A SATURATED MODEL OF AN UNSUPERSTABLE THEORY OF CARDINALITY GREATER THAN ITS THEORY HAS THE SMALL INDEX PROPERTY

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ABSTRACT

A model M of cardinality λ is said to have the small index property if for every $G \subseteq \mathrm{Aut}(M)$ such that $[\mathrm{Aut}(M) : G] \leqslant \lambda$ there is an $A \subseteq M$ with $|A| < \lambda$ such that $\mathrm{Aut}_A(M) \subseteq G$. We show that if M^* is a saturated model of an unsuperstable theory of cardinality greater than $Th(M)$, then M^* has the small index property.

1. *Introduction*

Throughout the paper we work in $\mathfrak{C}^{\mathrm{eq}}$, and we assume that M^* is a saturated model of T of cardinality λ. We denote the set of automorphisms of M^* by $\mathrm{Aut}(M^*)$ and the set of automorphisms of M^* fixing A pointwise by $\mathrm{Aut}_A(M^*)$. We say that M^* has the small index property if whenever G is a subgroup of $\mathrm{Aut}(M^*)$ with index not larger than λ then for some $A \subset M^*$ with $|A| < \lambda$, $\mathrm{Aut}_A(M^*) \subseteq G$. The main theorem of this paper is the following result of Shelah: if M^* is a saturated model of cardinality $\lambda > |T|$ and there is a tree of height some uncountable regular cardinal $\kappa \geqslant \kappa_r(T)$ with μ branches but at most λ nodes where $\mu > \lambda$, then M^* has the small index property; in fact

$$[\mathrm{Aut}(M^*) : G] \geqslant \mu$$

for any subgroup G of $\mathrm{Aut}(M^*)$ such that for no $A \subseteq M^*$ with $|A| < \lambda$ is $\mathrm{Aut}_A(M^*) \subseteq G$. By a result of Shelah on cardinal arithmetic this implies that if $\mathrm{Aut}(M^*)$ does not have the small index property, then for some strong limit μ such that $cf\mu = \aleph_0$,

$$\mu < \lambda < 2^{\mu}.$$

So in particular, if T is unsuperstable, M^* has the small index property.

In the paper [2] 'Uncountable saturated structures have the small index property' by Lascar and Shelah, the following result was obtained:

THEOREM 1.1. *Let M^* be a saturated model of cardinality λ with $\lambda > |T|$ and $\lambda^{<\lambda} = \lambda$. Then if G is a subgroup of* $\mathrm{Aut}(M^*)$ *such that for no $A \subseteq M^*$ with $|A| < \lambda$ is* $\mathrm{Aut}_A(M^*) \subseteq G$ *then* $[\mathrm{Aut}(M^*) : G] = \lambda^{\lambda}$.

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Proof. See [**2**].

COROLLARY 1.2. *Let M^* be a saturated model of cardinality λ with $\lambda > |T|$ and $\lambda^{<\lambda} = \lambda$. Then M^* has the small index property.*

THEOREM 1.3. *A theory T has a saturated model of cardinality λ if and only if $\lambda = \lambda^{<\lambda} + D(T)$ or T is stable in λ.*

Proof. See [**3**, Chapter VIII].

So we can assume in the rest of this paper that T is stable in λ.

THEOREM 1.4. *The theory T is stable in μ if and only if $\mu = \mu_0 + \mu^{<\kappa(T)}$ where μ_0 is the first cardinal in which T is stable.*

Proof. See [**3**, Chapter III].

Since T is stable in λ, we must have $\lambda = \lambda^{<\kappa(T)}$, so $cf\lambda \geqslant \kappa(T)$. Since the first cardinal κ such that $\lambda^\kappa > \lambda$ is regular, we also know that $cf\lambda \geqslant \kappa_r(T)$.

DEFINITION 1.5. Let Tr be a tree. If $\eta, \nu \in Tr$, then $\gamma[\eta, \nu]$ is the least γ such that $\eta(\gamma) \neq \nu(\gamma)$; otherwise it is min(height(η), height(ν)).

