of cardinals $\langle \lambda_n : n < \omega \rangle$ such that $\lambda = \bigcup_{n<\omega} \lambda_n$).

The end of a proof is denoted by ▌; the end of the proof of Claim 1.7 is denoted by $\blacksquare_{1.7}$.

We are grateful to Gregory Cherlin for reading carefully this paper, making grammatical corrections, and rewriting parts of our proofs.

1. THE MAIN THEOREM

THEOREM 1.0. *Let λ be a strong limit cardinal of cofinality $\aleph_0$. For every torsion free abelian group G of cardinality λ and prime p, $|\mathrm{Ext}_p(G, \mathbf{Z})| < \lambda$ or $|\mathrm{Ext}_p(G, \mathbf{Z})| = 2^\lambda$.*

Notation 1.1. Pick $\{\lambda_n < \lambda : n < \omega\}$ satisfying $\lambda = \sum_{n<\omega} \lambda_n$, and for all $n < \omega$, λ_n is regular and $2^{\lambda_n} < \lambda_{n+1}$. Let $\{G_n : n < \omega\}$ be an increasing chain of subgroups of G such that $G_0 = \{0\}$, $G = \bigcup_{n<\omega} G_n$, and $|G_{n+1}| = \lambda_{n+1}$.

Notation 1.2. Given a group H let $H^* = \mathrm{Hom}(H, \mathbf{Z})$, and let $H^p = \mathrm{Hom}(H, \mathbf{Z}/p\mathbf{Z})$. For $h \in H^*$ let h/p be the following element of H^p defined as $(h/p)(x) =^{\mathrm{def}} h(x) + p\mathbf{Z}$. (It is easy to show that $h \to h/p$ is a homomorphism of H^* into H^p.) For $Y \subseteq H^*$ let $Y/p = \{h/p : h \in Y\}$. So H^*/p is a subgroup of H^p.

We are interested in the cardinality of $G^p/(G^*/p)$. This group is interesting because of the following basic observation (see Nunke [13, p. 265] or [16]).

Fact 1.3. For an abelian torsion free group G, and a prime number p we have that $\mathrm{Ext}_p(G, \mathbf{Z}) \cong G^p/(G^*/p)$. ▌

Using Fact 1.3 it is easy to verify that Theorem 1.0 follows from the following theorem.

MAIN THEOREM 1.4. *For any abelian group G of cardinality λ we have $[G^p : (G^*/p)] \geqslant \lambda \Rightarrow [G^p : (G^*/p)] \geqslant \lambda^{\aleph_0}$ (notice that since λ is strong limit of cofinality $\aleph_0$ by cardinal arithmetic (see [11, (6.21)]) we have $2^\lambda = \lambda^{\aleph_0}$).*

We will now describe the rank function used in the proof of the main theorem.

DEFINITION 1.5. Let $f \in G^p$ and let $g \in G_n^*$.

(1) If $g/p = f \restriction G_n$, we say that (g, f) is *a nice pair.*

(2) Define a ranking function $\mathrm{rk}(g, f)$. First by induction on α, we define when $\mathrm{rk}(g, f) \geqslant \alpha$ *simultaneously for all* $g \in \bigcup_{n<\omega} G_n^*$:

(a) $\mathrm{rk}(g,f) \geqslant 0$ iff (g,f) is a nice pair;

(b) $\mathrm{rk}(g,f) \geqslant \delta$ for a limit ordinal δ iff for every $\beta < \delta$ $\mathrm{rk}(g,f) \geqslant \beta$;

(c) $\mathrm{rk}(g,f) \geqslant \beta + 1$ iff (g,f) is a nice pair, and for the value of n which $g \in G_n$, there exists $g' \in G^*_{n+1}$ extending g such that $\mathrm{rk}(g',f) \geqslant \beta$;

(d) $\mathrm{rk}(g,f) \geqslant -1$.

(3) $\mathrm{rk}(g,f) = \alpha$ iff $\mathrm{rk}(g,f) \geqslant \alpha$ and it is false that $\mathrm{rk}(g,f) \geqslant \alpha + 1$.

(4) $\mathrm{rk}(g,f) = \infty$ iff for every ordinal α we have $\mathrm{rk}(g,f) \geqslant \alpha$.

