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REMARKS IN ABSTRACT MODEL THEORY

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PART I. ON THE BETH PROPERTY

Usually the Beth property lives in the shadow of the interpolation property. The main results of [12] indicate their affinity (assuming compactness, they are equivalent if a preservation theorem holds for trees). It was asserted that the original problem was Beth, and in fact weak Beth, (deducing from this their importance) (see Feferman [7], [8], [9], Friedman [10]).

Our reasons for dealing with the Beth property are:

(A) Every logic has a Beth closure, so we have an interesting operation on logics.

(B) the question "is the Beth closure of $\mathscr{L}$ compact" is more explicit then "is there a compact extension of $\mathscr{L}$ satisfying interpolation", and gives information concerning it.

(C) We have more to say on it.

We have been interested for a long time whether there is a compact logic satisfying interpolation ($\neq L_{\omega,\omega}$). Our main result here is that there is a compact logic satisfying Beth, even an easily definable one — the Beth closure of $L(Q^{\mathrm{cf}}_{\leq 2^{\aleph_0}})$. It does not satisfy INT, so those properties are distinct even for compact logic. In fact it satisfies PPP, hence the condition used in [12] for proving their equivalence for compact logic (preservation for tree sum) is reasonable.

We use notation from Makowsky [5].

1. On Beth closures

In this section we define some variants of the Beth closure of a logic $\mathscr{L}-\mathscr{L}^{\mathrm{B}}$ (one time), $\mathscr{L}^{\mathrm{Beth}}$ (which satisfies Beth). We then gain some sufficient conditions for the compactness of the logic $\mathscr{L}^{\mathrm{B}}$ and for the Weak Beth property of $\mathscr{L}$ (see Mekler and Shelah [16] for the proof for $L(Q)$ (consistency with ZFC)).

1.1. Notation. $\mathscr{L}$ will be a logic. If the occurrence number is $\kappa \geq \aleph_0$ we demand closure under $(\forall x_0, \ldots, x_i, \ldots)_{i<\alpha}$ for $\alpha < \kappa$, and the relations and functions have arity $<\kappa$ (as well as the predicates and function symbols). An $\mathscr{L}$-formula is defined naturally (with $<\kappa$ free variables), $\mathrm{Th}_{\mathscr{L}}(M) = \{\psi \in \mathscr{L}(\tau_M) : M \models \psi\}$ where τ_M is the vocabulary of M. We denote predicates (and relations) by P, Q, R; but when treating predicates as variables we write $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$. A bar denotes this is a sequence.

1.2. Definition. (1) $\Delta(\mathscr{L}) = (\mathscr{L})^{\Delta}$ is the Δ-closure.

[1] Meanwhile we have shown that this logic satisfies e.g. INT (the interpolation theorem) and ROB, PPP.

(2) $\psi(P;\bar R)$ is a *Beth sentence* if $\forall\bar R\,\exists^{\le 1}\boldsymbol{P}\ \psi(\boldsymbol{P},\bar R)$. $\Phi_{\psi(\boldsymbol{P},\bar{\boldsymbol{R}})}(\bar R)$ means $(\forall\bar{\boldsymbol{R}})(\exists^{\le 1}\boldsymbol{P})\ \psi(\boldsymbol{P},\bar{\boldsymbol{R}})$ and $(\exists\boldsymbol{P})\ \psi(\boldsymbol{P},\bar R)$.

(3) $\mathscr{L}^{\mathrm{B}}$ is the *closure of* $\mathscr{L}^{\mathrm{atB}}=\mathscr{L}\cup\{\Phi_{\psi(\boldsymbol{P},\bar{\boldsymbol{R}})}(\bar R):\psi(\boldsymbol{P},\bar{\boldsymbol{R}})$ a Beth sentence$\}$ under first-order operation and $(\forall x_0,\dots,x_i,\dots)_{i<\alpha}$ when $\alpha<\kappa$ (see 1.1) and, more generally, substitution.

(4) $\mathscr{L}^{\mathrm{Beth}}$ is the *Beth closure of* $\mathscr{L}$, i.e., define

$$\mathscr{L}_0^{\mathrm{Beth}}=\mathscr{L},\qquad \mathscr{L}_{\alpha+1}^{\mathrm{Beth}}=(\mathscr{L}_\alpha^{\mathrm{Beth}})^{\mathrm{B}},$$

$$\mathscr{L}_\delta^{\mathrm{Beth}}=\bigcup_{\alpha<\delta}\mathscr{L}_\alpha^{\mathrm{Beth}},\quad\text{and}\quad \mathscr{L}^{\mathrm{Beth}}=\bigcup_{\alpha}\mathscr{L}_\alpha^{\mathrm{Beth}}.$$

(5) $\mathscr{L}^{\Delta\text{-Beth}}$ is the *closure of* $\mathscr{L}$ under Δ and Beth, i.e., $\bigcup\mathscr{L}_\alpha^{\Delta\text{-Beth}}$,

$$\mathscr{L}_0^{\Delta\text{-Beth}}=\mathscr{L},\qquad \mathscr{L}_{2\alpha+1}^{\Delta\text{-Beth}}=(\mathscr{L}_{2\alpha}^{\Delta\text{-Beth}})^{\Delta},$$

$$\mathscr{L}_{2\alpha+2}^{\Delta\text{-Beth}}=(\mathscr{L}_{2\alpha+1}^{\Delta\text{-Beth}})^{\mathrm{B}},\qquad \mathscr{L}_\delta^{\Delta\text{-Beth}}=\bigcup_{\alpha<\delta}\mathscr{L}_\alpha^{\Delta\text{-Beth}}.$$

(6) For a model M, $M_{\mathscr{L}}^{m}$ is the *expansion of* M by a relation for every $\mathscr{L}$-formula. Such a model is called $\mathscr{L}$-Morleyized; similarly for the theory.

1.3. Claim. (1) *If $\psi(P,\bar R)\in\mathscr{L}$ is a Beth sentence, then for some $\theta(\bar x,\bar R)\in\mathscr{L}^{\mathrm{B}}$ (P an $l(\bar x)$-place predicate)*

$$M\vDash\psi(P,\bar R)\to(\forall x)[P(\bar x)\equiv\theta(\bar x,\bar R)],$$
$$M\vDash\neg(\exists\boldsymbol{P})\ \psi(\boldsymbol{P},\bar R)\to\neg(\exists\bar x)\ \theta(\bar x,\bar R).$$

(2) *If μ is the first regular cardinal $\geqslant\mathrm{oc}(\mathscr{L})$, then $\mathscr{L}^{\mathrm{Beth}}=\mathscr{L}_\mu^{\mathrm{Beth}}$ (in 1.2(4)'s notation) and $\mathscr{L}^{\Delta\text{-Beth}}=\mathscr{L}_\mu^{\Delta\text{-Beth}}$ (in 1.2(5)'s notation); in both cases $\mathscr{L}_\alpha=\mathscr{L}_{\alpha+2}\Rightarrow(\forall\beta\geqslant\alpha)\ \mathscr{L}_\alpha=\mathscr{L}_\beta$.*

Note that for compact $\mathscr{L}$, $\mu=\aleph_0$; and

$$\mathscr{L}_\alpha^{\mathrm{Beth}}\subseteq\mathscr{L}_\beta^{\mathrm{Beth}},\quad \mathscr{L}_\alpha^{\Delta\text{-Beth}}\subseteq\mathscr{L}_\beta^{\Delta\text{-Beth}}\quad\textit{for } \alpha\leqslant\beta.$$

Proof. (1) Let $\psi_1=\psi_1(P;\bar c,\bar R)=[\psi(P,\bar R)\wedge P(\bar c)]$. Put $\theta(\bar c,\bar R)=\Phi_{\psi_1(\boldsymbol{P},\bar c,\bar R)}(\bar c,\bar R)$.

(2) Immediate.

1.4. Definition. (1) An *$\mathscr{L}$-Bethless model M* is a model such that: *for every* relation P on M, not definable by any $\mathscr{L}$-formula with parameters in M, the theory $\mathrm{Th}_{\mathscr{L}}((M,P))$ has two models (M',P_1), (M',P_2) with $P_1\ne P_2$.

(2) We define by induction on α, when M is an *$(\mathscr{L},\alpha)$-Bethless model*: for every $\beta<\alpha$ and relation P on M, not definable by any $\mathscr{L}$-formula with parameters from M, $\mathrm{TH}_{\mathscr{L}}((M,P)^{m}_{\mathscr{L}_\beta^{\mathrm{Beth}}})$ has two $(\mathscr{L},\beta)$-Bethless models $(M',P_1,\dots)$, $(M',P_2,\dots)$, $P_1\ne P_2$ (M' is the common τ_M-reduct).

1.4A. Remark. Note that any model is an $(\mathscr{L},0)$-Bethless model and $\mathscr{L}$-Bethless is equivalent to $(\mathscr{L},1)$-Bethless.

1.5. Lemma. *Suppose* $\lambda = \lambda^{<\mathrm{oc}(\mathscr{L})}$ *and* $|\tau| \leqslant \lambda \Rightarrow |\mathscr{L}(\tau)| \leqslant \lambda$.

(1) *Suppose* $\mathscr{L}$ *satisfies, for every* $\xi \leqslant \zeta$:

$(*)_{\lambda,\xi}$ *If* T *is a complete theory in* $\mathscr{L}(\tau)$ *which has an* $\mathscr{L}_\xi^{\mathrm{Beth}}$*-Morleyized model,* $|T| \leqslant \lambda$, *then* T *has an* $(\mathscr{L}, \xi)$*-Bethless model.*

Then also $\mathscr{L}_\alpha^{\mathrm{Beth}}$ *(and even* $\mathscr{L}_\alpha^{\Delta\text{-Beth}}$*) satisfies* $(*)_{\lambda,\beta}$ *when* $\alpha + \beta \leqslant \zeta$.

(2) If $\mathscr{L}$ *is* (λ, μ)*-compact and satisfies* $(*)_{\lambda,\alpha}$, *then* $\mathscr{L}_\alpha^{\mathrm{Beth}}$ *(and* $\mathscr{L}_\alpha^{\Delta\text{-Beth}}$*) is* (λ, μ)*-compact.*

Proof. (1) Clearly we can concentrate on the $\mathscr{L}^{\mathrm{Beth}}$ case, the proof of $\mathscr{L}_\alpha^{\Delta\text{-Beth}}$ is similar. We now prove by induction on $i \leqslant \alpha$, for all complete $T \subseteq \mathscr{L}_i^{\mathrm{Beth}}(\tau')$, $|T| \leqslant \lambda$, which has an $\mathscr{L}_j^{\mathrm{Beth}}$-Morleyized model, $j \geqslant i$, that every $(\mathscr{L}, i)$-Bethless model of $T \cap \mathscr{L}$ is a model of $T_i \stackrel{\mathrm{def}}{=} T \cap \mathscr{L}_i^{\mathrm{Beth}}$ and then prove by induction on $j \leqslant \beta$, that any $(\mathscr{L}, \alpha + j)$-Bethless model of $T \cap \mathscr{L}_{\alpha+j}^{\mathrm{Beth}}$ is a $(\mathscr{L}, j)$-Bethless model of T when T is a complete theory in some $\mathscr{L}_\alpha^{\mathrm{Beth}}(\tau')$, $|T| \leqslant \lambda$ which has an $\mathscr{L}_{\alpha+j}^{\mathrm{Beth}}$-Morleyized model.

For $i = 0$ this is trivial. For $i = \delta$ a limit ordinal a $(\mathscr{L}, \delta)$-Bethless model of T_0 is for any $\gamma < \delta$ an $(\mathscr{L}, \gamma)$-Bethless model of T_0 hence M is a model of T_γ, but $T_\delta = \bigcup_{\gamma<\delta} T_\gamma$ hence we finish. So suppose $i = \gamma + 1$. First suppose $\psi(\boldsymbol{P}, \bar{\boldsymbol{c}}, \bar{\boldsymbol{R}}) \in \mathscr{L}_\gamma^{\mathrm{Beth}}$ $(\bar{R} \subseteq \tau,$ $\bar{\boldsymbol{c}}$ a finite sequence of individual constants): $(\forall x)[\theta(\bar{x}) \equiv \Phi_{\psi(\boldsymbol{P},\bar{\boldsymbol{c}},\bar{\boldsymbol{R}})}(\bar{x}, \bar{R})]$ belongs to T and ψ is an atomic formula. We shall show that: for $\bar{c} \in M$: $M \models \theta[\bar{c}]$ *iff* $M \models (\exists \boldsymbol{P})\ \psi(\boldsymbol{P}, \bar{c}, \bar{R})$. The implication $\Rightarrow$ follows by 1.3(1) and the choice of T. So suppose that the implication $\Leftarrow$ fails for some $\bar{c}$. So for some P, $(M, P) \models \psi[\boldsymbol{P}, \bar{c}, \bar{R}]$. Let $T' = \mathrm{Th}_{\mathscr{L}_\gamma^{\mathrm{Beth}}}((M, P)^{\mathrm{m}}_{\mathscr{L}_\gamma^{\mathrm{Beth}}})$, so as M was $(\mathscr{L}, \gamma + 1)$-Bethless, there are models $(M', P_l, \ldots)$ $(l = 1, 2)$ of $T' \cap \mathscr{L}$ which are $(\mathscr{L}, \gamma)$-Bethless, $P_1 \neq P_2$. By the induction hypothesis $(M', P, \ldots)$ is a model of T', and we get a contradiction to "$\psi(\boldsymbol{P}, \bar{c}, R)$ is a Beth sentence". As $\mathscr{L}_i^{\mathrm{Beth}} =$ the closure of $(\mathscr{L}_\gamma)^{\mathrm{atB}}$ by substitution, we have carried the induction on i. The induction on j is similar.

(2) Easy.

1.6. Definition. (1) A *Sk.f. like* (Skolem function like) $\mathscr{L}$-function F is a pair of functions F_0, F_1 such that: for every vocabulary τ, $F_0(\tau)$ is a vocabulary extending τ, maybe with new sorts; $F_1(\tau)$ is a theory in $\mathscr{L}(F_0(\tau))$ such that:

(*) Any τ-model can be expanded to an $F_0(\tau)$-model of $F_1(\tau)$, and $F_1(\tau)$ is $\mathscr{L}$-Morleyized.

(2) An $\mathscr{L}$-theory T *has* Sk.f. (F_0, F_1) if for some τ, $F_1(\tau) \subseteq T \subseteq \mathscr{L}(F_0(\tau))$; T is (λ, F_0, F_1)*-Skolemized* if T has Sk.f. (F_0, F_1), $|\mathscr{L}(F_0(\tau)| < \lambda$ and every finite subset of T has a model. We call (F_0, F_1) λ*-bounded* if $|\tau| < \lambda$ Implies $|\mathscr{L}(F_0(\tau)| < \lambda$.

Hypothesis. Finite occurrence numbers are assumed for the rest of the section.

1.7. Lemma. *Suppose $\mathscr{L}$ satisfies:*

(**) *If T is a complete (λ, F_0, F_1)-Skolemized theory in $\mathscr{L}(\tau)$, then T has an $(\mathscr{L}, \omega)$-Bethless model (where (F_0, F_1) is λ-bounded, of course).*

Then $\mathscr{L}^{\mathrm{Beth}}$, and even $\mathscr{L}^{\Delta\text{-Beth}}$, satisfies (**) (hence is λ-compact) for suitable F'_0, F'_1 (where $|\tau|<\lambda \Rightarrow \mathscr{L}(F'_0(\tau))$ has power $<\lambda$.)

Proof. Like the proof of Lemma 1.5.

Remark. Instead of Skolemization we can use devices like 3.1(4).

1.8. Definition. (1) A model M is $(\mathscr{L}, \aleph_0)$-*strongly homogeneous*, if for every finite sequences $\bar{a}$, $\bar{b}$ from M, if they realize the same $\mathscr{L}$-formulas in M, then some automorphism of M maps $\bar{a}$ to $\bar{b}$. We can replace $\aleph_0$ by any λ, and then $l(\bar{a}), l(\bar{b})<\lambda$.

(2) A model M is $(\mathscr{D}, \aleph_0)$-*homogeneous* ($\mathscr{D}$ a set of types $p(\bar{x})$, usually complete in some logic), if: (a) every $\bar{a}\in M$ realizes some $p\in\mathscr{D}$, and (b) if $\bar{a}$ realizes $p(\bar{x}, \bar{y})\restriction \bar{x}$, $p(\bar{x}, \bar{y})\in\mathscr{D}$ then for some $\bar{b}\in M$, $\bar{a}^\frown\bar{b}$ realizes $p(\bar{x}, \bar{y})$.

(3) A model M is $(\mathscr{L}, \aleph_0)$-*saturated* if every $\mathscr{L}$-type with finitely many parameters from M, finitely satisfiable in M, is realized in M (such a model is $(\mathscr{D}, \aleph_0)$-homogeneous for some $\mathscr{D}$).

1.9. Claim. *A sufficient condition for $\mathscr{L}^{\mathrm{B}}$ to be $(<\lambda^*, \aleph_0)$-compact is that for some logic $\mathscr{L}^*$, $\mathscr{L}\subseteq\mathscr{L}^*$, and λ^*-bounded Skolem function like $\mathscr{L}^*$-functions (F_0, F_1) the following holds:*

(*) *For every complete (λ^*, F_0, F_1)-Skolemized $T\subseteq\mathscr{L}^*$ we have a model M_T such that:*

(a) *M_T is a model of $T\cap\mathscr{L}$ and is $\mathscr{L}$-Morleyized,*

(b) *each M_T is $(\mathscr{L}, \aleph_0)$-saturated,*

(c) *M_T is $(\mathscr{L}, \aleph_0)$-strongly homogeneous,*

(d) *if $T_1\subseteq T_2$, $\tau_{T_2}=F_0(\tau_{T_1}+\bar{P})$, $F_1(\tau_{T_1}+\bar{P})\subseteq T_2$ ($\bar{P}$ finite), then $M_{T_1}, M_{t_2}\restriction\tau_{T_1}$ are $\mathscr{L}_{\infty,\omega}$-equivalent.*

Remark. $\mathscr{L}_{\infty,\omega}$ is defined naturally.