NOTATION 1.6. Let Tr be a tree. If $h \in \mathrm{Aut}(M^*)$ and $\alpha < \mathrm{height}(Tr)$, $\eta, \nu \in Tr$, then

$$h^{\eta(\alpha)<\nu(\alpha)} = \begin{cases} h & \text{if } \eta(\alpha) < \nu(\alpha), \\ \mathrm{id}_{M^*} & \text{otherwise.} \end{cases}$$

LEMMA 1.7. *Let $\{C_i \mid i \in I\}$ be independent over A and let $\{D_i \mid i \in I\}$ be independent over B. Suppose that for each $i \in I$, $tp(C_i/A)$ is stationary. Let f be an elementary map from A onto B, and for each $i \in I$, let f_i be an elementary map extending f which sends C_i onto D_i. Then*

$$\bigcup_{i \in I} f_i$$

is an elementary map from $\bigcup_{i \in I} C_i$ onto $\bigcup_{i \in I} D_i$.

Proof. This is left to the reader.

LEMMA 1.8. *Let $|T| < \lambda$. Let Tr be a tree of height ω with κ_n nodes of height n for some $\kappa_n < \lambda$. Let $n < \omega$ and let $\langle M_i \mid i \leqslant n\rangle$ be an increasing chain of models. Let $M_n \subseteq N_0 \subseteq N_1 \subseteq M^*$ with $|N_1| < \lambda$. Suppose $\langle h_i \mid i \leqslant n\rangle$ are automorphisms of M^* such that*

(1) $h_i = \mathrm{id}_{M_i}$,

(2) $h_i[N_j] = N_j$ *for* $j \leqslant 1$,

(3) $h_i[M_k] = M_k$ *for* $k \leqslant n$.

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For each $\nu \in Tr \restriction \text{level}(n+1)$ *let* m_ν, l_ν *be automorphisms of* N_0. *Let* $n \in Tr \restriction \text{level}(n+1)$. *Suppose* $g_\eta \in \text{Aut}(N_0)$ *such that, for all* $\nu \in Tr \restriction \text{level}(n+1)$,

$$g_\eta m_\eta (m_\nu)^{-1}(g_\eta)^{-1} = l_\eta (l_\nu)^{-1} h_{\gamma[\eta,\nu]}^{\eta(\gamma[\eta,\nu])<\nu(\gamma[\eta,\nu])}.$$

Let m_ν^+, l_ν^+ *be extensions of* m_ν *and* l_ν *to automorphisms of* N_1 *for all* $\nu \in Tr \restriction \text{level}(n+1)$. *Then there exist a model* $N_2 \subseteq M^*$ *containing* N_1 *such that* $|N_2| \leqslant |N_1| + |T| + \kappa_{n+1}$ *and* $h_i[N_2] = N_2$ *for* $i \leqslant n$, *and a* $g'_\eta \in \text{Aut}(N_2)$ *extending* g_η *and for all* $\nu \in Tr \restriction \text{level}(\alpha+1)$ *automorphisms of* N_2, m'_ν *and* l'_ν *extending* m_ν^+ *and* l_ν^+ *respectively such that*

$$g'_\eta m'_\eta (m'_\nu)^{-1}(g'_\eta)^{-1} = l'_\eta (l'_\nu)^{-1} h_{\gamma[\eta,\nu]}^{\eta(\gamma[\eta,\nu])<\nu(\gamma[\eta,\nu])}.$$

Proof. Let g_η^+ be a map with domain N_1 such that $g_\eta^+(N_1) \Cup_{N_0} N_1$, $g_\eta^+(N_1) \subseteq M^*$ and g_η^+ extends g_η. Let g_η^{++} be a map extending g_η such that the domain of $(g_\eta^{++})^{-1}$ is N_1, $(g_\eta^{++})^{-1}(N_1) \subseteq M^*$ and $(g_\eta^{++})^{-1}(N_1) \Cup_{N_0} N_1$. So $g_\eta^+ \cup g_\eta^{++}$ is an elementary map. Let l''_η and m''_η be extensions of l_η^+ and m_η^+ to automorphisms of M^*. Let

$$m_\nu^{++} = (g_\eta^{++})^{-1}(h_{\eta,\nu})^{-1} l_\nu (l_\eta)^{-1} g_\eta^{++} m''_\eta \restriction (m'_\eta)^{-1}[(g^{++})^{-1}[N_1]].$$

