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Singly Cogenerated Annihilator Classes

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We concern ourselves with the question of when an annihilator class (a class closed under formation of arbitrary products and subgroups) of abelian groups is cogenerated by a single group. We show that the class of p^λ-reduced groups (where p is a prime and λ is an ordinal greater than ω) is not singly cogenerated. If G is a cotorsion-free group, we show that the torsion-free class cogenerated by G is not singly cogenerated as an annihilator class. This result permits identification of all singly cogenerated annihilator classes which are also closed under formation of extensions, and so we characterize those singly generated radicals which are idempotent. They are precisely the radicals determined by annihilator classes singly cogenerated by a pure injective. © 1987 Academic Press, Inc.

I. INTRODUCTION

Classes of abelian groups closed under formation of products and subgroups are important classes; these classes are sometimes called annihilator classes. They are, of course, epireflective subcategories of the category of abelian groups. More generally, if $\mathscr{A}$ is a class of groups, define for each group G, $R_{\mathscr{A}}G = \bigcap \{\ker f : f \in \mathrm{Hom}(G, X), X \in \mathscr{A}\}$. This defines a subfunctor of the identity $R_{\mathscr{A}}$ which is a radical; that is, $R_{\mathscr{A}}(G/R_{\mathscr{A}}G) = 0$ for

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all groups G. The annihilator class $\{G: R_{\mathscr{A}}G=0\}$ is the smallest class containing $\mathscr{A}$ which is closed under formation of products and subgroups. The correspondence between annihilator classes and radicals is bijective.

It is of interest to know when such an annihilator class is cogenerated by a single group (equivalently, by a set of groups). An annihilator class $\mathscr{A}$ is cogenerated by a group X provided G belongs to $\mathscr{A}$ precisely when G can be embedded in a product of copies of X. In this case, we write R_X instead of $R_{\mathscr{A}}$ or $R_{\{X\}}$. The class of all torsion-free groups is cogenerated by the group of rationals $\mathbb{Q}$, and thus the torsion subgroup functor can be described as $R_{\mathbb{Q}}$. Note that, in this case, the radical $R_{\mathbb{Q}}$ is idempotent. If R is an arbitrary radical, it need not be idempotent; but it will determine an idempotent radical in the following way: inductively define for each ordinal α, R^α by, for each group G, $R^{\alpha+1}G=R(R^\alpha G)$ and $R^\alpha G=\bigcap_{\beta<\alpha} R^\beta G$ if α is a limit. For each group G, there is an ordinal $\mu_G=\mu$ so that $R^{\mu+1}G=R^\mu G$. Define $R^\infty G=R^\mu G$ for each G. Then it follows that R^∞ is an idempotent radical with $R^\infty G\leqslant RG$ for all G. Moreover, R^∞ determines a torsion-theory as the annihilator class of R^∞ is the smallest class containing the annihilator class of R closed under formation of products, subgroups, and extensions.

If p is a prime, then multiplication by p forms a radical, and for each ordinal λ, p^λ is also a radical. The annihilator classes of p^λ-reduced groups $\{G: p^\lambda G=0\}$ are important classes of groups. It is clear that $p^n=R_{\mathbb{Z}(p^n)}$ where n is a positive integer and $\mathbb{Z}(p^n)$ denotes the cyclic group of order p^n. It follows that $p^\omega=R_{\Pi\mathbb{Z}(p^n)}=R_{\oplus\mathbb{Z}(p^n)}$. In [FOW2], it was shown that $p^\infty\neq R_X$ for any X, that is the class $\{G\mid p^\infty G=0\}$ is not cogenerated by a single group, and the question of whether or not the class of p^λ-reduced groups for $\lambda\geqslant\omega+1$ is cogenerated by a single group was raised as an open question. In this paper we shall show the answer is no; that is, if λ is an ordinal greater than ω, then $p^\lambda\neq R_X$ for any group X.

In a second section we shall show that for many radicals R_X, determined by a single group, the idempotent radical R_X^∞ is not determined by a single group. Thus even though a class may be singly cogenerated to begin with, the process of closing up the class under products, subgroups and extensions often destroys the single cogeneratedness. In particular we show that if G is cotorsion-free, then $R_G^\infty\neq R_X$ for any X. This result is used to show that $R_X=R_X^2$ if and only if $R_X=R_Y$ where Y is divisible or, if X is reduced, Y is algebraically compact.

We conclude the paper with a section containing some results concerning when a torsion-theory is singly cogenerated; these results are dependent upon the set-theoretic assumption that there do not exist measurable cardinals ($\nexists\aleph_m$). All of the previously mentioned results above hold in ZFC.