Proof. Clearly by (*)(b) $\mathscr{L}$ is λ^*-compact, and (by the λ^*-boundedness) $|\tau|<\lambda^*\Rightarrow|\mathscr{L}(\tau)|<\lambda^*$. We shall prove for $n=0, 1$ (letting $\mathbf{T}$ be the family of $T\subseteq(\mathscr{L}^*)^{\mathrm{Beth}}$, T a complete (λ, F_0, F_1)-Skolemized theory, $M_T\overset{\mathrm{def}}{=}M_{T\cap\mathscr{L}^*}$)

$(\mathrm{A})_n$ For $T\in\mathbf{T}$, M_T is a model of $T\cap\mathscr{L}_n^{\mathrm{Beth}}$.

For $n=0$ there are no problems: $(\mathrm{A})_0$ is one of the demands.

For $n=1$: It suffices to prove that for $T\in\mathbf{T}$, and $\psi(\boldsymbol{P}, \bar{\boldsymbol{c}}, \bar{\boldsymbol{R}})\in\mathscr{L}$, and $\bar{R}$ from τ_T, if $(\forall x)[\theta(\bar{x}) \equiv \Phi_{\psi(\boldsymbol{P},\bar{c},\bar{\boldsymbol{R}})}(\bar{x}, \bar{R})]$ belongs to T, θ atomic, then for $\bar{c} \in M_T$: $M_T \models (\exists \boldsymbol{P})\, \psi(\boldsymbol{P}, \bar{c}, \bar{R})$ iff $M \models \theta[\bar{c}]$. The implication $\Leftarrow$ is easy by 1.3(1).

As for the implication $\Rightarrow$, suppose it fails. So for some P, $(M_T, P) \models \psi(P, \bar{c}, \bar{R})$ but $M \models \neg\theta(\bar{c}]$. Hence P is not definable by any $\mathscr{L}$-formula with parameter in M_T. On the other hand, as $\psi(\boldsymbol{P}, \bar{\boldsymbol{c}}, \bar{\boldsymbol{R}})$ is a Beth sentence, P is preserved by automorphisms of M_T. As M_T is $(\mathscr{L}, \aleph_0)$-strongly homogeneous, if $\bar{a}$, $\bar{b}$ realize the same $\mathscr{L}$-type in M_T, then there is an automorphism of M_T taking $\bar{a}$ to $\bar{b}$, hence P is definable by some $\mathscr{L}_{\infty,\omega}$-formula. Now $(M_T, P, \bar{c})$ can be expanded to an $F_0(\tau_T + P + \bar{c})$-model of $F_1(\tau_T + P + \bar{c})$ which we call M^*, and let $T_2 = \mathrm{Th}_{\mathscr{L}^*}(M^*)$. Now we know that M_T and $M_{T_2} \restriction \tau_T$ are $\mathscr{L}_{\infty,\omega}$-equivalent, hence also $(M_T, \bar{c})$, $(M_{T_2} \restriction \tau_T, \bar{c})$ are $\mathscr{L}_{\infty,\omega}$-equivalent, So for $(M_{T_2} \restriction \tau_T, \bar{c})$ there is an $\mathscr{L}_{\infty,\omega}$-formula defining a relation P' such that $M_{T_2} \models \psi[P', \bar{c}, \bar{R}]$ (use the definition we have found for M_T, and (*)(d) which says that M_T, $M_{T_2} \restriction \tau_T$ are $(\mathscr{L})_{\infty,\omega}$ equivalent).

However, also $M_{T_2} \models \psi[P'', \bar{c}, \bar{R}]$ where P'' is the interpretation of P in M_{T_2}. By $\psi(\boldsymbol{P}, \bar{\boldsymbol{c}}, \bar{\boldsymbol{R}})$ being a Beth sentence, $P' = P''$. However, we shall now prove that P'' is not definable by an $\mathscr{L}_{\infty,\omega}(\tau_T + \bar{c})$-formula. For this it is enough to find sequences $\bar{b}, \bar{a} \in M_{T_2}$ realizing the same $\mathscr{L}$-type over $\bar{c}$ in $M_{T_2} \restriction \tau_T$ such that $\neg P''(\bar{b}) \equiv P''(\bar{a})$. If we restrict ourselves to $\bigwedge_{l<m} \theta_i(\bar{b}, \bar{c}) \equiv \theta_i(\bar{a}, \bar{c})$ for finitely many $\mathscr{L}(\tau_T)$-formulas, we can find such $\bar{a}$, $\bar{b}$ in M_T as P is not definable there. As T_2 is the $\mathscr{L}^*$-theory of an expansion of M_T and M_{T_2} is $(\mathscr{L}, \aleph_0)$-saturated, the existence of $\bar{b}$, $\bar{c}$ is clear.

We get a contradiction, hence prove $(\mathrm{A})_1$.

1.10. Fact. *From the hypothesis of* 1.9 *we can conclude that $\mathscr{L}$ has the weak Beth property.*

Remark. Compare with Mekler–Shelah [16]; essentially this is an abstract version of the result in §2 there; this is clearer in 1.11, 1.12, 1.13.

Proof. As we know $|\tau| < \lambda^* \Rightarrow |\mathscr{L}(\tau)| < \lambda^*$ and $\mathscr{L}$ is $(<\lambda^*, \aleph_0)$-compact, it suffices to prove that: if T is complete, $\psi(\boldsymbol{P}; \bar{\boldsymbol{R}})$ a weak-Berth sentence for $\bar{R} \in \tau_T$, then for some formula $\phi(\bar{x}, \bar{R})$, $\psi(\{\bar{x} : \phi(\bar{x}, \bar{R})\}, \bar{R}) \in T$. Suppose not; let $P \subseteq M_T$ be such that $(M_T, P) \models \psi[P, \bar{R}]$; P is not definable (even with parameters) in M_T (if we use some parameter to define P, we can eliminate it by P's uniqueness). Now we continue as in the proof of 1.9.

1.11. Claim. *We can weaken the hypothesis of* 1.10 *as follows:*

(*)′ *For every complete (λ^*, F_0, F_1)-Skolemized $T \subseteq \mathscr{L}^*$ we have a class of models K_T such that:*
- (a) *Each $M \in K_T$ is a model of $T \cap \mathscr{L}$.*
- (b) *For some $\mu \leq \lambda^*$, if $T \in \mathbf{T}$, $\tau \subseteq \tau_T$ is such that $|\tau| < \mu$, $T \cap \mathscr{L}^*(\tau)$ is Morleyized, then $M_T \restriction \tau$ is $(\mathscr{L}, \aleph_0)$-strongly homogeneous.*
- (c) *If $T_1 \subseteq T_2$, $\tau_{T_2} = F_0(\tau_{T_1} + \bar{P})$, $F_1(\tau_{T_1} + \bar{P}) \subseteq T_2$, ($\bar{P}$ finite), $\tau'_1 \subseteq \tau_1$ is as in* (b),

p a set of formulas of $\mathscr{L}(\tau_1)$ with the free variables $\bar{x}=\langle x_0,\ldots,x_{n-1}\rangle$, $\exists\bar{x}\,[\bigwedge p']\in T\cap\mathscr{L}(\tau_1)$ for finite $p'\subseteq p$, then there are $M_l\in K_{T_l}$, M_1, $M_2\restriction\tau_1$ are $\mathscr{L}_{\infty,\omega}$-equivalent, M_2 realizing p.

Remark. We can usually replace $(\mathscr{L},\aleph_0)$-strongly homogeneous by:

1.13. Definition. *M is $(\mathscr{L},\aleph_0)$-ps-strongly homogeneous* if for every $\bar{a},\bar{b}\in B$ realizing the same $\mathscr{L}$-type, there is a class $V^*\in V$, which is an inner model, V a generic extension of V^*, $\mathscr{L}(\tau_\mu)\in V^*$; so we can look at V and hence M, as a Boolean-valued model M, $M\models$"$\bar{a},\bar{b}$ realize the same $\mathscr{L}$-type" (i.e., this is forced) and $\mathrm{Th}_{\mathscr{L}}(M,\bar{a},\bar{b})\in V^*$, and M has an automorphism which takes $\bar{a}$ to $\bar{b}$ (and may move truth values) (we can assume for simplicitly that the universe of the model is an ordinal).

1.14. Remark. In the applications, we can ask more things to be in V^*.

1.15. Observations. Suppose the vocabulary of $\mathscr{L}$ is recursive, the same is true for $\mathscr{L}^{\mathrm{Beth}}$, provided we make the following minor change. $\Phi_{\psi(\mathbf{P},\bar{\mathbf{R}})}(\bar{R})$ is defined for every ψ; the demand on Bethness of ψ is delayed to the satisfaction (or Φ should contain a proof of ψ being Beth). Also we have a completeness theorem for $\mathscr{L}^{\mathrm{B}}$: if e.g., 1.7(**) holds then $F_0(\tau)$, $F_1(\tau)$ are recursive.

2. Beth and PPP

Our main interest here is to give sufficient conditions for the one step Beth closure of a logic to satisfy the "pair preservation property" and "uniform reduct property for pairs", i.e., that we can compute the truth values of $M_0+M_1\models\psi$ (for $\psi\in\mathscr{L}$) from the truth values of $M_l\models\psi_k^l$ ($k<k_l$) (where the ψ_k^l do not depend on the M_l).

2.1. Definition. (1) $\mathscr{L}$ has the *PPP* if for every models M, N, $\mathrm{Th}_{\mathscr{L}}(M+N)$ is determined by $\mathrm{Th}_{\mathscr{L}}(M)$, $\mathrm{Th}_{\mathscr{L}}(N)$ ($M+N$—we have more sorts).

(2) $\mathscr{L}$ has the *URP_2-property* if for every vocabularies τ_1, τ_2 (disjoint w.l.o.g.) and sentence $\psi\in\mathscr{L}(\tau_1+\tau_2)$ there are sentences $\psi_i^l\in\mathscr{L}(\tau_l)$ ($i=1,\ldots,n_i$) such that the truth value of $(M_1+M_2)\models\psi$ is determined by the truth values of $M_l\models\psi_i^l$ (M_l a τ_l-model, M_1+M_2 a $(\tau_1+\tau_2)$-model).

Remark. We can reformulate (2) as: ψ is equivalent to a Boolean combination of the ψ_i^l's.

2.2. Claim. (1) $\mathrm{PPP}+|\mathscr{L}|^+$*-compact implies* URP_2 (*where* $|\mathscr{L}|=\sup|\{\mathscr{L}(\tau):\tau$ *a finite vocabulary*$\}|$).

(2) URP_2 *implies* PPP.

Proof. (1) Suppose $\psi\in\mathscr{L}(\tau_1+\tau_2)$ is a counterexample to URP$_2$. This means that for no finite $\Psi_1\subseteq\mathscr{L}(\tau_1)$, $\Psi_2\subseteq\mathscr{L}(\tau_2)$ we can compute the truth value of $M_1+M_2\models\psi$ from the truth values of $M_l\models\phi$, $(\phi\in\Psi_l)$. Note M_l is a τ_l-model, and that for notational simplicity the sets of sorts of τ_1 and τ_2 are disjoint. So $\mathscr{L}(\tau_1)$, $\mathscr{L}(\tau_2)$ are disjoint.

So for every finite sets $\Psi_l\subseteq\mathscr{L}(\tau_l)$ $(l=1,2)$ there is a function $h=h_{\Psi_1\cup\Psi_2}$ from $\Psi_1\cup\Psi_2$ to $\{\mathbf{t},\mathbf{f}\}$ (= the set of truth values) such that some models of $\Gamma_h=\{\phi\equiv h(\phi)\colon \phi\in\Psi_1\cup\Psi_2\}$ satisfies ψ and some satisfy $\neg\psi$. (Note that $\phi\equiv h(\phi)$ is equivalent to ϕ if $h(\phi)=\mathbf{t}$, and is equivalent to $\neg\phi$ if $h(\phi)=\mathbf{f}$). W.l.o.g. τ_1, τ_2 are finite, and it is well known that for some $h:\mathscr{L}(\tau_1)\cup\mathscr{L}(\tau_2)\to\{\mathbf{t},\mathbf{f}\}$, for every finite $\Psi^l\subseteq\mathscr{L}(\tau_l)$ $(l=1,2)$, for some finite Ψ_l, $\Psi^l\subseteq\Psi_l\subseteq\mathscr{L}(\tau_l)$ $(l=1,2)$ and $h\restriction(\Psi^1\cup\Psi^2)\subseteq h_{\Psi_1\cup\Psi_2}$. So every finite subset of $\Gamma_h\cup\{\psi\}$ has a model and also every finite subset of $\Gamma_h\cup\{\neg\psi\}$ has a model. By the $|\mathscr{L}|^+$-compactness, $\Gamma_h\cup\{\psi\}$ has a model, and let it be $M_1^++M_2^+$; similarly $\Gamma_h\cup\{\neg\psi\}$ has a model and let it be $M_1^-+M_2^-$. So $M_1^++M_2^+\models\psi$, $M_1^-+M_2^-\models\neg\psi$, but M_l^+, M_l^- are $\mathscr{L}$-equivalent (for $l=1,2$) by the definition of Γ_h. This contradicts the PPP.

(2) Easy.

2.3. Lemma. *Suppose (for some Sk.f.)*

(i) *$\mathscr{L}$ satisfies* URP$_2$.

(ii) *Every T as*[2] *in* 1.7 *for $\mathscr{L}^{\mathrm{B}}$, has a $(\mathscr{L}^{\mathrm{B}},\aleph_0)$-strongly homogeneous $(\mathscr{L}^{\mathrm{B}},\aleph_0)$-saturated model.*

Then $\mathscr{L}^{\mathrm{B}}$ satisfies URP$_2$ *too.*

2.4. Remark. We can apply this to $\mathscr{L}^{\mathrm{Beth}}$ by proving by induction for $\mathscr{L}^{\mathrm{B}}_n$.

Proof. Let $\phi=\Phi_{\psi(\mathbf{P},\bar{\mathbf{R}})}(\bar{R})$ where $\psi(\mathbf{P},\mathbf{R})\in\mathscr{L}$, $\bar{R}\subseteq\tau_1+\tau_2$ is a Beth sentence. It suffices to prove URP$_2$ for such sentences (then prove that the set of sentences satisfying the URP$_2$ is closed under substitution). Note $\bar{R}$ may contain individual constants. W.l.o.g. $\bar{R}$ lists all members of $\tau_1+\tau_2$. W.l.o.g., P is a $(2n)$-place relation, the first n places for elements of sorts of τ_1, the rest for elements of the sorts of τ_2 (which are disjoint). We write $P(\bar{x},\bar{y})$; now clearly:

Assertion. *If there are $\theta_i^l(\bar{x},\bar{z}_i^l,\bar{R})\in\mathscr{L}^{\mathrm{B}}(\tau_l)$ $(l=1,2,\ i<i_0<\omega)$ such that for every M_l (τ_l-models, $l=1,2$), and P:*

> *if $M_1+M_2\models\psi(P,\bar{R})$, then for some $\bar{c}_i^l\in M_l$, $P(\bar{x},\bar{y})$ is equivalent to a Boolean combination of the formulas $\theta_i^1(\bar{x},\bar{c}_i^1,\bar{R})$, $\theta_i^2(\bar{y},\bar{c}_i^2,\bar{R})$,*

then the desired conclusion holds.

So we shall suppose there are no $\bar{c}_i^l$, θ_i^l as above. Then there is a complete consistent $T\subseteq\mathscr{L}^{\mathrm{B}}(\tau^*)$, $\tau^*=F_0(\tau_1+\tau_2):T\supseteq F_1(\tau_1+\tau_2)$ such that

(a) $\Phi_{\psi(\mathbf{P},\bar{\mathbf{R}})}(\bar{R})\in T$, i.e., $T\vdash(\exists\mathbf{P})\psi(\mathbf{P},\bar{R})$.

(b) T "says" that for every θ_i^l, $\bar{z}_i^l$ as above, $P(\bar{x},\bar{y})$ is not defined as a Boolean combination of $\theta_i^1(\bar{x},\bar{z}_i^1,\bar{R})$, $\theta_i^2(\bar{x},\bar{z}_i^2,\bar{R})$.

[2] I.e. for some (F_0,F_1) Sk.f.like λ-bounded, T is a complete (λ,F_0,F_1)-skolemized theory in $L^{\mathrm{B}}(\tau_T)$.

Let M be a τ^*-model, $M \restriction (\tau_1+\tau_2) = M_1 + M_2$ such that M is a $(\mathscr{L}^{\mathrm{B}}, \aleph_0)$-strongly homogeneous, $(\mathscr{L}^{\mathrm{B}}, \aleph_0)$-saturated model of T. Now the P satisfying $\psi(-, \bar{R})$ is definable by an $\mathscr{L}^{\mathrm{B}}$-formula (see Claim 1.3). Also if $\bar{b}, \bar{c} \in M_1$ realize the same $\mathscr{L}^{\mathrm{B}}(\tau^*)$-type in M, then there is an automorphism of M which takes $\bar{b}$ to $\bar{c}$, hence there is an automorphism of M_1 which takes $\bar{b}$ to $\bar{c}$. [Note that realizing the same $\mathscr{L}^{\mathrm{B}}$-type in M_1 is not necessarily sufficient.] So there is an automorphism f of $M \restriction (\tau_1+\tau_2)$, $f(\bar{b}) = \bar{c}$, $f \restriction M_2 =$ identity. Remember $\psi(P, \bar{R})$ is a Beth sentence. So for any $\bar{d} \in M_2$, if $l(\bar{b}) = l(\bar{c}) = l(\bar{d}) = n$, then $P(\bar{b}, \bar{d}) \equiv P(\bar{c}, \bar{d})$. Similarly this holds interchanging M_1 and M_2. We can conclude

$$M \models (\forall \bar{x}\bar{y})\,[P(\bar{x}, \bar{y}) \equiv \bigvee_i \bigwedge_j (\theta^1_{i,j}(\bar{x}, \bar{R}) \wedge \theta^2_{i,j}(\bar{y}, \bar{R})]$$

(remember $\bar{R}$ contains everybody from $\tau_1+\tau_2$), where $\theta^l_{i,j} \in \mathscr{L}^{\mathrm{B}}(\tau^*)$ (not $\mathscr{L}(\tau_1+\tau_2)$) but $\bar{x}$ varies on M_1, $\bar{y}$ varies on M_2. But $P(\bar{x}, \bar{y})$ is also definable by an $\mathscr{L}^{\mathrm{B}}(\tau_1+\tau_2)$-formula.

