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Research Article

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No universal group in a cardinal

Abstract: For many classes of models, there are universal members in any cardinal λ which "essentially satisfies GCH, i.e., $\lambda = 2^{<\lambda}$," in particular for the class of models of a complete first-order T (well, if at least $\lambda > |T|$). But if the class is "complicated enough", e.g., the class of linear orders, we know that if λ is "regular and not so close to satisfying GCH", then there is no universal member. Here, we find new sufficient conditions (which we call the olive property), not covered by earlier cases (i.e., fail the so-called SOP_4). The advantage of those conditions is witnessed by proving that the class of groups satisfies one of those conditions.

Keywords: Model theory, universal models, olive property, group theory, non-structure, classification theory

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1 Introduction

1.1 Background and open questions

A natural and old question is how to characterize the class of cardinals λ in which a class $\mathfrak{k}$ has a universal member, where $\mathfrak{k}$ is, e.g., the class of models of a first-order theory T with elementary embeddings.

On history, see Kojman–Shelah [4] and later Džamonja [1]. Recall that if $\lambda = 2^{<\lambda} > \aleph_0$, then many classes have a universal member in λ, so assuming GCH, we know when there is a universal model in every $\lambda > \aleph_0$.

For transparency, we consider a first-order countable T. Now, the class $\mathfrak{k}_T = (\mathrm{Mod}_T, \prec)$ of models of T with elementary embeddings has the amalgamation property, the JEP and satisfies that $A \subseteq M \in \mathrm{Mod}_T \Rightarrow (\exists N \in \mathrm{Mod}_T)(A \subseteq N \prec M \wedge \|N\| = |A| + \aleph_0)$. From such classes or just a.e.c. with amalgamation and JEP with $\mathrm{LST}(\mathfrak{k})$ instead of $\aleph_0$, it follows that if $\lambda = \lambda^{<\lambda} > \aleph_0$, then there is a saturated (or universal homogeneous) $M \in \mathrm{Mod}_T$ of cardinality λ, which implies it is universal for $\mathfrak{k}_T$. If $\lambda = 2^{<\lambda} > \aleph_0$ does not satisfy $\lambda = \lambda^{<\lambda}$, there still is a so-called special model which is universal. So, we are interested in the cases where GCH fails.

Recall that on the one hand Kojman–Shelah [4] show that if T is the theory of dense linear orders or just T has the strict order property, then T fails (in a strong way) to have a universal member in regular cardinals in which cardinal arithmetic is "not close to GCH"; (for regular λ this means there is a regular μ such that $\mu^+ < \lambda < 2^\mu$, while for singular λ we need of course $\lambda < 2^{<\lambda}$ and a very weak pcf condition).

By [9], we can weaken "the strict order property" to the 4-strong order property SOP_4. On consistency, see Džamonja–Shelah [2].

Natural questions (we shall address some of them) are the following.

Question 1.1. (i) Is there a weaker condition (on T) than SOP_4 which suffices?
(ii) Can we find a best one?
(iii) Can we find such a condition satisfied for some theory T which is NSOP_3?

Question 1.2. (i) Is there T with the class $\mathrm{Univ}(T)\backslash(2^{\aleph_0})^+$ strictly smaller than the one for linear orderings, see Definition 1.12 (ii); is it better if we restrict ourselves to regular cardinals above $2^{\aleph_0}$?
(ii) Can we get the above to be $\{\lambda : \lambda = 2^{<\lambda}\}$?
(iii) What about singular cardinals?

Question 1.3. (i) Is it consistent that the class of linear orders has a universal member in λ such that $2^{<\lambda} > \lambda > 2^{\aleph_0}$. (For $\lambda = \aleph_1 < 2^{\aleph_0}$, the answer is yes, see [6]; a more detailed version is in preparation).
(ii) The same can be asked for some theory with SOP_4 or the olive property (defined below, e.g., in Definition 1.8).

Recall that by Shelah–Usvyatsov [16], the class of groups has NSOP_4 but has SOP_3, so it was not clear where it stands.

Question 1.4. (i) Where does the class of groups stand (concerning the existence of a universal member in a cardinal)?
(ii) Is it consistent that there is a universal locally finite group of cardinality $\aleph_1$? Of cardinality $\beth_\omega$? $\lambda = \beth_\omega^+$? Of regular cardinals $\lambda \in (\beth_\omega^+, \beth_{\omega+1})$? Of other cardinals $\lambda < \lambda^{\aleph_0}$? Recall (Grossberg–Shelah [3]) that if μ is strong limit of cofinality $\aleph_0$ above a compact cardinal, then there is a universal locally finite group of cardinality μ, but if $\mu = \mu^{\aleph_0}$, then there is no one.

Concerning singulars, we ask the following.

Question 1.5. Does $\theta = \mathrm{cf}(\theta)$ and $\theta^{+2} < \mathrm{cf}(\lambda) < \lambda < 2^\theta$ imply $\lambda < \mathrm{univ}(\lambda, T)$?

Question 1.6. (i) Characterize the failure of the criterion of [8], Džamonja–Shelah [2] (for consistency).
(ii) Does SOP_3 (or something weaker) suffice for no universal in λ when $\mu = \mu^{<\mu} \ll \lambda < 2^\mu$?
(iii) Which theories T fail to have a universal in λ when $\lambda = \mu^{++} = 2^\mu < 2^{\mu^+}$?
(iv) We may consider weaker properties of T for no universal in λ, $\mu = \mu^{<\mu} \ll \lambda < 2^\mu$.
(v) Sort out the variants of the olive property (defined below, e.g., in Definition 1.8).

Discussion 1.7. The case $\lambda = \mu^+$, $\lambda < 2^\mu$ and $2^{<\mu} \leq \lambda$ (e.g., for transparency $\mu = \mu^{<\mu}$) is not resolved as we do not necessarily have $\bar{C} = \langle C_\delta : \delta \in S^\lambda_\mu \rangle$ guessing clubs, see [4].

By some earlier results (see [8]), if $\mu = 2^\kappa$ (i.e., μ is not a strong limit cardinal) and there is no universal, then there is a sequence $\langle \Lambda_\delta : \delta \in S^\lambda_\mu \rangle$, $\Lambda_\delta \subseteq {}^{(C_\delta)}\mu$ of cardinality λ such that for every sequence $\langle \eta_\delta \subseteq {}^{(C_\delta)}\mu : \delta \in S^\lambda_\mu \rangle$ there is a club E of λ such that for every $\delta \in E \cap S^\lambda_\mu$ there is a $\nu \in \Lambda_\delta$ such that the functions η_δ, ν agree on $E \cap \mathrm{nacc}(C_\delta)$.

Using a more complicated T, we can replace ${}^{C_\delta}\mu$ by ${}^{(C_\delta \times D_\delta)}\mu$, so the agreement above is on the product $(E \cap \mathrm{nacc}(C_\delta)) \times (E \cap \mathrm{nacc}(C_\delta))$ but of unclear value.

On subsequent works on consistency, see [14, 15].

1.2 What is accomplished

What do we achieve? We introduce the "olive property", which is a sufficient condition for a class to have a universal member in λ only if λ is "close to satisfying GCH", similarly to the linear order case. This condition is weaker than SOP_4, hence gives a positive answer to Question 1.1 (i). But the condition implies SOP_3, so it does not answer Question 1.1 (iii) and it is also totally unclear whether it is best in any sense and whether its negation has interesting consequences.

However, it answers Question 1.4 (i) to a large extent because the class of groups has the olive property and we can also deal with locally finite groups answering some cases of Question 1.4 (ii); see Section 3. Also, we try to formalize conditions sufficient for non-existence, see Definition 2.6. As the reader may find the definition of the (variants of the) olive property opaque, we define a simple case used for the class of groups, and the reader may then look first at the class of groups in Section 3.

Definition 1.8. A (first-order) universal theory T has the olive property when there are $(\varphi_0, \varphi_1, \psi)$ and a model $\mathfrak{C}$ of T such that
(i) for some m, $\varphi_0 = \varphi_0(\bar{x}_{[m]}, \bar{y}_{[m]})$, $\varphi_1 = \varphi_1(\bar{x}_{[m]}, \bar{y}_{[m]})$, $\psi = \psi(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{z}_{[m]})$ are quantifier-free formulas (and $\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{z}_{[m]}$ are m-tuples of variables, see Notation 1.10 below);

(ii) for every k and $\bar{f} = \langle f_\alpha : \alpha < k \rangle$, where f_α is a function from α to $\{0,1\}$, we can find $\bar{a}_\alpha \in {}^m\mathfrak{C}$ for $\alpha < k$ such that
 (a) $\varphi_\iota[\bar{a}_\alpha, \bar{a}_\beta]$ for $\alpha < \beta < k$ when $\iota = f_\beta(\alpha)$,
 (b) $\psi[\bar{a}_\alpha, \bar{a}_\beta, \bar{a}_\gamma]$ when $\alpha < \beta < \lambda$ and $f_\gamma \restriction [\alpha, \beta]$ is constantly 0;

(iii) there are no $\bar{a}_\ell \in {}^m\mathfrak{C}$ for $\ell = 0,1,2,3$ such that the following conditions[1] are satisfied in $\mathfrak{C}$:
 (a) $\varphi_0[\bar{a}_0, \bar{a}_\ell]$ for $\ell = 1,2,3$, $\varphi_1[\bar{a}_1, \bar{a}_\ell]$ for $\ell = 1,2$ and $\varphi_0[\bar{a}_2, \bar{a}_3]$,
 (b) $\psi[\bar{a}_0, \bar{a}_2, \bar{a}_3]$.

Concluding Remarks 1.9. Concerning some things not addressed here, we note the following.