II. THE CLASSES OF p^λ-REDUCED GROUPS

As mentioned in the Introduction, the classes of p^λ-reduced groups, for $\lambda \leqslant \omega$, are each singly cogenerated. It is easy to see that the class of p^∞-reduced groups is not singly cogenerated, for if $p^\infty = R_X$, then, for some ordinal λ, $p^\lambda X = 0$, and $p^\infty G = 0$ would imply $p^\lambda G = 0$, which is absurd. A similar argument shows that the class of all reduced groups is not singly cogenerated.

In this section we show that for $\lambda > \omega$, p^λ is not of the form R_X for any group X. We begin by showing the existence of groups for which we have some control over their homomorphic images.

2.1. THEOREM. *Let λ be an infinite cardinal. Then there exists an abelian p-group $H = H_\lambda$ of cardinality $\lambda^{\aleph_0}$ such that*:

(a) $p^\omega H \cong \mathbb{Z}(p)$,

(b) *if A is any abelian group of cardinality less than λ, and $\phi \in \mathrm{Hom}(H, A)$, then $\phi(p^\omega H)$ is contained in the divisible part of A.*

Proof. As usual, we identify λ with $\{\alpha : \alpha < \lambda\}$, the set of all ordinals less than λ. Define the sets $T = \bigcup_{n<\omega} T_n$ and T^* by:

$$T_n = \{f\colon \{0, 1, \ldots, n-1\} \to \lambda \times \lambda\},$$

$$T^* = \{f\colon \omega \to (\lambda \times \lambda) \backslash \Delta\}.$$

We define the group $H = H_\lambda$ by the following generators and relations:

Generators: elements of the form $e_{g,\beta}$ where $g \in T$ and $\beta < \lambda$, elements of the form $a_{f,n}$ where $f \in T^*$ and $n < \omega$, and a distinguished element a.

Relations: (a) $p^{n+1} e_{g,\beta} = 0$ if $g \in T_n$,

(b) $pa_{f,0} = a$ for all $f \in T^*$

(c) $pa = 0$

(d) $pa_{f,n+1} = pa_{f,n} - (e_{f\restriction n, f^0(n)} - e_{f\restriction n, f^1(n)})$ for all $f \in T^*$ and $n < \omega$. Here $f \restriction n$ denotes the restriction of f to $n = \{0, 1, \ldots, n-1\}$ and $f(n) = (f^0(n), f^1(n))$.

It follows from the relations and by induction that $p^{n+2} a_{f,n+1} = a$ and hence $a \in p^\omega H$. Our first objective is to show that $p^\omega H = \langle a \rangle$.

Let $\bar{H} = H/\langle a \rangle$; it is clear that $\bar{H}$ is given by generators and relations:

Generators: elements $\bar{e}_{g,\beta}$ where $g \in T$ and $\beta < \lambda$, and elements $\bar{a}_{f,n}$ where $f \in T^*$ and $n < \omega$.

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Relations: (a) $p^{m+1}\bar{e}_{g,\beta}=0$ for $g\in T_m$,

(b) $p\bar{a}_{f,0}=0$ for all $f\in T^*$,

(c) $p\bar{a}_{f,n+1}=\bar{a}_{f,n}-(\bar{e}_{f\restriction n,f^0(n)}-\bar{e}_{f\restriction n,f^1(n)})$ for all $f\in T^*$ and $n<\omega$.

We will show that $\bar{H}$ is separable (i.e., $p^\omega\bar{H}=0$) and consequently $p^\omega H=\langle a\rangle\cong\mathbb{Z}(p)$.

Define $B=\bigoplus\langle\tilde{e}_{g,\beta}\rangle$ taken over all $g\in T$ and $\beta<\lambda$ where the order of $\tilde{e}_{g,\beta}$ is p^{m+1}, m being that (unique) integer for which $g\in T_m$. Let $\bar{B}$ denote the torsion-completion of B. The elements $\tilde{a}_{f,m}$ defined by the infinite sums $a_{f,m}=\sum_{n=m}^{\infty}(\tilde{e}_{f\restriction n,f^0(n)}-\tilde{e}_{f\restriction n,f^1(n)})\,p^{n-m}$ represent elements of $\bar{B}$. Note each term in the sum has order p^{m+1} and so $\tilde{a}_{f,m}$ has order p^{m+1}. The collection of all elements $\tilde{e}_{g,\beta}$ and $\tilde{a}_{f,m}$, for $g\in T$, $\beta<\lambda$, $f\in T^*$ and $m<\omega$, satisfy the relations (a), (b), and (c) with "$^-$" replaced by "$^\sim$."

Now let G be that subgroup of $\bar{B}$ generated by B and all the elements $\tilde{a}_{f,n}$. Clearly G/B is divisible, and, as B is pure in $\bar{B}$, G is pure in $\bar{B}$.