The following two claims give the principal properties of $\mathrm{rk}(g,f)$.

Claim 1.6. Let (g,f) be a nice pair.

(1) The following statements are equivalent:

(a) $\mathrm{rk}(g,f) = \infty$.

(b) There exists $g' \in G^*$ extending the function g such that $g'/p = f$.

(2) If $\mathrm{rk}(g,f) < \infty$ then $\mathrm{rk}(g,f) < \lambda^+$.

(3) If g' is a proper extension of g and (g',f) is also a nice pair then $\mathrm{rk}(g',f) \leqslant \mathrm{rk}(g,f)$, and if $\mathrm{rk}(g,f) < \infty$ then the inequality is strict.

Proof. (1) Statement (a) $\Rightarrow$ (b). Let n be the value such that $g \in G^*_n$. If we will be able to define $\{g_k \in G^*_{n+k} : k < \omega\}$ such that (i) $g_0 = g$, (ii) $g_k \subseteq g_{k+1}$, and (iii) $\mathrm{rk}(g_k, f) = \infty$ then clearly we will be done since $g' =^{\mathrm{def}} \bigcup g_k$ is as required. The definition is by induction on k.

For $k = 0$ let $g_0 = g$.

For $k > 0$, suppose g_k is defined. By (iii) we have $\mathrm{rk}(g_k, f) = \infty$, there exists $g^* \in G^*_{n+k+1}$ exending g_k such that $\mathrm{rk}(g^*, f) = \infty$, and let $g_{k+1} =^{\mathrm{def}} g^*$.

Statement (b) $\Rightarrow$ (a). Since $g \subseteq g'$, it is enough to prove by induction on α that for every $k \geqslant n$ when $g_k =^{\mathrm{def}} g' \restriction G_k$ we have that $\mathrm{rk}(g_k, f) \geqslant \alpha$.

For $\alpha = 0$, since $g'/p = f$ clearly for every $k g_k/p = f \restriction G_k$ so (g_k, f) is a nice pair.

For limit α, by the induction hypothesis for every $\beta < \alpha$ and every k, $\mathrm{rk}(g_k, f) \geqslant \beta$. Hence by Definition 1.5(2)(b), $\mathrm{rk}(g_k, f) \geqslant \alpha$.

For $\alpha = \beta + 1$, by the induction hypothesis for every k, $\mathrm{rk}(g_k, f) \geqslant \beta$. Let $k_0 \geqslant n$ be given. Since $g_{k_0} \subseteq g_{k_0+1}$, and $\mathrm{rk}(g_{k_0+1}, f) \geqslant \beta$. Definition 1.5(2)(c) implies that $\mathrm{rk}(g_{k_0}, f) \geqslant \beta + 1$; i.e., for every $k \geqslant n$ we have $\mathrm{rk}(g_k, f) \geqslant \alpha$.

(2) Let $g \in G^*_n$ and $f \in G^p$ be given. It is enough to prove that if $\mathrm{rk}(g,f) \geqslant \lambda^+$ then $\mathrm{rk}(g,f) = \infty$. Using part (1) it is enough to find $g' \in G^*$ such that $g \subseteq g'$ and $g'/p = f$.

We define by induction on $k < \omega$, $g_k \in G^*_{n+k}$ such that $g_k \subseteq g_{k+1}$, and $\mathrm{rk}(g_k, f) \geqslant \lambda^+$. For $k = 0$ let $g_k = g$. For $k + 1$, for every $\alpha < \lambda^+$, as

$\mathrm{rk}(g_k, f) > \alpha$ by 1.5(2)(c) there is $g_{k,\alpha} \in G_{n+k+1}$ extending g_k such that $\mathrm{rk}(g_{k,\alpha}, f) \geqslant \alpha$. But the number of possible $g_{k,\alpha}$ is $\leqslant |G^*_{n+k+1}| \leqslant 2^{\lambda_{n+k+1}} < \lambda^+$ hence there are a function g and a set $S \subseteq \lambda^+$ of cardinality λ^+ such that $\alpha \in S \Rightarrow g_{k,\alpha} = g$. Then take $g_{k+1} = g$.