Note that $m_\nu^+ \cup m_\nu^{++}$ is an elementary map. Let

$$l_\nu^{++} = (l''_\eta)^{-1} g_\eta^+ m_\eta^+ (m_\nu^+)^{-1} (g_\eta^+)^{-1} (h_{\eta,\nu})^{-1} \restriction h_{\eta,\nu}[g_\eta^+[N_1]].$$

Note that $l_\nu^+ \cup l_\nu^{++}$ is an elementary map. Let g''_η, m''_ν, l''_ν be elementary extensions to M^* of $g_\eta^+ \cup g_\eta^{++}$, $m_\nu^+ \cup m_\nu^{++}$, and $l_\nu^+ \cup l_\nu^{++}$. Let N_2 be a model of size $|N_1| + |T| + \kappa_{n+1}$ containing N_1 such that N_2 is closed under m''_η, g''_η, l''_η, all the $h_{\eta,\nu}$ and m''_η, l''_ν. Let m'_ν, l'_ν, g'_η, $h'_{\eta,\nu}$, m'_η, l'_η be the restrictions to N_2 of the m''_ν, l''_ν, g''_η, $h''_{\eta,\nu}$, m''_η, l''_η.

THEOREM 1.9. *If* $\lambda > |T|$, $cf\lambda = \omega$, M^* *is a saturated model of cardinality* λ, *and if* G *is a subgroup of* $\text{Aut}(M^*)$ *such that for no* $A \subseteq M^*$ *with* $|A| < \lambda$ *is* $\text{Aut}_A(M^*) \subseteq G$, *then* $[\text{Aut}(M^*) : G] = \lambda^\omega$.

Proof. Suppose that the theorem is not true. Let $\{\kappa_i \mid i < \omega\}$ be an increasing sequence of cardinals each greater than $|T|$ with $\sup = \lambda$. Let $Tr = \{\eta \in {}^{<\omega}\lambda \mid \eta(i) < \kappa_i\}$. Let $M^* = \bigcup_{i<\omega} B_i$ with $|B_i| \leqslant \kappa_i$. By induction on $n < \omega$ for every $\eta \in Tr \restriction \text{level}\, n$ we define models $N_n \subset M^*$ and $h_n \in \text{Aut}_{N_n}(M^*) - G$ such that $B_n \subseteq N_n$ and $|N_n| \leqslant \kappa_n$, and automorphisms g_η, m_η, l_η of N_n such that if $\rho \neq \nu$ then $l_\rho \neq l_\nu$ and

$$g_\rho m_\rho (m_\nu)^{-1}(g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h_{\gamma[\rho,\nu]}^{\rho(\gamma[\rho,\nu])<\nu(\gamma[\rho,\nu])}.$$

Suppose we have defined the g_η, m_η, l_η for height$(\eta) \leqslant m$, and N_j for $j \leqslant m$. If $n = m+1$, for each $i < \kappa_n$ we define models $N_{n,i}$ such that $B_n \subseteq N_{n,i}$, $N_m \subseteq N_{n,i}$, $\langle N_{n,i} \mid i < \kappa_n \rangle$ is increasing continuous, and for some $\eta_i \in Tr \restriction \text{level}\, n$, $g_{\eta_i} \in \text{Aut}(N_{n,i})$ such that for each $\eta \in Tr \restriction \text{level}\, n$, $\eta = \eta_i$ cofinally many times in κ_n, and for every $\nu \in Tr \restriction \text{level}\, n$, $m_\nu^i \neq l_\nu^i \in \text{Aut}(N_{n,i})$ such that

$$g_{\eta_i} m_{\eta_i}^i (m_\nu^i)^{-1} (g_{\eta_i})^{-1} = l_{\eta_i}^i (l_\nu^i)^{-1} h_{\gamma[\eta_i,\nu]}^{\eta_i(\gamma[\eta_i,\nu])<\nu(\gamma[\eta_i,\nu])}.$$

The g_{η_i}, m_ν^i, l_ν^i are easily defined by induction on $i < \kappa_n$ using Lemma 1.8, so that if $i_1 < i_2$ then $m_\nu^{i_1} \subseteq m_\nu^{i_2}$, $l_\nu^{i_1} \subseteq l_\nu^{i_2}$, and if $\eta_{i_1} = \eta_{i_2}$ then $g_{\eta_{i_1}} \subseteq g_{\eta_{i_2}}$. Then if we let

