ELSEVIER

Available online at www.sciencedirect.com
SCIENCE DIRECT®

Annals of Pure and Applied Logic 140 (2006) 60–71

ANNALS OF PURE AND APPLIED LOGIC

www.elsevier.com/locate/apal

Covering the Baire space by families which are not finitely dominating

Heike Mildenberger[a,*], Saharon Shelah[b,c], Boaz Tsaban[b]

[a] *Universität Wien, Institut für Formale Logik, Währinger Str. 25, 1090 Vienna, Austria*
[b] *Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Givat Ram, 91904 Jerusalem, Israel*
[c] *Mathematics Department, Rutgers University, 110 Frelinghuysen Road, NJ 08854-8019, USA*

Available online 19 October 2005

Abstract

It is consistent (relative to ZFC) that each union of $\max\{\mathfrak{b},\mathfrak{g}\}$ many families in the Baire space ${}^\omega\omega$ which are not finitely dominating is not dominating. In particular, it is consistent that for each nonprincipal ultrafilter $\mathcal{U}$, the cofinality of the reduced ultrapower ${}^\omega\omega/\mathcal{U}$ is greater than $\max\{\mathfrak{b},\mathfrak{g}\}$. The model is constructed by oracle chain condition forcing, to which we give a self-contained introduction.

MSC: 03E15; 03E17; 03E35; 03D65

Keywords: Finitely dominating families; Groupwise density number $\mathfrak{g}$; Unbounding number $\mathfrak{b}$; Cofinality of ultrapowers

1. Introduction

The undefined terminology used in this paper is as in [9,2]. A family $Y \subseteq {}^\omega\omega$ is *finitely dominating* if for each $g \in {}^\omega\omega$ there exist k and $f_1, \ldots, f_k \in Y$ such that $g(n) \leq \max\{f_1(n), \ldots, f_k(n)\}$ for all but finitely many n. The *additivity number* for classes $\mathfrak{Y} \subseteq \mathfrak{Z} \subseteq P({}^\omega\omega)$ with $\bigcup\mathfrak{Y} \notin \mathfrak{Z}$ is

$$\mathsf{add}(\mathfrak{Y},\mathfrak{Z}) = \min\left\{|\mathfrak{F}| : \mathfrak{F} \subseteq \mathfrak{Y} \text{ and } \bigcup\mathfrak{F} \notin \mathfrak{Z}\right\}.$$

Let $\mathfrak{D}$ (respectively, $\mathfrak{D}_{\mathrm{fin}}$) be the collection of all subsets of ${}^\omega\omega$ which are not dominating (respectively, finitely dominating). Define

$$\mathsf{cov}(\mathfrak{D}_{\mathrm{fin}}) = \min\left\{|\mathfrak{F}| : \mathfrak{F} \subseteq \mathfrak{D}_{\mathrm{fin}} \text{ and } \bigcup\mathfrak{F} = {}^\omega\omega\right\}.$$

It is easy to see that $\mathsf{add}(\mathfrak{D}_{\mathrm{fin}},\mathfrak{D}) = \mathsf{cov}(\mathfrak{D}_{\mathrm{fin}})$, so we will use this shorter notation.

* Corresponding author.
E-mail addresses: heike@logic.univie.ac.at (H. Mildenberger), shelah@math.huji.ac.il (S. Shelah), tsaban@math.huji.ac.il (B. Tsaban).
URL: http://www.cs.biu.ac.il/~tsaban (B. Tsaban).

0168-0072/$ - see front matter
doi:10.1016/j.apal.2005.09.008

In [8] it is pointed out that

$$\max\{\mathfrak{b}, \mathfrak{g}\} \leq \mathsf{cov}(\mathfrak{D}_{\text{fin}}),$$

the inequality $\mathfrak{b} \leq \mathsf{cov}(\mathfrak{D}_{\text{fin}})$ being immediate from the definitions, and the inequality $\mathfrak{g} \leq \mathsf{cov}(\mathfrak{D}_{\text{fin}})$ having been implicitly proved in [5, Theorem 2.2]. (For the reader's convenience, we give a short proof for this in Corollary 2.3.) In [8] it is shown that in all "standard" forcing extensions (e.g., those appearing in [2, Section 11]), equality holds. It is conjectured in [8] that this equality is not provable. We prove this conjecture. In fact, we prove a stronger result: Let $\mathcal{M}$ denote the ideal of meager sets of real numbers.

Theorem 1.1. *It is consistent (relative to ZFC) that* $\aleph_1 = \mathsf{non}(\mathcal{M}) = \mathfrak{g} < \mathsf{cov}(\mathfrak{D}_{\text{fin}}) = \mathsf{cov}(\mathcal{M}) = \mathfrak{c} = \aleph_2$.

The statement of Theorem 1.1 determines the values of almost all standard cardinal characteristics of the continuum in the model witnessing it: If $\mathcal{N}$ is the ideal of null sets of real numbers, then by provable inequalities (see [9,2]), we have that $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{h}$, $\mathsf{add}(\mathcal{N})$, $\mathsf{add}(\mathcal{M})$, $\mathfrak{b}$, $\mathfrak{s}$, $\mathsf{cov}(\mathcal{N})$, and $\mathsf{non}(\mathcal{M})$ are all equal to $\aleph_1$, and $\mathsf{cov}(\mathcal{M})$, $\mathsf{non}(\mathcal{N})$, $\mathfrak{r}$, $\mathfrak{d}$, $\mathfrak{u}$, $\mathfrak{i}$, $\mathsf{cof}(\mathcal{M})$, and $\mathsf{cof}(\mathcal{N})$ are all equal to $\aleph_2$ in this model.

In [8] it is shown that for each nonprincipal ultrafilter $\mathcal{U}$ on ω, $\mathsf{cov}(\mathfrak{D}_{\text{fin}}) \leq \mathsf{cof}({}^{\omega}\omega/\mathcal{U})$.

Corollary 1.2. *It is consistent (relative to ZFC) that for each nonprincipal ultrafilter $\mathcal{U}$ on ω,* $\max\{\mathfrak{b}, \mathfrak{g}\} < \mathsf{cof}({}^{\omega}\omega/\mathcal{U})$.

This corollary partially extends the closely related Theorems 3.1 and 3.2 of [7], which are proved using the same machinery: Oracle chain condition forcing.

2. Making $\mathsf{cov}(\mathfrak{D}_{\text{fin}})$ and $\mathsf{cov}(\mathcal{M})$ large

From now on, by *ultrafilter* we always mean a nonprincipal ultrafilter on ω. We will use the following convenient characterization. For functions $f, g \in {}^{\omega}\omega$ and an ultrafilter $\mathcal{U}$ we write $f \leq_{\mathcal{U}} g$ for $\{n : f(n) \leq g(n)\} \in \mathcal{U}$.

Lemma 2.1 ([8]). *For each cardinal number κ, the following are equivalent:*

(1) $\kappa < \mathsf{cov}(\mathfrak{D}_{\text{fin}})$;
(2) *For each κ-sequence $\langle(\mathcal{U}_\alpha, g_\alpha) : \alpha < \kappa\rangle$ with each $\mathcal{U}_\alpha$ an ultrafilter and each $g_\alpha \in {}^{\omega}\omega$ there exists $g \in {}^{\omega}\omega$ such that for each $\alpha < \kappa$, $g_\alpha \leq_{\mathcal{U}_\alpha} g$.*

We first show how this characterization easily implies an assertion made in the introduction.

Definition 2.2. For $A \in [\omega]^{\omega}$, define the function $A^+ \in {}^{\omega}\omega$ by $A^+(n) = \min\{k \in A : n < k\}$ for all n.

Corollary 2.3 ([5]). $\mathfrak{g} \leq \mathsf{cov}(\mathfrak{D}_{\text{fin}})$.

Proof. We use Lemma 2.1. Assume that $\kappa < \mathfrak{g}$, and $(\mathcal{U}_\alpha, g_\alpha)$, $\alpha < \kappa$, are given with each $\mathcal{U}_\alpha$ an ultrafilter and each $g_\alpha \in {}^{\omega}\omega$. We must show that there exists $g \in {}^{\omega}\omega$ such that for each $\alpha < \kappa$, $g_\alpha \leq_{\mathcal{U}_\alpha} g$. We will use the following "morphism".

Lemma 2.4. *For each $f \in {}^{\omega}\omega$ and each ultrafilter $\mathcal{U}$,*

$$\mathcal{G}_{\mathcal{U},f} = \{A \in [\omega]^{\omega} : f \leq_{\mathcal{U}} A^+\}$$

is groupwise dense.

Proof. Clearly, $\mathcal{G}_{\mathcal{U},f}$ is closed under taking almost subsets. Assume that $\{[a_n, a_{n+1}) : n \in \omega\}$ is an interval partition of ω. By merging consecutive intervals we may assume that for each n, and each $k \in [a_n, a_{n+1})$, $f(k) \leq a_{n+2}$.

Since $\mathcal{U}$ is an ultrafilter, there exists $\ell \in \{0, 1, 2\}$ such that

$$A_\ell = \bigcup_n [a_{3n+\ell}, a_{3n+\ell+1}) \in \mathcal{U}$$

Take $A = A_{\ell+2 \bmod 3}$. For each $k \in A_\ell$, let n be such that $k \in [a_{3n+\ell}, a_{3n+\ell+1})$. Then $f(k) \leq a_{3n+\ell+2} = A^+(k)$. Thus $A \in \mathcal{G}_{\mathcal{U},f}$. $\square$

Thus, we can take $A \in \bigcap_{\alpha<\kappa} \mathcal{G}_{\mathcal{U}_\alpha, g_\alpha}$ and $g = A^+$. $\square$

How are we going force a large value for $\mathsf{cov}(\mathfrak{D}_{\mathrm{fin}})$? If $\mathsf{cov}(\mathfrak{D}_{\mathrm{fin}}) = \aleph_1$, then by Lemma 2.1 this is witnessed by a sequence $\langle(\mathcal{U}_\alpha, g_\alpha) : \alpha < \aleph_1\rangle$. To refute a single such witness, we will use the following forcing notion, where $A_\alpha \in \mathcal{U}_\alpha$ for each $\alpha < \aleph_1$.