(i) Concerning the proof here of "there is no universal", we can carry it via defining invariants parallel to Kojman–Shelah [4] such that[2]
 (*) (a) if $M \in \mathrm{Mod}_{T,\lambda}$, then $\mathrm{INV}_\lambda(M)$ is a set of cardinality $\le \lambda$ or just $\le \chi < 2^\lambda$;
 (b) if $M_1, M_2 \in \mathrm{Mod}_{T,\lambda}$ and M_1 is embeddable into M_2, then $\mathrm{INV}_\lambda(M_1) \subseteq \mathrm{INV}_\lambda(M_2)$;
 (c) there is a set of 2^λ objects $\mathbf{x}$ such that $(\exists M \in \mathrm{Mod}_{T,\lambda})(\mathbf{x} \in \mathrm{INV}_\lambda(M))$.

(ii) We can use more complicated versions of the olive property. In the proof we use one δ and then one $\alpha \in \mathrm{nacc}(C_\delta) \cap E$ (or less), but we may use several ordinals α resulting in more complicated versions. This will become more pressing if we have a complimentary property, guaranteeing "no universal" or some variant.

1.3 Preliminaries

Notation 1.10. (i) Let $\bar{x}_{[I]} = \langle x_t : t \in I \rangle$ and similarly $\bar{y}_{[I]}$, $\bar{x}_{[I],\alpha}$, etc. and $\bar{x}_{[I],\ell} = \langle x_{t,\ell} : t \in I \rangle$.
(ii) For a first-order complete T, $\mathfrak{C}_T$ is the "monster model of T".

Definition 1.11. (i) For a set A, $|A|$ is its cardinality, but for a structure M, its cardinality is $\|M\|$ while its universe is $|M|$; this applies, e.g., to groups.
(ii) We use G, H for groups and M, N for general models.
(iii) Let $\mathfrak{k}$ denote a pair $(K_\mathfrak{k}, \le_\mathfrak{k})$ or we may say a class (of models) $\mathfrak{k}$, where
 (a) $K_\mathfrak{k}$ is a class of $\tau_\mathfrak{k}$-structures, where $\tau_\mathfrak{k}$ is a vocabulary;
 (b) $\le_\mathfrak{k}$ is a partial order on $K_\mathfrak{k}$ such that $M \le_\mathfrak{k} N \Rightarrow M \subseteq N$;
 (c) both $K_\mathfrak{k}$ and $\le_\mathfrak{k}$ are closed under isomorphisms.
(iv) (a) We call $\mathfrak{k}$ universal and may write only $K_\mathfrak{k}$ when $\le_\mathfrak{k}$ is being a submodel.
 (b) We say $f : M \to N$ is a $\le_\mathfrak{k}$-embedding when f is an isomorphism from M onto some $M_1 \le_\mathfrak{k} N$.
(v) If T is a first-order theory, then Mod_T is the pair $(\mathrm{mod}_T, \le_T)$, where mod_T is the class of models of T and $\le_T$ is $\prec$ if T is complete and $\subseteq$ if T is not complete.
(vi) We may write T instead of Mod_T, e.g., in Definition 1.12 below.
(vii) For a class K of structures $K_\lambda = \{M \in K : \|M\| = \lambda\}$.

Definition 1.12. (i) For a class $\mathfrak{k}$ and a cardinal λ, a set $\{M_i : i < i^*\}$ of models from $K_\mathfrak{k}$ is jointly $(\lambda, \mathfrak{k})$-universal when for every $N \in K_\mathfrak{k}$ of size λ, there is an $i < i^*$ and an $\le_\mathfrak{k}$-embedding of N into M_i.
(ii) For $\mathfrak{k}$ and λ as above, let

$$\mathrm{univ}(\lambda, \mu, \mathfrak{k}) := \min\{|\mathscr{M}| : \mathscr{M} \text{ is a family of members of } K_\mathfrak{k} \text{ each of cardinality } \le \mu, \text{ which is jointly } \mathfrak{k}\text{-universal for } \lambda\}.$$

We note that if $\mu = \lambda$, we may omit μ, and we also let $\mathrm{Univ}(\mathfrak{k}) = \{\lambda : \mathrm{univ}(\lambda, \mathfrak{k}) = 1\}$.
(iii) For a pair[3] $\bar{\mathfrak{k}} = (\mathfrak{k}_1, \mathfrak{k}_2)$ of classes with $\mathfrak{k}_\iota = (K_{\mathfrak{k}_\iota}, \le_{\mathfrak{k}_\iota})$ as in Definition 1.11 (iii) for $\iota = 1,2$ such that $K_{\mathfrak{k}_1} \subseteq K_{\mathfrak{k}_2}$, let $\mathrm{univ}(\lambda, \mu, \bar{\mathfrak{k}})$ be the minimal $|\mathscr{M}|$ such that $\mathscr{M}$ is a family of members of $K_{\mathfrak{k}_2}$, each of cardinality $\le \mu$, such that every $M \in K_{\mathfrak{k}_1}$ of cardinality λ can be $\le_{\mathfrak{k}_2}$-embedded into some member of $\mathscr{M}$.

1 In the class of groups, in clause (α), $\varphi_0[\bar{a}_0, \bar{a}_1]$, $\varphi_1[a_1, a_2]$, $\varphi_1[a_1, \bar{a}_3]$ suffice.
2 For transparency, λ is regular uncountable, see Definition 1.11 (v),(vii).
3 This will be used for $\mathfrak{k}_1$ being the class of locally finite groups and $\mathfrak{k}_2$ being the class of groups.

Dealing with a.e.c.'s (see [13]), we have the following.

Definition 1.13. (i) We say that a formula $\varphi = \varphi(\bar{x}_{[I]})$, in any logic, is $\mathfrak{k}$-upward preserved when $\tau_\varphi \subseteq \tau_\mathfrak{k}$ and if $M \leq_\mathfrak{k} N$ and $\bar{a} \in {}^I M$, then $M \vDash \varphi[\bar{a}]$ implies $N \vDash \varphi[\bar{a}]$.
(ii) For $\bar{\mathfrak{k}}$ as in Definition 1.12 (iii), we say that a pair $\bar{\varphi}(\bar{x}_{[I]}) = (\varphi_1(\bar{x}_{[I]}), \varphi_2(\bar{x}_{[I]}))$ is $\bar{\mathfrak{k}}$-upward preserving when $\tau_{\varphi_1} \cup \tau_{\varphi_2} \subseteq \tau_{\mathfrak{k}_\iota}$ and if $M_\iota \in K_{\mathfrak{k}_\iota}$ for $\iota = 1, 2$, $\bar{a} \in {}^I(M_1)$ and $M_1 \leq_{\mathfrak{k}_2} M_2$, then $M_1 \vDash \varphi_1[\bar{a}]$ implies $M_2 \vDash \varphi_1[\bar{a}]$.
(iii) In part (ii), if $\varphi_0 = \varphi_1$, then we may write φ instead of $\bar{\varphi}$. Saying[4] that a sequence $\bar{\psi}$ is $\mathfrak{k}$-upward preserving means that every formula appearing in $\bar{\psi}$ is $\mathfrak{k}$-upward preserving.

Definition 1.14. (i) For an ideal J on a set A and a set B, let $\mathbf{U}_J(B) = \mathrm{Min}\{|\mathscr{P}| : \mathscr{P}$ is a family of subsets of B, each of cardinality $\leq |A|$, such that for every function f from A to B, for some $u \in \mathscr{P}$ we have $\{a \in A : f(a) \in u\} \in J^+\}$.
(ii) For an ideal J on a set A, a cardinal θ and a set B, let $\mathbf{U}^\theta_J(B) = \mathrm{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [B]^{\leq |A|}$ and if $f \in {}^A({}^\theta B)$, then for some $u \in \mathscr{P}$ we have $\{a \in A : \mathrm{Rang}(f(a)) \subseteq u\} \in J^+\}$. So, $\mathbf{U}^\theta_J(B) \leq \mathbf{U}_J(|N|^\theta)$.
(iii) Clearly, only $|B|$ matters, so we normally write $\mathbf{U}_J(\lambda)$, see [10].

1.4 Annotated content

The paper is organized as follows. In Section 2, we give definitions of some versions of the olive property and give an example failing the SOP_4. We phrase relevant set theoretic conditions like Qr_1 (slightly weaker than those used earlier). Then, we give a complete proof using $\mathrm{Qr}_1(\chi_2, \chi_1, \lambda)$ to deduce $\mathrm{Univ}(\chi_1, \lambda, \mathfrak{k}) \geq \chi_2$, so there is no universal in the class $\mathfrak{k}$ in the cardinal λ, when $\mathfrak{k}$ has the olive property. In Section 3, we prove that the class of groups has the olive property. We also deal with the non-existence of universal structures for pairs of classes, e.g., for the the pair (locally finite groups, groups).