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$g_\eta = \bigcup\{g_{\eta_i} \mid \eta_i = \eta\}$, $m_\eta = \bigcup_{i<\kappa_n} m^i_\eta$, $l_\eta = \bigcup_{i<\kappa_n} l^i_\eta$, $N_n = \bigcup_{i<\kappa_n} N_{n,i}$ and $h_n \in \mathrm{Aut}_{N_n}(M^*) - G$, we have finished. Let Br be the set of branches of Tr of height ω. For $\rho \in Br$ let $g_\rho = \bigcup\{g_\eta \mid \eta < \rho\}$, $m_\rho = \bigcup\{m_\eta \mid \eta < \rho\}$, and $l_\rho = \bigcup\{l_\eta \mid \eta < \rho\}$. If $\rho \neq \nu$, then $g_\rho \neq g_\nu$ since without loss of generality we may assume that $\rho(\gamma[\rho, \nu]) < \nu(\gamma[\rho, \nu])$ and

$$g_\rho m_\rho (m_\nu)^{-1} (g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h^{\rho(\gamma[\rho,\nu])<\nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]}$$

and

$$g_\nu m_\nu (m_\rho)^{-1} (g_\nu)^{-1} = l_\nu (l_\rho)^{-1}$$

implies

$$g_\rho (g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h^{\rho(\gamma[\rho,\nu])<\nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]}.$$

So if $g_\rho = g_\nu$, this would imply that $h^{\rho(\gamma[\rho,\nu])<(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]} = \mathrm{id}_{M^*}$, a contradiction. If

$$[\mathrm{Aut}(M^*) : G] < \lambda^\omega$$

then for some $\rho, \nu \in Br$ we must have $l_\rho (l_\nu)^{-1} \in G$ and $g_\rho (g_\nu)^{-1} \in G$, but then we get a contradiction as $g_\rho (g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} \in G$ and $l_\rho (l_\nu)^{-1} \in G$, but $h^{\rho(\gamma[\rho,\nu])<\nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]} \notin G$.

COROLLARY 1.10. *If $\lambda > |T|$, $cf\lambda = \omega$ and M^* is a saturated model of cardinality λ then M^* has the small index property.*

So we will assume in the remainder of the paper that in addition to T being stable, $cf\lambda \geq \kappa_r(T) + \aleph_1$ and T, M^*, and λ are constant.

2. *Constructing M^* as a chain from K_δ*

DEFINITION 2.1. Let $\delta < \lambda^+$, $cf\delta \geq \kappa_r(T)$. Define

$$K^s_\delta = \{\bar N \mid \bar N = \langle N_i \mid i \leq \delta\rangle, N_i \text{ is increasing continuous}, |N_i| = \lambda,$$
$$N_0 \text{ is saturated}, N_\delta = M^*, \text{ and } (N_{i+1}, c)_{c \in N_i} \text{ is saturated}\}.$$

For $\mu > \aleph_0$,

$$K^\mu_\delta = \{\bar A \mid \bar A = \langle A_i \mid i \leq \delta\rangle, A_i \text{ is increasing continuous}, |A_\delta| < \mu, \mathrm{acl}\, A_i = A_i\}.$$

If $\bar A \in K^{\lambda^+}_\delta$, then $f \in \mathrm{Aut}(\bar A)$ if f is an elementary permutation of A_δ and if $i \leq \delta$, then $f \restriction A_i$ is a permutation of A_i.

DEFINITION 2.2. Let $\bar A^0, \bar A^1 \in K^\mu_\delta$. Then $\bar A^0 \leq \bar A^1$ if and only if $\bigwedge_{i \leq \delta} A^0_i \subseteq A^1_i$ and

$$i < j \leq \delta \Rightarrow A^1_i \mathop{\cup\!\!\!\cup}_{A^0_i} A^0_j.$$

LEMMA 2.3. (1) *The order $(K^\mu_\delta, \leq)$ is partial.*

(2) *Let $\bar A^\zeta \in K^\mu_\delta$ for $\zeta < \zeta(*)$ and let $\xi < \zeta \Rightarrow \bar A^\xi \leq \bar A^\zeta$. If we let $A_i = \bigcup_{\zeta<\zeta(*)} A^\zeta_i$, and $|\bigcup_{\zeta<\zeta(*)} A^\delta_i| < \mu$, then*

$$\bar A = \langle A_i \mid i \leq \delta\rangle \in K^\mu_\delta,$$

and for every $\zeta < \zeta(*)$, $\bar A_\zeta \leq \bar A$.