Definition 2.5. Fix an ordinal γ. Assume that $A_\alpha \in [\omega]^\omega$ and $g_\alpha \in {}^\omega\omega$ for $\alpha < \gamma$. Define a forcing notion

$$\mathbb{Q} = \mathbb{Q}(A_\alpha, g_\alpha : \alpha < \gamma) = \{(n, h, F) : n \in \omega, h \in {}^n\omega, F \in [\gamma]^{<\aleph_0}\},$$

with $(n_1, h_1, F_1) \le (n_2, h_2, F_2)$ if $n_1 \le n_2$, $h_2 \restriction n_1 = h_1$, $F_1 \subseteq F_2$, and

$$\big(\forall\alpha \in F_1\big)\big(\forall n \in [n_1, n_2) \cap A_\alpha\big)\ g_\alpha(n) \le h_2(n).$$

Observe that $\mathbb{Q}$ is σ-centered. $\mathbb{Q}$ is a restricted variant of the Hechler forcing. Advanced readers are recommended to skip the proof of the following lemma, which is the same as for the Hechler forcing.

Lemma 2.6. *Assume that $A_\alpha \in [\omega]^\omega \cap V$ and $g_\alpha \in {}^\omega\omega \cap V$ for each $\alpha < \gamma$. Then for $\mathbb{Q} = \mathbb{Q}(A_\alpha, g_\alpha : \alpha < \gamma)$, $V^{\mathbb{Q}} \models \big(\exists g \in {}^\omega\omega\big)\big(\forall \alpha < \gamma\big)\ A_\alpha \subseteq^* \{n : g_\alpha(n) \le g(n)\}$.*

Proof. Assume that G is a $\mathbb{Q}$-generic filter over V. Let $g = \bigcup \pi_2[G]$, where π_2 denotes the projection on the second coordinate. Clearly, g is a partial function from ω to ω. By density arguments, we have that g is as required. To see this, consider first the sets

$$D_m = \{(n, h, F) \in \mathbb{Q} : m \le n\}$$

for $m \in \omega$. Each D_m is dense in $\mathbb{Q}$: Assume that $(n, h, F) \in \mathbb{Q}$. If $m \le n$ then $[n, m) = \emptyset$; therefore $(n, h, F) \le (n, h, F \cup \{\alpha\}) \in D_m$. Otherwise, define $h' : m \to \omega$ by $h'(k) = h(k)$ for $k < n$, and $h'(k) = \max\{f_\beta(k) : \beta \in F\}$ for $k \in [n, m)$. Then (m, h', F) is a member of $D_{m,\alpha}$ extending (n, h, F). The density of the sets D_m implies that $\mathrm{dom}(g) = \omega$. Moreover, for each $\alpha < \gamma$ the set

$$E_\alpha = \{(n, h, F) \in \mathbb{Q} : \alpha \in F\}$$

is dense in $\mathbb{Q}$ (for each condition (n, h, F), $(n, h, F \cup \{\alpha\})$ is a stronger condition which belongs to E_α). Now fix $\alpha < \gamma$ and choose an element $(n_0, h_0, F_0) \in G \cap E_\alpha$. For each $n \in A_\alpha \setminus n_0$ choose an element $(n_1, h_1, F_1) \in G \cap D_{n+1}$, and a common extension (n_2, h_2, F_2) of (n_0, h_0, F_0) and (n_1, h_1, F_1). As $\alpha \in F_0$ and $n \in [n_0, n_2) \cap A_\alpha$, we have that $g_\alpha(n) \le g(n)$. Since this holds for each $n \ge n_0$, we have that $A_\alpha \subseteq^* \{n : g_\alpha(n) \le g(n)\}$. $\square$

Consequently, doing an iteration of forcing notions with the above forcing used cofinally often, with $\gamma = \aleph_1$ and an appropriate book-keeping, will increase $\mathsf{cov}(\mathfrak{D}_{\mathrm{fin}})$. We will be more precise in the proof of Theorem 2.9.

Observe that the sets A_α played no special role and in fact we could take $A_\alpha = \omega$ for each α (in this case we obtain a dominating real). However, this freedom to choose A_α will play a crucial role in the following, where we would like to make sure that $\mathfrak{b}$ (or $\mathsf{non}(\mathcal{M})$) and $\mathfrak{g}$ remain small while we increase $\mathsf{cov}(\mathfrak{D}_{\mathrm{fin}})$.

We now make some easy observations concerning our planned forcing. We will construct our model by a finite support iteration $\langle\mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \aleph_2\rangle$ of c.c.c. forcing notions $\mathbb{Q}_\alpha$ which add reals for cofinally many $\alpha < \aleph_2$. Consequently, $V^{\mathbb{P}}$ satisfies $\mathfrak{c} \ge \aleph_2$, where $\mathbb{P} = \mathbb{P}_{\aleph_2} = \bigcup_{\alpha<\aleph_2} \mathbb{P}_\alpha$. The model V we begin with will satisfy $V = L$ (in fact, $\diamondsuit^*_{\aleph_1}$ and $\diamondsuit_{\aleph_2}(S^2_1)$, with $S^2_1 = \{\alpha < \aleph_2 : \mathsf{cf}(\alpha) = \aleph_1\}$, are enough). Consequently, V satisfies $|\mathbb{P}| = \aleph_2 = 2^{\aleph_1}$. Since $\mathbb{P}$ satisfies the c.c.c., (nice) $\mathbb{P}$-names for reals are countable and therefore there are at most $|\mathbb{P}|^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ names for reals in $\mathbb{P}$, so $V^{\mathbb{P}} \models \mathfrak{c} = \aleph_2$.

Since we are using a finite support iteration, Cohen reals are introduced cofinally often along the iteration, and this is well known to imply $\mathsf{cov}(\mathcal{M}) \ge \aleph_2$ in the final model (briefly: Each meager set in the final model is contained in an F_σ, and thus Borel, meager set. Each Borel set is coded by a real, and every real appears at a stage $\alpha < \aleph_2$, so Cohen reals added later will not belong to the Borel meager set which is the interpretation of this code, and since this property is absolute, they will not belong to the interpretation in the final model. Since $\aleph_2$ is regular, the codes for $\aleph_1$ many Borel meager sets all appear at an intermediate stage, so their union does not contain Cohen reals added later).

Corollary 2.7. *In the final model, $\mathsf{cov}(\mathcal{M}) = \mathfrak{c} = \aleph_2$ holds.*

Now we show how to impose some more constraints on our iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \aleph_2 \rangle$ so that in $V^{\mathbb{P}_{\aleph_2}}$, $\mathrm{cov}(\mathfrak{D}_{\mathrm{fin}}) = \aleph_2$. Our exposition follows closely the treatment of names given in [4].

Choice 2.8. We fix a $\diamondsuit_{\aleph_2}(S^2_1)$-sequence $\langle S_\delta : \delta \in S^2_1 \rangle$ in the ground model. The idea is that stationarily often S_δ will guess a function

$$f : (\aleph_1 \times \aleph_2) \cup \aleph_1 \to ([\aleph_2]^{\leq \aleph_0})^{\aleph_0}. \tag{1}$$

(So for each $\delta < \aleph_2$ of cofinality $\aleph_1$, $S_\delta : (\aleph_1 \times \delta) \cup \aleph_1 \to ([\delta]^{\leq \aleph_0})^{\aleph_0}$.)

We identify $\aleph_2$ with the partial order $\mathbb{P}_{\aleph_2}$ we are about to build. Then $[\aleph_2]^{\leq \aleph_0}$ contains all of the maximal antichains. Thus $([\aleph_2]^{\leq \aleph_0})^{\aleph_0}$ contains a name for each subset of ω (which corresponds to an element of ${}^\omega\omega$). Now any sequence

$$\langle (\mathcal{U}_\alpha, g_\alpha) : \alpha < \aleph_1 \rangle$$

in the extension has a ground model function $f : (\aleph_1 \times \aleph_2) \cup \aleph_1 \to ([\aleph_2]^{\leq \aleph_0})^{\aleph_0}$, such that $f(\alpha)$ is a name for g_α and $f(\alpha, \cdot)$ is a name for an enumeration of the elements of $\mathcal{U}_\alpha$.

For each f as in Eq. (1),

$$\{\delta \in S^2_1 : S_\delta = f \restriction \delta\}$$

is stationary in $\aleph_2$. We will inductively define an $\aleph_2$-stage finite support iteration and an injection function $F_\delta : \mathbb{P}_\delta \to \aleph_2$ for $\delta < \aleph_2$ such that the range of each F_δ is an initial segment of $\aleph_2$ which includes δ, and for $\varepsilon < \delta < \aleph_2$, $F_\varepsilon \subseteq F_\delta$.