As M is $(\mathscr{L}^{\mathrm{B}}, \aleph_0)$-saturated, usual compactness arguments give

$$P(\bar{x}, \bar{y}) \equiv \bigvee_{i<i_0} \bigwedge_{j<j_i} (\theta^1_{i,j}(\bar{x}, \bar{R}) \wedge \theta^2_{i,j}(\bar{y}, \bar{R}))$$

were i_0, j_i are finite, $\theta^l_{i,j} \in \mathscr{L}(\tau^*)$. Now we can forget τ^*, and look only at $M_1 + M_2$. Define a relation E_1 between n-tuples from M_1:

$$\bar{a} E_1 \bar{b} \quad \text{iff} \quad (\forall \bar{c} \in M_2)[P(\bar{a}, \bar{c}) \equiv P(\bar{b}, \bar{c})]$$

Similarly E_2 on M_2.

The $\theta_{i,j}$ above show that E_1, E_2 have finitely many equivalence classes. They are definable in M_1+M_2 by an $\mathscr{L}^{\mathrm{B}}(\tau_1+\tau_2)$-formula (we have just defined them). If each E_l is definable in M_l by an $\mathscr{L}^{\mathrm{B}}(\tau_l)$-formula, we get a contradiction to the choice of T.

Let $\tau_l^+ = \tau_l + \{E_l\}$. So $M_l^+ = (M_l, E_l)$ is a τ_l^+-model. Let $\psi_l \in \mathscr{L}(\tau_l^+)$ be such that $M_l^+ \models \psi_l$ $(l=1,2)$ and if $N_l^+ \models \psi_l$ $(l=1,2)$, then $N_1^+ + N_2^+$ satisfies all the (finitely) many relevant information from $\mathrm{Th}_{\mathscr{L}}(M_1^+ + M_2^+)$ (possible by URP_2 for $\mathscr{L}$).

Question. If $\psi_l = \psi_l(E_l, \tau_l)$ a Beth sentence (i.e., defining implicitly E_l)?

If the answer for $l=1,2$ is yes, the E_l are explicitly defined in M_l by an $\mathscr{L}^{\mathrm{B}}(\tau_l)$-formula, contradiction.

If for at least one l the answer is no, say for $l=1$ we can find $(N_1, E_1^a) \models \psi_1$, $(N_1, E_1^b) \models \psi_1$ but $E_1^a \neq E_1^b$. Now $(N_1, E_1^x) + M_2^+$ $(x=a,b)$, satisfies enough $\mathscr{L}$-sentences which $M_1^+ + M_2^+$ satisfies, to have a P solving $\psi(-, \bar{R})$. But for $x=a$, $x=b$, we get distinct P (look at E_1's definition). Contradiction to $\psi(\bar{\boldsymbol{P}}, \bar{\boldsymbol{R}})$ being a Beth sentence.

2.5. Lemma. *In* 2.3(ii) *we can omit "$(\mathscr{L}, \aleph_0)$-saturated" if "T is a τ^*-theory with sk.f." is preserved by adding finitely many individual constants to the signature and by completing, demanding the $(\mathscr{L}, \aleph_0)$-strongly homogeneous for the reduct to τ^*.*

Proof. The only need for $\aleph_0$-saturation is to replace $\bigvee_i \bigwedge_j (\theta^1_{i,j}(\bar{x}) \wedge \theta^2_{i,j}(\bar{y}))$ by a finite formula. Let

$$\begin{aligned} T' = T &\cup \{\theta(\bar{c}^1) \equiv \theta(\bar{c}^2) : \theta(\bar{x}) \in \mathscr{L}(\tau^*),\ \bar{x} \text{ of first sort}\} \\ &\cup \{\theta(\bar{d}^1) \equiv \theta(\bar{d}^2) : \theta(y) \in \mathscr{L}(\tau^*),\ \bar{y} \text{ of second sort}\} \\ &\cup \{P(\bar{c}^1, \bar{d}^2) \not\equiv P(\bar{c}^2, d^2)\}. \end{aligned}$$

If T' is consistent, we work as before and get a contradiction in the point where we use "$(\mathscr{L}, \aleph_0)$-saturated" (remember we demand in 1.4 the $(\mathscr{L}, \aleph_0)$-strongly homogeneous for the τ^*-reduct).

If T' is inconsistent, we work as there for T, having the $\theta^l_{i,j}$ by the above.

2.6. Remark. Another way to phrase the hypothesis is:

(1') For every T and $\mathscr{L}(\tau^*)$-type $p(\bar{x})$ consistent with T, T has an $(\mathscr{L}, \aleph_0)$-strongly homogeneous $\mathscr{L}$-Bethless model realizing p.

The proof of 2.3 really says (see Definition 3.1):

2.7. Lemma. (1) *Suppose* (i) *$\mathscr{L}$ has the* URP$_2$, (ii) *$\mathscr{L}$ has the weak homogeneity property. Then $\mathscr{L}^{\mathrm{B}}$ has the* URP$_2$.[3]

(2) *Suppose $\mathscr{L}$ is compact and has the homogeneity property. Then $\mathscr{L}$ has the weak homogeneity property.*

3. Automorphisms and definable logics

We define here homogeneity properties of a logic $\mathscr{L}$ (saying $\mathrm{Th}_{\mathscr{L}}(M)$ has models with automorphisms we require). We then prove some variant of it assuming $\mathscr{L}$ has the INT (= interpolation) property and PPP (and other variants of the assumptions and conclusion). At last we define "a *definable* logic" (i.e., by a set-theoretic formula with no parameters) and prove the consistency of "no definable logic extending $L(Q)$ has PPP and INT". (Note that when $V = L$ "definable" is an extremely weak restriction.)

We do not systematically deal with the "pair of logics" versions or the trivial implications involving those definitions. The definitions seem to us interesting though the results here are easy.

3.1. Definition. (1) $\mathscr{L}$ has the *super* [*strong*] *λ-homogeneity* property if for every τ-model M [there is an expansion M^*] such that $\mathrm{Th}_{\mathscr{L}}(M)$ [$\mathrm{Th}_{\mathscr{L}}(M^*)$] has a model N [whose τ-reduct is *a*] $(\mathscr{L}, \lambda)$-strongly homogeneous model (see Definition 1.8(1)).

(2) $\mathscr{L}$ has the *homogeneity* property if for every τ-model M, and $c_1, c_2 \in M$

[3] First show that if $(M_1 + M_2, P) \models \psi(P, \bar{R})$, then E_1 has finitely many equivalences classes (otherwise use the weak homogeneity property). Second, using the same property find a uniform bound $n(1)$ on the number of E_1-equivalence classes; the same holds for E_2. Now continue as in the proof of 2.3.

realizing the same $\mathscr{L}$-type in M, $\mathrm{Th}_{\mathscr{L}}(M, c_1, c_2)$ has a model (N, c_1, c_2) such that some automorphism of N maps c_1 to c_2. (We can use n-tuples $\bar{c}_l$ instead; this is equivalent.)

(3) $\mathscr{L}$ has the *weak homogeneity* property if for every τ-model M and infinite $P \subseteq M$, $\mathrm{Th}_{\mathscr{L}}(M, P)$ has some model (N, P') such that some automorphism of N is not the identity on P'.

(4) We add "local" if just for every sentence of the relevant theory there is a model N as required. We can add also "for finite vocabulary", etc.

(5) $(\mathscr{L}, \mathscr{L}^*)$ has the *super* λ-*homogeneity* property if for every τ-model M, whose $\mathscr{L}^*$-theory is Morleyized, $\mathrm{Th}_{\mathscr{L}}(M)$ has an $(\mathscr{L}, \lambda)$-homogeneous model. Similarly for the other properties.

3.2. Remarks. (1) By [18] any $\aleph_0$-compact logic $L(Q_n)_{n<\omega}$, the Q_n are cardinality quantifiers, has the weak homogeneity property.

(2) By [23], [24], [26] if we assume GCH, then some compact logic does not have even the weak homogeneity property (e.g. $L(Q)$, where Q the quantifier says two atomic Boolean algebras are isomorphic).

For the definition of FROB see notation of [5].

3.3. Claim. (1) *If $\mathscr{L}$ satisfies the* PPP *and* FROB, *then it has the homogeneity property.*

(2) *If $\mathscr{L}$ satisfies the* PPP *and* INT, *then it has the local homogeneity property.*

Proof. We prove only (1) (the other is similar). Suppose M, c_1, c_2 form a counterexample to the homogeneity property with finite occurrence. Let M', c_1', c_2' be a disjoint copy. Let $N = [M, M']$, $T = \mathrm{Th}_{\mathscr{L}}(N, c_1, c_2, c_1') = \mathrm{Th}_{\mathscr{L}}(N, c_1, c_2, c_2')$ (the equality is by the PPP, and c' denotes the name of c_1' or c_2' in τ_T). Let

ψ_l = "f is an isomorphism from the first sort to the second (ignoring the c's) mapping c' to c_l".

Clearly $T \cup \{\psi_1, \psi_2\}$ does not have a model, hence FROB fails.

3.4. Definition. (1) A logic $\mathscr{L}$ is called *definable* if the relations "$\psi \in \mathscr{L}(\tau)$", "$M \models \psi$" are definable (in set theory, without parameters). So $\psi \in \mathscr{L}(\tau)$, $M \models \psi$ are meaningful in any universe of set theory.

(2) A logic $\mathscr{L}$ is called λ-*definable* if for some $A \subseteq \lambda$, the relations "$\psi \in \mathscr{L}(\tau)$", "$M \models \psi$" are definable using A as the only parameter.

Remark. Most reasonable logics are definable: the exception is fragments of such logics (mainly $L_{\omega_1,\omega}$). So restriction by definability is reasonable.

3.5. Claim. *It is consistent that no definable extension $\mathscr{L}$ (or $\aleph_0$-definable extension) of $L(Q)$ has the* PPP *and* INT.

Proof. Let

$K=\{(A, P, Q_1 < c, f)$: A an uncountable set, P a countable subset, $<$ a linear order on $Q=A-P-\{c\}$, f a function from $Q\times Q$ to P such that $f^{-1}(\{p\})$ is a chain for every $p\in P\}$.

Clearly K is definable in $L(Q)$ by some ϕ, and by [22] it is not empty, and such linear order is not isomorphic to its inverse. For $M\in K$ let M^* be the same except inverting the order. Let us define a forcing notion which forces a member $\boldsymbol{M}$ of K. The universe of $\boldsymbol{M}$ will be ω_1, $P^M=\omega-\{0\}$, $c^M=0$, $Q^M=\omega_1-\omega$. A condition p consists of a finite subset w_p of $\omega_1-\omega$, a linear order $\leqslant^p$ on w_p, and a function $f_p: w_p\times w_p\to\omega-\{0\}$ so that $f_p(\langle\alpha,\beta\rangle)=f_p(\langle\alpha',\beta'\rangle)$ implies $\alpha\leqslant^p\alpha'\wedge\beta\leqslant^p\beta'$ or $\alpha'\leqslant^p\alpha\wedge\beta'\leqslant^p\beta$. We say $p\leqslant q$ if $w_p\subseteq w_q$, $\leqslant^p=\leqslant^q\restriction w_p$, $f_q=f_q\restriction w_p\times w_p$. We can prove (essentially as in [22]) that the forcing notion satisfies the c.c.c., and for a generic set G, $\dot{\boldsymbol{M}}[G]\in K$. Now there is a natural automorphism F of order 2 of the forcing notion: $P\to P^*$ where in p^* we just invert the order. Clearly $\boldsymbol{M}[F(G)]=(\boldsymbol{M}^*[G])^*$. Hence in $V[G]$, $M_0=\boldsymbol{M}[G]$, $M_1=(\boldsymbol{M}[G])^*$ are $\mathscr{L}$-equivalent (as $\mathscr{L}$ is definable, P is homogeneous). By the PPP, $[M_0, M_0]$, $[M_0, M_1]$ are $\mathscr{L}$-equivalent. Let ψ_1 $[\psi_2]$ say that the linear order in the two sorts are isomorphic [anti-isomorphic]. As explained in [22], $\psi_1\wedge\psi_2$ has no model in which each sort satisfies ϕ. So we have obtained a contradiction.

3.5A. Remark. In 3.5 we can replace INT by FROB.

3.6. Claim. *Suppose $\mathscr{L}$ satisfies* PPP, WB *and for countable* τ, $|\mathscr{L}(\tau)|<\beth_\alpha$. *Then the well order number of $\mathscr{L}$ (for one sentence) is* $<\omega+\alpha$.

Proof. Should be clear.

By 3.2(2) and easy manipulation:

3.6. Lemma. *Suppose $\mathscr{L}$ satisfies the* URP_2.

(1) *Then the following are equivalent* (i) ROB, (ii) FROB, (iii) *the homogeneity property for* PC (*see below*).

(2) *Also the following are equivalent:* (i) WFROB (*see* [5]), (ii) *the local homogeneity properly for* PC.

3.7. Definition. (1) $\mathscr{L}$ has the *homogeneity property for PC* when:
if for $\tau\subseteq\tau_M$, c_1 and $c_2\in M$ realize the same $\mathscr{L}$-type in $M\restriction\tau$, *then* $\mathrm{Th}_{\mathscr{L}}(M, c_1, c_2)$ has a model (N, c_1', c_2') such that $N\restriction\tau$ has an automorphism taking c_1' to c_2'.

3.8. Remark. I thank Makowsky for discussions concerning this lemma. He also showed that the weak homogeneity property implies $[\omega]$-compactness.

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4. Interpolation for cofinality logic in stationary logic

4.1. Definition. $L(\mathrm{aa})$ is defined as follows: defining the formulas we allow as variables monadic predicates; however we do not allow existential or universal quantification over them, but the quantifier $\mathrm{aa}P:(\mathrm{aa}P)\ \phi(P)$ is allowed, it bounds the variable P, and

$$M \models (\mathrm{aa}P)\ \phi(P) \quad \text{iff} \quad \{P \subseteq S_{<\aleph_1}(|M|) : M \models \phi[P]\}$$
$$\text{contains a closed unbounded subset of } S_{<\aleph_1}(|M|)$$

(closed means under countable increasing union, unbounded means every member of $S_{<\aleph_1}(|M|)$ is contained in some member of the subset and $S_{<\lambda}(A) = \{B : B \subseteq A, |B| < \lambda\}$).

The dual quantifier is $(\mathrm{st}P):(\mathrm{st}P)\ \phi = \neg(\mathrm{aa}P)\ \neg\phi$.

4.2. Definition. $L(Q^{\mathrm{cf}}_{\aleph_0})$ is first-order logic expanded by the quantifier $Q^{\mathrm{cf}}_{\aleph_0}$, which acts syntacticly as "if $\phi(x, y, \bar{z})$ is a formula (with $x, y, \bar{z}$ free) then so is $(Q^{\mathrm{cf}}_{\aleph_0}x, y)\ \phi(x, y; \bar{z})$ (with $\bar{z}$ free)".

Semanticly $M \models (Q^{\mathrm{cf}}_{\aleph_0}x, y)\ \phi(x, y, \bar{a})$ iff on $\mathrm{Dom}[\phi(x, y, \bar{a})] \stackrel{\mathrm{def}}{=} \{b \in M : M \models (\exists y)(b, y, \bar{a})\}$ the relation $\phi(x, y; \bar{a})$ defines a linear order with no last element (i.e., $x < y \stackrel{\mathrm{def}}{=} \phi(x, y, \bar{a})$) with cofinality $\aleph_0$.

4.3. Discussion. The confinality logic $L(Q^{\mathrm{cf}}_{\aleph_0})$ was introduced by Shelah [19], [20] as a solution to a problem of Friedman and Keisler: is there a logic, stronger than first-order, which is compact (and not just λ-compact for some λ). It also has reasonble axiomatization and it seemed weak.

In search for stronger logics, in Shelah [20] $L(\mathrm{aa})$ was introduced. Like second-order logic, in it formula free monadic predicates are allowed but the quantifier is different. We cannot say "for some P" but "for almost all P". This logic draws much attention. Barwise, Kaufman and Makkai [1] investigate it thoroughly; showing it has all the good properties known for $L(Q)$ and, of course, it seems considerably stronger, so Eklof and Mekler use it to investigate $\aleph_1$-abelian groups (see [2], [3]). Kaufman suggests and investigates determined structures. Kaufman and Kakuda investigate ZF(aa).

However only lately properties of the logic $L(\mathrm{aa})$ were found indicating it really inherits something from second-order logic. Here we show that the interpolation theorem holds for the pair of logics $(L(Q^{\mathrm{cf}}_{\aleph_0}), L(\mathrm{aa}))$. Considering that there has been much research efforts on interpolation (and related notions) for $\aleph_0$-compact logics, without having any example (even "pathological" one), and that the logics involved are reasonable and not invented for the example, this is nice though the proof is easy. (A drawback is our having a pair of logics, not one.)

This is the main result here. A subsequent result is the biggness of the Hanf number of $L(\mathrm{aa})$ (see Kaufman and Shelah [4]) which really shows that on models

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of power $>\aleph_1$, $L(\mathrm{aa})$ is really very strong, stronger than quantification on countable sets.

4.4. Claim. $L(Q^{\mathrm{cf}}_{\aleph_0}) \leq L(\mathrm{aa})$.

Proof. This is because in $L(\mathrm{aa})$ we can express "$\phi(x, y; \bar z)$ is a linear order of cofinality $\aleph_0$" by $\psi_\phi(\bar z) \overset{\mathrm{def}}{=} [\psi(x, y; \bar z)$ defines a linear order with no last element$] \wedge (\mathrm{aa}P)(\forall x)[(\exists y)\ \phi(x, y; \bar z) \to (\exists y \in P)\ (\phi(x, y; \bar z)]$.