2 The olive property

Definition 2.1. (i) Convention.
(a) Let T be a first order theory and $\mathfrak{C} = \mathfrak{C}_T$ a monster for T.
(b) (1) Let $\Delta \subseteq \mathbb{L}(\tau_T)$ be a set of formulas.
(2) Omitting Δ means $\Delta = \mathbb{L}(\tau_T)$ if T is complete, Δ = set of quantifiers-free formula otherwise, and we may write qf instead of Δ.
(c) (1) Let m and $n \geq k_\iota \geq 2$ for $\iota = 0, 1$, $n \geq k_0 + k_1 \geq 3$, $\eta \in {}^n 2$ be such that $\eta(0) = 0$ and $\eta^{-1}\{0\}$ is not an initial segment, and $\eta^{-1}\{\iota\}$ has $\geq k_\iota$ members for $\iota = 0, 1$.
(2) If $\eta(\ell) = \ell \bmod 2$ for $\ell < k$, we may write n instead of η.
(d) (1) If $\bar{k} = (k_0, k_1)$, $k_1 \leq k_0 + 1 \leq k_1 + 1$, we may write $k_0 + k_1$ instead of $\bar{k}$ and let $k(\iota) = k_\iota$ for $\iota = 0, 1$.
(2) Omitting m means "for some m".
(3) Omitting $n, \eta, \bar{k}$ means $n = 3, \eta = \langle 0, 1, 0 \rangle, \bar{k} = (2, 1))$ for some m.
(e) (1) Below we may write $\psi_\iota = \psi_{\iota, k_\iota}$ and $\varphi_\iota = \psi_{\iota,1}$ for $\iota = 0, 1$.
(2) If $\varphi_0 = \varphi_1 = \varphi$, we may write φ.
(3) We may omit $\psi_{3,k}$ when it is a logically true formula.
(ii) We say T has the $(\Delta, \eta, \bar{k}, m)$-olive property when there is a pair $(\bar{\psi}_0, \bar{\psi}_1)$ of sequences of formulas from Δ witnessing it, see (iii).
(iii) We say $(\bar{\psi}_0, \bar{\psi}_1)$ witnesses the $(\Delta, \eta, \bar{k}, m)$-olive property (for T, with the convention above) when[5]
(a) $\bar{\psi}_\iota = \langle \psi_{\iota,k}(\bar{x}_{[m],0}, \ldots, \bar{x}_{[m],k}) : k = 1, \ldots, k_\iota \rangle$ for $\iota = 0, 1$ with $\psi_{\iota,k} \in \Delta$;

4 Pedantically, a pair is a sequence of length 2, so (ii) and (iii) of Definition 1.13 are incompatible, but the intention should be clear from the context.
5 It is the case for every λ, but in this definition, by compactness, $\lambda = \aleph_0$ is enough.

(b)$_\lambda$ for every $\bar f = \langle f_\alpha : \alpha < \lambda\rangle$, where f_α is a function from α to $\{0,1\}$, we can find a model M of T and $\bar a_\alpha \in {}^m M$ for $\alpha < \lambda$ such that[6]
 (1) $\varphi_\iota[\bar a_\alpha, \bar a_\beta]$ for $\alpha < \beta < \lambda$ when $\iota = f_\beta(\alpha)$, recalling Definition 2.1 (i) (e) (1),
 (2) $\psi_{\iota,k}(\bar a_{\alpha_0}, \dots, \bar a_{\alpha_{k-1}}, \bar a_\beta)$ when $k \in \{2,\dots,k_\iota\}$ and $\alpha_0 < \cdots < \alpha_{k-1} < \beta < \lambda$ and $f_\beta \restriction [\alpha_0, \alpha_{k-1}]$ is constantly ι, so when $k = 1$, it holds trivially;

(c) there are no $\bar a_\ell \in {}^m\mathfrak{C}$ for $\ell < n+1$ such that
 (1) $\varphi_\iota[\bar a_i, \bar a_j]$ for $i < j < n+1$ and $\eta(i) = \iota$,
 (2) if $\iota \in \{0,1\}$, $k \in \{2,\dots,k_\iota\}$ and $\ell_0 < \cdots < \ell_{k-1}$ are from $\{\ell < n : \eta(\ell) = \iota\}$ and $\ell_{k-1} < \ell \le n$, then $\psi_{\iota,k}[\bar a_{\ell_0}, \dots, \bar a_{\ell_{k-1}}, \bar a_\ell]$.

Remark 2.2. This fits the classification of properties of such T in [11, 5.15–5.23].

Definition 2.3. (i) Let K be a universal class of τ-models, see Definition 1.11 (iv). We say that K has the $\lambda - (\eta, \bar k, m)$-olive property when some quantifier-free $(\bar\psi_0, \bar\psi_1)$ witnessing it, that is, (a), (b)$_\lambda$, (c) hold (replacing $\mathfrak{C}_T$ by "in some $M \in K_\lambda$").

(ii) We say that a class (e.g., an a.e.c.) $\mathfrak{k} = (K_{\mathfrak{k}}, \le_{\mathfrak{k}})$ has the $\lambda - (\eta, \bar k, m)$-property when there are $\bar\psi_0, \bar\psi_1$ which are $\mathfrak{k}$-upward preserved formulas in any logic (see Definition 1.13) and (a), (b)$_\lambda$, (c) of Definition 2.1 hold, replacing M by "some $\mathfrak{C} \in K_{\mathfrak{k}}$ of cardinality λ".

Remark 2.4. (i) Note that for T first-order complete, $\mathfrak{k} = \mathrm{Mod}_T = (\mathrm{mod}_T, \prec)$, Definition 2.3 (ii) gives Definition 2.1 and for T first-order universal not complete, $\mathfrak{k} = \mathrm{Mod}_T = (\mathrm{mod}_T, \subseteq)$, Definition 2.3 (ii) gives Definition 2.1. Similarly for Definition 2.3 (i).

(ii) Of course, for T first-order, the λ does not matter.

Claim 2.5. *Assume $n \ge k_0 + k_1 \ge 3$, $\eta \in {}^n 2$ and $|\eta^{-1}\{\iota\}| \ge k_\iota \ge 1$ for $\iota = 0, 1$. Then there is a complete first-order countable T having the $(\eta, \bar k, 1)$-olive property but T is* NSOP$_4$, *is* SOP$_3$ *and is categorical in $\aleph_0$.*

Proof. Let $\tau = \{P, Q_0, Q_1\}$, where P is a binary predicate and Q_ι is a $(k_\iota + 1)$-place predicate. Let $T^0_{\eta,\bar k}$ be the following universal theory in $\mathbb{L}(\tau)$.

$(*)_1$ A τ-model M is a model of $T^0_{\eta,\bar k}$ if and only if we cannot embed $N^*_{\eta,\bar k}$ into M where

$\oplus$ $N^*_{\eta,\bar k}$ is the τ-model with universe $\{a_0, \dots, a_n\}$ as in Definition 2.1 (iii) (c) (1)–(2) for $\varphi(x_0, x_1) = P(x_0, x_1)$, $\psi_\iota(x_0, \dots, x_{k(\iota)}) = Q_\iota(x_0, \dots, x_{k(\iota)})$ recalling Definition 2.1 (i) (e) (3) and $\ell < k \le n \Rightarrow a_\ell \ne a_k$.

Now:

$(*)_2$ $T^0_{\eta,\bar k}$ has the JEP and the amalgamation property by disjoint union.
 [Why? Assume that $M_0 \subseteq M_1, M_0 \subseteq M_2$ are models of T_0 (but abusing notation we allow M_0 to be empty) and $|M_1| \cap |M_2| = |M_0|$, we define $M = M_1 \cup M_2$ that is

$(*)_{2.1}$ (a) $|M| = |M_1| \cup |M_2|$,
 (b) $P^M = P^{M_1} \cup P^{M_2}$,
 (c) $Q^M_\iota = Q^{M_1}_\iota \cup Q^{M_2}_\iota$ for $\iota = 0, 1$.
 So, M is a τ-model, it is a model of T as in $(*)_1 \oplus$, hence any pair of distinct elements belongs to a relation, i.e., we have $\ell < k \le n \Rightarrow (a_\ell, a_k) \in P^M$.]

$(*)_3$ $T_{\eta,\bar k}$, the model completion of $T^0_{\eta,\bar k}$, is well defined and has elimination of quantifiers.
 [Why? As τ is finite with no function symbols and $(*)_2$.]

$(*)_4$ $T_{\eta,\bar k}$ is NSOP$_4$ (see [9, 2.5]).
 [Why? Because

$(*)_{4.1}$ if (A), then (B), where
 (A) (a) A_0, A_1, A_2, A_3 are disjoint sets,
 (b) M_ℓ is a model of $T^0_{\eta,\bar k}$ with universe A_ℓ for $\ell = 0, 1, 2, 3$,
 (c) if $\{\ell(1), \ell(2)\} \in \mathscr{W} := \{\{0,1\}, \{1,2\}, \{2,3\}, \{3,4\}\}$, then $M_{\{\ell(1),\ell(2)\}}$ is a model of $T^0_{k,n}$ with universe $A_{\ell(1)} \cup A_{\ell(2)}$ extending $M_{\ell(1)}$ and $M_{\ell(2)}$;

6 Actually clause (1) is a specific case of clause (2) provided that in clause (2) we allow $k = 1$. Similarly for clauses (c) (1) and (2).

(B) $M = \cup\{M_{\{\ell(1),\ell(2)\}} : \{\ell(1),\ell(2)\} \in \mathscr{W}\}$, where the union is defined as in the proof of $(*)_2$, is a model of $T^0_{k,n}$ extending all of them.]

[Why? Clearly, M is a τ-model and if f embeds $N^*_{\eta,\bar{k}}$ into M, as in $(*)_2$ we have $\mathrm{Rang}(f) \subseteq M_{\ell(1),\ell(2)}$ for some $\{\ell(1),\ell(2)\} \in \mathscr{W}$, a contradiction.]