For $\delta < \aleph_2$ we will denote by name(S_δ) the sequence of $\aleph_1$ sets of reals $\mathcal{U}_\alpha$ and of $\aleph_1$ reals g_α of the form

$$\left\langle \left(\left\{ \bigcup_{n\in\omega} \{n\} \times F_\delta^{-1}(S_\delta(\alpha, \xi)(n)) : \xi < \delta \right\}, \bigcup_{n\in\omega} \{n\} \times F_\delta^{-1}(S_\delta(\alpha)(n)) \right) : \alpha < \aleph_1 \right\rangle.$$

At stage $\delta \in S^2_1$ in the construction, if $\Vdash_{\mathbb{P}_\delta}$ "name(S_δ) is a sequence of $\aleph_1$ ultrafilters and $\aleph_1$ functions", then we can take $\mathbb{P}_\delta$-names A_α, $\alpha < \aleph_1$, such that $\Vdash_{\mathbb{P}_\delta} A_\alpha \in (\mathcal{U}_\alpha) \restriction \delta$, which means $\Vdash_{\mathbb{P}_\delta}$ "A_α is in the first component of name(S_δ)".

Theorem 2.9. *Let $V \models \diamondsuit_{\aleph_2}(S^2_1)$ and let $\mathbb{P}_{\aleph_2}$ be any forcing as in Choice 2.8. Then $V^{\mathbb{P}_{\aleph_2}} \models \mathrm{cov}(\mathfrak{D}_{\mathrm{fin}}) = \aleph_2$.*

Proof. If $\Vdash_{\mathbb{P}_{\aleph_2}}$ "$\langle (\mathcal{U}_\alpha, g_\alpha) : \alpha < \aleph_1 \rangle$ is a sequence of functions and ultrafilters", then at club many stages δ the restriction of the names to δ is also forced to be a sequence of ultrafilters in $V^{\mathbb{P}_\delta}$. For a proof of this (even in the countable support proper scenario) see [1]. But the restriction of the name to δ is guessed by name(S_δ) for stationarily many δ's in this club. So at such a stage δ the forcing $\mathbb{Q}_\delta$ adds a function h such that $g_\alpha \leq_{\mathcal{U}_\alpha} h$ for all $\alpha < \aleph_1$ and this shows that the sequence was not a witness for $\mathrm{cov}(\mathfrak{D}_{\mathrm{fin}}) = \aleph_1$. □

3. Interlude: Oracle chain condition forcing

Usually, the major difficulty in forcing inequalities between combinatorial cardinal characteristics of the continuum is to make sure that those which are required to be smaller ($\mathrm{non}(\mathcal{M})$ and $\mathfrak{g}$ in our case) do indeed remain small in the generic extension. In this section we describe one such method, which is suitable for our purposes: Oracle chain condition forcing [6, Chapter IV] (see also [3,4]).

Oracle chain condition forcing is a method for forcing with $\aleph_2$-stage finite support iteration, in such a way that some prescribed intersections of $\aleph_1$ many (descriptively nice) sets which are empty in an intermediate model remain empty in the final model.

Definition 3.1. An *oracle* (or $\aleph_1$-*oracle*) is a sequence $\bar{M} = \langle M_\delta : \delta \text{ limit} < \aleph_1 \rangle$ of countable transitive models of a sufficiently large finite portion of ZFC (henceforth denoted ZFC*), such that for each δ, $\delta \in M_\delta$ is countable in M_δ, and for each $A \subseteq \aleph_1$, the set

$$\mathrm{Trap}_{\bar{M}}(A) = \{\delta < \aleph_1 : \delta \text{ is a limit ordinal, and } A \cap \delta \in M_\delta\}$$

is a stationary subset of $\aleph_1$.

Clearly, $\diamondsuit$ implies the existence of an oracle. The sets $\mathrm{Trap}_{\bar{M}}(A)$ generate a filter $\mathrm{Trap}_{\bar{M}}$, which is normal and proper. Moreover, for each $A, B \subseteq \aleph_1$, there exists $C \subseteq \aleph_1$ such that $\mathrm{Trap}_{\bar{M}}(C) = \mathrm{Trap}_{\bar{M}}(A) \cap \mathrm{Trap}_{\bar{M}}(B)$.

Notation 3.2. Assume that $\mathbb{P} \subseteq \mathbb{Q}$ are forcing notions, and N is a set. Then $\mathbb{P} <_N \mathbb{Q}$ means: Every predense subset of $\mathbb{P}$ which belongs to N is predense in $\mathbb{Q}$.

Lemma 3.3. (1) *$<_N$ is transitive.*
(2) *If $N \subseteq N'$, then $\mathbb{P} <_{N'} \mathbb{Q}$ implies $\mathbb{P} <_N \mathbb{Q}$.*
(3) *If $\mathbb{Q} = \bigcup_{\alpha<\beta} \mathbb{Q}^\alpha$ and $\mathbb{P} <_N \mathbb{Q}^\alpha$ for each α, then $\mathbb{P} <_N \mathbb{Q}$.* □

Definition 3.4. Assume that $\bar{M}$ is an oracle. A forcing notion $\mathbb{P}$ satisfies the *$\bar{M}$-chain condition* if there exists an injection $\iota : \mathbb{P} \to \aleph_1$, such that

$$\{\delta < \aleph_1 : \delta \text{ is a limit ordinal, and } \iota^{-1}[\delta] <_{M_{\delta,\iota}} \mathbb{P}\} \in \mathrm{Trap}_{\bar{M}},$$

where $M_{\delta,\iota} = \{\iota^{-1}[A] : A \subseteq \delta \text{ and } A \in M_\delta\}$.

Thus each countable forcing notion satisfies the $\bar{M}$-chain condition, and if $\mathbb{P}$ satisfies the $\bar{M}$-chain condition, then $\mathbb{P}$ has the c.c.c., and $|\mathbb{P}| \le \aleph_1$. The definition of the $\bar{M}$-chain condition can be extended to forcing notions of cardinality $\aleph_2$ [6, IV.1.5]; however this is not needed here.

Proving the $\bar{M}$-chain condition according to Definition 3.4 is rather inconvenient. We give a useful method for verifying the $\bar{M}$-chain condition.

Proposition 3.5. *Assume that $\bar{M}$ is an oracle, $\mathbb{P} = \bigcup_{\alpha<\aleph_1} \mathbb{P}^\alpha$, for each $\alpha < \aleph_1$, ι_α is a bijection from $\mathbb{P}^\alpha$ onto a countable ordinal, and $\langle N_\alpha : \alpha < \aleph_1\rangle$ is a sequence of countable transitive models of ZFC*, such that the following conditions hold:*

(1) *For each $\alpha < \beta < \aleph_1$,*
  (a) *$\mathbb{P}^\alpha \subseteq \mathbb{P}^\beta$ with $\mathbb{P}^\beta \setminus \mathbb{P}^\alpha$ countably infinite,*
  (b) *$\iota_\alpha \subseteq \iota_\beta$, and*
  (c) *$N_\alpha \subseteq N_\beta$.*
(2) *For each (large enough) $\alpha < \aleph_1$,*
  (a) *$\iota_\alpha : \mathbb{P}^\alpha \to \omega\alpha$ is bijective,*
  (b) *$M_{\omega\alpha}, \langle \mathbb{P}^\alpha, \le_{\mathbb{P}^\alpha}\rangle, \iota_\alpha \in N_\alpha$, and*
  (c) *$\mathbb{P}^\alpha <_{N_\alpha} \mathbb{P}^{\alpha+1}$.*

Then $\mathbb{P}$ satisfies the $\bar{M}$-chain condition.

Proof. Using Lemma 3.3, we get by induction on β that for each $\alpha \le \beta \le \aleph_1$, $\mathbb{P}^\alpha <_{N_\alpha} \mathbb{P}^\beta$. In particular, $\mathbb{P}^\alpha <_{N_\alpha} \mathbb{P}$ for each α. Define $\iota = \bigcup_{\alpha<\aleph_1} \iota_\alpha$. Then $\iota : \mathbb{P} \to \aleph_1$ is an injection.

Assume that $\delta < \aleph_1$ is a (large enough) limit ordinal, and let α be such that $\delta = \omega\alpha$. Then

$$\iota^{-1}[\delta] = \iota^{-1}[\omega\alpha] = \iota_\alpha^{-1}[\omega\alpha] = \mathbb{P}^\alpha.$$

Assume that $A \subseteq \delta$, $A \in M_\delta$, and $\iota^{-1}[A] = \iota_\alpha^{-1}[A]$ is predense in $\mathbb{P}^\alpha$. As $\iota_\alpha \in N_\alpha$, $\iota_\alpha^{-1}[A] \in N_\alpha$. As $\mathbb{P}^\alpha <_{N_\alpha} \mathbb{P}$, $\iota_\alpha^{-1}[A]$ is predense in $\mathbb{P}$.

This shows that for *all* (large enough) limit ordinals $\delta < \aleph_1$, $\iota^{-1}[\delta] <_{M_{\delta,\iota}} \mathbb{P}$. Obviously, this implies the requirement in Definition 3.4. □

Proposition 3.5 gives us a recipe for verifying the $\bar{M}$-chain condition: Construct $\mathbb{P}$ by inductively constructing $\mathbb{P}^\beta$, such that (1)(a) holds. If β is a limit, take $\mathbb{P}^\beta = \bigcup_{\alpha<\beta} \mathbb{P}^\alpha$. Otherwise $\beta = \alpha + 1$ and $\mathbb{P}^\alpha$ is defined. Then there exists ι_β such that (1)(b) and (2)(a) hold. Choose N_α as in (1)(c) and (2)(b) (and containing some other elements if needed), and use N_α to define $\mathbb{P}^{\alpha+1}$ such that (2)(c) holds (this is the only tricky part in the construction). We can simplify the last step in this recipe a bit further.