For $x\in\bar{B}$, let $[x]$ denote the support of x; that is, $[x]\subseteq\bigcup_{n<\omega}(T_n\times\lambda)$ is the set of all indices for which the corresponding entry of x is nonzero. Let $k,m<\omega$ and $f,g\in T^*$ with $f\neq g$. Then for the generators $\tilde{a}_{f,m}$ and $\tilde{a}_{g,k}$ of G, we have:

$$[\tilde{a}_{f,m}]\cap[\tilde{a}_{g,k}] \qquad \text{is finite.} \tag{$*$}$$

For if this were infinite, then the set

$$\{n: \{(f\restriction n, f^0(n)), (f\restriction n, f^1(n))\}\cap\{(g\restriction n, g^0(n)), (g\restriction n, g^1(n))\}\neq\varnothing\}$$

is infinite. This would imply that $g\restriction n=f\restriction n$ for an infinite number of n's and therefore $f=g$ contrary to the choice of f and g. This shows that $\tilde{a}_{f,m}$ and $\tilde{a}_{g,k}$ are independent modulo B and consequently

$$G/B=\bigoplus_{f\in T^*}(\langle\{\tilde{a}_{f,m}\,|\,m<\omega\}\rangle+B/B).$$

Indeed, if $\phi: G\to G/B$ denotes the natural map, then $P\phi(\tilde{a}_{f,m+1})=\phi(\tilde{a}_{f,m})$ and the set of generators $\{\phi(\tilde{a}_{f,m}): m<\omega\}$ satisfy the appropriate relations showing $\langle\{\tilde{a}_{f,m}: m<\omega\}\rangle+B/B\cong\mathbb{Z}(p^\infty)$.

Define the homomorphisms $\sigma:\bar{H}\to G$ and $\tau: G\to\bar{H}$ by $\sigma(\bar{e}_{g,\beta})=\tilde{e}_{g,\beta}$, $\sigma(\bar{a}_{f,n})=\tilde{a}_{f,n}$, $\tau(\bar{e}_{g,\beta})=\bar{e}_{g,\beta}$, and $\tau(\tilde{a}_{f,n})=\bar{a}_{f,n}$. Clearly, σ is well defined, onto, and $(*)$ implies that τ is well defined. Since $\tau\sigma=1_{\bar{H}}$, we have σ is an isomorphism, and so $\bar{H}$ is separable.

Now let A be any group of cardinality less than λ, and let $\phi\in\operatorname{Hom}(H,A)$. Let n be fixed and choose $f_n\in T_n$. The set $\{e_{f_n,\beta}:\beta<\lambda\}$ has cardinality λ and, as $|A|<\lambda$, there exists a subset X_n of λ of cardinality λ for which $\phi(e_{f_n,\alpha})=\phi(e_{f_n,\beta})$ for all $\alpha,\beta\in X_n$. Define $f_{n+1}:\{0, 1,..., n\}\to\lambda\times\lambda$

by $f_{n+1} \upharpoonright n = f_n$ and $f_{n+1}(n) = (\alpha, \beta)$ where $\alpha, \beta \in X_n$, $\alpha \neq \beta$. By induction, and in this way, we can produce a function $f = \bigcup_{n<\omega} f_n$ belonging to T^* having the property that $\phi(e_{f \upharpoonright n, f^0(n)}) = \phi(e_{f \upharpoonright n, f^1(n)})$ for all $n < \omega$.

From the relations satisfied in H, we have $p\phi(a_{f,n+1}) = \phi(a_{f,n})$ and $\phi(a_{f,0}) = \phi(a)$. Consequently the subgroup generated by the images $\phi(a_{f,n})$ is either zero or a copy of $\mathbb{Z}(p^\infty)$.

2.2. Corollary. *Let α be an ordinal greater than ω. Then the radical p^α is not of the form R_X for any group X.*

Proof. Suppose that $p^\alpha = R_A$ for some group A. Then A must be reduced, since $p^\alpha A = 0$. Let λ be the cardinality of A (clearly $\lambda > \omega$) and let $H = H_\lambda$ be the group constructed in the theorem. Since $\alpha > \omega + 1$, $R_A H = p^\alpha H = 0$. But, by the theorem, every morphism $\phi: H \to A$ sends $p^\omega H = \langle a \rangle$ to zero and so $\langle a \rangle \subseteq R_A H$, a contradiction.

III. Classes Determined by Cotorsion-Free Groups

As already mentioned, the class of torsion-free groups is singly cogenerated by the group of rationals $\mathbb{Q}$. The class of all torsion-free groups G enjoying $p^\omega G = 0$ is a radical class singly cogenerated by the group of p-adic integers J_p. The class of $\aleph_1$-free groups is a radical class not cogenerated by a single group [DG] and the class of all cotorsion-free groups is not cogenerated by a single group [DG]. The class of torsionless groups is the annihilator class of the radical $R_{\mathbb{Z}}$ and hence is singly cogenerated. As a consequence of results in this section, the idempotent radical $R_{\mathbb{Z}}^\infty$ is not of the form R_X for any group X. This answers an open question posed in [FOW1]. Indeed, the main result of this section is that if G is a cotorsion-free group, then R_G^∞ is not of the form R_X for any X. Recall that a group G is cotorsion-free provided it is torsion-free, reduced, and for all primes p, $\mathrm{Hom}(J_p, G) = 0$.