(3) *If $\bar A^\zeta \leq \bar A^*$ for $\zeta < \zeta(*)$, and $\bar A$ is as above, then $\bar A \leq \bar A^*$.*

Proof. (1) This follows from the transitivity of non-forking.
(2) This can be proved using the finite character of forking.
(3) This also follows from the finite character of forking.

DEFINITION 2.4. Let $A \subseteq M$, with $|A| < \kappa_r(T)$ and let $p \in S(\operatorname{acl} A)$. Then $\dim(p, M)$ is the minimal cardinality of a maximal independent set of realizations of p inside M. If M is $\kappa_r^\varepsilon(T)$-saturated (κ_r^ε-saturated means $\aleph_\varepsilon$-saturated if $\kappa_r(T) = \aleph_0$, and $\kappa_r(T)$-saturated otherwise), then by [**3**, III, 3.9], $\dim(p, M)$ is the cardinality of any maximal independent set of realizations of p inside M.

LEMMA 2.5. *Let $|M| = \lambda$ and assume that M is $\kappa_r^\varepsilon(T)$-saturated. Then M is saturated if and only if for every $A \subseteq M$, with $|A| < \kappa_r(T)$ and $p \in S(\operatorname{acl} A)$,* $\dim(p, M) = \lambda$.

Proof. See [**3**, III, 3.10].

LEMMA 2.6. *Let $\langle \bar{A}^\alpha |\ \alpha < \lambda \rangle$ be an increasing continuous sequence of elements of $K_\delta^{\lambda^+}$ such that, for all $\gamma < \delta$, and all $A \subseteq \bigcup_{\alpha<\lambda} A_\gamma^\alpha$, if $|A| < \kappa_r(T)$ and $p \in S(\operatorname{acl} A)$ then for λ values of $\alpha < \lambda$,*

(1) *$A_\zeta^\alpha = A_\zeta^{\alpha+1}$ for all $\zeta \leq \gamma$,*

(2) *there exists $a \in A_{\gamma+1}^{\alpha+1}$ such that the type of $a / A_{\gamma+1}^\alpha$ is the stationarization of p.*

Then

$$\langle N_\gamma |\ \gamma < \delta \rangle \in K_\delta^s$$

where $N_\gamma = \bigcup_{\alpha<\lambda} A_\gamma^\alpha$.

Proof. It is enough to show, for all $\gamma < \delta$, that $(N_{\gamma+1}, c)_{c \in N_\gamma}$ is saturated. For this, by Lemma 2.5, it is enough to show, for all $A \subseteq N_{\gamma+1}$ such that $|A| < \kappa_r(T)$ and for every type $p \in S(\operatorname{acl} A \cup N_\gamma)$, that

$$\dim(p, N_{\gamma+1}) = \lambda.$$

By the assumption of the lemma, there exists $\{a_i |\ i < \lambda\}$ realizations of $p \upharpoonright \operatorname{acl} A$ and $\{A_{\gamma+1}^{\alpha_i} |\ i < \lambda\rangle$ such that for each $i < \lambda$, $a_i \in A_{\gamma+1}^{\alpha_i+1}$, $A_\gamma^{\alpha_i+1} = A_\gamma^{\alpha_i}$, and

$$a_i \perp\!\!\!\perp_A A_{\gamma+1}^{\alpha_i} \quad \text{and} \quad a_i \cup A_{\gamma+1}^{\alpha_i} \perp\!\!\!\perp_{A_\gamma^{\alpha_i}} N_\gamma,$$

which implies that

$$a_i \perp\!\!\!\perp_{A_{\gamma+1}^{\alpha_i}} N_\gamma \quad \text{and} \quad a_i \perp\!\!\!\perp_A N_\gamma.$$