Lemma 3.6. *Assume that N is a transitive model of ZFC*, such that $\langle \mathbb{P}, \le_{\mathbb{P}}\rangle \in N$. Then: $\mathbb{P} <_N \mathbb{Q}$ if, and only if, each open dense subset of $\mathbb{P}$ which belongs to N is predense in $\mathbb{Q}$.*

Proof. We need to prove ($\Leftarrow$). Assume that $I \in N$ is predense in $\mathbb{P}$. Then $I^* = \{p \in \mathbb{P} : (\exists q \in I)\ p \geq q\} \in N$, and is open and dense in $\mathbb{P}$. Thus, I^* is predense in $\mathbb{Q}$, and therefore I is predense in $\mathbb{Q}$ as well. □

Corollary 3.7. (2)(c) *in Proposition 3.5 can be replaced by:*

(2)(c′) *Each open dense subset of* $\mathbb{P}^\alpha$ *which belongs to* N_α *is predense in* $\mathbb{P}^{\alpha+1}$.

The following theorem exhibits the importance of the oracle chain condition for a single step forcing.

Theorem 3.8 (*[6, IV.2.1]*). *Assume that* $V \models \diamondsuit$, *and* $\varphi_\alpha(x)$, $\alpha < \aleph_1$, *are* Π^1_2 *formulas*[1] *(possibly with real parameters), and*

$$V \models \neg\,(\exists x)\,(\forall \alpha < \aleph_1)\ \varphi_\alpha(x).$$

If this continues to hold when we add a Cohen real to V, *then there exists an oracle* $\bar{M}$ *such that for each forcing notion* $\mathbb{P}$ *satisfying the* $\bar{M}$*-chain condition,* $V^{\mathbb{P}} \models \neg\,(\exists x)\,(\forall \alpha < \aleph_1)\ \varphi_\alpha(x)$.

The following consequence can be derived from Theorem 3.8.

Lemma 3.9 (*[6, IV.2.2]*). *Assume that* $\diamondsuit$ *holds in* V. *There is an oracle* $\bar{M}$ *in* V *such that for each* $\mathbb{P}$ *satisfying the* $\bar{M}$*-c.c., if, in* V, A *is a nonmeager set of reals, then* A *is nonmeager in* $V^{\mathbb{P}}$. *Consequently,* $V^{\mathbb{P}} \models \mathsf{non}(\mathcal{M}) = \aleph_1$.

Oracle chain condition can (and is intended to) be used with finite support iterations.

Lemma 3.10 (*[6, IV:3.2–3.3]*). *Assume that* $\bar{M}$ *is an oracle.*

(1) *For a finite support iteration* $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle$, *if each* $\mathbb{P}_\alpha$ *satisfies the* $\bar{M}$*-chain condition, then so does* $\mathbb{P}_\gamma = \bigcup_{\alpha<\gamma} \mathbb{P}_\alpha$.
(2) *If* $|\mathbb{P}| = \aleph_1$, *and* $\mathbb{P}$ *satisfies the* $\bar{M}$*-chain condition (in* V*), then in* $V^{\mathbb{P}}$ *there is an oracle* $\bar{M}^*$ *such that for each* $\mathbb{Q} \in V^{\mathbb{P}}$ *satisfying the* $\bar{M}^*$*-chain condition,* $\mathbb{P} \star \underset{\sim}{\mathbb{Q}}$ *satisfies the* $\bar{M}$*-chain condition (in* V*).*

Consider a finite support iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \aleph_2 \rangle$ of forcing notions, and let $\mathbb{P} = \bigcup_{\alpha<\aleph_2} \mathbb{P}_\alpha$. Assume that we wish to use Theorem 3.8 for $\mathbb{P}$. Then by Lemma 3.10(1), it suffices to make sure that each $\mathbb{P}_\alpha$ satisfies the $\bar{M}$-chain condition. By Lemma 3.10(2), this amounts to choosing each $\mathbb{Q}_\alpha$ in such a way that it satisfies the oracle chain condition for the oracle $\bar{M}^*$ corresponding to the oracle $\bar{M}$ given in Theorem 3.8 for $\mathbb{P}_\alpha$.

The nice thing is that we need not worry what exactly these oracles are, as long as we can make sure that for any prescribed oracle $\bar{M}$, the forcing notion $\mathbb{Q}_\alpha$ used in the iteration can be chosen so that it satisfies the $\bar{M}$-chain condition.

We sometimes have to make more than one oracle commitment. In fact, we may wish to add new commitments cofinally often along the iteration (indeed, we do that in the proof of Theorem 5.11). This can be achieved by coding all of the oracles of interest (those introduced in earlier stages of the iteration as well as the new ones required in the current iteration) in a single oracle. Since the length of the iteration is $\aleph_2$, the following lemma tells that this is possible.

Lemma 3.11 (*[6, IV.3.1]*). *If* $\bar{M}_\alpha$, $\alpha < \aleph_1$, *are oracles in* V, *then there exists a single oracle* $\bar{M}$ *such that for each* $\mathbb{P}$ *satisfying the* $\bar{M}$*-chain condition,* $\mathbb{P}$ *satisfies the* $\bar{M}_\alpha$*-chain condition for each* α.

4. Keeping $\mathsf{non}(\mathcal{M})$ small

The main lemma needed to carry out our constructions is the following.

Lemma 4.1. *Assume that* $\bar{M}$ *is an oracle, and for each* $\alpha < \aleph_1$, $\mathcal{U}_\alpha$ *is an ultrafilter and* $g_\alpha \in {}^\omega\omega$. *Then there exist sets* $A_\alpha \in \mathcal{U}_\alpha$, $\alpha < \aleph_1$, *such that* $\mathbb{Q} = \mathbb{Q}(A_\alpha, g_\alpha : \alpha < \aleph_1)$ *(Definition 2.5) satisfies the* $\bar{M}$*-chain condition.*

[1] That is, formulas of the form $(\forall a \in \mathbb{R})\,(\exists b \in \mathbb{R})\ \psi$, where $\psi \in L_{\aleph_1,\aleph_0}$ ($L_{\aleph_1,\aleph_0}$ is the extension of the first order language by allowing countable conjunctions).

Proof. We use Proposition 3.5 and the remarks following it (with $\mathbb{P}$ replaced by $\mathbb{Q}$ everywhere). We choose A_α by induction on α. At stage α we define

$$\mathbb{Q}^\alpha = \mathbb{Q}(A_\beta, g_\beta : \beta < \alpha)$$

(so at the end, $\mathbb{Q} = \bigcup_{\alpha<\aleph_1} \mathbb{Q}^\alpha$ and (1)(a) is guaranteed) and ι_α as in (1)(b) and (2)(a), then we choose N_α such that $N_\beta \subseteq N_\alpha$ for each $\beta < \alpha$, and $g_\alpha \in N_\alpha$ and (2)(b) holds.

Recall that N_α is countable, so we can choose an increasing sequence $\langle a_k : k \in \omega\rangle$ of natural numbers such that for each $g \in N_\alpha$, $g(a_k) < a_{k+1}$ for all but finitely many k (to obtain such a sequence, take an increasing function $f \in {}^\omega\omega$ which dominates all members of ${}^\omega\omega \cap N_\alpha$, and define $a_k = f^k(0)$). Since $\mathcal{U}_\alpha$ is an ultrafilter, there exists $\ell \in \{0, 1\}$ such that

$$A_\alpha := \bigcup_{k\in\omega} [a_{2k+\ell}, a_{2k+1+\ell}) \in \mathcal{U}_\alpha.$$

It remains to show that this definition guarantees (2)(c), that is, $\mathbb{Q}^\alpha <_{N_\alpha} \mathbb{Q}^{\alpha+1}$. We will use Corollary 3.7 for that. Assume that $D \in N_\alpha$ is an open dense subset of $\mathbb{Q}^\alpha$, and $p = (n, h, F) \in \mathbb{Q}^{\alpha+1} \setminus \mathbb{Q}^\alpha$ (so $\alpha \in F$). Define, for each $m > n$, $h_m : m \to \omega$ by

$$h_m(k) = \begin{cases} h(k) & k < n \\ \max\{g_\beta(k) : \beta \in F\} & n \le k. \end{cases}$$

Then $(n, h, F) \le (m, h_m, F)$, and in particular $(n, h, F \setminus \{\alpha\}) \le (m, h_m, F \setminus \{\alpha\})$. Note that the mapping $m \mapsto h_m$ belongs to N_α.

Define $f : \omega \to \omega$ by letting $f(k)$ be the minimal m such that there exists an element $(m, \tilde{h}, \tilde{F}) \in D$ which extends $(k, h_k, F \setminus \{\alpha\})$. Then $f \in N_\alpha$, so there exists k such that $m := f(a_{2k+\ell-1}) < a_{2k+\ell}$. Let $q_0 = (a_{2k+\ell-1}, h_{a_{2k+\ell-1}}, F \setminus \{\alpha\})$. By the definition of f, there exists $q_1 := (m, \tilde{h}, \tilde{F}) \in D$ which extends q_0. Let $q_2 = (m, \tilde{h}, \tilde{F} \cup \{\alpha\}) \in \mathbb{Q}^{\alpha+1}$.

Then $q_1 \le q_2$ since they share the same domain. Since $q_1 \in D$, it remains to show that $(n, h, F) \le q_2$. $(n, h, F \setminus \{\alpha\}) \le q_0 \le q_1$; thus $(n, h, F \setminus \{\alpha\}) \le q_2$, and hence it suffices to show that for each $i \in [n, m) \cap A_\alpha$, $g_\alpha(i) \le \tilde{h}(i)$. But since $A_\alpha \cap [a_{2k+\ell-1}, a_{2k+\ell}) = \emptyset$, $[n, m) \cap A_\alpha \subseteq [n, a_{2k+\ell-1})$, and if $i \in [n, a_{2k+\ell-1})$, then $\tilde{h}(i) = h_{a_{2k+\ell-1}}(i) = \max\{g_\beta(i) : \beta \in F\} \ge g_\alpha(i)$, since $\alpha \in F$, and we are done. □

By Lemma 3.10, Lemma 4.1 will enable us to keep $\mathrm{non}(\mathcal{M})$ small. We now turn to the problem of keeping $\mathfrak{g}$ small.