(3) Immediate. $\blacksquare_{1.6}$

Lemma 1.7. (1) *Let (g, f) be a nice pair, and let a be an integer. Then we have* $\mathrm{rk}(g, f) \leqslant \mathrm{rk}(ag, af)$.

(2) *For every nice pair (g, f) we have* $\mathrm{rk}(g, f) = \mathrm{rk}(-g, -f)$.

Proof. (1) By induction on α prove that $\mathrm{rk}(g, f) \geqslant \alpha \Rightarrow \mathrm{rk}(ag, af) \geqslant \alpha$ (see more details in Lemma 1.8).

(2) Apply part (1) twice. $\blacksquare_{1.7}$

Lemma 1.8. *Let $n < \omega$ be fixed, and let $g(g_1, f_1)$, (g_2, f_2) be nice pairs with $g_l \in G^*_n$ $(l = 1.2)$.*

(1) *If (g_1, f_1), and (g_2, f_2) are nice pairs then $(g_1 + g_2, f_1 + f_2)$ is a nice pair, and* $\mathrm{rk}(g_1 + g_2, f_1 + f_2) \geqslant \mathrm{Min}\{\mathrm{rk}(g_l, f_l) : l \leqslant 2\}$.

(2) *Let (n, f_1, g_1) and (n, f_2, g_2) be as above. If* $\mathrm{rk}(g_1, f_1) \neq \mathrm{rk}(g_2, f_2)$ *then* $\mathrm{rk}(g_1 + g_2, f_1 + f_2) = \mathrm{Min}\{\mathrm{rk}(g_l, f_l) : l \leqslant 2\}$.

Proof. (1) It is easy to show that the pair is nice. We show by induction on α simultaneously for all $n < \omega$, and every $g_1, g_2 \in G^*_n$ that $\mathrm{Min}\{\mathrm{rk}(g_l, f_l) : l \leqslant 2\} \geqslant \alpha$ implies that $\mathrm{rk}(g_1 + g_2, f_1 + f_2) \geqslant \alpha$.

When $\alpha = 0$ or α is a limit ordinal this is easy. Suppose $\alpha = \beta + 1$, and that $\mathrm{rk}(g_l, f_l) \geqslant \beta + 1$; by the definition of rank there exists $g'_l \in G^*_{n+1}$ extending g_l such that (g'_l, f_l) is a nice pair and $\mathrm{rk}(g'_l, f_l) \geqslant \beta$. By the induction assumption $\mathrm{rk}(g'_1 + g'_2, f_1 + f_2) \geqslant \beta$. Hence $g'_1 + g'_2$ is as required in the definition of $\mathrm{rk}(g_1 + g_2, f_1 + f_2) \geqslant \beta + 1$.

(2) Suppose w.l.o.g. that $\mathrm{rk}(g_1, f_1) < \mathrm{rk}(g_2, f_2)$, let $\alpha_1 = \mathrm{rk}(g_1, f_1)$, and let $\alpha_2 = \mathrm{rk}(g_2, f_2)$. By part (1), $\mathrm{rk}(g_1 + g_2, f_1 + f_2) \geqslant \alpha_1$, by Proposition 1.7, $\mathrm{rk}(-g_2, -f_2) = \alpha_2 > \alpha_1$. So we have

$$\begin{aligned}\alpha_1 &= \mathrm{rk}(g_1, f_1) = \mathrm{rk}(g_1 + g_2 - g_2, f_1 + f_2 - f_2)\\ &\geqslant \mathrm{Min}\{\mathrm{rk}(g_1 + g_2, f_1 + f_2), \mathrm{rk}(-g_2, -f_2)\}\\ &= \mathrm{rk}(g_1 + g_2, f_1 + f_2) \geqslant \alpha_1.\end{aligned}$$

Hence the conclusion follows. $\blacksquare_{1.8}$

Notation. By O_{G_n} we denote the constant function whose domain is G_n and its value is 0.