4.5. Convention. From now on we consider every sentence of $L(Q^{\mathrm{cf}}_{\aleph_0})(\tau)$ as a sentence of $L(\mathrm{aa})$.

4.6. Theorem. *The pair* $(L(Q^{\mathrm{cf}}_{\aleph_0}), L(\mathrm{aa}))$ *has the interpolation property*; *i.e., if* $\phi, \psi \in L(Q^{\mathrm{cf}}_{\aleph_0})$, $\vdash \phi \to \psi$ *(i.e., it is valid), then for some* $\theta \in L(\mathrm{aa})$, $\tau_\theta \subseteq \tau_\phi \cap \tau_\psi$ *and* $\vdash \psi \to \theta$ *and* $\vdash \theta \to \psi$.

As $L(\mathrm{aa})$ is $\aleph_0$-compact and $|\tau| \leq \aleph_0 \Rightarrow |L(\mathrm{aa})(\tau)| \leq \aleph_0$ (and the occurrence number is $\aleph_0$) the following lemma suffices:

4.7. Lemma. *Suppose that* $\tau_0 = \tau_1 \cap \tau_2$ *are countable vocabularies,* T_l *a complete theory in* $L(\mathrm{aa})(\tau_l)$ *(for* $l = 1, 2, 3$) *and* $T_1 \cap T_2 = T_0$.

Then $(T_1 \cap L(Q^{\mathrm{cf}}_{\aleph_0})(\tau_1)) \cup (T_2 \cap L(Q^{\mathrm{cf}}_{\aleph_0})(\tau_2))$ *has a model.*

Proof. We start with the following notation.

4.8. Notation. $\Gamma_l = \{\psi(P_{i_0}, \ldots, P_{i_1}) : i_0 < \cdots < i_n < \omega_1,\ \psi \in L(\mathrm{aa})[\tau_l]$ and $(\mathrm{aa}S_0), \ldots, (\mathrm{aa}S_n)\ \psi(S_0, \ldots, S_n) \in T_l\}$. Clearly $T_l \subset \Gamma_l$.

Given any model M_l and any ψ_l which is a finite conjunction of members of Γ_l it is easy to choose by induction $\langle P_i \subseteq M_l : i < n\rangle$ such that $M_l \models \psi(P_0, \ldots, P_{n-1})$. That is (as T_l has a model):

4.9. Fact. Γ_l *is consistent.*

Moreover, if we let Γ_0^+ be any completion (in $L(\mathrm{aa})[\tau_0]$) of Γ_0:

4.10. Fact. *For* $l = 1, 2$, $\Gamma_0^+ \cup \Gamma_l$ *is consistent.*

Proof. Let $\psi(P_{i_0}, \ldots, P_{i_{n-1}})$ be a conjunction of finitely many members of Γ_0^+. Then $(\neg(\mathrm{aa}S) \cdots (\mathrm{aa}S)\neg\psi) \in T_0$. (Otherwise $\neg\psi \in \Gamma_0 \subseteq \Gamma_0^+$, but Γ_0^+ is consistent.) Let $M_l \models T_l$ and suppose $\theta_0(P_{i_0}, \ldots, P_{i_n})$ is a finite conjunction of formulas in Γ_l. Now choose by induction sets P_i $(i < n)$ such that

(i) $M_l \models \neg(\mathrm{aa}S_{i+1}) \cdots (\mathrm{aa}S_n)\neg\psi(P_0, \ldots, P_i, S_{i+1}, \ldots, S_n)$.

(ii) For any formula $\theta \in L(\mathrm{aa})[\tau_l]$ if $M_l \models (\mathrm{aa}S)\,\theta(P_0, \ldots, P_{i-1}, S)$ then $M_l \models \theta(P_0, \ldots, P_{i-1}, P_i)$ (note that there are only countably many such θ's).

Let $\tau_l' = \tau_l + \{P_i : i < \omega_1\}$. So let Γ_l^+ be a complete consistent extension of $\Gamma_0^+ \cup \Gamma_l$, in $L(\mathrm{aa})[\tau_l']$. Let Γ_l^{f} be the extension of Γ_l^+ by giving name to every formula (R_ψ to ψ) with individual free variables only, so Γ_l^{f} is complete in $L(\mathrm{aa})[\tau_l^{\mathrm{f}}]$, $\tau_0^{\mathrm{f}} = \tau_1^{\mathrm{f}} \cap \tau_2^{\mathrm{f}}$. Clearly $\Gamma_l^* \stackrel{\mathrm{def}}{=} \Gamma_l^{\mathrm{f}} \cap L_{\omega,\omega}(\tau_l^{\mathrm{f}})$ is a complete theory in $L_{\omega,\omega}[\tau_l^{\mathrm{f}}]$, and $\Gamma_0^{\mathrm{f}} = \Gamma_1^{\mathrm{f}} \cap \Gamma_2^{\mathrm{f}}$, $\tau_0^{\mathrm{f}} = \tau_1^{\mathrm{f}} \cap \tau_2^{\mathrm{f}}$. So by Robinson's lemma (for first-order logic) $\Gamma_1^* \cup \Gamma_2^*$ is consistent.

4.11. Fact. *For any* $i < \omega_1$, $l < 3$, Γ_l (*and also* $\Gamma_l^+, \Gamma_l^{\mathrm{f}}$) "says" that P_i is an $\mathrm{L(aa)}$-*elementary submodel of the universe* (*restricting ourselves to the vocabulary* $\tau_l + \{P_j : j < i\}$). *Also it "says"* $P_i \subseteq P_j$ *for* $i < j$.

Let M be an $\aleph_2$-saturated model of $\Gamma_1^* \cup \Gamma_2^*$ and let N be the substructure with universe $\bigcup_{i<\aleph_1} P_i^M$. Note that by Fact 4.11 and unions of chains $N \models \Gamma_1^* \cup \Gamma_2^*$ (do it for each Γ_l^* separately).

4.12. Lemma. *Suppose* $\forall \bar z\, ((Q^{\mathrm{cf}}_{\aleph_0} x, y)(R(x, y, \bar z) \leftrightarrow S(\bar z)) \in \Gamma_l^*$ (R, S *are predicates*). *Fix* $\bar c \in N$, *and suppose* $R(x, u, \bar z)$ *is a linear order with no last element.*

(i) *If* $N \models \neg S(\bar c)$, *then* $N \models (Q^{\mathrm{cf}}_{\aleph_1} x, y) R(x, y, \bar c)$.

(ii) *If* $N \models S(\bar c)$, *then* $N \models (Q^{\mathrm{cf}}_{\geqslant\aleph_2} x, y) R(x, y, \bar c)$.

Proof. Choose i so that $\bar c \in P_i$ and if R is R_ϕ and P_j occurs in ϕ, then $j < i$.

(i) In this case, for every $j < i$ there is a $b_j \in P_{j+1} - P_j$ such that if $a \in P_j$ and a is in the field of $R(x, y, \bar c)$, then $N \models R(a, b_j, \bar c)$. This follows immediately from the assertion:

$$[(\forall \bar z)(\exists y_0)(\forall x)[(\exists y)(R(x, y, \bar z)) \wedge P_i(x) \rightarrow P_{i+1}(y_0) \wedge R(x, y_0, \bar z]] \in \Gamma_l$$

But this is clear, since if $M_l \models T_l$, and $\bar d \in M_l$, $R(x, y, \bar d)$ is a linear ordering of cofinality $> \aleph_0$ and P_i is countable, then the intersection of $P_i^{M_l}$ and the field of $R(x, y, \bar d)$ is bounded. But $P_{i+1}^{M_l}$ is an elementary submodel of M_l so there is a bound in $P_{i+1}^{M_l}$.

Now the sequence of $\{b_j : j < \aleph_1\}$ witnesses that the cofinality of $R(x, y, \bar c)$ is $\aleph_1$.

(ii) Note first that since M is $\aleph_2$-saturated, the cofinality of $R(x, y, \bar c)$ in P_i^M is $\geqslant \aleph_2$. But,

$$N \models (\forall x)[(\exists \tau) R(x, \tau, \bar c) \rightarrow (\exists y)(R(x, y, \bar c) \wedge P_i(y))]$$

as

$$(\forall \bar z)[(Q^{\mathrm{cf}}_{\aleph_0} x, y)\, R(x, y, \bar z) \rightarrow (\mathrm{aa}S)(\forall x)[(\exists y) R(x, y, \bar z) \\ \rightarrow (\exists y \in S) R(x, y, \bar z)]])] \in T_l.$$

So the intersection of the field of $R(x, y, \bar c)$ and P_i^M is unbounded in M, hence in N. Thus the linear order defined in N by $R(x, y, \bar c)$ has cofinality at least $\aleph_2$.

Now we reverse the cofinalities to get the required model.

4.13. Lemma. *There is an elementary submodel N^* of N such that N^* is a model of $\Gamma_l^* \cap L_l(Q^{\mathrm{cf}}_{\aleph_0})$ for $l = 1, 2$.*

Proof. We define by induction N_i for $i < \omega$ such that

(i) $|N_i| = \aleph_1$.

(ii) $N_i < N$.

(iii) For any linear order without endpoints, $<$, definable with parameters in N_i:

(a) If the cofinality of $<$ in N is $\leqslant \aleph_1$, then N_{i+1} contains a subset of the field of $<$ which is unbounded in N.

(b) If the cofinality of $<$ in N is $> \aleph_1$, then there is an element of N_{i+1} which bounds the intersection of N_i with the field of $<$.

But, the choice of the required N_i is easy, and from it the result is clear (the union $\bigcup_{i<\omega} N_i$ is as required).

Remark. (1) Looking at the proof of Theorem 4.6 we can see that:

(a) We can make the P_i's indescernible (using Ramsey theorem).

(b) We can find an interpolant of the form $(\mathrm{aa}P_1) \cdots (\mathrm{aa}P_n)\ \psi(P_1, \ldots, P_n)$, $\psi \in L(Q^{\mathrm{cf}}_{\aleph_0})$.

5. Higher cardinals and strongly homogeneous models

We deal with cofinality quantifiers and stationary logic for uncountable cardinals. Our result is that the pair $(L(Q^{\mathrm{cf}}_{\leqslant\lambda}), L(\mathrm{aa}_\lambda))$ satisfies the super $\aleph_0$-homogeneity property (see Definition 3.1(1)).

5.1. Definition. (1) For cardinals $\kappa \leqslant \lambda$, and set A we define a filter $\mathscr{E}^\kappa_\lambda(A)$ on $S_{\leqslant\lambda}(A)$. It is generated by sets of the following form: $\{\bigcup_{i<\kappa} A_i$: for $i < \kappa$, $A_i \subseteq A$, $|A_i| \leqslant \lambda$, and $F(\langle A_j : i < j\rangle) \subseteq A_{i+1}\}$ for some F.

(2) Suppose $\lambda_2 \leqslant \lambda_1$, $\kappa^1 \leqslant \lambda_1$, $\kappa^2 \leqslant \lambda_2$, then $\mathscr{E}^{\kappa^1,\kappa^2}_{\lambda_1,\lambda_2}(A)$ is the following filter on $S_{\leqslant\lambda_2}(A)$: $S \in \mathscr{E}^{\kappa^1,\kappa^2}_{\lambda_1,\lambda_2}(A)$ *iff* $\{B \subseteq A : |B| = \lambda_1$, and $S \cap S_{\leqslant\lambda_2}(B) \in \mathscr{E}^{\kappa^2}_{\lambda_2}(B)\}$ belong to $\mathscr{E}^{\kappa^1}_{\lambda_1}(A)$.

(3) the meaning of "the $\mathscr{E}$-majority of A ($\in S_{\leqslant\lambda}(A)$) satisfies ..." is "$\{A : A$ satisfies ...$\} \in \mathscr{E}$".

Remark. (1) In these filters we can replace κ by $\mathrm{cf}\,\kappa$.

(2) On such filters see [21], §3].

5.2. Definition. (1) For a class C of regular cardinals, $L(Q^{\mathrm{cf}}_C)$ is defined just like $L(Q^{\mathrm{cf}}_{\aleph_0})$, but $M \vDash (Q^{\mathrm{cf}}_C x, y)\, \phi(x, y, \bar a)$ *iff* $\phi(x, y, \bar a)$ defines on $\{b \in M : M \vDash (\exists y)\, \phi(b, y, \bar a)\}$ a linear order with no last element whose cofinality is in C. If $C = \{\mu : \mu$ regular, $\mu \leqslant \lambda\}$ we write $Q^{\mathrm{cf}}_{\leqslant\lambda}$ instead.

(2) For a cardinal λ we define $L(\mathrm{aa}_\lambda)$ just like $L(\mathrm{aa})$, but $M \vDash (\mathrm{aa}_\lambda P)\, \phi(P)$ iff $\{P \in S_{\leq\lambda}(|M|) : M \vDash \phi[P]\}$ belongs to $\mathscr{E}_\lambda^\lambda$ (i.e., to $\mathscr{E}_\lambda^\lambda(|M|)$. The dual quantifier to $(\mathrm{aa}_\lambda P)$ is $(\mathrm{st}_\lambda P)$.

(3) All the languages $L(Q_C^{\mathrm{cf}})$ have the same syntax, so for a sentence $\psi \in L(Q_{C_1}^{\mathrm{cf}})$ (or theory) it is clear what we mean by "M is a model of ψ in the C_2-interpretation, $M \vDash_{C_2} \psi$". We identify λ and $\{\lambda\}$ in this content.

Similarly for $\psi \in L(\mathrm{aa}_\mu)$, "$M$ is a model of ψ in the λ-interpretation, $M \vDash_\lambda \psi$" is defined.

(4) Dealing with $L(Q_C^{\mathrm{cf}})$ we ignore the trivial cases $C = \emptyset$ or $C = \{\mu : \mu$ a regular cardinal$\}$. We know (see Mekler–Shelah [17] for (2) and (3), [20] for (1) and 4.4 for (4)):

5.3. Theorem. (1) *For any C_1, C_2 (non-trivial), a theory T in $L(Q_{C_1}^{\mathrm{cf}})$ has a model iff it has a model in the C_2-interpretation (so all those logics are compact). In fact if $\lambda \in C_2$, $\lambda \notin C_2$, λ, μ regular, then T has a* $\mathrm{Min}\{\lambda, \mu\}$*-saturated model in which each definable linear order with no last element has cofinality λ or μ.*

(2) *For any λ and $\psi \in L(\mathrm{aa}_\lambda)$, if ψ has a model, then ψ has a model in the $\aleph_0$-interpretation.*

(3) *If $\lambda = \lambda^{<\lambda}$, $T \subseteq L(\mathrm{aa}_\lambda)$, $|T| \subseteq \lambda$, then T is consistent iff T is consistent in the $\aleph_0$-interpretation iff T has a λ^+-compact model of power λ^+.*

(4) $L(Q_{\leq\lambda}^{\mathrm{cf}}) \leq L(\mathrm{aa}_\lambda)$ *(and we adopt the convention $L(Q_{\leq\lambda}^{\mathrm{cf}}) \subseteq L(\mathrm{aa}_\lambda)$).*

5.4. Claim. *Let M be a model, $|\tau_M| + \kappa \leq \lambda$, κ is regular.*

(1) *Then for a set of $A \in S_{\leq\lambda}(|M|)$ which belongs to $\mathscr{E}_\lambda^\kappa$ the following holds:*

(*) *For any relation $R(x, y, \bar{z}) \in \tau_M$ and $\bar{c} \in A$, if $R(x, y, \bar{c})$ defines in M a linear order with domain* $\mathrm{Dom}\, R(x, y, \bar{c})$ *(and no last element) of cofinality μ, then $R(x, y; \bar{c})$ defines in $M \restriction A$ a linear order of cofinality μ' where:*

$$\mu > \lambda \Rightarrow \mu' = \kappa, \qquad \mu \leq \lambda \Rightarrow \mu' = \mu.$$

(2) *Suppose further that C is a class of regular cardinals, $\kappa \in C$, and every regular $\mu > \lambda$, $\mu \leq \|M\|$ belongs to C. Then for a set of $A \in S_{\leq\lambda_2}(|M|)$ which belongs to $\mathscr{E}_\lambda^\kappa$, $M \restriction A$ is a $L(Q_C^{\mathrm{cf}})$-elementary submodel of M.*

Proof. Easy.

5.5. Lemma. *The pair $(L(Q_{\leq\lambda}^{\mathrm{cf}}), L(\mathrm{aa}_\lambda))$ has the super $\aleph_0$-homogeneity property (getting even an $(L(Q_{\leq\lambda}^{\mathrm{cf}}), \aleph_0)$-saturated model).*

Remark. (1) The proof gives a little more; the $2^{|\tau_M|}$ is needed for the saturation only (otherwise $|\tau_M|$ suffices).

(2) We can get super μ-homogeneity if $\mu \leq \lambda$.

Proof. So let M be a model, Morleyized for $L(\mathrm{aa}_\lambda)$. Let μ_0 be regular such that

$M \in H(\mu_0)$, $\lambda^+ < \mu_0$, $2^{|\tau_M|} < \mu_0$ and let $\mathfrak{A}$ be $(H(\mu_0); \epsilon)$ expanded by M (i.e., its relation and a predicate for its universe) and Morleyized for $L(\mathrm{aa}_\lambda)$.

We shall build a model of $\mathrm{Th}_{L(Q^{\mathrm{cf}}_{\leq\kappa})}(M)$ of power $\mu \stackrel{\mathrm{def}}{=} \lambda^+ + 2^{|\tau_M|}$ so w.l.o.g. there is a cardinal $\chi = \chi^{<\chi} > \mu$ (by working inside the inner model $L[A]$ where $A \subseteq \chi$, χ regular for suitable A).