5. Keeping $\mathfrak{g}$ small

First we state a sufficient condition for $\mathfrak{g}$ being small.

Lemma 5.1. *Assume that* $\{Y_\zeta : \zeta < \mathfrak{c}\} \subseteq [\omega]^\omega$, *and* κ *is a cardinal such that:*

(1) *For each meager set* $\mathbf{B} \subseteq [\omega]^\omega$, $|\{\zeta : Y_\zeta \notin \mathbf{B}\}| = \mathfrak{c}$.
(2) *For each* $B \in [\omega]^\omega$, $|\{\zeta < \mathfrak{c} : B \subseteq^* Y_\zeta\}| < \kappa$.

Then $\mathfrak{g} \le \kappa$.

Proof. By a result of Blass [2], $\mathfrak{g} \le \mathrm{cf}(\mathfrak{c})$, so we can assume that $\kappa \le \mathrm{cf}(\mathfrak{c})$. We now define κ sets and then show that they are groupwise dense and that their intersection is empty.

Let $\langle \bar{n}^\zeta : \zeta < \mathfrak{c}\rangle$ list all strictly increasing sequences of natural numbers, each sequence appearing cofinally often. By induction on $\zeta < \mathfrak{c}$ we choose $\varepsilon_\zeta \le \kappa$, $\gamma_\zeta < \mathfrak{c}$ and $C_\zeta \in [\omega]^\omega$ as follows.

If there is some $\varepsilon < \kappa$ such that for each $\xi < \zeta$ with $\varepsilon_\xi = \varepsilon$ we have $[n^\zeta_i, n^\zeta_{i+1}) \not\subseteq C_\xi$ for all but finitely many i, then we take as ε_ζ the minimal such ε. By the assumption (1), we can choose γ_ζ to be the minimal $\gamma < \mathfrak{c}$ such that $\gamma \ne \gamma_\xi$ for all $\xi < \zeta$ and there are infinitely many i such that $[n^\zeta_i, n^\zeta_{i+1}) \subseteq Y_\gamma$. In this case we set $C_\zeta = \bigcup\{[n^\zeta_i, n^\zeta_{i+1}) : i \in \omega, [n^\zeta_i, n^\zeta_{i+1}) \subseteq Y_{\gamma_\zeta}\}$. Otherwise we set $\varepsilon_\zeta = \kappa$ and $C_\zeta = \omega$.

For each $\xi < \kappa$, define

$$\mathcal{G}_\xi = \{B \in [\omega]^\omega : (\exists\zeta < \mathfrak{c})\ \varepsilon_\zeta \ge \xi \text{ and } B \subseteq^* C_\zeta\}.$$

We show that each $\mathcal{G}_\xi$ is groupwise dense. Clearly, it is closed under almost subsets. Let an increasing sequence $\bar{n}$ be given. Then for each $\nu < \xi$, there is by our construction some $\zeta(\nu) < \mathfrak{c}$ such that $\varepsilon_{\zeta(\nu)} = \nu$ and $[n_i, n_{i+1}) \subseteq C_{\zeta(\nu)}$ for infinitely many i. As $\kappa \le \mathrm{cf}(\mathfrak{c})$, $\zeta(*) = \sup\{\zeta(\nu) : \nu < \xi\} < \mathfrak{c}$. By the choice of $\langle \bar{n}^\zeta : \zeta < \mathfrak{c}\rangle$ there is some $\beta \in (\zeta(*), \mathfrak{c})$ such that $\bar{n}^\beta = \bar{n}$. So $\varepsilon_\beta \ge \xi$, and $\bigcup\{[n_i^\beta, n_{i+1}^\beta) : [n_i^\beta, n_{i+1}^\beta) \subseteq Y_{\gamma_\beta}\} = C_\beta \in \mathcal{G}_\xi$.

To see that $\bigcap\{\mathcal{G}_\xi : \xi < \kappa\} = \emptyset$, assume that B is infinite and for each ξ, $B \in \mathcal{G}_\xi$. Then for each $\xi < \kappa$, there is $\beta_\xi < \mathfrak{c}$ such that $\varepsilon_{\beta_\xi} = \xi$ and $B \subseteq^* C_{\beta_\xi} \subseteq Y_{\gamma_{\beta_\xi}}$. Since κ is regular, we can thin out and assume that if $\xi_1 < \xi_2$, then $\varepsilon_{\beta_{\xi_1}} \ne \varepsilon_{\beta_{\xi_2}}$. Thus we have that for $\xi_1 < \xi_2$, $\beta_{\xi_1} \ne \beta_{\xi_2}$, and hence $\gamma_{\beta_{\xi_1}} \ne \gamma_{\beta_{\xi_2}}$. Consequently, $|\{\gamma_{\beta_\xi} : \xi < \kappa\}| = \kappa$. But $\{\gamma_{\beta_\xi} : \xi < \kappa\} \subseteq \{\zeta < \mathfrak{c} : B \subseteq^* Y_\zeta\}$, contradicting the assumption (2). $\square$

As we already stated in the previous sections, we shall use a finite support iteration $\langle \mathbb{P}_\delta, \mathbb{Q}_\delta, : \delta < \aleph_2\rangle$ of c.c.c. forcing notions, and choose constant or increasing oracles $\bar{M}^\delta$, such that $\mathbb{P}_\delta$ has the $\bar{M}^\delta$-chain condition for each δ. We start with a ground model satisfying $\diamondsuit^*_{\aleph_1}$ and $\diamondsuit_{\aleph_2}(S^2_1)$. Let $\langle S_\delta : \delta \in S^2_1\rangle$ be a $\diamondsuit_{\aleph_2}(S^2_1)$-sequence.

There are three possibilities for $\mathbb{Q}_\delta$. If $\mathrm{cf}(\delta) = \aleph_0$ or if δ is a successor, then $\mathbb{Q}_\delta$ is the Cohen forcing.

If $\mathrm{cf}(\delta) = \aleph_1$ and $\Vdash_{\mathbb{P}_\delta}$ "name(S_δ) is a sequence of ultrafilters $\mathcal{U}_\alpha$ and of functions g_α, $\alpha < \aleph_1$", then we choose A_α, $\alpha < \aleph_1$ as in Lemma 4.1 but with additional provisos and force with $\mathbb{Q}_\delta = \mathbb{Q}(\langle A_\alpha, g_\alpha : \alpha < \aleph_1\rangle)$. For the premise of this sentence we shortly say: S_δ guesses $\langle(\mathcal{U}_\alpha, g_\alpha) : \alpha < \aleph_1\rangle$. Otherwise, we set $\mathbb{Q}_\delta = \{0\}$.

Definition 5.2. For $\gamma \le \aleph_2$ we consider the class $\mathcal{K}_\gamma$ of γ-*approximations*

$$\langle(\mathbb{P}_\delta, \underset{\sim}{\mathbb{Q}}_\delta, \bar{M}^\delta, W_1, W_2) : \delta < \gamma\rangle$$

with the following properties:

(a) $\langle \mathbb{P}_\delta, \underset{\sim}{\mathbb{Q}}_\delta : \delta < \gamma\rangle$ is a finite support iteration of partial orders such that for each $\delta < \gamma$, $|\mathbb{P}_\delta| \le \aleph_1$.
(b) $\langle \bar{M}^\delta : \delta < \gamma\rangle$ is a constant sequence of oracles such that for all δ, $\mathbb{P}_\delta$ satisfies the $\bar{M}^\delta$-chain condition and for $\delta + 1 < \gamma$, $\Vdash_{\mathbb{P}_\delta}$ "$\underset{\sim}{\mathbb{Q}}_\delta$ satisfies the $(\bar{M}^{\delta+1})^*$-c.c." (as in Lemma 3.10(2)). The constant value of the oracle sequence is some oracle $\bar{M}$ as in Lemma 3.9, keeping $\mathrm{cov}(\mathcal{M}) = \aleph_1$.
(c) $W_1, W_2 \subseteq \aleph_2 \setminus S^2_1$, W_1 and W_2 are disjoint and if γ is a limit of cofinality $\aleph_1$, then $W_1 \cap \gamma$, $W_2 \cap \gamma$ are both cofinal in γ.
(d) If $\beta \in (W_1 \cup W_2) \cap \gamma$ then $\underset{\sim}{\mathbb{Q}}_\beta$ is the Cohen forcing adding the real $\underset{\sim}{r}_\beta \in {}^\omega 2$.
(e) If $\delta \in S^2_1 \cap \gamma$ and S_δ guesses $\langle(\mathcal{U}_\alpha(\delta), g_\alpha(\delta)) : \alpha < \aleph_1\rangle$, then there is some strictly increasing enumeration $\langle \zeta_\alpha(\delta) : \alpha < \aleph_1\rangle$ of a cofinal part of $W_2 \cap \delta$, and for every $\alpha < \aleph_1$ there is $\ell_{\zeta_\alpha(\delta)} \in \{0, 1\}$ such that $Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)} := r^{-1}_{\zeta_\alpha(\delta)}(\{\ell_{\zeta_\alpha(\delta)}\}) \in \mathcal{U}_\alpha$, and $\mathbb{Q}_\delta = \mathbb{Q}(Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)}, g_\alpha(\delta) : \alpha < \aleph_1)$.[2]
(f) For all $\delta \le \gamma$, $\Vdash_{\mathbb{P}_\delta}$ "$(\forall A \in [\omega]^\omega)$ $\{\beta \in W_1 \cap \delta : A \subseteq^* \underset{\sim}{Y}^1_\beta\}$ is at most countable".[3] Here, for $\delta = \gamma$ limit, $\mathbb{P}_\gamma$ is the direct limit of $\langle \mathbb{P}_\beta : \beta < \gamma\rangle$, and for $\delta = \gamma = \beta + 1$, $\mathbb{P}_\gamma = P_\beta \star \underset{\sim}{\mathbb{Q}}_\beta$.