The assumption of the theorem that $[G^p : (G^*/p)] \geqslant \lambda$ is used in Lemma 1.10 below. In order to formulate it we need a definition:

Definition 1.9. Let $\alpha_n =^{\mathrm{def}} \mathrm{Min}\{\alpha$: for every cardinal $\mu < \lambda$ there exists $\{f_i : i < \mu\} \subseteq G^p$ such that

(a) for every $i < \mu$ we have $f_i \restriction G_n = 0_{G_n}$,

(b) for every $i \neq j$, $\mathrm{rk}(0_{G_n}, f_i - f_j) \geqslant 0$ implies $\mathrm{rk}(0_{G_n}, f_i - f_j) < \alpha$, and

(c) $i \neq j \Rightarrow f_i - f_j \notin G^*/p.\}$

Lemma 1.10. (1) *α_n is well defined and is less than λ^+.*

(2) *$\alpha_n \geqslant \alpha_{n+1}$ for every $n < \omega$.*

(3) *There exists $n_0 < \omega$ and there exists a limit ordinal α such that for every $n > n_0$, $\alpha_n = \alpha$.*

(4) *If α_n is a limit then $cf\alpha_n < \lambda$.*

Proof. (1) We have to show that for every $\mu < \lambda$ there are $\{f_i \in G^p : i < \mu\}$ and an ordinal $\alpha < \lambda^+$ such that (a), (b), (c) of Definition 1.9 hold. Given $n < \omega$, $\mu < \lambda$ denote $\chi = (\mu + 2^{|G_n|})^+$. Since $[G^p : (G^*/p)] \geqslant \lambda$ there exists $\{g_i \in G^p : i < \chi\}$ such that

$$(\#)\ i \neq j \Rightarrow g_i - g_j \notin G^*/p.$$

Since χ is regular and greater than the number of functions from G_n into $\mathbf{Z}$ $(= 2^{|G_n|})$ there exists $S \subseteq \chi$ $|S| = \chi$ such that $i \neq j \in S \Rightarrow g_i \restriction G_n = g_j \restriction G_n$.

Fix $i_0 = \mathrm{Min}\, S$, pick $T \subseteq S - \{i_0\}$ of cardinality μ. Let $\{h_i : i < \mu\} = \{g_\xi : \xi \in T\}$. For $i < \mu$ define $f_i = h_i - g_{i_0}$. Clearly $\{f_i : i < \mu\}$ satisfies (a) and (c). Why do we also have (b)? Suppose $\mathrm{rk}(0_{G_n}, f_i - f_j) = \infty$ then by Claim 1.6(1) there exists $g' \supseteq 0_{G_n}$ in G^* such that $f_i - f_j = g'/p$ contradicting $(\#)$.

Let $\alpha =^{\mathrm{def}} \mathrm{Sup}\{\mathrm{rk}(0_{G_n}, f_i - f_j) : i, j < \mu\}$. Why is $\alpha < \lambda^+$? By Claim 1.6 $\mathrm{rk}(0_{G_n}, f_i - f_j) < \lambda^+$. Since $\mu < \lambda^+$ and λ^+ is regular we have that $\alpha < \lambda^+$.

(2) Given $\mu < \lambda$ let $\chi = (2^{|G_{n+1}|} + \mu)^+$, and let $\{f_i \in G^p : i < \chi\}$ exemplify α_n. As in (1) choose $\{f_i \in G^p : i < \mu\}$ such that $i \neq j \Rightarrow f_i \restriction G_{n+1} = 0_{G_{n+1}}$ and $f_i - f_j = g_{\xi_i} - g_{\xi_j}$. By Claim 1.6(3) $\mathrm{rk}(0_{G_{n+1}}, f_i - f_j) \leqslant \mathrm{rk}(0_{G_n}, f_i - f_j) = \mathrm{rk}(0_{G_n}, g_{\xi_i} - g_{\xi_j}) \leqslant \alpha_n$. Hence $\{f_i : G^p : i < \mu\}$ exemplify $\alpha_{n+1} \leqslant \alpha_n$.

(3) Since there is no infinite descending sequence of ordinals there exists $n_0 < \omega$ such that for every $n, k > n_0$ we have $\alpha_n = \alpha_k$. Taking the second clause of Claim 1.6(3) into account it follows easily that if $\alpha_n = \alpha_{n+1}$ then α_n is a limit ordinal.

(4) Since λ is singular and $\alpha_n < \lambda^+$ (by (1)) $cf\alpha_n < \lambda$.