By 5.3(2), (3) $\mathrm{Th}_{L(\mathrm{aa}_\lambda)}(\mathfrak{A})$ has a model $\mathfrak{B}$ in the χ-interpretation. Of course in $\mathfrak{B}$ we can interpret a model of $\mathrm{Th}_{L(Q^{\mathrm{cf}}_{\leq\kappa})}(M)$ in the χ-interpretation which is $(L(\mathrm{aa}_\chi), \chi)$-saturated. Let $C = \{\kappa \text{ regular, } \kappa \leq \lambda \text{ or } \kappa = \chi\}$ so by saturation, as $\lambda^+ < \chi$, no $L(\mathrm{aa}_\chi)$-formula with parameters defines in $\mathfrak{B}$ a linear order (with no last element and) with cofinality $< \chi$. So the C-interpretation and χ-interpretation coincide. By applying twice Claim 5.4(2) for an $\mathscr{E}^{\lambda^+,\aleph_0}_{\chi,\mu}$-majority of the $A \in S_{\leq\mu}(|\mathfrak{B}|)$, $\mathfrak{B} \restriction A$ is an $L(Q^{\mathrm{cf}}_C)$-elementary submodel of $\mathfrak{B}$ (remember $\mu < \chi$).

Clearly it suffices to prove the following fact (the κ which interests us is λ^+).

5.6. Fact. *Suppose N_0, N_1 are models definable in $\mathfrak{B}$ (i.e., their universe and relations are first-order definable with parameter in $\mathfrak{B}$), have the same vocabulary, and are $L(\mathrm{aa}_\chi)$-equivalent and let $\kappa < \chi$. Then for an $\mathscr{E}^{\kappa}_{\chi}$-majority of $A \in S_{\leq\chi}(|\mathfrak{B}|)$, $N_0 \restriction A \cong N_1 \restriction A$ (i.e., $N_l \restriction (|N_l| \cap A)$).*

5.6A. Remark. So surely this holds for a $\mathscr{E}^{\kappa,\kappa^1}_{\chi,\chi_1}$-majority of $A \in S_{\leq\lambda}(|\mathfrak{B}|)$, when $\chi_1 < \chi$, $\kappa^1 \leq \chi_1$.

Proof. As in the proof of 4.11 there is a complete theory Γ in $L(\mathrm{aa}_\chi)$ $[\tau + \{P_i : i < \kappa\}]$ such that $\psi(P_{i_1}, \ldots, P_{i_n}) \in \Gamma$, $i_1 < \cdots < i_n < \kappa$ implies $N_l \models (\mathrm{st}_\chi P_{i_1})$ $(\mathrm{st}_\chi P_{i_2}) \cdots (\mathrm{st}_\chi P_{i_n})\, \psi(P_{i_1}, \ldots, P_{i_n})$ (note Γ is closed by finite conjunction).

Let Γ_l be Γ when we replace the predicates from τ_{N_l} by their defining $L(\mathrm{aa}_\chi)[\tau_{\mathfrak{B}}]$-formulas with parameters and restrict everything by the formula defining $|N_l|$. Clearly, for every $\psi = \psi(P_{i_1}, \ldots, P_{i_n})$ $(i_1 < \cdots < i_n < \kappa)$.

$$\mathfrak{B} \models (\mathrm{st}_\chi P_{i_1}) \cdots (\mathrm{st}_\chi P_{i_n})\, \psi(P_{k_1}, \ldots, P_{i_n})$$

Clearly $\mathfrak{A} \models (\mathrm{aa}_\lambda P)(\exists x)[(\forall y)(P(y) \equiv y \in x)]$ (because every subset of $H(\mu_0)$ of power $\leq \lambda$ belongs to $H(\mu_0)$). Hence also $\mathfrak{B}$ satisfies this (in the χ-interpretation). So we can find $a_i \in \mathfrak{B}(i < \chi^+)$ such that:

(*) $A_i = \{b : \mathfrak{B} \models \text{“}b \in a_i\text{”}\}$ has power χ, it is increasing, for δ of cofinality χ, $A_\delta = \bigcup_{i<\delta} A_i$, and $\{A_i : i < \chi^+\} \in \mathscr{E}^{\chi}_{\chi}(|\mathfrak{B}|)$.

Hence we can define by induction on $\gamma < \kappa$, for every $\eta \in {}^{\gamma}(\chi^+)$ an ordinal $i(\eta, l)$ such that:

(a) $i(\eta \restriction \beta, l) < i(\eta, l) < \chi^+$ and $i(\eta, l) \geq \mathrm{Sup}\{\eta(j) : j < l(\eta)\}$ for $\beta < l(\eta)$.

(b) For every $\psi = \psi(P_{i_1}, \ldots, P_{i_n}, P_{j_1}, \ldots, P_{j_m}) \in \Gamma_l$, $i_1 < \cdots < i_n \leq \gamma < j_1 < \cdots < j_n$

$$\mathfrak{B} \models (\mathrm{st}_\chi P_{j_1}) \cdots (\mathrm{st}_\chi P_{j_m})\, \psi(A_{i(\eta \restriction i_1, l)}, \ldots, A_{i(\eta \restriction i_n, l)}, P_{j_1}, \ldots, P_{j_m}).$$

There is no problem in this. It is also clear that for every $\eta, \nu \in {}^{\kappa}(\kappa^+)$,

$(N_1, A_{i(\eta\restriction\gamma,l)} \cap N_1)_{\gamma<\kappa}$ and $(N_2, A_{i(\eta\restriction\gamma,l)} \cap N_2)_{\gamma<\kappa}$ are $L(\mathrm{aa}_\chi)$-equivalent, $L(\mathrm{aa}_\chi)$-saturated, and as in 5.4, $(N_1 \restriction \bigcup_{\gamma<\gamma} A_{i(\eta\restriction\gamma,l)}, A_{i(\eta\restriction\gamma,l)} \cap N_1)_{\gamma<\kappa}$ and $(N_2 \restriction \bigcup_{\gamma<\kappa} A_{i(\nu\restriction\gamma,l)}, A_{i(\nu\restriction\gamma,l)} \cap N_2)_{\gamma<\kappa}$ are isomorphic.

However, $S=\{i<\chi^+$: if $j<i$, $\eta \in {}^\gamma j$, $\gamma<\kappa$, $l=1,2$ then $i(\eta,l)<i\}$ is a closed unbounded subset of χ^+. Now if $\delta \in S$, $\mathrm{cf}\,\delta=\kappa$, we can easily find η which is increasing and converge to δ, then clearly $\langle i(\eta\restriction\gamma, l):\gamma<\kappa\rangle$ is increasing and converge to δ. It is also clear that $\bigcup_{\gamma<\kappa} A_{i(\eta\restriction\gamma,l)} = \bigcup_{i<\delta} A_i$. Hence by the previous paragraph $N_0 \restriction \bigcup_{i<\delta} A_i \cong N_1 \restriction \bigcup_{i<\delta} A_i$. But

$$\left\{\bigcup_{i<\delta} A_i : \delta \in S, \mathrm{cf}\,\delta=\kappa\right\} \in \mathscr{E}^\kappa_\chi(|\mathfrak{B}|)$$

so we finish proving the fact, hence the theorem.

6. A compact logic with the Beth property

We prove here (in ZFC) the existence of a compact logic satisfying the Beth property (and which is stronger than first-order logic). Moreover this logic has a reasonable description: it is the Beth closure of $L(Q^{\mathrm{cf}}_{\leqslant(2^{\aleph_0})})$, and it has the URP_2 but not the Craig property, thus it shows that in the main theorem of Makowsky–Shelah [11] the preservation theorem for trees cannot be replaced by the preservation theorem for the sum of two models.

Really we deal mainly with $\mathscr{L}^*_\kappa$, a sublogic of $L(\mathrm{aa}_\kappa)$ and deduce the properties of $L(Q^{\mathrm{cf}}_{\leqslant\kappa})^{\mathrm{Beth}}$ from it. We rely heavily on Sections 2, 4 and 5.

6.1. Definition. $\mathrm{CF}\,\mathscr{D}(\lambda)$ is the family of regular cardinals μ, such that for some μ^+-saturated model M, τ_M of power $\leqslant\lambda$, some $L_{\infty,\omega}$-formula $\phi(x,y)$ defines on $\{b \in M:(\exists y)\,\phi(x,y)\}$ a linear order of cofinality μ. (We can allow quasi-linear order and replace x, y be sequences of length $n<\omega$.)

6.2. Claim. (1) *If M, N are μ^+-saturated elementary equivalent, $\tau_M=\tau_N$ has power $\leqslant\lambda$, $\phi(x,y)\in L_{\infty,\omega}$ defines a linear order of cofinality μ on $\{x:(\exists y)\,\phi(x,y)\}$ in M, then the same holds in N.*

(2) *If λ, M, $\phi(x,y)$, μ are as in Definition* 6.1, *then $\mu \leqslant 2^\lambda$.*

(3) *Moreover in* (2), *there is no cofinal sequence in* $\mathrm{Dom}\,\phi(x,y)$ (*in M*) *of elements realizing the same strong type* (*over $\emptyset$*). (See [25, Ch. III].)

Proof. (1) It is well known that M, N are L_{∞,μ^+}-equivalent.

(2) Let M_0 be an elementary submodel of M of power λ, $\langle a_i:i<\mu\rangle$ a cofinal sequence in the linear order $\phi(x,y)$ (in M). Suppose $\mu>2^\lambda$; then w.l.o.g. $p=\mathrm{tp}(a_i, M_0)$ is constant. By [25, VII 4.1, p. 406] there is an elementary mapping f (whose domain and range are $\subseteq M$) such that $f\restriction M_0=$ the identity, and $\mathrm{tp}_*(\{a_i:i<\mu\}, M_0 \cup \{f(a_i):i<\mu\})$ is finitely satisfiable in M_0.

Clearly each $f(a_i)$ realizes p, hence is in the domain of $\phi(x, y)$. By the choice of a_i $(i<\mu)$ for some $i<\mu$, $M\models\phi(f(a_0), a_i)$. As $\mathrm{tp}(a_i, M_0\cup\{f(a_j):j<\mu\}$ is finitely satisfiable in M_0 it does not split over M_0 (see [25, I.2.6, p. 11]); hence as all $f(a_j)$ realize the same type over M_0 they realize also the same type over $M_0\cup\{a_i\}$. Hence for every j, $M\models\phi[f(a_j), a_i]$. Let g be an elementary mapping such that $g\restriction M_0=$ the identity, $g(f(a_j))=a_j$ and whose domain includes a_i. Then $M\models \phi[a_j, g(a_i)]$ for every j. Contradiction to the cofinality of $\langle a_j:j<\mu\rangle$.

(3) Suppose $\langle a_i:i<\mu\rangle$ is cofinal, all a_i's realizing the same strong type over $\emptyset$. W.l.o.g. M is $(\mu+\lambda)^+$-saturated (by (1)), and $M_0\prec M$ has power λ. Let

$$\Gamma=\{\phi(x_{i_1},\ldots,x_{i_n}):i_1,\ldots,i_n<\mu,\models\phi[a_{i_1},\ldots,a_{i_n}]\}$$
$$\cup\{\phi(x_i,\bar c)\equiv\phi(x_j,\bar c):\bar c\in M_0,\ \phi \text{ a formula}\}.$$

If Γ is consistent, let the assignment $x_i\to a_i'$ realize it; as in the proof of (1) also $\langle a_i':i<\mu\rangle$ is cofinal, and $\mathrm{tp}(a_i', M_0)$ does not depend on i (by the choice of Γ). So we can continue as in (2).

Γ is consistent by the hypothesis.

6.3. Definition. We define a logic $\mathscr{L}^*_\kappa$. $\mathscr{L}^*[\tau]$ is the set of sentences $\psi\in L(\mathrm{aa}_\kappa)[\tau]$ s.t.: for any $\lambda>\kappa$, expansion $\mathfrak{A}$ of $(H(\mu_0),\in)$ (for μ_0 regular $>\kappa^+$), $\tau_{\mathfrak{A}}$ countable and any $(\mathscr{L}(\mathrm{aa}_\chi),\chi)$-saturated model $\mathfrak{B}$ of $\mathrm{Th}_{\mathscr{L}(\mathrm{aa}_\kappa)}(\mathfrak{A})$ in the χ-interpretation, for $\chi=\chi^{<\chi}$ and any N interpretable in $\mathfrak{B}$, by an $L_{\infty,\omega}(\mathrm{aa}_\chi)$-formula with finitely many parameters $|\tau_N|\leqslant\aleph_0:N\models_\chi\psi$ *iff* for a $\mathscr{E}^{\kappa^+,\kappa}_{\chi,\lambda}$-majority of $A\in S_{\leqslant\lambda}(|\mathfrak{B}|)$, $N\restriction A\models_\kappa\psi$; where we make the hypothesis:

6.4. Hypothesis. For arbitrarily large cardinals χ, $\chi=\chi^{<\chi}$.

6.5. Discussion. (1) The hypothesis is just for convenience. Other wise we should say: for any set A of ordinals s.t. $H(\mu_0+\lambda)\in L[A]$ the requirement of the definition holds in $L[A]$, for every large enough regular cardinal (for $L[A]$).

(2) We can also wave the role of $\mathfrak{A}$. Instead we should demand that $\mathfrak{B}$ is a model of $\mathrm{ZFC}^-(\mathrm{aa}_\chi)$ (Zermelo–Fraenkel set theory with choice but without the power set axiom, and any image of a set by an $L(\mathrm{aa}_\chi)$-definable function is a set and $\mathfrak{B}\models(\mathrm{aa}_\chi P)(\exists x)(\forall y)[P(y)\equiv y\in x]$. we can also make $\tau_{\mathfrak{B}}=\{\in\}$ without changing anything and/or $\mathfrak{B}\models$"$|N|$ is included in some set").

6.6. Claim. *If* $\kappa\geqslant 2^{\aleph_0}$, *then* $L(Q^{\mathrm{cf}}_{\leqslant\kappa})\leqslant\mathscr{L}^*_\kappa$.

Proof. The point is that by 6.1(2) any $L(\mathrm{aa}_\chi)$-definable linear order in N, has cofinality $\leqslant 2^{\aleph_0}$ or $\geqslant\chi$. Hence by 5.4(2) any $\phi\in L(Q^{\mathrm{cf}}_{\leqslant\kappa})$ satisfies the requirement in Definition 6.3.

6.7. Claim. $\mathscr{L}^*_\kappa$ *is a regular logic.*

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Proof. Easy (for the universal quantifier use normality of the filter $\mathscr{E}_{\chi,\lambda}^{\kappa^+,\kappa}$, i.e., if $S_a \in \mathscr{E}_{\chi,\chi}^{\kappa^+,\kappa}(A)$ for $a \in A$, then

$$\{B \in S_{\leqslant\lambda}(A)\text{: for every } a \in b, B \in S_a\} \in \mathscr{E}_{\chi,\lambda}^{\kappa^+,\kappa}$$

which follows from the normality of every $\mathscr{E}_{\mu(0)}^{\mu(1)}(A)$).

6.8. Claim. *$\mathscr{L}_\kappa^*$ is a compact logic.*

Proof. Let Γ be $\subseteq \mathscr{L}_\kappa^*[\tau]$, any finite $T \subseteq \Gamma$ has a model M_T and $\lambda = |\Gamma| + \kappa^+$. So for some regular μ_0, $\{M_T : T \subseteq \Gamma, T \text{ finite}\}$ and τ belongs to $H(\mu_0)$. Let $\mathfrak{A}$ be $(H(\mu_0), \in, \kappa)$, and let $\chi = \chi^{<\chi} > \|\mathfrak{A}\|$, $\mathfrak{B}$ an $(L(\mathrm{aa}_\chi), \chi)$-saturated model of $\mathrm{Th}_{L(\mathrm{aa}_\kappa)}(\mathfrak{A})$ in the χ-interpretation. Clearly for any finite $T \subseteq \Gamma$, $\mathfrak{B} \vDash$ "there is a model of T in the χ-interpretation." As $\mathfrak{B}$ is $(L(\mathrm{aa}_\lambda), \chi)$-saturated, we can find an $L(\mathrm{aa}_\chi)$-formula with parameters from $\mathfrak{B}$, defining a model $(N, R^N)_{R \in \tau_T}$ s.t., for any $\psi \in T$, $\tau_\psi = \{R_1, \ldots, R_n\}$

$$\mathfrak{B} \vDash \text{“}(N, R_1^N, \ldots, R_n^N) \vDash_\lambda \psi\text{”}$$

By Definition 6.3, for every such ψ, for an $\mathscr{E}_{\chi,\lambda}^{\kappa^+,\kappa}$-majority of $A \in S_{\leqslant\lambda}(N)$, $(N, R_1^N, \ldots, R_n^N) \restriction A \vDash_\kappa \psi$. By the λ-completeness of $\mathscr{E}_{\chi,\lambda}^{\kappa^+,\kappa}$ there is a model of Γ.

6.9. Claim. (1) *The pair $(\mathscr{L}_\kappa^*, L(\mathrm{aa}_\kappa))$ has the interpolation property.*

(2) *If $M = [N_1, N_2, N_3; \bar{R}]$, N_1 N_2 are $L(\mathrm{aa}_\kappa)$-equivalent, then for some $\mathscr{L}_\kappa^*$-equivalent model $M' = [N_1', N_2', N_3'; \bar{R}']$, N_1' is isomorphic to N_2'.*

(3) *The pair $(\mathscr{L}_\kappa^*, L(\mathrm{aa}_\kappa))$ has the super $\aleph_0$-homogeneity property.*

Proof. (1) By the compactness it suffices to prove that if the complete $L(\mathrm{aa}_\kappa)$-theories T_l, satisfy

$$T_1 \cap T_2 = T_1 \cap L(\mathrm{aa}_\kappa)[\tau_{T_1} \cap \tau_{T_2}] = \tau_2 \cap L(\mathrm{aa}_\kappa)[\tau_{T_1} \cap \tau_{T_2}],$$

then $(T_1 \cup T_2) \cap \mathscr{L}_\kappa^*[\tau_{T_1} \cup \tau_{T_2}]$ has a model. This follows by (2).

(2) The proof is like 5.6 (see 5.6(A)).

(3) The proof is like 5.5.

Up to now, all we have proved on $\mathscr{L}_\kappa^*$ is satisfied by $L(Q_{\leqslant\kappa}^{\mathrm{cf}})$.