$(*)_5$ $T_{\eta,\bar{k}}$ (and $\mathrm{Mod}_{T^0_{\eta,\bar{k}}}$) has the $(\eta,\bar{k})$-olive property as witnessed by

$$\varphi(x_0,x_1) = P(x_0,x_1), \quad \psi_\iota(x_0,\dots,x_{k(\iota)}) = Q_\iota(x_0,\dots,x_{k(\iota)}).$$

[Why? In Definition 2.1 (iii), clause (a) holds trivially and clause (c) is obvious from the choice of $T^0_{\eta,\bar{k}}$. For clause $(b)_\lambda$ we are given $\langle f_\alpha : \alpha < \lambda\rangle$, where f_α is a function from α to $\{0,1\}$, and we have to find M as there. We define a τ-model M with

- universe $\{a^*_\alpha : \alpha < \lambda\}$ such that $\alpha < \beta \Rightarrow a^*_\alpha \neq a^*_\beta$;
- $P^M = \{(a^*_\alpha, a^*_\beta) : \alpha < \beta < \lambda\}$;
- $Q^M_\iota = \{(a^*_{\alpha_0},\dots,a^*_{\alpha_{k(\iota)-1}}, a_\beta) : \alpha_0 < \dots < \alpha_{k(\iota)-1} < \beta$ and $f_\beta \restriction [\alpha_0, \alpha_{k(\iota)-1}]$ is constantly $\iota\}$.

It suffices to prove that M is a model of $T^0_{\eta,\bar{k}}$. So, toward a contradiction, assume h embeds $N^*_{\eta,\bar{k}}$ into M, so let $h(a^*_\ell) = a_{g(\ell)}$, where $g : \{0,\dots,n\} \to \lambda$; recalling $\ell_1 < \ell_2 \le n \Rightarrow (a^*_{\ell_1}, a^*_{\ell_2}) \in P^{N^*_{\eta,\bar{k}}}$ and the choice of P^M, then necessarily g is a one-to-one function. For $\ell < n$, recall that $N^*_{\eta,\bar{k}} \models$ "$P(a^*_\ell, a^*_{\ell+1})$" but h is an embedding so $M \models$ "$P[a^*_{g(\ell)}, a^*_{g(\ell+1)}]$", but if $g(\ell) \ge g(\ell+1)$, this fails by the choice of P^M hence $g(\ell) < g(\ell+1)$. Now, let $i_* = \min\{i : \eta(i) = 1\}$. Let $i_0 < \dots < i_{k(0)-1}$ be from $\eta^{-1}\{0\}$ such that $i_0 = 0$ and $i_{k(\iota)-1}$ is maximal, hence $i_* \in [i_0, i_{k(0)-1})$. Now, $N^*_{\eta,\bar{k}} \models Q_0[a^*_{i_0},\dots,a^*_{i_{k(0)-1}}, a^*_n]$, hence $M \models Q_0[a^*_{g(i_0)},\dots,a^*_{g(i_{k(0)-1})}, a^*_{g(n)}]$ and this implies that $f_{g(n)} \restriction [g(i_0), g(i_{k(0)-1})]$ is constantly 0, hence $f_{g(n)}(g(i_*)) = 0$. Similarly, let $j_0 < \dots < j_{k(1)-1}$ be from $\eta^{-1}\{1\}$ such that $j_0 = i_*$; now $N^*_{\eta,\bar{k}} \models Q_1[a^*_{j_0},\dots,a^*_{j_{k(1)-1}}, a^*_n]$, hence $M \models Q_1[a_{g(0)},\dots,a_{g(j_{k(1)-1})}, a_n]$, hence $f_{g(n)} \restriction [g(j_0), g(j_{k(1)-1})]$ is constantly 1, hence $f_{g(n)}(g(i_*)) = 1$, a contradiction, so $(*)_5$ holds indeed.]

$(*)_6$ $T_{\eta,\bar{k}}$ has the SOP_3.

[Why? Again let $i_* = \min\{i : \eta(i) = 1\}$, $u_0 = \{0,\dots,i_*-1\}$, $u_1 = \{i_*\}$, $u_2 = \{i_*+1,\dots\}$.]

Note that

$(*)_{6.1}$ (u_0,u_1,u_2) is a partition of $\{0,\dots,n\}$;

$(*)_{6.2}$ if $\iota < 2$ and $(a_{\ell_0},\dots,a_{\ell_{k(\iota)}}) \in Q_\iota^{N^*_{\eta(\iota),\bar{k}}}$, then $\{\ell_0,\dots,\ell_{k(\iota)}\} \cap u_j = \emptyset$ for some j

[Why? Otherwise, as $\ell_0 < \ell_1 < \cdots$, necessarily $\ell_0 < i_*$, $\ell_{k(\iota)} > i_*$ and $i_* = \ell_k$ for some $k \in (0, k(\iota))$. But then $\eta(\ell_0) = 0 \neq 1 = \eta(k)$, in contradiction to Definition 2.1 (iii) (c).]

The rest should be clear by considering the proof of the model completion of the theory of triangle-free graphs having SOP_3, see [9, Section 2]. □

As in earlier cases we apply a kind of guessing of clubs (almost suitable also for them, i.e., for the proof with strict order and SOP_4). An unexpected gain is that here we use a weaker version: there is no requirement $\alpha < \lambda \Rightarrow \lambda > |\{C_\delta \cap \alpha : \delta \in S$ satisfies $\alpha \in \mathrm{nacc}(C_\delta)\}|$ but it is unclear how this helps. Also, here the use of the pair $(\bar{\mathscr{A}}, \bar{\mathbf{g}})$ may be helpful.

Definition 2.6. (i) For λ regular uncountable and $\chi_2 > \chi_1 \ge \lambda$ let $\mathrm{Qr}_1(\chi_2,\chi_1,\lambda)$ mean that there are $S, \bar{C}, I, \bar{\mathscr{A}}, \bar{\mathbf{g}}$ witnessing it, which means that

⊞ (a) $S \subseteq \lambda$ and I is an ideal on S;
(b) $\bar{C} = \langle C_\delta : \delta \in S\rangle$;
(c) $C_\delta \subseteq \delta$, note that possibly $\sup(C_\delta) < \delta$;
(d) (1) $\bar{\mathbf{g}} = \langle \bar{g}_j : j < \chi_2\rangle$,
(2) $\bar{g}_j = \langle g_{j,\delta} : \delta \in S\rangle$,
(3) $g_{j,\delta} : C_\delta \to \{0,1\}$;
(e) (1) $\bar{\mathscr{A}} = \langle \bar{\mathscr{A}}_j : j < \chi_2\rangle$,
(2) $\bar{\mathscr{A}}_j = \langle \mathscr{A}_{j,\delta} : \delta \in S\rangle$,
(3) $\mathscr{A}_{j,\delta} \subseteq \mathscr{P}(\mathrm{nacc}(C_\delta))$;
(f) $\mathbf{U}_I(\chi_1) < \chi_2$; see Definition 1.14, if $\chi_1 = \lambda$, then we stipulate $\mathbf{U}_I(\chi_1) = \chi_1$, hence this means $\chi_1 < \chi_2$;
(g) if $j_1 \neq j_2$ are $< \chi_2$, $\delta \in S$, $A_1 \in \mathscr{A}_{j_1,\delta}$ and $A_2 \in \mathscr{A}_{j_2,\delta}$, then there is $\gamma \in A_1 \cap A_2$ such that $g_{j_1,\delta}(\gamma) \neq g_{j_2,\delta}(\gamma)$;
(h) if $j < \chi_2$ and E is a club of λ, then for some $Y \in I^+$, hence $Y \subseteq S$ for every $\delta \in Y$ we have $\mathrm{nacc}(C_\delta) \cap E \in \mathscr{A}_{j,\delta}$.

(ii) For $\ell = 1, 2, 3$ let $\mathrm{Qr}_\ell(\chi_2, \chi_1, \lambda)$ be defined by
- if $\ell = 1$ as above;
- if $\ell = 2$ as above, but there is a sequence $\langle J_\delta : \delta \in S\rangle$ of ideals on $\mathrm{nacc}(C_\delta)$ such that $\mathscr{A}_{j,\delta} = \{\mathrm{nacc}(C_\delta)\backslash X : X \in J_\delta\}$;
- if $\ell = 3$ we use clauses (a)–(g) from part (i) and

$(h)^-$ if E_j is a club of λ for $j < \chi_2$ and $\langle \xi_j : j < \chi_2\rangle$ is a sequence of ordinals with $\sup\{\xi_j : j < \chi_2\} < \chi_2$, then we can find $j_1 < j_2 < \chi_2$, $\delta \in S$ and $\gamma \in \mathrm{nacc}(C_\delta)$ such that $\xi_{j_1} = \xi_{j_2}$, $\gamma \in E_{j_1} \cap E_{j_2}$ and $g_{j_1,\delta}(\gamma) \neq g_{j_2,\delta}(\gamma)$.

(iii) $\mathrm{Qr}_{\ell,\iota}(\chi_2, \chi_1, \lambda)$ is defined as in $\mathrm{Qr}_\ell(\chi_2, \chi_1, \lambda)$ but $g_{j,\delta} : \mathrm{nacc}(C_\delta) \to \iota$, etc.

Remark 2.7. Can we weaken the conclusion of clause (h) of Definition 2.6 (i), etc. to
- $\{\alpha \in \mathrm{nacc}(C_\delta) : \sup(\alpha \cap E) > \max(C_\delta \cap \alpha)\} \in \mathscr{A}_{j,\delta}$.

That is, this suffices in Theorem 2.9 but there is no clear gain so we have not looked into it.

Fact 2.8. (i) *We have* $\mathrm{Qr}_2(\chi_2, \chi_1, \lambda) \Rightarrow \mathrm{Qr}_1(\chi_2, \chi_1, \lambda) \Rightarrow \mathrm{Qr}_3(\chi_2, \chi_1, \lambda)$.