3.1. Theorem. *Let λ be an infinite cardinal. Then there exists a short exact sequence.*

$$0 \to \mathbb{Z} \to A \to C \to 0,$$

*where $C \leqslant \Pi_\lambda \mathbb{Z}$, and if G is any group Hausdorff in the $\mathbb{Z}$-*adic *topology and having $|G| < \lambda$, then for every $\phi \in$* Hom(A, G), *$\phi(\mathbb{Z})$ is contained in a cotorsion subgroup of G.*

Proof. Let $T_n = \{f: \{0, 1, ..., n-1\} \to \lambda \times \lambda \times \mathbb{Z}\}$ and $T = \bigcup_{n<\omega} T_n$. Let $T^* = \{f: \omega \to ((\lambda \times \lambda) - \Delta) \times \mathbb{Z}\}$. We generate a free abelian group F by using generators $e_{g,\alpha,n}$ for $g\in T$, $\alpha<\lambda$, and $n\in\mathbb{Z}$ and a distinguished element e. We write

$$F=e\mathbb{Z}\oplus\bigoplus_{T\times\lambda\times\mathbb{Z}} e_{g,\alpha,n}\mathbb{Z}.$$

Define the group P by

$$P=e\hat{\mathbb{Z}}\oplus\prod_{T\times\lambda\times\mathbb{Z}} e_{g,\alpha,n}\mathbb{Z}.$$

For $f\in T^*$, let $f(i)=(f^0(i), f^1(i), f^2(i))$ for $i\in\omega$, $f^0(i)$, $f^1(i)\in\lambda$, and $f^2(i)\in\mathbb{Z}$. Define x_f by $x_f=e\cdot\sum_{i=0}^{\infty} f^2(i)\cdot i!+\sum_{i=0}^{\infty}(e_{f\restriction i,f^0(i),f^2(i)}-e_{f\restriction i,f^1(i),f^2(i)})\,i!$ and observe that $x_f\in P$. We let A be the pure subgroup of P generated by F and by $\{x_f: f\in T^*\}$. Since the x_f's are pairwise almost disjoint, it follows that $e\mathbb{Z}=e\hat{\mathbb{Z}}\cap A$. Hence $A/e\mathbb{Z}=A/e\hat{\mathbb{Z}}\cap A=A+e\hat{\mathbb{Z}}/e\hat{\mathbb{Z}}\leqslant\Pi_\lambda\mathbb{Z}$ and thus we have the exact sequence $0\to e\mathbb{Z}\to A\to C\to 0$ with $e\mathbb{Z}\cong\mathbb{Z}$ and $C\leqslant\Pi_\lambda\mathbb{Z}$.

Next let $\pi\in\hat{\mathbb{Z}}$ be a p-adic number. Then π has the representation $\pi=\sum_{i=1}^{\infty} t_i\cdot i!$, $t_i\in\mathbb{Z}$. Let G be a group of cardinality less than λ and let $\phi\in\mathrm{Hom}(A, G)$. Using induction in a similar way as done in the proof of Theorem 2.1, one can show that there exists an $f\in T^*$ so that $f^2(i)=t_i$ for all $i\in\omega$ and that

$$\phi(e_{f\restriction i,f^0(i),t_i})=\phi(e_{f\restriction i,f^1(i),t_i})$$

for all $i\in\omega$.

Therefore $\phi(x_f)\equiv\phi(e\pi)$ mod $i!\,G$ for all $i\in N$. Since G is Hausdorff in the $\mathbb{Z}$-adic topology, limits are unique, and it is clear that we may define $\phi^*: e\hat{\mathbb{Z}}\to G$ by defining $\phi^*(\pi)=\phi(x_f)$ where f is constructed as above. We obtain $\phi^*(e\hat{\mathbb{Z}})$ is an epimorphic image of an algebraically compact group and thus ϕ carries $e\mathbb{Z}$ into a cotorsion subgroup of G.