6.10. Theorem. *$\mathscr{L}_\kappa^*$ has the Beth property.*

Proof. Suppose $\phi(P, \bar{Q})$ is a Beth sentence, P a monadic predicate for simplicity (i.e., for every model $(A, \bar{Q})$ for at most one $P \subseteq A$, $(A, P, Q) \vDash \psi[P, Q]$). So

$$\psi(P^0, \bar{Q}) \cap P^0(c) \to (\phi(P^1, \bar{Q}) \to P^1(c)).$$

So by 6.9(2) there is $\theta(x, \bar{Q}) \in L(\mathrm{aa})(\bar{Q})$ which is an interpolant hence defines P (when it exists). (This repeats the proof INT $\to$ Beth). W.l.o.g., $\neg\phi(\{x : \psi(x, \bar{Q})\}, \bar{Q}) \to \neg\,\theta(y, \bar{Q})$. Clearly it suffices to prove that $\theta(c, \bar{Q}) \in \mathscr{L}_\kappa^*$.

Let $\mathfrak{A}$, $\mathfrak{B}$, χ, $N=(|N|,\bar{Q})$, λ be as in Definition 6.3, $\lambda\geq\beth_\lambda$. Then for $\mathscr{E}^{\kappa^+,\kappa}_{\chi,\lambda}$-majority of $A\in S_{\leq\lambda}(|\mathfrak{B}|)$, $(\mathfrak{B},\bar{Q})\restriction A$ is $\aleph_0$-strongly homogeneous.

Let $\bar{c}^*\in\mathfrak{B}$ be a finite sequence in which all parameters used in defining $|N|$ and Q_i^N (by $L_{\infty,\omega}(\mathrm{aa}_\chi)$-formulas, in $\mathfrak{B}$) appear.

Our problem is that maybe $\psi(P,\bar{Q})$ in $(N,\bar{Q})\restriction A$ has a solution, whereas in $(N,\bar{Q})$ it does not have (the other direction is easy). Now every automorphism of $(\mathfrak{B}\restriction A,\bar{c}^*)$ maps the $|N|, Q_l$ to themselves (as they are definable in $\mathfrak{B}$ by an $L_{\infty,\omega}$-formula with parameters $\subseteq\bar{c}^*$) hence maps P to itself (otherwise $\psi(P,\bar{Q})$ is not a Beth sentence). As $(\mathfrak{B}\restriction A,\bar{c}^*)$ is $\aleph_0$-strongly homogeneous, P is necessarily also defined in $\mathfrak{B}$ by an $L_{\infty,\omega}$-formula with $\bar{c}^*$ as parameter. The same formula defines a monadic relation P' on N. Now the number of such formulas is $\leq 2^{2^{\aleph_0}}$ (the number of complete n-types, $n<\omega$, for $\mathrm{Th}_{\mathscr{L}_\kappa}(N)$ is $\leq 2^{\aleph_0}$). As $\lambda\geq 2^{2^{\aleph_0}}$ and $\psi(P,\bar{Q})\in\mathscr{L}^*$, for $\mathscr{E}^{\kappa^+,\kappa}_{\chi,\lambda}$-majority of $A\in S_{\leq\lambda}(N)$ for every $L_{\infty,\omega}$-definable P, $(N,P,\bar{Q})\models_\chi\psi(P,\bar{Q})$ iff $(N,P,\bar{Q})\restriction A\models_\kappa\psi(P,\bar{Q})$, contradiction.

6.11. Conclusion. *Let $\kappa\geq 2^{\aleph_0}$.*

(1) *$L(Q^{\mathrm{cf}}_{\leq\kappa})^{\mathrm{Beth}}$ (the Beth closure of $L(Q^{\mathrm{cf}}_{\leq\kappa})$ is $\leq\mathscr{L}^*_\kappa$.*

(2) *$L(Q^{\mathrm{cf}}_{\leq\kappa})^{\mathrm{Beth}}$ is compact, $(L(Q^{\mathrm{cf}}_{\leq\kappa})^{\mathrm{Beth}}, L(\mathrm{aa}_\kappa))$ has the interpolation property and the super $\aleph_0$-homogeneity property (getting, in fact, an $\aleph_0$-saturated model) and trivially it has the Beth property.*

(3) *$L(Q^{\mathrm{cf}}_{\leq\kappa})^{\mathrm{Beth}}$ has the* PPP, *and*

(4) *$L(Q^{\mathrm{cf}}_{\leq\kappa})^{\mathrm{Beth}}$ does not have the interpolation property nor even the Δ-*INT *property.*

Proof. (1) By 6.6, $L(Q^{\mathrm{cf}}_{\leq\kappa})\leq\mathscr{L}^*_\kappa$, hence our conclusion follows by 6.10.

(2) Follows by (1) and corresponding claims on $(\mathscr{L}^*_\kappa, L(\mathrm{aa}_\kappa))$ in 6.7, 6.9(2), 6.9(3).

(3) We prove it by induction on n for $L(Q^{\mathrm{cf}}_{\leq\kappa})^{\mathrm{Beth}}_n$; using 2.3.

(4) It is enough to find N_1, N_2 such that

(a) $N_1\equiv_{L(Q^{\mathrm{cf}}_{\leq\kappa})}N_2$.

(b) For every n, only finitely many complete n-types (in $L(Q^{\mathrm{cf}}_{\leq\kappa})[\tau_{N_l}]$) are realized in N_l.

(c) Each N_l is $\aleph_0$-strongly homogeneous.

(d) N_1, N_2 belong to disjoint $\mathrm{PC}(L(Q^{\mathrm{cf}}_{\leq\kappa}))$ class.

By (b), (c) every $L(Q^{\mathrm{cf}}_{\leq\kappa})^{\mathrm{Beth}}$-formula is equivalent in N_l to an $L(Q^{\mathrm{cf}}_{\leq\kappa})$-formula (prove by induction on n for formula of $L(Q^{\mathrm{cf}}_{\leq\kappa})^{\mathrm{Beth}}_n$).

As those equivalence are the same for N_1 and N_2, N_1, N_2 are $L(Q^{\mathrm{cf}}_{\leq\kappa})^{\mathrm{Beth}}$-equivalent. Now (d) finishes the proof of the conclusion (we can omit (b) if we strengthen (a), by adding: even if we expand the N_l by all $L_{\infty,\omega}(Q^{\mathrm{cf}}_{\leq\kappa})$-definable relations). Let T_0 be the model completion of the theory of partial order. Now T_0 exists, has eliminations of quantifiers, and all its models are directed.

We can find a λ-strongly homogeneous, λ-saturated model M of T_0. Let

$\langle a_i : i < \lambda\rangle$ be an increasing sequence of membership of M, and for every $\mu < \lambda$ let

$$M_\mu = M \restriction \{b : (\exists i < \mu)\, b < a_i\}$$

Clearly each M_μ is $\aleph_0$-strongly homogeneous (even cf μ-strongly homogeneous) and $\aleph_0$-saturated and all M_μ are $L(Q^{\mathrm{cf}}_{\leqslant\kappa})$-equivalent and $\mathrm{Th}_{L(Q^{\mathrm{cf}}_{\leqslant\kappa})}(M_\mu)$ has eliminations of quantifiers. As for (d):

Fact. *$K_l = \{(A, <) : (A, <)$ is a directed partial order, and there is an increasing sequence $\langle a_i : i < \delta\rangle$, $[\mathrm{cf}\, \delta \leqslant \kappa$ iff $l = 0]$ and $(\forall a \in A)(\exists i < \delta)\, a \leqslant a_i\}$ is a $\mathrm{PC}(L(Q^{\mathrm{cf}}_{\leqslant\kappa}))$-class and they are disjoint.*

This is enough to contradict INT. To contradict Δ-INT use partial orders with no three pairwise incomparable elements.

6.12. Claim. (1) *$\mathcal{L}^*_\kappa$ has the Δ-*INT *property.*

(2) 6.11 *holds for $L(Q^{\mathrm{cf}}_{\leqslant\kappa})^{\Delta\text{-Beth}}$ (except on* INT.)

Proof. (1) Like the proof of 6.10 but easier.

PART II. COMPACTNESS VERSUS OCCURRENCE

In the first section we give more restrictions on the compactness spectrum of a logic.

In the second section we introduce some solutions to $?/[\lambda]$-compact $= \lambda \leqslant$ occurrence no./λ-compact. We then prove that if a logic $\mathcal{L}$ has the amalgamation property, then $\mathcal{L}$ is $[\lambda]$-compact iff λ is an occurrence cardinal of $\mathcal{L}$ (for regular λ).

In the third section we prove that if $0 < \kappa_0 < \kappa_1$ are compact cardinals, then there is an $[\omega]$-compact, non-compact logic having the amalgamation property.

1. Compactness revisited

1.1. Observation. *The following are equivalent:*

(1) *$\mathcal{L}$ is $[\lambda]$-compact for every regular $\lambda \geqslant \lambda_0$ (i.e., is eventually compact).*

(2) *$\mathcal{L}$ is $[\lambda]$-compact for every $\lambda \geqslant \lambda_0$.*

(3) *$\mathcal{L}$ is $[\infty, \lambda_0]$-compact.*

(4) *$\mathcal{L}$ is (∞, λ_0)-compact.*

Remark. See 4.3.6(ii) of [5]. For original references to the facts we shall use, see [5].

Proof. (2) $\Rightarrow$ (3) by [5, 1.1.7(1)], (3) $\Leftrightarrow$ (4) by [5, 1.1.7(ii)], (3) $\Rightarrow$ (1) by [5, 1.1.6(i)(ii)]. Assume (1) and let us prove (2). If $\lambda = \lambda_0$ is regular this is clear by (1); if λ is singular, by (1), L is $[\lambda^+]$-compact, hence by [5, 1.4.9(ii)] there is a uniform ultrafilter F on λ^+ which belongs to $\mathrm{UF}(\mathscr{L})$, so by 1.4.11(i) of [5] F is (λ^+, λ)-regular, hence by [5, 1.4.9(i)] L is $[\lambda^+, \lambda]$-compact hence is $[\lambda]$-compact.

1.2. Conclusion. (1) In [5, 4.3.8] we can conclude $\mathscr{L}$ is (∞, ω_α)-compact.

(2) In [5, 3.3.1] we can conclude L is (∞, λ)-compact.

The following lemma, by [5, 2.1], can be rephrased as a pure set-theoretic lemma on ultrafilters.

1.3. Lemma. *Suppose λ is singular, $\kappa = \mathrm{cf}\,\lambda$, $\mathscr{L}$ is $[\lambda]$-compact but not $[\kappa]$-compact and $\lambda = \sum_{i<\kappa} \lambda_i$, $\lambda_i < \lambda$, each λ_i regular.*

Suppose μ is a regular cardinal $>\lambda$.

(1) *Suppose there are $f_\alpha \in \prod_{i<\kappa} \lambda_i$ (for $\alpha < \mu$) such that for every $\alpha < \beta$, $f_\alpha <_{\mathscr{D}_{<\kappa}} f_\beta$ (see below) and for every $f \in \prod_{i<\kappa} \lambda_i$ for some α, $f <_{\mathscr{D}_{<\kappa}} f_\alpha$. Then $\mathscr{L}$ is $[\mu]$-compact.*

(2) *If there are $f_\alpha \in \prod_{i<\kappa} \lambda_i$ $(\alpha < \mu)$ such that $f_\alpha <_{\mathscr{D}_\kappa} f_\beta$ for $\alpha < \beta < \mu$ and for every $f \in \prod_{i<\kappa} \lambda_i$ for some $\alpha < \mu$, $f <_{\mathscr{D}_\kappa} f_\alpha$ and χ is a regular cardinal $<\kappa$ (so $\kappa > \aleph_0$), then $\mathscr{L}$ is $[\mu]$-compact or $[\chi]$-compact where*

1.4. Notation (1) $\mathscr{D}_{<\kappa} = \{A \subseteq \kappa : |\kappa - A| < \kappa\}$. $\mathscr{D}_\kappa$ is the filter of closed unbounded subsets of λ.

(2) For $f, g \in \prod_{i<\kappa}, \lambda_i$, $g <_{\mathscr{D}} g$ if $\{i < \kappa : f(i) < g(i)\} \in \mathscr{D}$ (more formally we should write $f(\lambda_i) < g(\lambda_i)$.

1.5. Remark. (1) So really the hypothesis of 1.3(1),(2) above speaks about the cofinality of $\prod_{i<\kappa} \lambda_i/\mathscr{D}$. See [27]; [28, Ch. XIII, §5, §6]. We can get cases where the hypothesis of 1.3(1) or (2) holds for some λ_i (given λ, μ). E.g.

1.6. Lemma. (1) *Suppose λ is singular, $\kappa = \mathrm{cf}\,\lambda$, $(\forall \chi < \lambda)\ \chi^\kappa < \lambda$, κ is uncountable, $\mu = \lambda^+$. Then we can find λ_i $(i < \kappa)$, f_α $(\alpha < \mu)$ as in* 1.3(2). [Let $\langle \lambda_i^0 : i < \kappa \rangle$ be increasing continuous, $\sum \lambda_i^0 = \lambda$; $\lambda_i = (\lambda_i^0)^+$, $f_\alpha \in \prod_{i<\alpha} \lambda_i$, $f_\alpha <_{\mathscr{D}_\kappa} f_\beta$ for $\alpha < \beta$.]

(2) *Suppose λ is singular, $\kappa = \mathrm{cf}\,\lambda$, $\mu = \lambda^+$, $(\forall \lambda_0 < \lambda)(\lambda_0^\kappa < \lambda)$. Then we can find λ_i $(i < \kappa)$ and f_α $(\alpha < \lambda^+)$ as required in* 1.3(1). [See [28, Ch. XIII, §5].]

1.7. Remark. In 1.3 we can use other filters (instead $\mathscr{D}_{<\kappa}$); we can use a filter $\mathscr{D}$ on κ if

(*) $(\kappa, \mathscr{D}, \{A : \kappa - A \notin \mathscr{D}\})$ is $\mathscr{L}$-characterizable (see 2.3) (and $\mathscr{D}$ extends $\mathscr{D}_{<\kappa}$).

By [28] in case (2) of 1.6, for every regular $\mu, \lambda \leq \mu \leq \lambda^\kappa$ $[\kappa > \aleph_0]$ for some [normal] ultrafilter $\mathscr{D}$ on κ, there are f_α $(\alpha < \mu)$ as required. So if e.g. $2^\kappa = \kappa^+$, (*) will hold.

Proof of 1.3. Let λ' be a large enough regular cardinal, and N an expansion of $(H(\lambda'), \in)$ which $\mathscr{L}$-cofinably characterizes any cardinal $\geqslant\mu$ which can be $\mathscr{L}$-cofinally characterized. In particular we shall assume the conclusion fails, i.e., μ is $\mathscr{L}$-cofinally characterizable.

So N has an elementary extension N^*, and there is $a^* \in N^*$ such that

$$N^* \models \text{“}a^* \text{ is a subset of } \lambda \text{ of power } <\lambda\text{”}.$$

but for every $i<\lambda$, $N^* \models$ “$i \in a^*$”.

As $\mathscr{L}$ is not $[\kappa]$-compact, N^* $\mathscr{L}$-cofinally characterized κ, hence $\{i : i<\kappa\}$ is unbounded in κ^{N^*}, hence for some $i(*)<\kappa$

$$N^* \models \text{“}a^* \text{ has power } <\lambda_{i(*)}\text{”}.$$

Now there is $f^* \in N^*$ s.t.

$$N^* \models \text{“}f^* \in \prod_{i<\kappa} \lambda_i \quad \text{and for } i(*) \leqslant i < \kappa, f^*(i) = \sup(a^* \cap \lambda_i)\text{”}$$

(clearly f^* exists — the sentence asserting it is satisfied by N).

Hence $N^* \models$ “for some $\alpha < \mu$, $f^* <_{\mathscr{D}} f_\alpha$” (where $\mathscr{D} = \mathscr{D}_{<\kappa}$ or $\mathscr{D} = \mathscr{D}_\kappa$ according to the case). As N $\mathscr{L}$-cofinally characterizes μ, and

$$N^* \models \text{“}\{f_\alpha : \alpha < \mu\rangle \text{ is } <_{\mathscr{D}}\text{-increasing, } <_{\mathscr{D}} \text{ a partial order}$$
$$\text{and for every } f \in \prod_{i<\kappa} \lambda_i \text{ for some } \alpha, f <_{\mathscr{D}} f_\alpha\text{”}$$

there is $\alpha < \mu$ s.t.

$$N^* \models \text{“}f^* <_{\mathscr{D}} f^*_\alpha, \text{ i.e., } \{i < \kappa : f^*(i) < f_\alpha(i)\} \in \mathscr{D}\text{”}.$$

As $f_\alpha \in N$, and by the choice of a^*, as we may increase $i(*)$ w.l.o.g., for every i

$$i(*) \leqslant i < \kappa, \quad N^* \models \text{“}f_\alpha(i) \in a^*\text{”} \quad \text{hence} \quad N^* \models \text{“}f^*(i) > f_\alpha(i)\text{”}.$$

We can conclude that for some $b \in N^*$, $N^* \models$ “$b \in \mathscr{D}$ and $i \notin \mathscr{D}$” for every $i<\kappa$ (remember $N^* \models$ “every cobounded subset of κ belong to $\mathscr{D}$”).

So really the requirement from 1.7 suffices. Why it holds: For $\mathscr{D} = \mathscr{D}_{<\kappa}$, this is very easy: $\{i : i<\kappa\}$ is an unbounded subset of κ $(\kappa^{N^*}, <^{N^*})$ (as $\mathscr{L}$ is not $[\kappa]$-compact). For $\mathscr{D} = \mathscr{D}_\kappa$ we use the failure of $[\kappa]$-compactness and of $[\chi]$-compactness of $\mathscr{L}$.

2. Amalgamation implies [λ]-compactness for λ an occurrence cardinal

We generalize here the main result of Makowsky–Shelah [14]. For this we analyze more closely the occurrence cardinal; just as previously compactness was sliced to $[\lambda]$-compactness we suggest here some interpretation to “$[\lambda]$-occurrence cardinals”.