With the help of several lemmas we will prove the following.

Theorem 5.3. *If $V \models \diamondsuit^*_{\aleph_1}$ and $\diamondsuit_{\aleph_2}(S^2_1)$, then for each $\gamma \le \aleph_2$, $\mathcal{K}_\gamma$ is not empty.*

Let V fulfill the premises and let $\mathbb{P}_{\aleph_2}$ be the direct limit of the first components of an $\aleph_2$-approximation. If G is a $\mathbb{P}_{\aleph_2}$-generic filter and $\underset{\sim}{Y}^1_\zeta[G_{\aleph_2}] = Y_\zeta$ for $\zeta \in W_1$, then we have in the final model a sequence $\langle Y_\zeta : \zeta < \mathfrak{c}\rangle$ as in Lemma 5.1 with $\kappa = \aleph_1$.

Corollary 5.4. $V^{\mathbb{P}_{\aleph_2}} \models \mathrm{cov}(\mathcal{M}) = \mathfrak{g} = \aleph_1 < \mathrm{cov}(\mathfrak{D}_{\mathrm{fin}}) = \aleph_2$.

We prove Theorem 5.3 by induction on γ and we shall work with end extensions. For some γ's, one has to work to show item (e). We will do this in our first lemma. For all γ's but maybe the successor steps of points not in S^2_1, one has to work to show that item (f) can be preserved in the induction. This will be done in the last three lemmas.

---

[2] The $\zeta_\alpha(\delta)$, $\alpha < \aleph_1$, chosen here do not have to be coherent when regarding different δ's and we index them with δ because we need it. Strictly speaking the $\ell_{\zeta_\alpha(\delta)}$ is a function $\ell_{\zeta_\alpha(\delta)}(\delta)$. And also strictly speaking we should index by γ as well, but we are suppressing this because we are anyway only working with end extensions when increasing γ.

[3] Here it is W_1. We use the Cohens in W_2 to build the forcings of type $\mathbb{Q}_\delta = \mathbb{Q}(Y^{\ell_{\zeta_\alpha}(\delta)}_{\zeta_\alpha(\delta)}, g_\alpha(\delta) : \alpha < \aleph_1)$ and the Cohens $\underset{\sim}{Y}^1_\zeta$, $\zeta \in W_1$, to build the Y_ζ's as in Lemma 5.1.

Lemma 5.5. *Consider a successor $\gamma = \delta + 1$, $\delta \in S^2_1$. Given any $\aleph_1$-oracle $(\bar{M}^{\delta+1})^*$, the sequence $\langle \zeta_\alpha(\delta) : \alpha < \aleph_1 \rangle$ can be chosen as in* (e) *so that the forcings given in item* (e) *have the $(\bar{M}^{\delta+1})^*$-c.c.*

Proof. This is a variation of Lemma 4.1. We suppress some of the δ's. We choose $\langle \zeta_\alpha : \alpha < \aleph_1 \rangle$ enumerating $W_2 \cap \delta$ so that, given the oracle $(\bar{M}^{\delta+1})^* = \langle N_\alpha : \alpha < \aleph_1 \rangle$, the Cohen real r_{ζ_α} is generic over N_α. For this it suffices that the countable model $N_\alpha \in V^{\mathbb{P}_{\zeta_\alpha}}$, which means that ζ_α just has to be sufficiently large. Let the a_k be chosen as in the proof of Lemma 4.1. Then there are infinitely many k such that

$$r^{-1}_{\zeta_\alpha}(\{\ell_{\zeta_\alpha}\}) \cap [a_{2k+\ell-1}, a_{2k+\ell}) = \emptyset,$$

and as in the proof of Lemma 4.1 this suffices. □

Choice 5.6. We start with $\bar{M}$ as described. By Lemma 3.10, all the $\mathbb{P}_\delta$, $\delta \le \aleph_2$, have the $\bar{M}$-chain condition as soon as we can arrange that all the $\mathbb{Q}_\delta$ have the $(\bar{M})^*$-chain condition in $V^{\mathbb{P}_\delta}$. The Cohen forcing has the $\bar{M}$-chain condition for any $\bar{M}$. The $\mathbb{Q}_\delta$ in the steps $\delta \in S^2_1$ can be chosen by the previous lemma so that they have the $(\bar{M})^*$-c.c.

Lemma 5.7. *If $\delta \in S^2_1$, $\mathbb{Q}_\delta$ is chosen as in Lemma 5.5, and $\mathbb{P}_\delta$ satisfies* (f) *of Definition 5.2, then $\mathbb{P}_{\delta+1}$ has the property stated in item* (f)*.*

Proof. Suppose that $p \Vdash_{\mathbb{P}_{\delta+1}}$ "$\underset{\sim}{A} \in [\omega]^\omega$ and $|\{\zeta \in W_1 \cap \delta : \underset{\sim}{A} \subseteq^* \underset{\sim}{Y}^{\ell_\zeta}_\zeta\}| = \aleph_1$", and w.l.o.g. $p \Vdash_{\mathbb{P}_{\delta+1}}$ "$\underset{\sim}{A} \in [\omega]^\omega$ and $\{\zeta \in W_1 \cap \delta : \underset{\sim}{A} \subseteq^* \underset{\sim}{Y}^{\ell_\zeta}_\zeta\}$ is increasingly enumerated by $\{\xi_\alpha : \alpha < \aleph_1\} = W_1(A)$".

We take for $n \in \omega$ a maximal antichain $\{p_{n,i} : i \in \omega\}$ above p deciding the statements $\check{n} \in \underset{\sim}{A}$ with truth value $t_{n,i}$. Let $C_{n,i} = \{\varepsilon \le \delta : p_{n,i}(\varepsilon) \ne 1\}$. For $\varepsilon \in C_{n,i} \cap S^2_1$ with $\mathbb{Q}_\varepsilon \ne \{0\}$, let $p_{n,i}(\varepsilon) = (m_{n,i}(\varepsilon), h_{n,i}(\varepsilon), F_{n,i}(\varepsilon))$. Let $F'_{n,i}(\varepsilon) = \{\zeta_\alpha(\varepsilon) : \alpha \in F_{n,i}(\varepsilon)\}$. We assume that all these are objects not just names. For $\varepsilon \in C_{n,i} \setminus S^2_1$ let $p_{n,i}(\varepsilon) = h_{n,i}(\varepsilon)$, $m_{n,i}(\varepsilon) = |h_{n,i}(\varepsilon)|$ and set the other two components for simplicity to zero. Set $m_{n,i} = \max\{m_{n,i}(\varepsilon) : \varepsilon \in C_{n,i}\}$. Set

$$\bar{C} = \langle\langle (m_{n,i}(\varepsilon), h_{n,i}(\varepsilon), F_{n,i}(\varepsilon), F'_{n,i}(\varepsilon), \langle g_\alpha(\varepsilon) \restriction m_{n,i} : \alpha \in F_{n,i}(\varepsilon)\rangle) : \varepsilon \in C_{n,i}\rangle : n, i \in \omega\rangle.$$

For each $\beta \in \aleph_1$, let $p_\beta \ge p$, $p_\beta \Vdash_{\mathbb{P}_{\delta+1}}$ "$\underset{\sim}{A} \cap [s_\beta, \infty) \subseteq \underset{\sim}{Y}^{\ell_{\xi_\beta}}_{\xi_\beta}$" and p_β shall decide the value of $\ell_{\xi_\beta} \in 2$ and $s_\beta \in \omega$. For $\beta < \aleph_1$ we set $C_\beta = \{\varepsilon \le \delta : p_\beta(\varepsilon) \ne 1\}$. If $\varepsilon \in C_\beta \cap S^2_1$, then $p_\beta(\varepsilon) = (m_\beta(\varepsilon), h_\beta(\varepsilon), F_\beta(\varepsilon))$. If $\varepsilon \in C_\beta \setminus S^2_1$, then $p_\beta(\varepsilon) = h_\beta(\varepsilon)$, ${}_\beta(\varepsilon) = |h_\beta(\varepsilon)|$ and $F_\beta(\varepsilon) = \emptyset$. For all β, $\varepsilon \in C_\beta$, let $F'_\beta(\varepsilon) = \{\zeta_\alpha(\varepsilon) : \alpha \in F_\beta(\varepsilon)\} \subseteq W_2$.

Set

$$R_\beta(m) = \langle (m_\beta(\varepsilon), h_\beta(\varepsilon), F_\beta(\varepsilon), F'_\beta(\varepsilon), \langle g_\alpha(\varepsilon) \restriction m : \alpha \in F_\beta(\varepsilon)\rangle) : \varepsilon \in C_\beta\rangle.$$

These are finite arrays of finite sets.