3.2. Corollary. *If G is a cotorsion-free, then R_G^∞ is not of the form R_X for any X.*

Proof. Suppose that $R_G^\infty=R_X$ for some group X. We first argue that X must be cotorsion-free. From [DG] there exists an $\aleph_1$-free H such that $\mathrm{Hom}(H, G)=0$. Assume $0\neq C\subset X$ is cotorsion. Then there exists an algebraically compact group A and an epimorphism $\phi: A\to C$. Let F be a nontrivial pure free subgroup of H and $\psi: F\to A$ be a nonzero homomorphism with the property that $\phi\psi: F\to C$ is nonzero. Since A is pure injective, there exists an extension of ψ to all of H giving a nonzero homomorphism from H to C and hence from H to X. This implies

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$R_X H \neq H$. But $R_G H = H$, so $R_G^\infty H = H$, a contradiction. Thus X is cotorsion-free.

Let λ be an infinite cardinal larger than the cardinality of X, and let A be the group of the theorem. Since $A/e\mathbb{Z} \leqslant \Pi_\lambda \mathbb{Z}$, $R_G(A/e\mathbb{Z}) \leqslant R_G(\Pi_\lambda \mathbb{Z}) = 0$. Let $\phi: A \to A/e\mathbb{Z}$ be the natural map; if $a \in R_G A$, then $\phi(a) \in R_G(A/e\mathbb{Z}) = 0$, and so $a \in e\mathbb{Z}$. Thus $R_G A = e\mathbb{Z}$. Consequently, $R_G^2 A = R_G^\infty A = 0$. But $R_X A = e\mathbb{Z}$, a contradiction.

We adopt some notation: if R is a radical, then $\mathscr{A}_R$ denotes the annihilator class of R. We shall denote the p-torsion and torsion subgroup functors by τ_p and τ, respectively. The divisible p-torsion subgroup functor is denoted by $d\tau_p$. If M is a group then the socle determined by M, S_M, is defined by $S_M A = \sum \{\operatorname{Im} f : f \in \operatorname{Hom}(M, A)\}$.

3.3. Lemma. *Let* $Y = \mathbb{Q} \oplus \prod_{p \in \pi_0} \mathbb{Z}(p^\infty) \oplus \prod_{p \in \pi_1 - \pi_0} \mathbb{Z}(p)$ *and* $M = \bigoplus_{p \in \pi_1 - \pi_0} \mathbb{Z}(p^\infty) \oplus (\bigoplus_{p \notin \pi_1} (\bigoplus_n \mathbb{Z}(p^n)))$ *where* $\varnothing \neq \pi_0 \subseteq \pi_1 \subseteq \{p: p \text{ prime}\}$. *Then*

$$R_Y^\infty = S_M = \sum_{q \notin \pi_0} \tau_q \cap \left(\bigcap_{p \in \pi_1 - \pi_0} p^\infty \right) = \sum_{q \notin \pi_1} \tau_q + \sum_{p \in \pi_1 - \pi_0} d\tau_p.$$

Proof. We have $R_Y = R_{\mathbb{Q}} \cap (\bigcap_{p \in \pi_0} R_{\mathbb{Z}(p^\infty)}) \cap (\bigcap_{p \in \pi_1 - \pi_0} R_{\mathbb{Z}(p)}) = \tau \cap (\sum_{q \notin \pi_0} \tau_q) \cap (\bigcap_{p \in \pi_1 - \pi_0} p)$, and R_Y is not idempotent for $R_Y \mathbb{Z}(p^2) \neq R_Y^2 \mathbb{Z}(p^2) = 0$ for $p \in \pi_1 - \pi_0$. Clearly, $\sum_{q \notin \pi_0} \tau_q \cap (\bigcap_{p \in \pi_1 - \pi_0} p^\infty) = \sum_{q \notin \pi_1} \tau_q + \sum_{p \in \pi_1 - \pi_0} d\tau_p = S_M$. Since $R_Y^\infty \leqslant R_Y$, $R_Y^\infty \leqslant \bigcap_{p \in \pi_1 - \pi_0} p^\infty$ and so $R_Y^\infty \leqslant S_M$. But as $\operatorname{Hom}(M, Y) = 0$, $Y \in \mathscr{A}_{S_M}$, hence $\mathscr{A}_{R_Y} \subseteq \mathscr{A}_{S_M}$ and, as S_M is a radical, $S_M \leqslant R_Y$. It follows that $S_M \leqslant R_Y^\infty$ since S_M is a socle contained in R_Y.

3.4. Lemma. *Let* $Z = \prod_{p \in \pi_2} \mathbb{Z}(p) \oplus \prod_{p \in \pi_3 - \pi_2} J_p$ *and* $N = Q^{(\pi_3)} \oplus (\bigoplus_{p \in \pi_3 - \pi_2} (\bigoplus_M \mathbb{Z}(p^M)))$ *where* $\pi_3 \neq 0$ *and* $\pi_2 \subseteq \pi_3 \subseteq \{p: p \text{ prime}\}$, *and* $Q^{(\pi_3)} = \{a/b \in Q: (b, p) = 1 \text{ if } p \notin \pi_3\}$. *Then* $R_Z^\infty = S_N^{(\infty)} = (\bigcap_{p \in \pi_2} p^\infty) \cap (\bigcap_{p \in \pi_3 - \pi_2} (p^\omega : \tau))$, *where* $S_N^{(\infty)}$ *denotes the smallest radical containing* S_N (*see* [FOW2]).