(ii) *We have* $\mathrm{Qr}_1(\chi_2, \chi_1, \lambda)$ *and even* $\mathrm{Qr}_2(\chi_2, \chi_1, \lambda)$ *when*
 (a) $\kappa^+ < \lambda \leq \chi_1 < \chi_2 < 2^\kappa$;
 (b) $\kappa = \mathrm{cf}(\kappa)$, $\lambda = \mathrm{cf}(\lambda)$;
 (c) $\mathbf{U}_I(\chi_1) < \chi_2$ *when* $\lambda < \chi_1$ *and* I *an ideal on* S *so* $S \notin I$;
 (d) $S \subseteq S^\lambda_\kappa$ *is stationary,* $\bar{C} = \langle C_\delta : \delta \in S\rangle$ *guess clubs,* $C_\delta \subseteq \delta$, $\mathrm{otp}(C_\delta) = \kappa$;
 (e) $I = \{A \subseteq S$: *for some club* E *of* λ *for no* $\delta \in S$ *do we have* $\mathrm{nacc}(C_\delta) \cap E \in J^{\mathrm{bd}}_{C_\delta}\}$.

(iii) *If clauses* (a)–(b) *of part* (ii) *hold and* $S \subseteq S^\lambda_\kappa$ *is stationary, then there is* $\bar{C}$ *as required in clause* (d).

Proof. Clause (i) is easy to prove. The proof of (ii) is straightforward.

(iii) Clause (d) follows by [7, Section 2]. □

Theorem 2.9. (i) *If* T *is complete, with the* $(\eta, \bar{k})$*-olive property and* $\lambda > \kappa^+$ *and* λ, κ *are regular,* $2^\kappa > \lambda \geq \kappa^{++} + |T|$, *then* T *has no universal model in* λ *(for* $\prec$*).*

(ii) *If* T *is complete, with the* $(\eta, \bar{k}, m)$*-olive property and* $\lambda = \mathrm{cf}(\lambda) \geq |T|$ *and* $\mathrm{Qr}_1(\chi_2, \chi_1, \lambda)$, *then* $\mathrm{univ}(\chi_1, \lambda, T) \geq \chi_2$.

(iii) *Similarly, for class* $\mathfrak{k}$ *of models with the* $\lambda - (\mathrm{qf}, \eta, \bar{k})$*-olive property, see Definition 2.3* (ii), *so, e.g., for universal* K *with the* JEP *and the* $\lambda - (\mathrm{qf}, \eta, \bar{k})$*-olive property.*

(iv) *Like part* (iii) *for a pair* $\bar{\mathfrak{k}}$.

(v) *We can weaken* $\mathrm{Qr}_1(\chi_2, \chi_1, \lambda)$ *to* $\mathrm{Qr}_{1,\theta}(\chi_2, \chi_1, \lambda)$ *when* $\theta = 2^\partial$, $\chi_1 = \chi_1^\partial$, $\partial < \lambda$.

Remark 2.10. (i) We can use Qr_3 instead of Qr_1 by the same proof but the gain is not clear.

(ii) If, e.g., $\lambda = \mu^+$, $\mu = \mu^{<\mu} = 2^\partial$, $\chi_1 = \lambda = \chi_2$, T has a universal in λ, T as in Theorem 2.9 (i), then failure of $\mathrm{Qr}_1(\lambda, \lambda, \lambda)$ implies: there is $\mathscr{F} \subseteq {}^\mu\mu$ such that $(\forall \eta \in {}^\mu\mu)(\exists \nu \in \mathscr{F})(\exists^\mu i < \mu)(\eta(i) = \nu(i))$.

Proof of Theorem 2.9. (i) It follows from (ii) by Fact 2.8 (ii).

(ii) Let $(\bar{\psi}_0, \bar{\psi}_1)$, i.e., $\bar{\psi}_\iota = \langle \psi_{\iota,k}(\bar{x}_0, \ldots, \bar{x}_k) : k = 1, \ldots, k_\iota\rangle$ for $\iota = 0, 1$ witness the $(\eta, \bar{k}, m)$-olive property. For simplicity, we can assume without loss of generality that $m = 1$ and the formulas $\psi_{\iota,k}$ are quantifier-free and T has only predicates and its vocabulary is finite. To make this proof also be a proof of Theorem 2.9 (iii), let $\leq_{\mathfrak{k}}$ be $\prec$ on mod_T. Let $S, \bar{C}, \bar{\mathscr{A}}, \bar{\mathbf{g}}$ witness $\mathrm{Qr}_1(\chi_2, \chi_1, \lambda)$. For each $j < \chi_2$ we define $\bar{f}_j$ by

$(*)_1$ (a) $\bar{f}_j = \langle f_{j,\alpha} : \alpha < \lambda\rangle$;
 (b) $f_{j,\alpha} : \alpha \to \{0, 1\}$ is defined by
  (1) if $\beta < \alpha \in S$, then $f_{j,\alpha}(\beta) = g_{j,\alpha}(\min(C_\alpha\backslash\beta))$,
  (2) if $\beta < \alpha \in \lambda\backslash S$, then $f_{j,\alpha}(\beta) = 0$.

For each $j < \chi_2$ we can find $M_j \models T$ of cardinality λ and pairwise distinct elements $\langle a_{j,\alpha} : \alpha < \lambda\rangle$ of M_j satisfying $(\mathrm{b})_\lambda$ of Definition 2.1 for $\bar{f}_j$. Let

$$M_{j,\alpha} = M_j \restriction \cup \{a_{j,\beta} : \beta < \alpha\}.$$

Let the function $h^0_j : \lambda \to M_j$ be defined by $h^0_j(\alpha) = a_{j,\alpha}$. Let $\mathscr{P} \subseteq [\chi_1]^\lambda$ witness $\mathbf{U}_J(\chi_1) < \chi_2$, so if $\lambda = \chi_1$, we use $\mathscr{P} = \{\lambda\}$; without loss of generality $u \in \mathscr{P} \wedge \alpha < \chi_1 \wedge |u \cap \alpha| = \lambda \Rightarrow u \cap \alpha \in \mathscr{P}$. For $u \in \mathscr{P}$ or just $u \in [\chi_1]^\lambda$ let h^1_u be a one-to-one function from u onto λ.

Toward a contradiction, assume that there are $\xi_* < \chi_2$ and a sequence $\langle \mathfrak{A}_\xi : \xi < \xi_* \rangle$ of models of T, each of cardinality $\leq \chi_1$ witnessing $\mathrm{univ}(\chi_1, \lambda, T) < \chi_2$, even equal to $|\xi_*|$. Without loss of generality, the universe of $\mathfrak{A}_\xi$ is $\alpha_\xi \leq \chi_1$ for $\xi < \xi_*$. So, for every $j < \chi_2$ there are $\xi = \xi_j < \xi_*$ and[7] an $\leq_\mathfrak{k}$-embedding h^2_j of M_j into $\mathfrak{A}_\xi$, hence there is $u_j \in \mathscr{P}$ such that $W_j := \{\alpha \in S : h^2_j(a_{j,\alpha}) \in u_j\} \in I^+$ and let $v_j \supseteq u_j \cup \mathrm{Rang}(h^2_j)$ be such that $v_j \in [\alpha_{\xi_j}]^\lambda \subseteq [\chi_1]^\lambda$ and $\mathfrak{A}_j \restriction v_j \prec \mathfrak{A}_j$ and let $\langle \gamma_{j,\alpha} : \alpha < \lambda \rangle$ list the members of v_j.

Let $h'_j = h^2_{v_j} \circ (h^0_j \restriction W_j)$ and $h_j = h^1_{v_j} \circ h^2_j \circ (h^0_j \restriction W_j)$ be functions from W_j into $\mathfrak{A}_{\xi_j}$ and λ, respectively. Let

$$N_j = (\mathfrak{A}_{\xi_j} \restriction v_j, P^{N_j}_*)$$

be the expansion of $\mathfrak{A}_{\xi_j} \restriction v_j$ by the relation $P^{N_j}_* = \mathrm{Rang}(h'_j)$ and let

$$E_j = \{\delta < \lambda : \delta \text{ is a limit ordinal}, (\forall \alpha < \lambda)((h_{v_j})^{-1}(\alpha) \in \{\gamma_{j,\beta} : \beta < \delta\} \equiv \alpha < \delta) \text{ and } N_j \restriction \{\gamma_{j,\alpha} : \alpha < \delta\} \prec N_j\},$$

which clearly is a club in λ. Hence, by clause (h) of Definition 2.6 (i) there is an ordinal $\delta_j \in E_j \cap S$ such that $A_j := \mathrm{nacc}(C_\delta) \cap E_j$ belongs to $\mathscr{A}_{j,\delta}$.

As $\xi_* < \chi_2$, $|\mathscr{P}| < \chi_2$ and $|\{h_j(a_{j,\delta}) : j < \chi_2, \delta \in S\}| < \sup\{\|\mathfrak{A}_\xi\| : \xi < \xi_*\} \leq \chi_1 < \chi_2$ by the pigeonhole principle there are j_1, j_2 such that

$(*)_2$ (a) $j_1 = j(1) < j_2 = j(2)$;
(b) $\xi_{j(1)} = \xi_{j(2)}$;
(c) $\delta_{j_1} = \delta_{j_2}$ call it δ (so $\delta \in S$);
(d) $u_{j_1} = u_{j_2}$ call it u, so $u = u_{j_\iota} \subseteq |N_{j_\iota}|$ for $\iota = 1, 2$;
(e) $h^2_{j_1}(a_{j_1,\delta}) = h^2_{j_2}(a_{j_2,\delta})$ call it b, so $b \in \mathrm{Rang}(h^2_{j_1}) \cap \mathrm{Rang}(h^2_{j_2})$.

By clause (g) of Definition 2.6 (i) there is $\gamma \in A_{j_1} \cap A_{j_2}$ such that $g_{j_1,\delta}(\gamma) \neq g_{j_2,\delta}(\gamma)$.