We can generalize this to the context of abstract elementary submodel relations, as in Makowsky–Mundici [6]. For this $[\lambda, \kappa]$-compactness will be reinterpreted as "$(\lambda, S_{<\kappa}(\lambda), \{A \subseteq \lambda : A \neq \lambda\})$-characterization" (see Definition 2.3) and $[\lambda]$-occurrence can be interpreted by "if $M \subseteq N$, $\bigcup_{i<\lambda} A_i \subseteq |M|$ and for every $S \subseteq \lambda$, $|S| < \lambda$ the models M, N are isomorphic over $\bigcup_{i\in S} A_i$, then M, N are isomorphic over $\bigcup_{i<\lambda} A_i$" (it is more reasonable to define $[\lambda]$-occurrence by "if $\tau = \bigcup_{i<\lambda} \tau_i$, M, N τ-models, $M \restriction \bigcup_{i\in S} \tau_i \equiv N \restriction \bigcup_{i\in S} \tau_i$ for every $S \subseteq \lambda$, $|S| < \lambda$, then $M \equiv N$" where $\equiv$ is either a basic relation (which we axiomatize or interpret as having a common elementary extension).

2.1. Definition. (1) A cardinal λ is an *occurrence number* (or cardinal) of (the logic) $\mathscr{L}$ if for every sentence $\psi = \psi(\ldots, R_t, \ldots)_{t\in J}$, $J \subseteq \lambda \times I$ for any model M and relations R_t^0, R_t^1 $(t \in J)$ over it (with the right arity, etc.) for some $S \subseteq \lambda$, $|S| < \lambda$, if $S \subseteq T \subseteq \lambda$, $|T| < \lambda$ then

$$(*) \qquad M \models \psi(\ldots, R_t^0, \ldots)_{t\in J} \equiv \psi(\ldots, R_t^0, \ldots, R_s^1, \ldots)_{\substack{t\in J\cap T\times I \\ s\in J\cap(\lambda - T)\times I}} .$$

(2) We call λ a *strong* occurrence number of $\mathscr{L}$, if S above depends on $\psi(\ldots, R_t, \ldots)$ (and not on M and the relation R_t^l interpreting the predicates R_t).

(3) We call λ a *weak* occurrence number of L, if for every ψ, J, I, M, R_t^l $(l = 0, 1, t \in J)$ as in (1), for every $S \subseteq \lambda$, $|S| < \lambda$, there is $T, S \subseteq T \subseteq \lambda$, $|T| < \lambda$, satisfying $(*)$ of (1).

Note

2.2. Fact. (1) *The following implication holds*: "$\mathscr{L}$ *is* $[\lambda]$-*compact*" $\Rightarrow$ "λ *is a strong occurrence number of* $\mathscr{L}$" $\Rightarrow$ "λ *is an occurrence number of* $\mathscr{L}$" $\Rightarrow$ "λ *is a weak occurrence number of* $\mathscr{L}$".

(2) $\mathrm{oc}(\mathscr{L}) = \mathrm{Min}\{\lambda :$ *every* $\mu \geq \lambda$ *is a strong occurrence number of* $\mathscr{L}\}$.

2.3. Definition. (1) For families $\mathscr{P}_l \subseteq \mathscr{P}(\lambda)$ we say $(\lambda, \mathscr{P}_1, \mathscr{P}_2)$ is $\mathscr{L}$-*characterizable* if: *for some* model M expanding some $(H(\mu), \in)$ $(\mu > 2^\lambda)$ *for every* $\mathscr{L}$-elementary extension N of M, and a s.t. $N \models$ "$a \in \mathscr{P}_1$" the set $\{\alpha < \lambda : N \models \alpha \in A\}$ belongs to $\mathscr{P}_2$.

(2) Under such circumstances we say N $\mathscr{L}$-*characterizes* $(\lambda, \mathscr{P}_1, \mathscr{P}_2)$.

(3) If $\mathscr{P} = \mathscr{P}_1 = \mathscr{P}_2$ (the case which interests us) we write $(\lambda, \mathscr{P})$ instead of $(\lambda, \mathscr{P}_1, \mathscr{P}_2)$.

2.4. Definition. We say $(\lambda, \mathscr{P})$ (where $\mathscr{P} \subseteq \mathscr{P}(\lambda)$) is $\mathscr{L}$-*oc-characterizable* if for some ψ, I, S, M, R_t^l the following hold, where $J \subseteq \lambda \times I$, $\psi = \psi(R_t)_{t\in J}$ and R_t^l are relations on M. For every $A \subseteq \lambda$ we define $R_{\langle\alpha,s\rangle}^A$ by: $R_{\langle\alpha,s\rangle}^A$ is $R_{\langle\alpha,s\rangle}^0$ if $\alpha \in A$ and $R_{\langle\alpha,s\rangle}^1$ if $\alpha \notin A$. Then

$$\mathscr{P} = \{A \subseteq \lambda : (M, R_t^l)_{t\in J, l=1,2} \models \psi(R_t^A)_{t\in J}\}.$$

2.5. Claim. *If $(\lambda, \mathscr{P})$ is $\mathscr{L}$-oc-characterizable, then $(\lambda, \mathscr{P})$ is $\mathscr{L}$-characterizable.*

2.6. Fact. (1) *Suppose $(\lambda, \mathscr{P})$ is $\mathscr{L}$-characterizable, F a compact ultrafilter of $\mathscr{L}$ on μ. Then $\mathscr{P}(\lambda)$ and $\mathscr{P}(\lambda)-R$ are F-closed, i.e., if $A_\alpha \subseteq \lambda$ (for $\alpha < \mu$), then:*

$$\mathrm{Lim}_F\langle A_\alpha : \alpha < \mu\rangle = \{i < \lambda : \{\alpha < \mu : i \in A_\alpha\} \in F\}$$

belongs to $\mathscr{P}$ iff $\{\alpha \in \mu : A_\alpha \in \mathscr{P}\}$ belongs to F.

(2) *If $(\lambda, \mathscr{P}_1, \mathscr{P}_2)$ is $\mathscr{L}$-characterizable, F a compact ultrafilter of $\mathscr{L}$ on μ, then $\{\alpha < \mu : A_\alpha \in \mathscr{P}_1\} \in F \Rightarrow (\mathrm{Lim}_F\langle A_\alpha : \alpha < \mu\rangle) \in \mathscr{P}_2$.*

(3) *The converses of* (1), (2) *holds; $(\lambda, \mathscr{P}_1, \mathscr{P}_2)$ is $\mathscr{L}$-characterizable iff*

$$\mathscr{P}_2 \text{ includes } \mathrm{cl}_{\mathscr{L}}(\mathscr{P}_1) = \{\mathrm{Lim}_F\langle A_\alpha : \alpha < \mu\rangle : A_\alpha \in \mathscr{P}, F \in \mathrm{UF}(\mathscr{L})\}.$$

(4) *$\mathscr{L}$ is $[\lambda]$-compact iff $(\lambda, \{A \subseteq \lambda : |A| < \lambda\})$ is not $\mathscr{L}$-characterizable.*

Proof. (4) If $\mathscr{L}$ is not $[\lambda]$-compact, there are sets of sentences Γ_i $(i < \lambda)$ (from $\mathscr{L}$) such that $\bigcup_{i<\lambda} \Gamma_i$ has no model, but $\bigcup_{i \in A} \Gamma_i$ has a model for every $A \subseteq \lambda$ of cardinality $< \lambda$; let M_A be a model of $\bigcup_{i \in A} \Gamma_i$. Suppose M expands $(H(\mu), \in)$, μ large enough, N is an $\mathscr{L}$-elementary extension of M, $N \vDash$ "$a \subseteq \lambda$" and $A \stackrel{\text{def}}{=} \{i < \lambda : N \vDash \text{“}i \in a\text{”}\}$ has power λ.

In $H(\mu)$ we can find Γ_i^1 $(i \in A)$, a set of $\mathscr{L}$-sentences such that $\bigcup_{i \in A} \Gamma_i^1$ has no model, and we can find M'_B, a model of $\bigcup_{i \in B} \Gamma_i^1$ for $B \subseteq A$, $|B| < \lambda$. The function M'_B (i.e. $B \mapsto M'_B$) is in $H(\mu)$, and so $M'_{a \cap \{i : N \vDash i \in A\}}$ is well defined and is a model of $\bigcup_{i \in A} \Gamma_i^1$ (check for each sentence). Contradiction.

If $\mathscr{L}$ is $[\lambda]$-compact, the proof is easy too.

2.7. Claim. *For any ultrafilter F on μ and a logic $\mathscr{L}$* (1) $\Rightarrow$ (2) *where:*

(1) *For every $\mathscr{L}$-oc-characterizable $(\lambda, \mathscr{P})$, and $A_\alpha \subseteq \lambda$ $(\alpha < \mu)$, $\{A_\alpha : A_\alpha \in \mathscr{P}, \alpha < \mu\} \in F$ iff $(\mathrm{Lim}_F\langle A_\alpha : \alpha < \mu\rangle) \in \mathscr{P}$.*

(2) $\mathrm{Min}\{|A| : A \in F\}$ *is an occurrence cardinal of $\mathscr{L}$.*

Proof. W.l.o.g. $\mu = \mathrm{Min}\{|A| : A \in F\}$ as if $B \in F$, $(\forall A \in F)(|B| \leq |A|)$ then for our purposes F and $F \restriction B = F \cap \mathscr{P}(B)$ are equivalent.

Let us check (1) $\Rightarrow$ (2). Suppose $\psi = \psi(R_t)$, $t \in \mu \times I$, M, R_t^l $(t \in J, l = 0, 1)$ are as in Definition 2.1(1). Suppose (2) fails this instance. Then for every $S \subseteq \lambda$, $|S| < \mu$ there is T, $S \subseteq T \subseteq \mu$, $|T| < \mu$ and

$$(*)_T \qquad M \vDash \psi(\ldots, R_t^0, \ldots)_t \equiv \neg\psi(\ldots, R_t^0, \ldots, R_s^1, \ldots)_{\substack{t \in J \cap (T \times I) \\ s \in J \cap ((\lambda - T) \times I}} .$$

Hence we can find $A_\alpha \subseteq \mu$, $|A_\alpha| < \mu$ for $\alpha < \mathrm{cf}\,\mu$, such that $A_\alpha \subseteq A_\beta$ for $\alpha < \beta$ and $(*)_{A_\alpha}$ holds. Now (letting μ be regular for notational simplicity) and by (1)

$$M \vDash \psi[\ldots, R_t^0, \ldots] \quad \text{iff} \quad \{\alpha : M \vDash \psi(\ldots, R_s^0, \ldots, R_t^1)_{\substack{s \in J \cap A_\alpha \times I \\ t \in J \cap ((\mu - A_\alpha) \times I)}}\} \in F.$$

But the last set is empty.

2.8. Main Theorem. *Suppose $\mathscr{L}$ has the amalgamation property. Then for every weak occurrence cardinal λ of $\mathscr{L}$, $\mathscr{L}$ is $[\lambda]$-compact provided that λ is regular.*

Remark. So we have considered various compactness demands. (We consider occurrence restriction as very weak compactness demands.) By the theorem they coincide for logics with the amalgamation property.

Proof. We assume the conclusion fails.

Part A. There is a model M, expanding some $(H(\lambda'), \epsilon)$, $2^\lambda < \lambda'$, which $\mathscr{L}$-characterize $(\lambda, \{A \subseteq \lambda : |A| < \lambda\})$. Let λ be an individual constant of M.

Part B. We now define a class $K(M)$. A model of $K(M)$ has the form $\mathfrak{A} = \langle A_0, A_1, A_2; \bar{Q}, R, F\rangle$ s.t.

(K1) $(A_2, \bar{Q})$ is M.

(K2) $R \subseteq A_1$ is a one-place relation.

(K3) F is a partial two-place function, with $F(x, y)$ is defined iff $x \in A_1$, $y \in A_2$, $M \vDash$ "$y < \lambda$", and $F(x, y) \in A_0$ when defined.

(K4) For every $x \in A_1$, $\lambda > i \neq j \Rightarrow F(x, i) \neq F(x, j)$.

(K5) For every $x, y \in A_1$

$$\begin{aligned} x = y \quad &\text{iff} \quad \{F(x, i) : i < \lambda\} \cap \{F(y, i) : i < \lambda\} \text{ has power } \geqslant \lambda \\ &\text{iff} \quad \{i < \lambda : F(x, i) \neq F(y, i)\} \text{ has power } < \lambda. \end{aligned}$$

We use $\mathfrak{A}$, $\mathfrak{B}$ to denote members of $K(M)$.

Part C. We say that $\mathfrak{A} \subseteq_K \mathfrak{B}$ if $\mathfrak{A} \subseteq \mathfrak{B}$ (i.e., $\mathfrak{A}$ is a submodel of $\mathfrak{B}$) and for every $x \in A_1^{\mathfrak{B}} - A_1^{\mathfrak{A}}$, $\{i < \lambda : F(x, i) \in A_0^{\mathfrak{A}}\}$ has power $< \lambda$.

For $c \in A_1^{\mathfrak{A}}$, $(\mathfrak{A} \in K(M))$, then let $\mathfrak{A}^{[c]}$ be a model equal to $\mathfrak{A}$ except for the relation R which satisfies:

if $\mathfrak{A} \vDash x \neq c \wedge x \in A_1^{\mathfrak{A}}$, then $x \in R^{\mathfrak{A}} \Leftrightarrow x \in R^{\mathfrak{A}^{[c]}}$,

if $\mathfrak{A} \vDash x = c$, then $x \in R^{\mathfrak{A}} \Leftrightarrow x \notin R^{\mathfrak{A}^{[c]}}$.

Part D. We say that $(\mathfrak{A}, \mathfrak{B}, c, C_i)_{i<\lambda}$ is a special sequence if:

(a) $\mathfrak{A} \subseteq \mathfrak{B}$, $c \in A_1^{\mathfrak{B}} - A_1^{\mathfrak{A}}$ and $\langle C_i : i < \lambda\rangle$ is a partition of $A_1^{\mathfrak{A}} \cup A_0^{\mathfrak{A}}$, $F(x, i) \in C_i$ for $i < \lambda$,

(b) for every $S \subseteq \lambda$, $|S| < \lambda$, there is an isomorphism g_S^0 $[g_S^1]$ from $\mathfrak{B}$ $[\mathfrak{B}^{[c]}]$ onto $\mathfrak{A}$, which is the identity on $A_2^{\mathfrak{A}} \cup \bigcup_{i \in S} C_i$.

For a while, we shall investigate special sequences, and draw the conclusion. Later we shall build such a sequence.

Part E. $\mathfrak{A}$ is an $\mathscr{L}$-elementary submodel of $\mathfrak{B}$. So let ψ be a sentence in the vocabulary $\tau_{\mathfrak{A}} + |\mathfrak{A}|$; and we should prove $\mathfrak{A} \vDash \psi \Rightarrow \mathfrak{B} \vDash \psi$. For clarity we explicate the dependence of ψ on the elements of $A_0^{\mathfrak{A}} \cup A_1^{\mathfrak{A}}$ and suppress the rest (in the notation). Let $C_i = \{d_{i,\alpha}^0 : \alpha < \alpha_i\}$ (with not repetition); $d_{i,\alpha}^1 = d_{0,0}^0$, and for $S \subseteq \lambda$ let

$$d_{i,\alpha}^S = \begin{cases} d_{i,\alpha}^0, & i \in S, \\ d_{i,\alpha}^1, & i \notin S. \end{cases}$$

Now we let $\psi=\psi(\ldots,d^0_{i,\alpha},\ldots)_{i<\lambda,\alpha<\alpha_i}$. For $S\subseteq\lambda$, $|S|<\lambda$, applying g^0_S (see (b) in Part D):

$$\mathfrak{A}\models\psi(\ldots,d^S_{i,\alpha},\ldots)\Leftrightarrow\mathfrak{B}\models\psi(\ldots,d^S_{i,\alpha},\ldots)$$

We now want to find $S\subseteq\lambda$, $|S|<\lambda$, s.t.

$$\mathfrak{A}\models\psi(\ldots,d^S_{1,\alpha},\ldots)\equiv\psi(\ldots,d^0_{i,\alpha},\ldots),$$
$$\mathfrak{B}\models\psi(\ldots,d^S_{i,\alpha},\ldots)\equiv\psi(\ldots,d^0_{i,\alpha},\ldots).$$

At first glance the definition of the weak occurrence number guarantees the existence of an S satisfying each one of those demands, but why both? As we can use conjunction: let $\phi_0,\phi_1\in\{\psi,\neg\psi\}$, suppose

$$\mathfrak{A}\models\phi_0[\ldots,d^0_{i,\alpha},\ldots],\qquad\mathfrak{B}\models\phi_1[\ldots,d^0_{i,\alpha},\ldots],$$

so (with changes of names) apply the definitions to the model $[\mathfrak{A},\mathfrak{B}]$, to the conjunction of those sentences.

Part F. $\mathfrak{A}$ is an $\mathscr{L}$-elementary submodel of $\mathfrak{B}^{[c]}$. Use g^1_S instead of g^0_S above.

Part G. The following diagram cannot be completed by $\mathscr{L}$-embedding, i.e., we cannot find $\mathfrak{A}^*$, h_0, h_1 like that.

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathfrak{B} \arrow[d, dashed, "h_0"] \\
\mathfrak{B}^c \arrow[r, dashed, "h_1"] & \mathfrak{A}^*
\end{tikzcd}$$

W.l.o.g., $h_0\restriction|\mathfrak{A}|=h_1\restriction|\mathfrak{A}|$ is the identity. Now we shall prove $h_0(c)$ and $h_1(c)$ are equal. If not, then

$$\mathfrak{A}^*\models\text{“}\{i<\lambda:F(h_0(c),i)=F(h_1(c),i)\}\text{ has power }<\lambda\text{”}.$$

By the definition of $K(M)$ this implies $\{i<\lambda:\mathfrak{A}^*\models F(h_0(c),i)=F(h_1(c),i)\}$ has power $<\lambda$; but we know (by (a) of Part D), $F(h_i(c),i)\in C_i\subseteq|\mathfrak{A}|$, hence $F(h_l(c),i)=F(c,i)$, hence that this set is λ itself. So a contradiction, hence $\mathfrak{A}^*\models h_0=h_1(c)$, hence

$$h_0(c)\in R^{\mathfrak{A}^*}\Leftrightarrow h_1(c)\in R^{\mathfrak{A}^*}$$

but $h_0(c)\in R^{\mathfrak{A}^*}\Leftrightarrow c\in R^{\mathfrak{B}}\Leftrightarrow c\notin R^{\mathfrak{B}^{[c]}}\Leftrightarrow h_1(c)\notin R^{\mathfrak{A}^*}$, a contradiction.