Proof. We have $R_Z = (\bigcap_{p \in \pi_2} p) \cap (\bigcap_{p \in \pi_3 - \pi_2} p^\omega : \tau)$, and R_Z is not a socle as $R_Y \mathbb{Z}(p^2) \neq R_Y^2 \mathbb{Z}(p^2) = 0$ for all $p \in \pi_2$. We also have $S_N = S_Q(\pi_3) + \sum_{p \in \pi_3 - \pi_2} \tau_p \geqslant \sum_{p \notin \pi_2} \tau_p$ since $S_{Q^{(\pi_3)}}$ fixes $\mathbb{Z}(q^n)$ for all $q \notin \pi_3$ and $n \geqslant 1$. Let $S = S_N^{(\infty)}$. If A is fixed by R_Z^∞ (equivalently by R_Z), then, without loss of generality we may assume A is reduced, and we have $p^\infty A = pA = A$ for all $p \in \pi_2$ and $\operatorname{Hom}(A/\tau A, Jp) = 0$ for all $p \in \pi_3 - \pi_2$. The latter implies $A/\tau A$ is p-divisible for all $p \in \pi_3 - \pi_2$ and so it follows that $A/\tau A$ is p-divisible for all $p \in \pi_3$. Thus $A/\tau A$ is fixed by $S_Q(\pi_3)$ and hence by S. Now if $p \in \pi_2$, then $\tau_p A = 0$ so $\tau A = \sum_{p \notin \pi_2} \tau_p A \leqslant S_Q(\pi_3) A + \sum_{p \in \pi_3 - \pi_2} \tau_p A$ so that τA is fixed by S. The stabilizer class of S is the smallest stabilizer class containing

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the stabilizer class of S_N which is also closed under formation of extensions, and thus, as $A/\tau A$ and τA are fixed by S_N, it follows that S fixes A. This means $R_Z^\infty \leqslant S$. On the other hand, as $\mathrm{Hom}(N, Z)=0$, $S_N \leqslant R_Z^\infty$ and so $S=S_N^{(\infty)} \leqslant R_Z^\infty$. This shows that $R_Z^\infty = (\bigcap_{p\in\pi_2} p^\infty) \cap (\bigcap_{p\in\pi_3-\pi_2} (p^\omega : \tau))$.

3.5. THEOREM. *A radical of the form R_X is idempotent if and only if $R_X = R_Y$ where $Y=\mathbb{Q}$, $\prod_{p\in\pi} \mathbb{Z}(p^\infty)$, or $\prod_{p\in\pi} J_p$ for some nonempty set of primes π.*