Now, we shall choose α_ℓ by induction on $\ell < n$ such that

$(*)_3$ (a) $\alpha_\ell \in W_{j_{\eta(\ell)}}$;
(b) $\alpha_\ell < \gamma$ but $\alpha_\ell > \sup(C_\delta \cap \gamma)$;
(c) $\langle \alpha_0, \dots, \alpha_\ell \rangle$ is increasing;
(d) in the model $N_{j_{\eta(\ell)+1}}$ the elements $h^2_{j_{\eta(\ell)+1}}(a_{j_1,\delta}) = b$ and $h^1_{j_{\eta(\ell)+1}}(a_{j_1,\alpha_\ell})$ realize the same quantifier type over

$$\{h_{j_{\eta(\ell(1))+1}}(a_{j_\eta,\alpha_{\ell(1)}}) : \ell(1) < \ell\}$$

or at least for all relevant (finitely many) formulas.

If we succeed, then in the model $\mathfrak{A}_{\xi_*}$, which extends N_{j_1} and N_{j_2}, the sequence

$$\langle h^2_{j_{\eta(\ell)}}(a_{j_{\eta(\ell)},\alpha_\ell}) : \ell < n \rangle {}^\frown \langle b \rangle$$

realizes the “forbidden” type that is the one from clause (c) of Definition 2.1, which is a contradiction.

As $\delta \in W_j \cap E_{j_{\eta(\ell)}}$, by the choice of $E_{j_{\eta(\ell)}}$ we can carry the induction.

(iii) Similarly.

(iv) As in [8] and the above, just use ∂-tuples of elements $\bar{a}$. □

A sufficient condition for cases of Qr_i is the following.

Definition 2.11. Let $\mathrm{Qr}_4(\lambda)$ mean $\lambda = \mu^+$ and $\langle C_\delta, D_\delta : \delta \in S \rangle$ satisfies $C_\delta \subseteq \delta$, where D_δ is a filter on $\mathrm{nacc}(C_\delta)$ such that $\mathscr{P}(\mathrm{nacc}(C_\delta))/D_\delta$ satisfies the 2^μ-c.c. and for every club E of λ for some $\delta \in S$, $E \cap \mathrm{nacc}(C_\delta) \in D^+_\delta$.

3 The class of groups has the olive property

3.1 General groups

We shall try to prove that the class of groups has a universal member almost only when cardinal arithmetic is close to GCH. This is done by the following theorem.

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7 Recall that now $\leq_\mathfrak{k} = \prec \restriction \mathrm{mod}_T$.

Theorem 3.1. *The class of groups has the olive property, see Definition 1.8 or Definition 2.1* (i) (d) (3), *in fact, the* $(\eta, \bar{k}, m)$*-olive property, where* $\eta = \langle 0, 1, 0\rangle$, $\bar{k} = (2, 1)$, $m = 6$.

Why Theorem 3.1 suffices? As then we can use Theorem 2.9 (iii); or see Conclusions 3.16 and 3.17. We break the proof into a series of definitions and claims; we may replace the use of HNN extensions (in Claim 3.13) and free amalgamation (in Claim 3.12) by the proof of Claim 3.14.

Definition 3.2. Let $\bar{\psi} = \bar{\psi}^{\rm grp}_{\rm olive}$ be $(\psi_{0,1}, \psi_{0,2}, \psi_{1,1})$ defined as follows (letting $m = 6$).
(i) $\psi_{0,1} = \varphi_0 = \varphi_0(\bar{x}_{[m]}, \bar{y}_{[m]}) = y_5^{-1} x_0 y_5 = x_2$;
(ii) $\psi_{1,1} = \varphi_1 = \varphi_1(\bar{x}_{[m]}, \bar{y}_{[m]}) = x_5^{-1} y_1 x_5 = y_3 \wedge x_5^{-1} y_4 x_5 = y_4$;
(iii) $\psi_{0,2} = \psi(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{z}_{[m]}) = (\sigma_*(x_0, y_1, z_4) = e \wedge \sigma_*(x_2, y_3, z_4) \neq e)$, on σ_* see below.

Definition/Claim 3.3. There is a $\sigma_* = \sigma_*(x, y, z)$ such that
(i) σ_* is a group word;
(ii) for some group G and $a, b, c \in G$ we have "$\sigma_*(a, b, c) \neq e_G$";
(iii) for any group G and $a, b, c \in G$ we have $e \in \{a, b, c\} \Rightarrow \sigma^G(a, b, c) = e_G$.

Proof. Straightforward, e.g., $(x^{-1}y^{-1}xy)^{-1}z^{-1}(x^{-1}y^{-1}xy)z$. □

Claim 3.4. *The* $\bar{\psi}$ *from Definition 3.2 satisfies clause* (iii) *of Definition 1.8, i.e., for no group* G *and* $\bar{a}_\ell \in {}^mG$ *for* $\ell < 4$ *do the formulas there hold.*

Remark 3.5. We prove more: There are no group G and $\bar{a}_\ell \in {}^mG$ for $\ell = 0, 1, 2, 3$ such that $\varphi_0[\bar{a}_0, \bar{a}_1]$, $\varphi_1[\bar{a}_1, \bar{a}_2]$, $\varphi_1[\bar{a}_1, \bar{a}_3]$ and $\psi[\bar{a}_0, \bar{a}_2, \bar{a}_3]$.

Proof. Assume toward a contradiction that $G, \langle \bar{a}_\ell : \ell < 4\rangle$ forms a counterexample; now conjugation by $a_{1,5}$ is an automorphism of G which we call g.

Now,
- $g(a_{0,0}) = a_{0,2}$ by (a) of 3.2 as $G \models \varphi_0[\bar{a}_0, \bar{a}_1]$;
- $g(a_{2,1}) = a_{2,3}$ by the first conjunct of (ii) of Definition 3.2 as $G \models \varphi_1[\bar{a}_1, \bar{a}_2]$;
- $g(a_{3,4}) = a_{3,4}$ by the second conjunct of (ii) of Definition 3.2 as $G \models \varphi_1[\bar{a}_1, \bar{a}_3]$.

Together, we have
- $g(\sigma_*(a_{0,0}, a_{2,1}, a_{3,4})) = \sigma_*(a_{0,2}, a_{2,3}, a_{3,4})$,

but this contradicts $G \models \psi[\bar{a}_0, \bar{a}_2, \bar{a}_3]$, see clause (iii) of Definition 3.2. □

Definition 3.6. Let $\bar{f} \in \mathbf{F}_\lambda$, i.e., $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$, $f_\alpha : \alpha \to \{0, 1\}$.
(i) Let $X_{\bar{f}} = X_{\bar{f},m}$, where we let $X_{\bar{f},k} = \{x_{\alpha,\ell} : \alpha < \lambda,\ \ell < k\}$ for $k \leq m$; recall that here $m = 6$.
(ii) Let $\bar{x}_{\alpha,k} = \langle x_{\alpha,\ell} : \ell < k\rangle$ for $k \leq m$ and let $\bar{x}_\alpha = \bar{x}_{\alpha,m}$.
(iii) For $\ell = 0, 1$ we define the set $\Gamma^\ell_{\bar{f}}$ of equations (pedantically, for $\ell = 1$ conjunctions of two equations)

$$\{\varphi_\ell(\bar{x}_\alpha, \bar{x}_\beta) : \alpha < \beta < \lambda \text{ and } f_\beta(\alpha) = \ell\}.$$

(iv) We define the set $\Gamma^2_{\bar{f}}$ of equations

$$\{\sigma_*(x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}) = e : \alpha < \beta < \gamma < \lambda \text{ and } f_\gamma{\restriction}[\alpha, \beta] \text{ is constantly } 0\}.$$

(v) Let $G^5_{\bar{f}}$ be the group generated by $X_{\bar{f},5}$ freely except for the equations in $\Gamma^2_{\bar{f}}$; note that the elements $x_{\alpha,5}$ are not mentioned in $\Gamma^2_{\bar{f}}$.
(vi) Let $G^6_{\bar{f}}$ be the group generated by $X_{\bar{f},6}$ freely except for the equations in $\Gamma^0_{\bar{f}} \cup \Gamma^1_{\bar{f}} \cup \Gamma^2_{\bar{f}}$.

Discussion 3.7. For our purpose we have to show that for $\alpha < \beta < \gamma$ (and $\bar{f} \in \mathbf{F}_\lambda$) we have $G_{\bar{f},6} \models$ "$\psi[\bar{x}_\alpha, \bar{x}_\beta, \bar{x}_\gamma]$" if and only if $f_\gamma{\restriction}[\alpha, \beta] = 0_{[\alpha,\beta]}$. For proving the "if" implication, assume $f_\gamma{\restriction}[\alpha, \beta] = 0_{[\alpha,\beta]}$. Now, the satisfaction of "$\sigma_*(x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}) = e$" is obvious by the role of $\Gamma^2_{\bar{f}}$, the analysis below is intended to prove the other half "$\sigma_*(x_{\alpha,2}, x_{\beta,3}, x_{\gamma,4}) \neq e$". For proving the "only if" implication, it suffices to prove that "$\sigma_*(x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}) \neq e$" when $f_\gamma{\restriction}[\alpha, \beta] \neq 0_{[\alpha,\beta]}$. For both cases, we prove that this holds in $G^5_{\bar{f}}$ and then prove that $G^5_{\bar{f}} \subseteq G^6_{\bar{f}}$ in the natural way.