Remarks. (1) We change the enumeration of the sequence $\{\alpha_n : n < \omega\}$ omitting the first n_0 elements. So using Lemma 1.10(3) we may assume that all members of $\{\alpha_n : n < \omega\}$ are the constant limit ordinal α.

(2) In part (4) of Lemma 1.10 it is possible to show that $cf\alpha_n = \aleph_0$, but since we do not use this, we skip its proof.

Proof of the Main Theorem 1.4. For $n < \omega$ let $T_n = \mathsf{X}_{k \leqslant n} \lambda_k$, $T =^{\mathrm{def}} \bigcup_n T_n$. We will construct $\{f_\eta \in G^p : \eta \in B(T)\}$ such that $\eta \neq \nu \in B(T) \Rightarrow f_\eta - f_\nu \notin G^*/p$. We define by induction on $n < \omega$ $\{g_{n,i} \in G^p : i < \lambda_n\}$ and an ordinal $\gamma_n < \alpha$ (α is the ordinal from Lemma 1.10(3)) such that

(1) $g_{n,i} \mid G_n = 0_{G_n}$ for all $i < \lambda_n$;

(2) for all $h \in G_n^*$ and $i < j < \lambda_n$ if $\mathrm{rk}(h, g_{n,i} - g_{n,j}) < \infty$ then $\mathrm{rk}(h, g_{n,i} - g_{n,j}) \leqslant \gamma_n$;

(3) $\mathrm{rk}(0_{G_n}, g_{n,i} - g_{n,j}) > \gamma_{n-1}$ for $i < j < \lambda_n$.

Having done so, we will set $f_\eta = \sum_{l<n} g_{l,\eta[l]}$ for $\eta \in T_n$. Then define f_η for $\eta \in B(T)$ using $\{f_\eta : \eta \in T_n,\ n < \omega\}$ as follows: Given $\eta \in B(T)$, f_η is the element of G^p satisfying $f_\eta \mid G_n = f_{\eta \restriction n}$. We show first that the construction is sufficient, and then that it can in fact be carried out.

The Construction Is Sufficient

PROPOSITION 1.11. *Let $\eta, \nu \in B(T)$. If $\eta \neq \nu$ then $f_\eta - f_\nu \notin G^*/p$.*

Proof. Suppose toward contradiction that for some $g \in G^*$ we have $f_\eta - f_\nu = g/p$. Let $k = h(\eta, \nu)$. For $l \geqslant k$ let ξ^l be $\mathrm{rk}(g \mid G_l, f_{\eta \restriction l+1} - f_{\nu \restriction l+1})$. We will reach a contradiction by showing that $\{\xi^l : k < l < \omega\}$ is a strictly decreasing sequence of ordinals.

For $l = k$, we show that $\xi^k \leqslant \gamma_k$. Let $i = \eta[k]$, $j = \nu[k]$. By the choice of k $i \neq j$. In this case $\xi^k = \mathrm{rk}(g \mid G_k, g_{k,i} - g_{k,j}) \leqslant \gamma_k$ by (2).

Now we proceed inductively. We assume that $\xi^l \leqslant \xi^k$ and show that $\xi^{l+1} < \xi^l$. Let $i = \eta[l+1]$, $j = \nu[l+1]$, and $\xi_1 = \mathrm{rk}(g \mid G_{l+1}, f_{\eta \restriction l+1} - f_{\nu \restriction l+1})$. Observe: $\xi_1 < \mathrm{rk}(g \mid G_l, f_{\eta \restriction l+1} - f_{\nu \restriction l+1}) = \xi^l$.

If $i = j$, then $f_{\eta \restriction l+2} - f_{\nu \restriction l+2} = f_{\eta \restriction l+1} - f_{\nu \restriction l+1}$ and hence $\xi^{l+1} = \xi_1 < \xi^l$.

Suppose therefore that $i \neq j$. Then $\xi^{l+1} = \mathrm{rk}(g \mid G_{l+1} + 0_{G_{l+1}}, (f_{\eta \restriction l+1} - f_{\nu \restriction l+1}) + (g_{l+1,i} - g_{l+1,j})) \geqslant \mathrm{Min}\{\xi_1, \mathrm{rk}(0_{G_{l+1}}, g_{l+1,i} - g_{l+1,j})\}$ *with equality* if the last two ordinals differ. Since $\xi_1 < \xi^l \leqslant \xi^k \leqslant \gamma_k < \gamma_{l+1} \leqslant \mathrm{rk}(0_{G_{l+1}}, g_{l+1,i} - g_{l+1,j})$ (by (3)), we again find $\xi^{l+1} = \xi_1 < \xi^l$.