Proof. We divide the proof into several cases. First assume X is torsion-free. If X is not reduced, then $X=dX\oplus A$ where $dX=\bigoplus \mathbb{Q}$ and A is reduced. Then $R_X = R_{\mathbb{Q}} \cap R_A = \tau \cap R_A$. But as $\mathrm{Hom}(\mathbb{Z}(p^n), A)=0$ for all p and all n, $\tau_p \leqslant R_A$ for all p, so $\tau \leqslant R_A$ and $R_X = R_{\mathbb{Q}}$. If X is reduced, then X cannot be cotorsion-free, so let $\pi=\{p : J_p \leqslant X\}$ and set $Y=\prod_\pi J_p$. Then R_Y is an idempotent radical and $R_X \leqslant R_Y$ as $Y\in \mathscr{A}_{R_X}$. We have $R_Y X = \bigcap_\pi p^\omega X$. Set $R_Y X = Q$ and observe that $\mathrm{Hom}_{\mathbb{Z}}(M, X) = \mathrm{Hom}_{\mathbb{Q}(\pi)}(M, Q)$ for all $\mathbb{Q}^{(\pi)}$-modules M. Thus $R_X = R_Q$ on the category of $\mathbb{Q}^{(\pi)}$-modules. But as Q is cotorsion-free, this implies $R_Q \neq R_Q^2$, if $Q\neq 0$ and so $R_Y X=0$ which implies $R_Y \leqslant R_X$, and we have equality. If X is mixed, suppose X is not reduced. Let $\pi_0=\{p : \mathbb{Z}(p^\infty) \leqslant X\}$ and $\pi_1 = \{p : \tau_p X \neq 0\}$; we have $\pi_0 \subseteq \pi_1$. Set $Y=\mathbb{Q}\oplus \prod_{p\in\rho_0} \mathbb{Z}(p^\infty) \oplus \prod_{p\in\pi_1-\pi_0} \mathbb{Z}(p)$ as in Lemma 3.4. Then $R_Y^\infty = S_M = \sum_{q\in\pi_1} \tau_q + \sum_{p\in\pi_1-\pi_0} d\tau_p$. Clearly $R_X \leqslant R_Y$ so $R_Y^\infty \geqslant R_X$. But as $\mathrm{Hom}(M, X)=0$, if $\pi_1-\pi_2\neq 0$, we have $S_M \leqslant R_X$ and so $R_X = S_M = R_Y^\infty$. But R_Y^∞ is not of the form R_A for any A. For if so, thus letting λ_p be the p-length of A for each prime p, we have $\bigcap_{p\in\pi_1-\pi_0} p^\infty A = \bigcap_{p\in\pi_1-\pi_0} p^{\lambda_p} A$. Let $q\in\pi_1-\pi_0$ and let $H=H_{\lambda q+1}$ be Nunke's simply presented q-group of length $\lambda q+1$. Then as $\sum_{p\notin\pi_0} \tau_p H = H$ and $\bigcap_{p\in\pi_1-\pi_0} p^\infty H = q^\infty H = 0$, we have $R_X H=0$ and so $H\leqslant \prod_I A$. But $0\neq \mathbb{Z}(q) = q^{\lambda q} H = \sum_{p\notin\pi_0} \tau_p H \cap (\bigcap_{p\in\pi_1-\pi_0} p^{\lambda p} H) \leqslant (\sum_{p\in\pi_0} \tau_p \cap (\bigcap_{p\in\pi_1-\pi_0} p^{\lambda p})(\prod_I A) = \sum_{p\notin\pi_0} \tau_p \cap (\bigcap_{p\in\pi_1-\pi_0} p^\infty)(\prod_I A)=0$. Thus it must be the case that $\pi_1=\pi_0$ and $R_X = \tau \cap R_{\prod_{\pi_0} \mathbb{Z}(p^\infty)} = \sum_{p\notin\pi_0} \tau_p = R_{\prod_{\pi_0}\mathbb{Z}(p^\infty)}$. If X is mixed, and reduced, then set $\pi_2=\{p : \mathbb{Z}(p) \leqslant X\}$ and $\pi_3 = \{p : J_p \leqslant \prod_I X\}$. As X is not cotorsion-free, both π_2 and π_3 cannot be empty and $\pi_2 \subseteq \pi_3$. Set N and Z as in Lemma 3.4, and $S=S_N^{(\infty)}$. Since $Z\in \mathscr{A}_{R_X}$, $R_X \leqslant R_Z$ so $R_X \leqslant R_Z^\infty$. But as $\mathrm{Hom}(N, X)=0$, we have $S_N \leqslant R_X$ so $S_N^{(\infty)} \leqslant R_X$ and hence $R_X = R_Z^\infty$. If $\pi_2 \neq 0$, then R_Z^∞ is not of the form R_A for any A (repeat the argument in the preceding case with $H=H_{\lambda q+1}$ for $q\in\pi_2$). Thus it must be the case that $\pi_2=0$ and so $R_X = R_Z$ where $Z=\prod_{\pi_3} J_p$.

Conversely, $R_{\mathbb{Q}}=\tau$ is idempotent, and $R_{\prod_\pi \mathbb{Z}(p^\infty)} = \sum_{p\notin\pi} \tau_p$ is idempotent. If $Y=\prod_\pi J_p$, then for any A, $R_Y A$ is pure in A as $A/R_Y A$ is a subgroup of a product of copies of Y and hence is torsion-free. Hence any map $f: R_Y A\to Y$ extends to a map $\hat{f}: A\to Y$ which implies $f=0$. Thus $R_Y^2 A = R_Y A$ and R_Y is idempotent.

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The next proposition shows that cotorsion-free groups are not the only groups which lead to non-singly cogenerated radical classes.

3.6. PROPOSITION. *There exists a non-cotorsion-free group G with R_G^∞ not of the form R_X for any X.*

Proof. Let p and q be distinct primes and $\mathbb{Z}_p$ and $\mathbb{Z}_q$ denote the localizations of $\mathbb{Z}$ at p and q respectively. Let $\hat{\mathbb{Z}}_q$ denote the p-adic completion of $\mathbb{Z}_q$ and set $G=\hat{\mathbb{Z}}_q \oplus \mathbb{Z}_p$. If M is a $\mathbb{Z}_p$-module, then $\mathrm{Hom}_{\mathbb{Z}}(M, G)=\mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ and we may repeat the arguments above to show that R_G^∞ is not of the form R_X in the category of $\mathbb{Z}_p$-modules. This at the same time implies R_G^∞ is not of the form R_X in the category of abelian groups.