Claim 3.8. (i) *If $\alpha < \beta < \gamma < \lambda$ and $f_\gamma \restriction [\alpha, \beta] \neq 0_{[\alpha,\beta]}$, then $G^5_{\bar f} \models$ "$\sigma_*[x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}] \neq e$".*
(ii) *If $\alpha < \beta < \gamma < \lambda$, then $G^5_{\bar f} \models$ "$\sigma_*(x_{\alpha,2}, x_{\beta,3}, x_{\gamma,4}) \neq e$".*

Proof. (i) Use Observation 3.9 (ii) below with $X = \{x_{\xi,\ell} : \xi \in \{\alpha, \beta, \gamma\}$ and $\ell < 5\}$.
(ii) Use Observation 3.9 (ii) below with $X = \{x_{\xi,\ell} : \xi < \lambda,\ \ell < 5$ and $\ell > 0\}$. □

Observation 3.9. (i) *If $x_{\alpha,\ell}, x_{\beta,k} \in X^5_{\bar f}$ and $(\alpha, \ell) \neq (\beta, k)$, then $G^5_{\bar f} \models$ "$x_{\alpha,\ell} \neq x_{\beta,k}$".*
(ii) *If $X \subseteq X^5_{\bar f}$ and $(\sigma_*(x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}) = e) \in \Gamma^2_{\bar f} \Rightarrow \{x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}\} \nsubseteq X$, then X generates freely a subgroup of $G^5_{\bar f}$.*

Proof. (i) Let $G' = \oplus\{\mathbb{Z}x : x \in X^5_{\bar f}\}$, which is an abelian group; let $G'' = \oplus\{\mathbb{Z}x_{\alpha,i} : \alpha < \lambda, i \notin \{\ell, k\}\}$ be a subgroup. So, G'/G'' by clause (iii) of Definition 3.3 because $\{0, 1, 4\} \nsubseteq \{\ell, k\}$ satisfies all the equations in $\Gamma^2_{\bar f}$ and it satisfies the desired inequality. As $G^5_{\bar f}$ is generated by $X^5_{\bar f}$ freely except for the equations in $\Gamma^2_{\bar f}$, the desired result follows. Alternatively, use part (ii).

(ii) Let $H = H_X$ be the group generated by X freely. We define a function F from $X^5_{\bar f}$ into H by

- $F(x)$ is x if $x \in X$ and is e_H if $x \in X^5_{\bar f} \setminus X$.

Now F respects every equation form $\Gamma^2_{\bar f}$ by clause (iii) of Claim 3.3, hence f induces a homomorphism from $G^5_{\bar f}$ into H, really onto. Hence the desired conclusion follows. □

Definition 3.10. For $\beta < \lambda$ we define a partial function F_β from $X^5_{\bar f}$ to $X^5_{\bar f}$ as follows.

- If $\alpha < \beta$ and $f_\beta(\alpha) = 0$, then $F_\beta(x_{\alpha,0}) = x_{\alpha,2}$.
- If $\gamma > \beta$ and $f_\gamma(\beta) = 1$, then $F_\beta(x_{\gamma,1}) = x_{\gamma,3}$, $F_\beta(x_{\gamma,4}) = x_{\gamma,4}$.

Claim 3.11. (i) *F_β is a well-defined partial one-to-one function from $X^5_{\bar f}$ to $X^5_{\bar f}$.*
(ii) *The domain and the range of F_β satisfies the criterion of Observation 3.9* (ii).

Proof. For (i), it is a function as no $x_{\alpha,\ell}$ appears in two cases. Also, if $F_\beta(x_{\alpha_1,\ell}) = x_{\alpha_2,k}$, then

$$\alpha_1 = \alpha_2 \wedge (\ell, k) \in \{(0, 2), (1, 3), (4, 4)\},$$

so F_β is one-to-one.

For (ii), assume $[\sigma_*(x_{\alpha_1,0}, x_{\alpha_2,1}, x_{\alpha_3,4}) = e] \in \Gamma^2_{\bar f}$ so

$(\star)_1$ $\alpha_1 < \alpha_2 < \alpha_3$

and

$(\star)_2$ $f_{\alpha,3} \restriction [\alpha_1, \alpha_2] = 0_{[\alpha_1,\alpha_2]}$.

First, toward contradiction assume $\{x_{\alpha_1,0}, x_{\alpha_2,1}, x_{\alpha_3,4}\} \subseteq \mathrm{Dom}(F_\beta)$. Now, if $\alpha_1 \geq \beta$, then $x_{\alpha_1,0} \notin \mathrm{Dom}(F_\beta)$, just inspect Definition 3.10 so necessarily $\alpha_1 < \beta$ and similarly $f_\beta(\alpha_1) = 0$ (but not used).

If $\alpha_2 \leq \beta$, then $x_{\alpha_2,1} \notin \mathrm{Dom}(F_\beta)$, so $\beta < \alpha_2$ and similarly $f_{\alpha_2}(\beta) = 1$ (again not used), so together $\alpha_1 < \beta < \alpha_2$. Also, as $x_{\alpha_3,4} \in \mathrm{Dom}(F_\beta)$, it follows that ($\beta < \alpha_3$, which follows by earlier inequalities and) $f_{\alpha_3}(\beta) = 1$, so together β witnesses that $f_{\alpha_3} \restriction [\alpha_1, \alpha_2]$ is not constantly zero; but this is a contradiction to $[\sigma(x_{\alpha_1,0}, x_{\alpha_2,0}, x_{\alpha_3,0}) = e] \in \Gamma^2_{\bar f}$.

Second, assume toward a contradiction that $\{x_{\alpha_2,0}, x_{\alpha_1,2}, x_{\alpha_3,4}\} \subseteq \mathrm{Rang}(F_\beta)$, but "$x_{\alpha_2,0} \in \mathrm{Rang}(F_\beta)$" is impossible by Definition 3.10. □

Claim 3.12. *To prove $G^5_{\bar f} \subseteq G^6_{\bar f}$, any of the following conditions suffices.*
(i) *There are a group H extending $G^5_{\bar f}$ and $y_\zeta \in G$ for $\zeta < \lambda$ such that*

$$\zeta < \lambda \wedge F_\zeta(x_{\varepsilon_1,\ell_1}) = x_{\varepsilon_2,\ell_2} \Rightarrow H \models \text{"}y_\zeta^{-1} x_{\varepsilon_1,\ell_1} y_\zeta = x_{\varepsilon_2,\ell_2}\text{"}.$$

(ii) *For each $\zeta < \lambda$ there is a group H extending $G^5_{\bar f}$ and $y \in G$ such that*

$$F_\zeta(x_{\varepsilon_1,\ell_1}) = x_{\varepsilon_2,\ell_2} \Rightarrow H \models \text{"}y^{-1} x_{\varepsilon_1,\ell_1} y = x_{\varepsilon_2,\ell_2}\text{"}.$$

Proof. For clause (i) to suffice, we define a function F from $X^6_{\bar f}$ into H by

- $F(x_{\varepsilon,\ell})$ is $x_{\varepsilon,\ell} \in G^5_{\bar f} \subseteq H$ if $\ell < 5 \wedge \varepsilon < \lambda$ and is y_ζ if $\ell = 5 \wedge \varepsilon = \zeta < \lambda$.

Check that the mapping F respects the equations in $\Gamma^0_{\bar f} \cup \Gamma^1_{\bar f} \cup \Gamma^2_{\bar f}$, hence it induces a homomorphism F^1 from $G^6_{\bar f}$ into H, for every group word $\sigma = \sigma(\dots, x_{\varepsilon_i,\ell_i}, \dots)_{i<n}$, $x_{\varepsilon_i,\ell_i} \in X^5_{\bar f}$, we have $G^6_{\bar f} \models$ "$\sigma = e$" $\Rightarrow G^5_{\bar f} \models$ "$\sigma = e$", so we are done.

For clause (ii) to suffice, let (H_ζ, y_ζ) for $\zeta < \lambda$ be as guaranteed by the assumption, i.e., clause (ii). Without loss of generality, $\zeta \neq \xi < \lambda = G_\zeta \cap G_\xi = G^5_{\bar f}$. Now, clause (i) follows by using free amalgamation of $\langle H_\zeta : \zeta < \lambda\rangle$ over $G^5_{\bar f}$, we know it is as required in clause (i), see, e.g., [5]. □

Claim 3.13. (i) *Clause* (ii) *of Claim 3.12 holds.*
(ii) *The conclusion of Claim 3.12 holds also for* $G^6_{\bar f}$.
(iii) *The conclusions of Claim 3.8 hold also for* $G^6_{\bar f}$.

Proof. (i) By the theorems on HNN extensions (see [5]) applied with the group being $G^5_{\bar f}$ and the partial automorphism π_ζ being the one F_ζ induced, i.e.,

- $\mathrm{Dom}(\pi_\zeta)$ is the subgroup of $G^5_{\bar f}$ generated by $\mathrm{Dom}(F_\zeta)$;
- $\pi_\zeta(x_{\varepsilon,\ell}) = F_\zeta(x_{\varepsilon,\ell})$ for $x_{\varepsilon,\ell} \in \mathrm{Dom}(F_\zeta)$.

By Claim 3.11 (ii) and Observation 3.9 (ii), we know that π_ζ is indeed an isomorphism.

(ii) This follows by Claims 3.12 and 3.13 (i).

(iii) By Claims 3.8 and 3.13 (ii). □

The proof of Theorem 3.1 should be clear by now.

3.2 Locally finite groups

Claim 3.14. *The pair* $(K_{\mathrm{lfgr}}, K_{\mathrm{gr}})$ *of classes, i.e. (locally finite groups, groups), has the olive property, as witnessed by* $\bar\varphi$ *from Definition 3.2.*

Proof. We rely on Observation 3.15 below and use its notation. Let

$$J = \{(\alpha,\beta,\gamma) : \alpha < \beta < \gamma < \lambda \quad \text{and} \quad f_\gamma \restriction [\alpha,\beta] = 0_{[\alpha,\beta]}\}.$$

Let $G^5_{\bar f}, G^6_{\bar f}$ be as in the proof of Theorem 3.1, that in Definition 3.6. Now, for $\bar\alpha = (\alpha_0,\alpha_1,\alpha_2) \in J$ let $\pi^5_{\bar\alpha}$ be the function from $X_{\bar f,5}$ (see Definition 3.6 (v)) into K defined as follows.