The Construction

Fix $\gamma_{n-1} < \alpha$ (for $n = 1$, let $\gamma_0 = 0$). We will construct a family $\{g_i : i < \lambda_n\}$ and an ordinal γ_n satisfying the conditions (1), (2), (3).

We begin by fixing a sequence $\langle f_{\mu,i} \in G^p : i < \mu \rangle$, for each $\mu < \lambda$, satisfying the conditions (a), (b), (c) from the definition of $\alpha_n (= \alpha)$.

Claim 1.12. Let $l < \omega$. For every cardinal χ there exists a cardinal μ such that $\chi < \mu < \lambda$ and there exists $T \subseteq \mu$ $|T| = \chi^+$ such that for every $i, j \in S$, $i < j \Rightarrow \mathrm{rk}(0_{G_{l+1}}, f_{\mu,i} - f_{\mu,j}) > \gamma_l$.

Proof. For every $\kappa \geqslant \chi$, such that $\kappa < \lambda$ let $\mu = \mu_\kappa \geqslant (\beth_1(\kappa))^+$. Note that since λ is a strong limit we have $\mu < \lambda$. Define a coloring $F_\mu\colon [\mu]^2 \to \{T, F\}$ as

$$F_\mu(i, j) = T \Leftrightarrow \mathrm{rk}(0_{G_{l+1}}, f_{i,\mu} - f_{j,\mu}) > \gamma_l.$$

By the Erdös Rado Theorem ([5], or see [11, Theorem 69]) there exists $S \subseteq \mu$ of cardinality κ^+ such that exactly one of the following possibilities holds:

$(T)_\mu$ for every $i, j \in S$ $\quad i < j \Rightarrow \mathrm{rk}(0_{G_{l+1}}, f_{i,\mu} - f_{j,\mu}) > \gamma_l$

$(F)_\mu$ for every $i, j \in S$ $\quad i < j \Rightarrow \mathrm{rk}(0_{G_{l+1}}, f_{i,\mu} - f_{j,\mu}) \leqslant \gamma_l$.

Clearly if there exists a cardinal κ, and $\mu = \mu_\kappa$ such that $(T)_\mu$ holds, we are done. Otherwise, for every $\kappa \geqslant \chi$ we have a cardinal $\mu = \mu_\kappa$ such that $(F)_\mu$; i.e., there exists a family of functions $\{g_i\colon i < \kappa^+\} \subseteq \{f_{j,\mu}\colon j < \mu\}$ such that $\mathrm{rk}(0_{G_{l+1}}, g_i - g_j) \leqslant \gamma_l < l \leqslant \alpha_{l+1}$ violating the definition of α_{l+1} as the first ordinal with this property. $\blacksquare_{1.12}$

To construct the family $\{g_i\colon i < \lambda_n\}$ we will combine Claim 1.12 with a second application of the Erdös Rado Theorem.

Let $\kappa = (\mathrm{Max}\{2^{\lambda_n}, cf\alpha\})^+ < \lambda$. Let $\chi =^{\mathrm{def}} \beth_2(\kappa)^+$. Apply Claim 1.12 to get a family $\{f_i\colon i \in I\}$ satisfying:

(a) $f_i \restriction G_n = 0_{G_n}$,

(b) for $i \neq j$, $\gamma_{n-1} < \mathrm{rk}(0_{G_n}, f_i - f_j) < \alpha$ with $I \subseteq \lambda$, $|I| = \chi$.

For $g \in G_n^*$ such that $g/p = O_{G_n}$ define a coloring F_g of $[I]^3$ by two colors according to the following scheme: (i, j, k) is

red $\quad$ if $\mathrm{rk}(g, f_i - f_j) \leqslant \mathrm{rk}(g, f_j - f_k)$;

green $\quad$ if $\mathrm{rk}(g, f_i - f_j) > \mathrm{rk}(g, f_j - f_k)$.