Our next result identifies the idempotent radicals that can appear as R_G^∞ with G a non-cotorsion-free $\mathbb{Z}_p$-module. We denote the torsion subgroup functor by τ, the divisible subgroup functor by d, and the functor $d\colon\tau$ is defined by $d\colon\tau$ is that subgroup of A containing τA satisfying the equation $(d\colon\tau A)/\tau A=d(A/\tau A)$.

3.7. THEOREM. *If G is a non-cotorsion-free $\mathbb{Z}_p$-module, then*

$$R_G^\infty \in \{0, \tau, d, d\tau, d\colon\tau\}.$$

Proof. Since G is not cotorsion-free, G contains at least one of the following groups: $\mathbb{Q}$, $\mathbb{Z}^{(p^\infty)}$, $\mathbb{Z}(p)$, Jp.

If $\mathbb{Z}(p^\infty)\leqslant G$, then $R_G=R_G^\infty=0$. If $\mathbb{Q}\leqslant G$, then $R_G\leqslant R_{\mathbb{Q}}=\tau$; if G is torsion-free, then $R_G=R_G^\infty=\tau$; if G is not torsion-free, then $\mathbb{Z}(p)\leqslant G$ so $R_G\leqslant p\tau$ and so $R_G^\infty=d\tau$.

If G is reduced and $\mathbb{Z}(p)\leqslant G$, then $R_G^\infty=d$. If G is reduced and torsion-free, then $J_p\leqslant G$ and $\tau\leqslant R_G$. Let X be any group, $Y=X/\tau X$ and $Y=dY\oplus Z$. The group Z is torsion-free reduced and hence can be embedded in a product of copies of J_p, and thus $R_G Z=0$, so $R_G(Y)=dY$, and consequently, $R_G^\infty=d\colon\tau$.

3.8. *Remark.* If G is a non-cotorsion-free S-module where S is any discrete valuation domain which is not a field, nor complete, then the above theorem remains valid in the category of S-modules.

3.9. COROLLARY. *Let S be a proper subring of $\mathbb{Q}$ containing $\mathbb{Z}$.*

(a) *If G is a cotorsion-free S-module, then R_G^∞ is not of the form R_X for any X.*

(b) *R_G^∞ not of the form R_X for any X holds if and only if G is cotorsion-free is equivalent to S being local.*

IV. Almost Strongly Cotorsion-Free Groups

In this section, under the additional set-theoretic assumption that there exists no measurable cardinal, we show that torsion theories singly cogenerated by an almost strongly cotorsion-free group are not singly generated. We also give some open problems of a set-theoretic nature.

A group X is *almost strongly cotorsion-free* if there exists a cardinal κ_0, less than the first measurable cardinal $\aleph_m$, such that $\mathrm{Hom}(\mathbb{Z}_\kappa, X)=0$ for all $\kappa \geqslant \kappa_0$, where $\mathbb{Z}_K = \mathbb{Z}^\kappa / \mathbb{Z}^{<\kappa}$ and $\mathbb{Z}^{<\kappa} = \{f \in \mathbb{Z}^\kappa: |\text{support of } f| < \kappa\}$.

5.1. Theorem. ($\nexists \aleph_m$) *Assume there is no measurable cardinal, and let X be almost strongly cotorsion-free. Then the torsion theory T_X singly cogenerated by X is not singly generated.*

Proof. Assume T_X is generated by H. Since X is almost strongly cotorsion-free, $\mathrm{Hom}(\mathbb{Z}_\kappa, X)=0$ for all regular cardinals $\kappa \geqslant \kappa_0$. Therefore $\mathbb{Z}_\kappa$ belongs to the torsion class generated by H, which implies $\mathrm{Hom}(H, \mathbb{Z}_\kappa) \neq 0$. Now by choosing κ big enough and applying the Wald–Los Lemma ([W1], see also Lemma 2.6 of [DG]), we obtain that $\mathbb{Z}$ is a summand of H. This means $T_X = (\mathbb{Z}\text{-mod}, 0)$ a contradiction.

5.2. Corollary. ($\nexists \aleph_m$) *Assume there exists no measurable cardinal. If the torsion theory*

$$T = (\{\oplus, E, Q\}\,(A), \{\textstyle\prod, E, S\}\,(B))$$

is singly generated and singly cogenerated, then $\mathrm{Hom}(\mathbb{Z}_\kappa, B) \neq 0$ *for arbitrarily large* κ.

We conclude with some open problems. It is known [W2] that under the assumption $V=L$, cotorsion-free is equivalent to almost strongly cotorsion-free.

5.3. Question. Can $(V=L)$ be replaced by $(\nexists \aleph_m)$?

5.4. Question. Is there a "minimal" set-theoretic axiom making the torsion theory $T_{\mathbb{Z}}$ not singly generated? That is, is there an axiom equivalent to $T_{\mathbb{Z}}$ being singly generated?

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