Now we thin out: First we assume that for some $k \in \omega$ for all $\beta < \aleph_1$, $|C_\beta| = k$, $s_\beta \le k$. We apply the delta system lemma to C_β, $\beta \in \aleph_1$, and get a root C. We assume that $\delta \in C$, as this is the difficult case. We apply the delta lemma for each $\varepsilon \in C$ to the $F_\beta(\varepsilon)$, $\beta \in \aleph_1$, and get a root $F(\varepsilon)$, and to $F'_\beta(\varepsilon)$, $\beta \in \aleph_1$, and get a root $F'(\varepsilon)$. We further assume that for each β in the delta system and for all $\varepsilon \in C$, all $F_\beta(\varepsilon) \setminus F(\varepsilon)$ are above $\max(\bigcup_{\varepsilon' \in C}(F(\varepsilon')) \cup (C \setminus \{\delta\}))$ and the same for the primed ones. We thin out further and assume that there are $(m(\varepsilon), h(\varepsilon), F(\varepsilon))$ such that for all $\beta < \aleph_1$, for all $\varepsilon \in C$, $m_\beta(\varepsilon) = m(\varepsilon)$, $h_\beta(\varepsilon) = h(\varepsilon) \in {}^{m(\varepsilon)}\omega$, and for the $\varepsilon \in C_\beta \setminus C$, the increasingly enumerated ε's in $C_\beta = \{\varepsilon^\beta_i : i < k\}$ are isomorphic to the lexicographically first $\langle \varepsilon_i : i < k\rangle$, i.e., $m_\beta(\varepsilon^\beta_i) = m(\varepsilon_i)$, $h_\beta(\varepsilon^\beta_i) = h(\varepsilon_i) \in {}^{m(\varepsilon_i)}\omega$, and we use a delta system argument on the $F_\beta(\varepsilon^\beta_i)$ giving a root $F(\varepsilon_i)$ and again impose on the parts $F_\beta(\varepsilon^\beta_i) \setminus F(\varepsilon_i)$ that they have to lie above $\bigcup_{i<k} F(\varepsilon_i)$ and are all of the same size. The analogous thinning out is done for the primed parts, that have to lie above $\max(\bigcup_{i<k}(F'(\varepsilon_i)) \cup (C \setminus \{\delta\}))$, be for all i of the same size $|F'_\beta(\varepsilon^\beta_i)|$ independently of β (but depending on i), and all of the $\langle F'_\beta(\varepsilon^\beta_i) : i < k\rangle$ shall have the same $\le$ or $\ge$-relations with the members of $C_\beta(\varepsilon_i)$. Moreover, if ε is a Cohen coordinate in C_β, then $p_\beta(\varepsilon)$ does not depend on β.

We let $m_{\max}$ be the maximum of the $m(\varepsilon)$ and of the lengths of all the finitely many Cohen coordinates for all β in the delta system. Let $\lhd$ denote the initial segment relation for finite sequences. We thin out further and assume that all the $R_\beta(m_{\max})$ have the same quantifier free $(<_{\aleph_1}, \lhd)$-type over $\mathrm{Ran}(\bar{C}) \cup \mathrm{Ran}(\mathrm{Ran}(\bar{C}))$. Speaking about components of five tuples $(m, h, F, F', \bar{g})$ separately is allowed as well as evaluating $\bar{g}$ and the members of all involved finite sets.

There are only countably many quantifier types in this language that can be fulfilled by a (finite) sequence $R_\beta(m_{\max})$ in our delta system.

Let G_δ be a subset of $\mathbb{P}_\delta$ that is generic over V such that $W^* = \{\gamma \in W_1(A) \cap \delta : p_\gamma \restriction \delta \in G_\delta\}$ is uncountable.

For $\gamma \in W^*$, let in $V[G_\delta]$,

$$B_\gamma = \{n \in \omega : \exists p' \in \mathbb{P}_{\delta+1},\ p' \geq p_\gamma,\ p' \restriction \delta \in G_\delta, \text{ and } p' \Vdash_{\mathbb{P}_{\delta+1}} n \in A\}.$$

$B_\gamma \subseteq^* Y^{\ell_{\xi_\alpha}}_{\xi_\alpha}[G]$, and the latter is fully evaluated by G, because $\xi_\alpha \in W_1 \subseteq \delta + 1$ for $\alpha < \aleph_1$, and $\delta \notin W_1$.

We shall show that for $\beta, \gamma \in W^*$, $B_\beta \cap [k, \infty) = B_\gamma \cap [k, \infty) = B \in V[G]$. Then B is a counterexample to $\langle(\mathbb{P}_\varepsilon, \mathbb{Q}_\beta, M^\varepsilon, W_1, W_2) : \varepsilon \leq \delta, \beta < \delta\rangle \in \mathcal{K}_\delta$.

Let $||_{\mathbb{P}_{\delta+1}}$ denote the compatibility relation in $\mathbb{P}_{\delta+1}$. If $n \in B_\beta$, then $p_\beta ||_{\mathbb{P}_{\delta+1}} p_{n,i}$ for the one i such that $p_{n,i} \in G$, and for this i we have $t_{n,i} = \mathit{true}$. The same holds for $n \notin B_\beta$ with *false*. So our claim that $B_\beta \cap [k, \infty) = B_\gamma \cap [k, \infty)$ for all $\beta, \gamma \in W^*$ now follows from

Claim 5.8. *For all β, γ in W^*:*

$$p_\beta ||_{\mathbb{P}_{\delta+1}} p_{n,i} \qquad \textit{iff} \qquad p_\gamma ||_{\mathbb{P}_{\delta+1}} p_{n,i}.$$

Proof. The point is the coordinate δ, since the restrictions to δ are in G_δ, and hence compatible. Assume $p_{n,i}(\delta) = (m_{n,i}, h_{n,i}, F_{n,i})$, $p_\beta(\delta) = (m_\beta, h_\beta, F_\beta)$, $p_\gamma(\delta) = (m_\gamma, h_\gamma, F_\gamma)$. We do not write the δ at these points, but will not suppress it completely. We assume that $p_\beta(\delta)$ is compatible with $p_{n,i}(\delta)$.

First case: $m_\beta \geq m_{n,i}$. Then $p_\beta || p_{n,i}$ means $h_\beta \rhd h_{n,i}$ and for all $\alpha \in F_\beta \cup F_{n,i}$ for all $m \in [m_{n,i}, m_\beta) \cap Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)}$, $(h_\beta(m) \geq g_\alpha(\delta)(m))$.

We have to show that the same holds for p_γ. First, by our thinning out $m_\beta = m_\gamma$, $h_\beta = h_\gamma$, and hence $h_\gamma \rhd h_{n,i}$, and $F_\beta \cap F_{n,i} = F_\gamma \cap F_{n,i}$.

1(a) We have to show: For all $\alpha \in F_{n,i}$ for all $m \in [m_{n,i}, m_\gamma) \cap Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)}$ $(h_\gamma(m) \geq g_\alpha(\delta)(m))$.

And since $h_\beta = h_\gamma$, for all $\alpha \in F_{n,i}$ for all $m \in [m_{n,i}, m_\gamma) \cap Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)}$, $(h_\gamma(m) \geq g_\alpha(\delta)(m))$.

1(b) We also have to show: For all $\alpha \in F_\gamma$ for all $m \in [m_{n,i}, m_\gamma) \cap Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)}$ $(h_\gamma(m) \geq g_\alpha(\delta)(m))$. For $\alpha \in F_\gamma \cap F_\beta$ the latter requirement is clearly fulfilled, as $h_\beta = h_\gamma$. For the part $F_\gamma \setminus F(\delta)$ we need to look closer: Suppose some condition in p_γ forced something about $Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)}$. Then $p_\gamma(\zeta_\alpha(\delta)) \neq 1$ and hence $\zeta_\alpha(\delta) \in C_\gamma \cap W_2$. But then because of the indiscernibility over $m_\gamma = m_\beta \leq m_{\max}$ (which is a component of $\bar{C}$), $\zeta_\alpha(\delta) \in C_\beta$ and hence it is in the root C. So p_β forced by our thinning out the same fact about $Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)} \cap m_{\max}$. Hence, for all $\alpha \in F_\gamma$ for all $m \in [m_{n,i}, m_\gamma) \cap Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)}$, $(h_\gamma(m) \geq g_\alpha(\delta)(m))$. So, taking 1(a) and 1(b) together, $p_\gamma || p_{n,i}$.

Second case: $m_\beta \leq m_{n,i}$. Then $h_\beta \lhd h_{n,i}$, and $p_\beta || p_{n,i}$ means that for all $\alpha \in F_\beta \cup F_{n,i}$ for all $m \in [m_\beta, m_{n,i}) \cap Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)}$, $(h_{n,i}(m) \geq g_\alpha(\delta)(m))$. This latter statement does hold also for F_γ instead of F_β and m_γ instead of m_β, because $m_\gamma = m_\beta$ and $(F_\beta, \langle g_\alpha(\delta) \restriction m_{n,i} : \alpha \in F_\beta\rangle)$ and $(F_\gamma, \langle g_\alpha(\delta) \restriction m_{n,i} : \alpha \in F_\gamma\rangle)$ are part of $R_\beta(m_{\max})$ and $R_\gamma(m_{\max})$ and hence indiscernible over $h_{n,i}$ for arguments $m \in Y^{\ell_{\zeta_\alpha(\delta)}}_{\zeta_\alpha(\delta)}$, as for these m's, that are forced to be in a Cohen part, $\zeta_\alpha(\delta) \in C$ and hence by our thinning out we have $m_{\max} \geq m$. Also $h_\gamma \lhd h_{n,i}$, and hence $p_\gamma || p_{n,i}$.