By the Erdös Rado Theorem there is a set $J \subseteq I$, $|J| = \kappa$ such that each coloring is constant on $[J]^3$. Let the value of F_g on $[J]^3$ be denoted c_g. Observe that c_g is never *green* as this would produce a descending sequence of ordinals. We claim that $\{f_i\colon i \in J_0\}$ $(J_0 \subseteq J, |J_0| = \lambda_n)$ provides a set that can play the role of $\{g_i\colon i < \lambda_n\}$. We show first

$$\mathrm{rk}(g, f_i - f_j) \leqslant \mathrm{rk}(0_{G_n}, f_i - f_k) \qquad \text{for} \quad i < j < k \quad \text{in } J. \tag{$*$}$$

Indeed: $\mathrm{rk}(g, f_i - f_j) = \mathrm{rk}(g + 0_{G_n}, (f_k - f_j) + (f_i - f_k)) \geqslant \mathrm{Min}\{\mathrm{rk}(g, f_k - f_j),$ $\mathrm{rk}(0_{G_n}, f_i - f_k)) = \mathrm{Min}\{\mathrm{rk}(g, f_j - f_k), \mathrm{rk}(0_{G_n}, f_i - f_k)\}$ and we have equality unless $\mathrm{rk}(g, f_j - f_k) = \mathrm{rk}(0_{G_n}, f_i - f_k)$, in which case as $\mathrm{rk}(g, f_i - f_j) \leqslant \mathrm{rk}(g, f_j - f_k)$ $(*)$ hold.

Accordingly it will be sufficient to find $\gamma_n < \alpha$ such that $\mathrm{rk}(0_{G_n}, f_i - f_j) \leqslant \gamma_n$ for all $i, j \in J$. Observe first that $\mathrm{rk}(0_{G_n}, f_i - f_j) \leqslant \mathrm{rk}(0_{G_n}, f_k - f_l)$ for $i < j < k < l$ in J. For $\beta < \kappa$, clearly choose $i(\beta) < j(\beta)$ in J so that $\beta < \gamma \Rightarrow j(\beta) < i(\gamma)$. Then $\mathrm{rk}(0_{G_n}, f_{i(\beta)} - f_{j(\beta)})$ is a monotonically nondecreasing sequence of length κ below α. Since $cf\alpha < \kappa$ this sequence is bounded below α by some γ_n. It is easy to see that $\mathrm{rk}(0_{G_n}, f_i - f_j) \leqslant \gamma_n$ for all $i, j \in J$. $\blacksquare_{1.4}$

2. GENERALIZATIONS

The fact that we worked in Theorem 1.3 with $\mathbf{Z}$ or $p\mathbf{Z}$ and the specific mapping h/p are not important to the proof of the theorem. The following theorem is true, and has essentially the same proof as Theorem 1.3.

Notation. Let H, H' be abelian groups, suppose that $\varphi\colon H \to H' \to 0$ is exact. For a group G let $\varphi\colon \mathrm{Hom}(G, H) \to \mathrm{Hom}(G, H')$ be the induced homomorphism by φ. Denote $H^p_\varphi = \mathrm{Hom}(G, H')$, and let $H^* = \mathrm{Hom}(G, H)$.

THEOREM 2.1. *For every H, H', φ and G as above and satisfying $|G| > |H| \cdot |H'|$, if $|G|$ is a strong limit of cofinality $\aleph_0$ then $[H^p_\varphi : H^*] \geqslant |G| \Rightarrow [H^p_\varphi : H^*] = 2^{|G|}$.*

The last theorem can be generalized to non-abelian groups. The more general statement is given below as Theorem 2.2. Notice that there is no reference to the group G.