We finish the proof of the theorem (as we have assumed analgamation) modulo the construction of the special sequence.

We shall have $A^{\mathfrak{A}}_0\cup A^{\mathfrak{A}}_1=\lambda$, $C_i=\{i\}$. So clearly it is enough for (b) of Part D to have:

(b′) for every $S\subseteq\lambda$, $|S|<\lambda$, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{B}^{[c]}$ are isomorphic over S.

For regular λ it is enough to have

(b″) for every $\alpha<\lambda$, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{B}^{[c]}$ are isomorphic over α.

Part I. Construction (alternatively see Part J). We shall define by induction on $\alpha<\lambda$, models $\mathfrak{A}^\alpha$, $\mathfrak{B}^\alpha\in K$ with $c\in A^{\mathfrak{B}^\alpha}_1-A^{\mathfrak{A}^\alpha}_1$, functions $g^l_{\alpha,\beta}$ $(\beta<\alpha)$ such that

(1) $\mathfrak{A}^\alpha$, $\mathfrak{B}^\alpha$, c satisfy (a) and (b) of Part D, except that $F(x, i)$ is defined for $i<\alpha$ only.

(2) $A_0^{\mathfrak{B}^\alpha} \cup A_1^{\mathfrak{B}^\alpha}$ has power $<\lambda$.

(3) $\langle \mathfrak{A}^{\mathfrak{B}} : \beta<\alpha\rangle$ is increasing continuous (by $\subseteq$ not $\subseteq_K$).

(4) If $\beta \leq i<\alpha$, $b, a \in A_1^{\mathfrak{B}^\beta}$; then $\mathfrak{B}^\beta \models F(a, i) \neq F(b, i)$ (if $a \neq b$).

(5) $g^0_{\alpha,\beta}$ is a partial isomorphism from $\mathfrak{B}^\alpha$ into $\mathfrak{A}^\alpha$, hence if $i<\alpha, x \in A_1^{\mathfrak{B}^\alpha} \cap \operatorname{dom} g^0_{\alpha,\beta}$ then $(F(x, i) = F(g^0_{\alpha,\beta}(x, i)))$.

(6) $g^1_{\alpha,\beta}$ is a partial isomorphism from $(\mathfrak{B}^\alpha)^{[c]}$ into $\mathfrak{A}^\alpha$, hence if $i<\alpha$. $x \in A_1^{\mathfrak{B}^\alpha} \cap \operatorname{Dom} g^1_{\alpha,\beta}$ then $(F(x, i)) = F(g^1_{\alpha|\beta}(x), i)$.

(7) $g^l_{\alpha,\beta}$ is the identity over $|\mathfrak{A}^\beta|$.

(8) For $\beta \leq \alpha(0) \leq \alpha(1)$, $g^l_{\alpha(0),\beta} \subseteq g^l_{\alpha(1),\beta}$.

(9) For every $l=0, 1, \beta \leq \alpha < \lambda$ for some $\gamma \geq \alpha$, $|\mathfrak{B}^\alpha| \subseteq \operatorname{Dom} g^l_{\gamma,\beta}$.

(10) For every $l=0, 1, \beta \leq \alpha<\lambda$, for some $\gamma \geq \alpha$, $|\mathfrak{A}^\alpha| \subseteq \operatorname{Rang} g^l_{\gamma,\beta}$.

There is no problem in the proof.

3. A strange logic with the JEP

In this section we give an $[\omega]$-compact logic satisfying the JEP, and which is stronger than first-order logic. This contradicts previous hopes. Really if λ is a compact cardinal, $\mathcal{D}$ a family of ultrafilters on cardinals $<\lambda$ we can define $L_{\lambda,\lambda}/\mathcal{D}$ as we defined $\mathcal{L}'$ (in 3.1's proof) allowing $\bigwedge^E_{i<\mu}$ for any $E \in \mathcal{P}(\mu)$ s.t. E and $\mathcal{P}(\mu)-E$ are $\mathcal{D}$-closed [i.e., E is $\mathcal{D}$-closed if $A_i \in E$, $i<\chi<\lambda$, $\mathcal{D} \in \mathcal{D}$ an ultrafilter on χ implies $\lim_{\mathcal{D}} A_i \in E$]. By 3.4, if some non-uniform ultrafilter on ω belongs to $\mathcal{D}$, then we can w.l.o.g. restrict ourselves to E which are ultrafilters on some $\mu<\lambda$. In any case for $L_{\lambda,\lambda}/\mathcal{D}$ to satisfy JEP (hence AM) it suffices to prove the parallel of subclaim 3.2: if $\mu<\lambda$, $E \subseteq \mathcal{P}(\mu)$ is $\mathcal{D}$-closed, $\mu \notin E$, then for some $E_1 \subseteq \mathcal{P}(\mu)-E$, $\mu \in E_1$, and E_1, $\mathcal{P}(\mu)-E_1$ are $\mathcal{D}$-closed.

By Claim 3.4 if $\mathcal{L}$ is $[\omega]$-compact, the dependency of the sentence $\psi(\ldots, \bar{R}_i, \ldots)$ on the choice of the $\bar{R}_i$'s is a finite sum of $\aleph_1$-complete ultrafilters and singletons.

3.1. Theorem. *Suppose $\aleph_0<\kappa<\lambda$ and κ, λ are compact cardinals. Let the logic $\mathcal{L}$ be the following sublogic of $L_{\lambda,\lambda}$: the formulas are the closure of the atomic formulas by:*

$$\neg\psi_1, \quad \psi_1 \wedge \psi_2, \quad (\exists x_0, \ldots, x_i, \ldots)_{i<\mu}\psi \quad \text{when } \mu<\lambda, \quad \text{and} \quad \bigwedge_{i<\mu}^{\mathcal{D}} \psi_i$$

where $\mathcal{D}$ is a κ-complete ultrafilter on μ, $\mu<\lambda$, and $\bigwedge^{\mathcal{D}}_{i<\mu}\psi_i$ means $\{i : \psi_i \text{ holds}\} \in \mathcal{D}$ (i.e., $\bigvee_{A \in \mathcal{D}} \bigwedge_{i \in A} \psi_i$). Then $\mathcal{L}$ is $[\lambda, <\infty]$-compact, satisfies Łos theorem for any ultrafilter $\mathcal{D}$ on any $\mu<\kappa$ (hence is $[\omega]$-compact), has the JEP *(hence the amalgamation property) but is stronger than first-order logic.*

Proof. For a cardinal μ and family $E \subseteq \mathscr{P}(\mu)$ define

$$\bigwedge_{i<\mu}^{E} \psi_i = \bigvee_{A \in E} \left(\bigwedge_{i \in A} \psi_i \wedge \bigwedge_{i \in \lambda - A} \neg \psi_i \right).$$

We call E $(<\kappa)$-closed if for every $\chi < \kappa$ and ultrafilter $\mathscr{D}$ on χ and $A_i \subseteq \mu$ $(i < \chi)$

$$\{i : A_i \in E\} \in \mathscr{D}$$

implies

$$\lim_{\mathscr{D}} A_i = \{\alpha < \mu : \{i < \chi : \alpha \in A_i\} \in \mathscr{D}\} \in E.$$

We call E $(<\kappa)$-bi-closed if E, $\mathscr{P}(\mu) - E$ are $(<\kappa)$-closed. Clearly if E is a κ-complete ultrafilter, then E is $(<\kappa)$-bi-closed. Define a logic $\mathscr{L}'$ like $\mathscr{L}$ but we allow $\bigwedge_{i<\mu}^{E} \psi_i$ for every $(<\kappa)$-bi-closed E (for $\mu < \lambda$). We shall prove that $\mathscr{L}'$ is as required, and then it follows by Claim 3.4 that $\mathscr{L}$, $\mathscr{L}'$ are (essentially) equal.

A. *Fact.* $\mathscr{L}'$ is $[\lambda, <\infty]$-compact. This is so because $\mathscr{L}' \subseteq \mathscr{L}_{\lambda,\lambda}$.

B. *Fact.* $\mathscr{L}'$ satisfies Łos theorem for any ultrafilter $\mathscr{D}$ on $\chi < \kappa$. This follows by direct checking (the definition of $(<\kappa)$-closed is tailor-made for this).

C. *Fact.* $\mathscr{L}'$ has the JEP.

Let M_1, M_2 be $\mathscr{L}'$-equivalent. It is enough to show that $C\mathscr{D}_{\mathscr{L}'}(M_1) \cup C\mathscr{D}_{\mathscr{L}'}(M_2)$ has a model (where $C\mathscr{D}_{\mathscr{L}'}(M)$ is the complete $\mathscr{L}'$-theory of $(M, c)_{c \in |M|}$). By Fact A it is enough to show that if $\Gamma_l \subseteq C\mathscr{D}_{\mathscr{L}'}(M_l)$, $|\Gamma_l| < \lambda$ for $l = 1, 2$, then $\Gamma_1 \cup \Gamma_2$ has a model. W.l.o.g. $|M_1|$, $|M_2|$ are disjoint. Let $\Gamma_2 = \{\phi_i(\bar{a}) : i < \mu\}$ and

$$E_0 = \{A \subseteq \mu : \Gamma_1 \cup \{\phi_i^l : i \in A\} \text{ has a model}\}.$$

If $\mu \in E_0$ we finish, so assume $\mu \notin E_0$, and it is also clear that $\emptyset \in E_0$ (as M_1 is a model of Γ_1 if we expand it by suitable individual constants).

By Fact B, E is $(<\kappa)$-closed. We shall later prove

3.2. Subclaim. *If $E \subseteq \mathscr{P}(\mu)$ $(\mu < \lambda)$ is $(<\kappa)$-closed, $\mu \notin E$, then there is an $E_1 \subseteq \mathscr{P}(\mu) - E_0$, $\mu \in E_1$ such that E_1 is $(<\kappa)$-bi-closed.*

Now $M_2 \models \bigwedge_{i<\mu} \phi_i(\bar{a})$ (note that $\bigwedge_{i<\mu} \phi_i(\bar{x})$ is in $L_{\lambda,\lambda}$ but not necessarily in $\mathscr{L}'$). As $\mu \in E_1$, $M_2 \models \bigwedge_{i<\mu}^{E^1} \phi_i(\bar{a})$, hence $M_2 \models (\exists \bar{x})[\bigwedge_{i<\mu}^{E^1} \phi_i(\bar{x})]$ (as $|\Gamma_2| < \lambda$ w.l.o.g. $\bar{x}$ has length $<\lambda$). But $(\exists \bar{x})[\bigwedge_{i<\mu}^{E^1} \phi_i(\bar{x})]$ belongs to $\mathscr{L}'$, hence also M_1 satisfies this sentence, hence for some $\bar{b}$ from M_1, $M_1 \models \bigwedge_{i<\mu}^{E^1} \phi_i[\bar{b}]$, hence for some $A \in E^1$, $M_1 \models \bigwedge_{i \in A} \phi_i(\bar{b})$, so $\Gamma_1 \cup \{\phi_1(\bar{a}) : i \in A\}$ has a model, hence $A \in E$, contradicting "$E^1 \subseteq \mathscr{P}(\mu) - E$".

Proof of Subclaim 3.2. E^1 exists iff the following set of sentences in the $L_{\kappa,\kappa}$-propositional calculus has a model, (let p_A $(A \in \mathscr{P}(\mu))$ stand for the truth value of $A \in E^1$):

$$\neg p_A \quad (A \in E),$$

$$p_\mu,$$

$$p_C \equiv \bigvee_{B \in \mathscr{D}} \bigwedge_{i \in B} p_{A_i} \quad \text{when } C = \lim_{\mathscr{D}} A_i,\ A_i \in \mathscr{P}(\mu) \text{ for } i < \chi,\ \chi < \kappa.$$

As κ is compact, we can look at any subset of power $<\kappa$, so it involves $<\kappa$ A's and there is an equivalence relation E on μ with $<\kappa$ equivalence classes, such that we may consider only $A = \bigcup_{a \in A} a/E$. So we reduce the problem to the case $\mu < \kappa$. Now there is a finite $w \subseteq \mu$ s.t. $(\forall A \in E)\ w \nsubseteq A$ [because otherwise for any finite $w \subseteq \mu$, $A_w \in E$ s.t. $w \subseteq A_w$ let $I = \{w \subseteq \mu : w \text{ finite}\}$, $\mathscr{D}$ an ultrafilter on I s.t. $\{u \in I : w \subseteq u\} \in J$ for every w, then $\lim_{\mathscr{D}} A_w = \mu \notin E_0$ but $A_w \in E$, a contradiction]. Let

$$E_1 = \{A \subseteq \mu : w \subseteq A\}.$$

It is easy to check all the demands.

3.3. Claim. *Suppose $\mathscr{D}$ is a filter on λ (i.e., a dual to an ideal of the Boolean algebra $\mathscr{P}(\lambda)$) and suppose*

(*) *if $A_n \subseteq \lambda$, $A_n \notin \mathscr{D}$ for $n < \omega$ and* Lim A_n *exists, then it is not in $\mathscr{D}$.*

Then there is a partition of λ to finitely many sets, $\langle A_l : l < n \rangle$ for $l > n$, and an $\aleph_1$-complete ultrafilter $\mathscr{D}_l$ on A_l (A_l may be a singleton and then $\mathscr{D}_l = \{A_l\}$) s.t. $\mathscr{D} = \{B \subseteq \lambda : (\forall l < n)(B \cap A_l \in \mathscr{D}_l)\}$.

Proof. Let $I = \{\lambda - A : A \in \mathscr{D}\}$, so I is an ideal. We shall prove that $\mathscr{P}(\lambda)/I$ is finite. Otherwise the Boolean algebra $\mathscr{P}(\lambda)/I$ has infinitely many pairwise disjoint non-zero elements A_l/I $(l < \omega)$, i.e., $A_l \notin I$, $A_l \cap A_m \in I$ for $l \neq m$. As

$$A_l - \left(A_l - \bigcup_{m<l} A_m\right) \subseteq \bigcap_{m<l} (A_l \cap A_m) \in I,$$

w.l.o.g. $A_l \cap A_m = \emptyset$ for $l \neq m$. Now $\lambda - A_l \notin \mathscr{D}$, hence by (*), $\lim_l (\lambda - A_l) \notin \mathscr{D}$ but $\lim_l (\lambda - A_l) = \lambda$ (as every i belongs to at most one A_m), contradiction.

So $\mathscr{P}(\lambda)/I$ is finite; let A_l/I $(l < n)$ be its atoms, w.l.o.g. $\langle A_l : l < n \rangle$ is a partition of λ. So $I_l = I \cap \mathscr{P}(A_l)$ is a maximal ideal of $\mathscr{P}(A_l)$, $\mathscr{D}_l = \mathscr{P}(A_l) - I_l$ is an ultrafilter on A_l. Now $\mathscr{D}_l$ is $\aleph_1$-complete, otherwise there are $B_k \in I_l$ $(k < \omega)$ with $\bigcup_{k<\omega} B_k = A_l$. W.l.o.g. $B_k \subseteq B_{k+1}$. So

$$B'_k \stackrel{\text{def}}{=} B_k \cup \bigcup_{\substack{l \neq k \\ l < n}} A_l$$

is not in $\mathscr{D}$, but $\lim_{k<w} B'_\omega = \lambda$ again a contradiction to (*).

3.4. Claim. *If $\mathscr{L}$ is an $[\omega]$-compact logic, and $\psi = \psi(\ldots, \bar{R}_i, \ldots)_{i<\lambda}$ is a sentence* *of $\mathscr{L}$, then for some partition $A_0, \ldots, A_{n-1}$ of λ, and $\aleph_1$-complete ultrafilters $\mathscr{D}_l$ on A_l (maybe $A_l = \{i\}$, $D_l = \{A_l\}$), the following holds:*

(+) *If $\bar{R}_i$, $\bar{R}'_i$ (for $i > \lambda$) are sequences of relations of the right arity on B, and $\{i \in A_l : \bar{R}_i = \bar{R}_i\} \in \mathscr{D}_l$ for $l = 0, n-1$, then*

$$(B, \ldots, \bar{R}_i, \ldots) \models \psi \quad \textit{iff} \quad (B, \ldots, \bar{R}'_i, \ldots) \models \psi.$$

Proof. Let $\mathscr{D}$ be the family of $A \subseteq \lambda$ such that for any $\bar{R}_i$, $\bar{R}'_i$, B as in (+), if $A \subseteq \{i < \lambda : \bar{R}_i = \bar{R}'_i\}$, then

$$(B, \ldots, \bar{R}_i, \ldots) \models \psi \quad \text{iff} \quad (B, \ldots, \bar{R}'_i, \ldots) \models \psi.$$

Now if $A_0, A_1 \in \mathscr{D}$, then $A = A_0 \cap A_1 \in \mathscr{D}$ (for any B, $\bar{R}$, $\bar{R}'_i$ as above define $\bar{R}''_i$ as R_i if $i \in A_0$ and as R'_i if $i \in \lambda - A_0$ and apply $\mathscr{D}$'s definition). Also if $A_n \in \mathscr{P}(\lambda) - \mathscr{D}$, $A = \lim_n A_n$ (and its exists) we let B_n, $\bar{R}_{n,i}$, $\bar{R}'_{n,i}$ exemplify $A_n \notin \mathscr{D}$, i.e.,

$$(B_n, \ldots, \bar{R}_{n,i}, \ldots) \models \psi(\ldots, \bar{R}_{n,i}, \ldots),$$
$$(B_n, \ldots, \bar{R}'_{n,i}, \ldots) \models \neg\psi(\ldots, \bar{R}'_{n,i}, \ldots).$$

By the $[\omega]$-compactness of $\mathscr{L}$ we get easily a counterexample showing $A \notin \mathscr{D}$. So we can apply Claim 3.3.

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