Since $cf\lambda \geq \kappa_r(T)$, without loss of generality we may assume that $A \subseteq A_{\gamma+1}^{\alpha_0}$. We must show that the $\langle a_i |\ i < \lambda \rangle$ are independent over $N_\gamma \cup A$. By induction on $i < \lambda$, we show that

$$\langle a_j |\ j \leq i \rangle$$

are independent over $A \cup \{A_\gamma^{\alpha_j} |\ j \leq i\}$. This is enough as

$$\{a_j |\ j \leq i\} \perp\!\!\!\perp_{A \cup \{A_\gamma^{\alpha_j} |\ j \leq i\}} N_\gamma.$$

Since $\langle a_j |\ j < i \rangle$ are independent over $A \cup \{A_\gamma^{\alpha_j} |\ j < i\}$, and

$$\{a_j |\ j < i\} \perp\!\!\!\perp_{A \cup \{A_\gamma^{\alpha_j} |\ j < i\}} A_\gamma^{\alpha_i},$$

the $\langle a_j |\ j < i\rangle$ are independent over $A \cup A^{\alpha_i}_{\gamma}$. Since $a_i \Downarrow_{A\cup A^{\alpha_i}_{\gamma}} A^{\alpha_i}_{\gamma+1}$, we have

$$a_i \Downarrow_{A\cup A^{\alpha_i}_{\gamma}} \{a_j |\ j < i\}.$$

LEMMA 2.7. *Let $\langle \bar{N}^{\alpha} |\ \alpha < \delta\rangle$ be an increasing continuous sequence of elements of $K^{\mu^+}_{\delta}$ such that $\bigcup_{\alpha<\delta} N^{\alpha}_{\delta} = M^*$ and for every $\gamma < \delta$, and $\alpha < \delta$,*

$$(N^{\alpha+1}_{\gamma+1}, c)_{c\in N^{\alpha}_{\gamma+1}\cup N^{\alpha+1}_{\gamma}}$$

and

$$(N^{\alpha+1}_{0}, c)_{c\in N^{\alpha}_{0}}$$

are saturated of cardinality λ. Then

$$\langle N_{\gamma} |\ \alpha < \delta\rangle \in K^{s}_{\delta}$$

where $N_{\gamma} = \bigcup_{\alpha<\delta} N^{\alpha}_{\gamma}$.

Proof. This is similar to the proof of the previous lemma.

LEMMA 2.8. *Let $cf\delta \geq \kappa_r(T) + \aleph_1$. Let $\bar{M} \in K^{s}_{\delta}$. Let $A_{\delta} \subseteq M^*$ such that $|A_{\delta}| < \lambda$ and $A_{\delta} = \bigcup_{i<\delta} A_i$ where $\langle A_i |\ i < \delta\rangle$ is an increasing continuous chain. Suppose that, for all $\beta < \delta$ and $i < \delta$,*

$$M_{\beta} \Downarrow_{A_i\cap M_{\beta}} A_i.$$

Let $a \subseteq M_{\beta^*}$ such that $|a| < \kappa_r(T)$. Then there exist a continuous increasing sequence $\langle A'_i |\ i < \delta\rangle$ and a set B such that $|B| < \kappa_r(T)$, $A_i \subset A'_i$, $a \subset \bigcup A'_i = A'_{\delta}$, $|A'_{\delta}| < \lambda$, for some non-limit $i^* < \delta$, $A'_i = A_i$ if $i < i^*$, and $A'_i = A_i \cup B$ if $i^* \leq i$ and for all i, $\beta < \delta$,

$$M_{\beta} \Downarrow_{A'_i\cap M_{\beta}} A'_i$$

and for all i, $\beta < \delta$,

$$M_{\beta} \cup (M_{\beta+1} \cap A_{\delta}) \Downarrow_{M_{\beta}\cup(M_{\beta+1}\cap A_i)} A'_i \cap M_{\beta+1}$$

and $A_{\delta} \Downarrow_{A_i} A'_i$.