THEOREM 2.2. (1) *Suppose λ is strong limit of cofinality $\aleph_0$ (i.e., λ is countable or a singular strong limit cardinal). Let $\langle G_m, pr_{m,n}\colon n \leqslant m < \omega\rangle$ be an inverse system whose inverse limit is G_ω such that $|G_n| < \lambda$ and $|G_\omega| \geqslant \lambda$.*

(2) *Let $\mathbf{I}$ be an index set of cardinality less than λ. For every $t \in \mathbf{I}$, let $\langle H^t_m, h_{m,n}\colon n \leqslant m < \omega\rangle$ be an inverse system of groups and H^t_ω be the corresponding inverse limits.*

(3) *Let for every $t \in \mathbf{I}$, $h^t_n\colon H^t_n \to G_n$ be homomorphisms such that all diagrams commute, and let h^t_ω be the induced homomorphism from H^t_ω into G_ω.*

(4) *If for every $\mu < \lambda$ there are $\langle f_i \in G_\omega\colon i < \mu\rangle$ such that for every $i \neq j$ $(\forall t \in \mathbf{I})[f_i - f_j \notin \mathrm{Range}(h^t_\omega)]$, then there are $\langle f_i \in G_\omega\colon i < \lambda^{\aleph_0}\rangle$ such that for every $i \neq j$ $(\forall t \in \mathbf{I})[f_i - f_j \notin \mathrm{Range}(h^t_\omega)]$.*

Proof. Similar to the proof of Theorem 1.4.

REFERENCES

1. S. CHASE, On group extensions and a problem of J. H. C. Whitehead, *in* "Topics in Abelian Groups," pp. 173–197, Scott, Foresman, Glenview, IL, 1963.
2. P. C. EKLOF, "Set Theoretic Methods in Homological Algebra and Abelian Groups," SMS 69, Les Presses de lwuniversité de Montréal, 1980.
3. P. C. EKLOF AND M. HUBER, On the rank of Ext, *Math. Z.* **174** (1980), 159–185.
4. P. C. EKLOF AND M. HUBER, On the rank of Ext. *in* "Abelian Group Theory" (R. Göbel and E. Walker, Eds.), Lecture Notes in Mathematics, Vol. 874, Springer-Verlag, New York/Berlin, 1981.
5. P. ERDÖS AND R. RADO, A partition calculus in set theory, *Bull. Amer. Math. Soc.* **62**, 427–489.
6. L. FUCH, "Infinite Abelian Groups," Vol. I, Academic Press, Orlando, FL, 1970.
7. L. FUCHS, "Infinite Abelian Groups," Vol. II, Academic Press, Orlando, FL, 1973.
8. H. HILLER, M. HUBER, AND S. SHELAH, The structure of Ext(A, Z) and $V = L$, *Math. Z.* **162** (1978), 39–50.
9. H. L. HILLER AND S. SHELAH, Singular cohomology in L, *Israel J. Math.* **28** (1977), 32–44.
10. P. HILTON AND U. STAMBACH, "A Course in Homological Algebra," Springer-Verlag, Berlin, 1971.
11. T. JECH, "Set Theory," Academic Press, Orlando, FL, 1978.
12. A. MEKLER AND S. SHELAH,
13. R. J. NUNKE, Whitehead's problem, *in* "Abelian Group Theory" (A. Dold and B. Eckmann, Eds.), Lecture Notes in Mathematics, Vol. 616, Springer-Verlag, New York/Berlin, 1977.
14. F. P. RAMSEY, On a problem in formal logic, *Proc. London Math. Soc. (2)* **30**, 264–286.
15. G. SAGEEV AND S. SHELAH, Weak compactness and the structure of Ext(G, Z), *in* "Abelian Group Theory" (R. Göbel and E. Walker, Eds.), Lecture Notes in Mathematics, Vol. 874, Springer-Verlag, New York/Berlin, 1981.
16. G. SAGEEV AND S. SHELAH, On the structure of Ext. *J. Symbolic Logic* **50** (1985), 302–315.
17. S. SHELAH, Infinite abelian groups, Whitehead problem and some constructions, *Israel J. Math.* **18** (1974), 243–256.
18. S. SHELAH, On the structure of Ext, unpublished manuscript, 1979.
19. S. SHELAH, On uncountable abelian groups, *Israel J. Math.* **32** (1970), 311–330.
20. . SHELAH, "Proper Forcing," Lecture Notes in Mathematics, Vol. 940, Springer-Verlag, New York/Berlin, 1982.
21. S. SHELAH, The consistency of Ext$(G, \mathbf{Z}) = \mathbf{Q}$, *Israel J. Math.* **39** (1981), 74–82.