So the claim is proved and with it also Lemma 5.7. □

Lemma 5.9. (1) *If* $\mathrm{cf}(\gamma) = \aleph_1$ *and* $\underset{\sim}{\mathbb{Q}}$ *and* $\bar{M}^\gamma$ *are as in the previous lemma and if* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \in \mathcal{K}_\gamma$, *then*

$$\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \hat{\ } \langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}, \bar{M}^\gamma\rangle \in \mathcal{K}_{\gamma+1}.$$

(2) *If* $\mathrm{cf}(\gamma) = \aleph_0$ *and if* $\langle \mathbb{P}_\delta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \in \mathcal{K}_\gamma$, *then*

$$\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \hat{\ } \langle \mathbb{P}_\gamma, \mathbb{C}, \bar{M}^\gamma\rangle \in \mathcal{K}_{\gamma+1}.$$

(3) *If* $\mathrm{cf}(\gamma) = \aleph_0$ *and if* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \restriction \beta \in \mathcal{K}_\beta$ *for each* $\beta < \gamma$, *then* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \in \mathcal{K}_\gamma$.

(4) *If* $\mathrm{cf}(\gamma) = \aleph_1$ *or* $\gamma = \aleph_2$, *and if* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \restriction \beta \in \mathcal{K}_\beta$ *for each* $\beta < \gamma$, *then* $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \bar{M}^\beta, W_1, W_2) : \beta < \gamma\rangle \in \mathcal{K}_\gamma$.

Proof. (1) This was proved in Lemma 5.7.

(2) If A is an almost subset of uncountably many Y_ζ's, then there is some $\gamma_0 < \gamma$ such that there are uncountably many such ζ below γ_0. A is possibly a name using the last, new forcing. But this is just Cohen forcing. So there is some finite part of a Cohen condition forcing that $\underset{\sim}{A}$ is in uncountably many Y_ζ's. But then also the forcing $\mathbb{P}_\gamma$ already contains a name for some infinite $B \subseteq \omega$ almost contained in the intersection of uncountably many Y_ζ's with $\zeta < \gamma_0$. So P_γ does not fulfill property (f) and hence the induction hypothesis is not fulfilled.

(3) First we use the pigeonhole principle for the Y_ζ's as in the previous item. Then we use the following

Lemma 5.10. *Assume*

(a) $\langle \mathbb{P}_n : n \in \omega \rangle$ *is a* $\lessdot$*-increasing sequence of c.c.c. forcing notions with union* $\mathbb{P}$,
(b) $\mathcal{Y}$ *is a set of* $\mathbb{P}_0$*-names of infinite subsets of* ω,
(c) *for* $n \in \omega$ *we have* $\Vdash_{\mathbb{P}_n}$ "$\kappa = \mathrm{cf}(\kappa) > |\{\underset{\sim}{Y} \in \mathcal{Y} : \underset{\sim}{B} \subseteq^* \underset{\sim}{Y}\}|$", *whenever* $\underset{\sim}{B}$ *is a* $\mathbb{P}_n$*-name of an infinite subset of* ω.

Then condition (c) *holds for* $\mathbb{P}$ *too.*

Proof. Since $\mathbb{P}$ is a c.c.c. forcing notion, also in $V^{\mathbb{P}}$ we have κ is a regular cardinal.

If the desired conclusion fails, then we can find a $\mathbb{P}$-name $\underset{\sim}{B}$ of an infinite subset of ω and a sequence $\langle (p_\alpha, \underset{\sim}{Y}_\alpha, m_\alpha) : \alpha < \kappa \rangle$ such that

(α) $m_\alpha \in \omega$,
(β) $\underset{\sim}{Y}_\alpha \in \mathcal{Y}$ without repetitions,
(γ) $p_\alpha \in \mathbb{P}$, $p_\alpha \Vdash_{\mathbb{P}} \underset{\sim}{B} \setminus m_\alpha \subseteq \underset{\sim}{Y}_\alpha$.

Since $\mathrm{cf}(\kappa) > \aleph_0$, for some $n(*), m(*) \in \omega$ the set $S \stackrel{\mathrm{df}}{=} \{\alpha < \kappa : p_\alpha \in \mathbb{P}_{n(*)}, m_\alpha = m(*)\}$ has cardinality κ. We identify it with κ.

Now for every large enough $\alpha \in S$ we have

$$p_\alpha \Vdash_{\mathbb{P}} \kappa = |\{\beta \in S : p_\beta \in \underset{\sim}{G}_{\mathbb{P}_{n(*)}}\}|.$$

Why? Else for an end segment of $\alpha < \kappa$ there is $q_\alpha \geq p_\alpha$ such that for all but $< \kappa$ many $\beta \in S$, $q_\alpha \Vdash p_\beta \notin \underset{\sim}{G}_{\mathbb{P}_{n(*)}}$. That means that for an end segment of $\alpha < \kappa$, w.l.o.g., for all $\alpha \in \kappa$, $\mathrm{Perp}_\alpha := \{\beta \in S : q_\beta \perp q_\alpha\}$ contains an end segment of S. Then we take the diagonal intersection D of all these end segments of S. Since κ is regular, D contains a club in κ. But then $\{q_\beta : \beta \in D\}$ is an antichain in $\mathbb{P}_{n(*)}$ of size κ. Contradiction.

Let $G_{n(*)}$ be a subset of $\mathbb{P}_{n(*)}$ generic over V, and let $S_* := \{\beta \in S : p_\beta \in G_{n(*)}\}$. We choose $G_{n(*)}$ such that $|S_*| = \kappa$. We let $B' = \cap\{\underset{\sim}{Y}_\beta \setminus m(*) : \beta \in S_*\}$. Then in $V[G_{n(*)}]$, B' is an infinite subset of ω included in κ members of $\mathcal{Y}$, contradicting the assumption. So Lemma 5.10 is proved. □

(4) If $\mathbb{P}_\delta$ adds some A, then this already comes earlier, say in $V^{\mathbb{P}_\varepsilon}$, $\varepsilon < \delta$, because $A \subseteq \omega$ and because of the c.c.c. If $A \subseteq^* Y_\zeta$ is forced, then $\zeta < \varepsilon$. This contradicts the induction hypothesis for $\mathbb{P}_\varepsilon$. This completes the proof of Lemma 5.9. □

The lemmas together give that there is an $\aleph_2$-approximation, and the proof of Theorem 5.3 is completed. □

With some extra care our proof can be modified to yield the following (cf. [7,4]).

Theorem 5.11. *It is consistent (relative to ZFC) that all of the following assertions hold:*

(1) *Each unbounded set of* ${}^\omega\omega$ *contains an unbounded subset of size* $\aleph_1$.
(2) *Each nonmeager subset of* ${}^\omega\omega$ *contains a nonmeager subset of size* $\aleph_1$.
(3) $\mathfrak{g} = \aleph_1$.
(4) $\mathrm{cov}(\mathfrak{D}_{\mathrm{fin}}) = \mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \aleph_2$.

Proof. This time we work with a version of $\mathcal{K}_\gamma$ with increasing oracles, which means that the $\bar{M}^\varepsilon$-chain condition implies $\bar{M}^\delta$-chain condition for $\varepsilon > \delta$ and that $\mathbb{P}_\delta \Vdash$ "$\mathbb{P}_{[\delta,\varepsilon)}$ has the $\underset{\sim}{\bar{M}}^{\delta+1}$-c.c.", though the initial segment need not yet fulfill it, and the name for this new oracle may not yet have an evaluation in an initial segment $\mathbb{P}_\gamma$, $\gamma < \delta$. The new parts of the oracles take care of the unbounded and the nonmeager families that appear later in the iteration and that are frozen by the next step if their intersection with $V^{\mathbb{P}_\delta}$ is guessed by the diamond sequence and happens to be unbounded or nonmeager at the current stage δ: The conservation of the unboundedness and nonmeagerness of the intersection is written into all the oracles from δ onwards. □

Acknowledgements

The authors were partially supported by: The Austrian "Fonds zur wissenschaftlichen Förderung", grant no. 16334, and the University of Helsinki (first author), the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation, Germany (first and third authors), the United States–Israel Binational Science Foundation Grant no. 2002323 (second author), and the Golda Meir Fund (third author). This is the second author's publication 847.

References

[1] A.R. Blass, S. Shelah, There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points, and the Rudin–Keisler ordering may be downward directed, Annals of Pure and Applied Logic 33 (1987) 213–243.

[2] A.R. Blass, Combinatorial cardinal characteristics of the continuum, in: M. Foreman, A. Kanamori, M. Magidor (Eds.), Handbook of Set Theory, Kluwer Academic Publishers, Dordrecht (in press).

[3] M. Burke, Liftings for Lebesgue measure, Israel Mathematical Conference Proceedings 6 (1993) 119–150.

[4] M. Burke, A.W. Miller, Models in which every nonmeager set is nonmeager in a nowhere dense Cantor set, Canadian Journal of Mathematics (in press). http://arxiv.org/abs/math.LO/0311443.

[5] H. Mildenberger, Groupwise dense families, Archive for Mathematical Logic 40 (2001) 93–112.

[6] S. Shelah, Proper and Improper Forcing, second ed., Springer, 1998.

[7] S. Shelah, J. Steprāns, Maximal Chains in ${}^{\omega}\omega$ and Ultrapowers of the Integers, Archive for Mathematical Logic 32 (1993) 305–319.

[8] S. Shelah, B. Tsaban, Critical cardinalities and additivity properties of combinatorial notions of smallness, Journal of Applied Analysis 9 (2003) 149–162.

[9] J. Vaughan, Small uncountable cardinals and topology, in: J. van Mill, G.M. Reed (Eds.), Open Problems in Topology, North-Holland, Amsterdam, 1990, pp. 195–218.