$(*)_1$ $\pi^5_{\bar\alpha}(x_{\beta,k})$ is

- e_K if $\beta \notin \{\alpha_0,\alpha_1,\alpha_2\}$;
- $z_{\ell,k}$ if $\beta = \alpha_\ell$.

Now:

$(*)_2$ $\pi^5_{\bar\alpha}$ respects the equations from $\Gamma^2_{\bar f}$.

[Why? The equation $\sigma_*(x_{\alpha_0,0}, x_{\alpha_1,1}, x_{\alpha_2,4}) = e$ holds as K satisfies $\sigma_*(z_{0,0}, z_{1,1}, z_{2,4}) = e$. For the other equations, see Definition 3.3 (iii); recall that the equations for the cases of φ_0 and φ_1 do not appear, see Definition 3.6 (v).]

Let $\pi^6_{\bar\alpha}$ be the following function from $X^6_{\bar f}$ into K.

$(*)_3$ $\pi^6_{\bar\alpha}(x)$ is

- $\pi^5_{\bar\alpha}(x)$ when $x \in X^5_{\bar f}$;
- z_s when $x = x_{\beta,5}$, $\beta < \lambda$ and $s = s_{\bar\alpha,\beta} := (\{\ell \le 2 : \alpha_\ell < \beta \text{ and } f_\beta(\alpha_\ell) = 0\}, \{\ell \le 2 : \beta < \alpha_\ell \text{ and } f_{\alpha_\ell}(\beta) = 1\})$.

[Why is $\pi^6_{\bar\alpha}$ as required? The least obvious point is: why $s \in S_*$? Let $s = (u_1,u_2)$; now

$$\ell_1 \in u_1 \wedge \ell_2 \in u_2 \Rightarrow \alpha_{\ell_1} < \beta < \alpha_{\ell_2} \Rightarrow \ell_1 < \ell_2$$

and $(\{0\},\{1,2\}) \ne s$ because $f_{\alpha_2}\restriction[\alpha_0,\alpha_1]$ is constantly zero.]

$(*)_4$ $\pi^6_{\bar\alpha}$ respects the equations in $\Gamma^0_{\bar f} \cup \Gamma^1_{\bar f}$.

[Why? Check the definitions.]

By $(*)_2$, $(*)_4$ there is a homomorphism $\pi_{\bar\alpha}$ from $G_{\bar f}$ into K extending $\pi^6_{\bar\alpha}$. Let G_* be the product of J-copies of K, i.e.,

$(*)_5$ (a) the set of elements of G_* is the set of functions g from J into K;
(b) $G_* \models$ "$g_1 g_2 = g_3$" if and only if $\bar\alpha \in J \Rightarrow K \models$ "$g_1(\bar\alpha)g_2(\bar\alpha) = g_3(\bar\alpha)$".

Now:

$(*)_6$ G_* is a locally finite group.

$(*)_7$ For $\alpha < \lambda, k < m$ let $\bar g_\beta = \langle g_{\beta,k} : k < m\rangle$, where $g^*_{\beta,k} \in G_*$ be defined by $(g_{\beta,k}(\bar\alpha))(x) = \pi^6_{\bar\alpha}(x_{\beta,k})$.

$(*)_8$ $G_*, \langle \bar g_\beta : \beta < \lambda\rangle$ witnesses the olive property.

[Why? Check.]

So we are done. □

Observation 3.15. *There are $K, z_{i,k}$ for $i < 3$, $k < m$ and $\langle \pi_s : s \in S_* \rangle$ such that*
(i) *K is a finite group;*
(ii) *$z_{i,k} \in K$;*
(iii) *$\sigma_*(z_{0,0}, z_{1,2}, z_{2,4}) = e$ but $\sigma_*(z_{0,2}, z_{1,3}, z_{2,4}) \neq e$;*
(iv) *$S_* = \{(u_1, u_2) : u_1, u_2 \subseteq \{0,1,2\}$ and $(\forall \ell_1 \in u_1)(\forall \ell_2 \in u_2)(\ell_1 < \ell_2)$ but $(u_1, u_2) \neq (\{0\}, \{1,2\})\}$;*
(v) *for $s = (u_1, u_2) \in S_*$ we have that π_s is a partial automorphism of K such that*
  (a) *if $\ell \in u_1$, then $\pi_s(x_{\ell,0}) = x_{\ell,2}$,*
  (b) *if $\ell \in u_2$, then $\pi_s(x_{\ell,1}) = x_{\ell,2}, \pi_s(z_{\ell,4}) = z_{\ell,4}$;*
(vi) *moreover, there are $z_s \in K$ for $s \in S_*$ such that $(\forall x \in \mathrm{Dom}(\pi_s))(\pi_s(x) = z_s^{-1} x z_s)$.*

Proof. First, we ignore clause (vi). We use finite nilpotent groups. Let $n_2 = 6m$, $n_1 = \binom{n_2}{2}$, $n_0 = \binom{n_1}{2}$ and let $f_\ell : [n_{\ell+1}]^2 \to n_\ell$ be one-to-one for $\ell = 0, 1$.

Let K_1 be the group generated by $\{y_{j,\ell} : j \le 2, \ell < n_j\}$ freely except for the equations
$(*)_1$ (a) $y_{j,\ell} \cdot y_{j,\ell} = e$;
 (b) $[y_{j+1,\ell_1}, y_{j+1,\ell_2}] = y_{j,f\{\ell_1,\ell_2\}}$, i.e., $y_{j+1,\ell_1}^{-1} y_{j+1,\ell_2}^{-1} y_{j+1,\ell_1} y_{j+1,\ell_2} = y_{j,f\{\ell_1,\ell_2\}}$ when $j < 2, \ell_1 < \ell_2 < n_{j+1}$;
 (c) $[y_{j_1,\ell_1}, y_{j_2,\ell_2}] = e$ when $(j_1 = 0 = j_2) \vee (j_1 \neq j_2 \le 2)$ and $\ell_1 < n_{j_1}, \ell_2 < n_{j_2}$.

Clearly, K_1 is finite.

Let $z'_{i,\ell} = y_{2,6i+\ell}$ for $i < 3$, $\ell < m$ and let ℓ_* be such that
$$[[z'_{0,0}, z'_{1,1}], z'_{2,4}] = y_{0,\ell_*}.$$
Let K_0 be the subgroup $\{e, y_{0,\ell_*}\}$ of K; it is a normal subgroup as it is included in the center of K_1 and let $K_2 = K_1/K_0$ and we define $z_{i,\ell}$ as $z'_{i,\ell}/K_0$.

Now,
$(*)_2$ $K_2, \langle z_{i,\ell} : i \le 2, \ell < m \rangle$ are as required in (i)–(v) of the claim.
 [Why? We should just check that for $s \in S_*$ there is π_s as required, i.e., that some subgroups of K_2 generated by subsets of $\langle z_{i,\ell} : i \le 2, \ell < m \rangle$ are isomorphic, but as none of them included $\{z_{0,0}, z_{1,1}, z_{2,4}\}$ and the way K_2 was defined this is straightforward.]

Lastly, there is a finite group K extending K_2 and $z_s \in K$ for $s \in S$ such that $x \in \mathrm{Dom}(\pi_s) \Rightarrow z_s^{-1} x z_s = \pi_s(x)$. Why? Simply because K_2 can be considered as a group of permutations of the set K_2 (e.g., multiplying from the right) and it is easy to find $z_s \in \mathrm{Sym}(K_2)$ as required. □

Conclusion 3.16. *Assume $\mathrm{Qr}_1(\chi_1, \chi_2, \lambda)$. Then, there is no sequence $\langle G_\alpha : \alpha < \alpha_* \rangle$ of length $< \chi_2$ of groups of cardinality $\le \chi_1$ such that any locally finite group H of cardinality λ can be embedded into at least one of them.*

The following is an example.

Conclusion 3.17. (i) *If $\mu = \mathrm{cf}(\mu)$, $\mu^+ < \lambda = \mathrm{cf}(\lambda) < 2^\mu$, then there is no group of cardinality λ universal for the class of locally finite groups.*
(ii) *For example, if $\aleph_2 \le \lambda = \mathrm{cf}(\lambda) < 2^{\aleph_0}$, this applies.*

3.3 The class of groups is not amenable

We have claimed (in earlier versions of [12]) that the class of groups is amenable (see Džamonja–Shelah [2]) but this is not true. An easy way to prove it is the following claim.

Claim 3.18. *For some group G_0, if $G \supseteq G_0$, $T = \mathrm{Th}(G)$, then G is not amenable.*

Proof. In any forcing extension $\mathbf{V}_1 = \mathbf{V}^{\mathbb{P}}$ of $\mathbf{V}$, we have
$(*)$ if $\mathbf{V}_1 \models$ "$\lambda = \mu^+ = 2^\mu + \diamondsuit_{S^\lambda_\mu}$, $\mathbb{Q}$ is μ^+-c.c., $(<\mu)$-complete",
then in $\mathbf{V}_1^{\mathbb{Q}}$, T has no universal member in λ and, moreover,
$$\mathrm{univ}(\lambda, \lambda, T) \le \lambda^+.$$
[Why? Because in $\mathbf{V}_1$ there is $\bar{C} = \langle C_\delta : \delta \in S^\lambda_\mu \rangle$ guessing clubs, hence this holds also in $\mathbf{V}_1^{\mathbb{Q}}$, so we can apply Theorems 2.9 and 3.1 or directly Conclusion 3.16.] So, by [2], we get a contradiction to amenability (using a suitable $\mathbb{Q}$). □

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