Proof. First by induction on $n \in \omega$, we define $\langle B_n |\ n < \omega\rangle$ such that $B_0 = a$, $|B_n| < \kappa_r(T)$ *for all $i < \delta$ and $\beta < \delta$,*

$$B_n \Downarrow_{(M_{\beta}\cap(A_i\cup B_{n+1}))\cup A_i} M_{\beta} \cup A_i.$$

So suppose B_n has been defined. By induction on $m < \omega$ we define subsets C_1 and C_2 of δ such that $0 \in C_i$, $|C_i| \leq \kappa_r(T)$ and such that if (a_1, b_1), (a_2, b_1), (a_1, b_2), (a_2, b_2) are four neighbouring points in $C_1 \times C_2$ with $a_1 < a_2$ and $b_1 < b_2$, then, for all i, j such that $a_1 \leq i < a_2$ and $b_1 \leq j < b_2$,

$$B_n \Downarrow_{M_{a_1}\cup A_{b_1}} M_{a_1+i} \cup A_{b_1+j}.$$

So it is enough to find $|B_{n+1}| < \kappa_r(T)$ such that for every $(a, b) \in C_1 \times C_2$,

$$B_n \Downarrow_{(M_a\cap(A_b\cup B_{n+1}))\cup A_b} M_a \cup A_b.$$

As $|C_1 \times C_2| < \kappa_r(T)$, this is possible. Let $B = \bigcup_{n\in\omega} B_n$. (If $\kappa_r(T) = \aleph_0$ then without loss of generality we can define the B_n such that for some $k < \omega$, $\bigcup_{n\in\omega} B_n = \bigcup_{n\in k} B_n$.) It is enough to prove the following statement.

There exists a non-limit $i^* < \delta$ such that if $A'_i = A_i$ for $i < i^*$, and $A'_i = A_i \cup B$ for $i \geq i^*$ then the conditions of the theorem hold.

Proof. For all $\beta < \delta$ and all $i < \delta$, if $A'_i = A_i \cup B$, then since

$$B \mathop{\smile\!\!\!\!\!\smile}\nolimits_{(M_\beta \cap (A_i \cup B)) \cup A_i} M_\beta \cup A_i,$$

we have

$$A'_i \mathop{\smile\!\!\!\!\!\smile}\nolimits_{A'_i \cap M_\beta} M_\beta.$$

Let $i^{**} < \delta$ such that for all $i \geq i^{**}$,

$$A_\delta \mathop{\smile\!\!\!\!\!\smile}\nolimits_{A_i} A'_i.$$

It is enough to find $i^{**} \leq i^* < \delta$ such that for all $\beta < \delta$,

$$B \mathop{\smile\!\!\!\!\!\smile}\nolimits_{M_\beta \cup (M_{\beta+1} \cap A_{i^*})} M_\beta \cup (M_{\beta+1} \cup A_\delta).$$

Let $\langle \beta_\alpha \mid \alpha \in \gamma \rangle$ where $\gamma < \kappa_r(T)$ be the set of all places such that

$$B \not\mathop{\smile\!\!\!\!\!\smile}\nolimits_{M_{\beta_\alpha - 1} \cup (M_{\beta_\alpha} \cap A_\delta)} M_{\beta_\alpha} \cup (M_{\beta_\alpha + 1} \cap A_\delta).$$

For each $\beta \in \langle \beta_\alpha \mid \alpha \in \gamma \rangle$ let i_α be such that

$$B \mathop{\smile\!\!\!\!\!\smile}\nolimits_{M_{\beta_\alpha} \cup (M_{\beta_\alpha + 1} \cap A_{i_\alpha})} M_{\beta - 1} \cup (M_{\beta_\alpha + 1} \cap A_\delta).$$

Let i_γ be such that

$$B \mathop{\smile\!\!\!\!\!\smile}\nolimits_{M_0 \cup (M_1 \cap A_{i_\gamma})} M_0 \cup (M_1 \cap A_\delta).$$

Let $i^* = \sup\{i_\alpha \mid \alpha \in \gamma + 1\} + 1 + i^{**}$. As $|B| < \kappa_r(T)$ and $cf\delta \geq \kappa_r(T)$, $i^* < \delta$, so there is no problem.

LEMMA 2.9. *Let $\bar{M} \in K^s_\delta$. Let $A \subseteq M^*$ such that $|A| < \lambda$ and $A = \bigcup_{i<\delta} A_i$ where $\langle A_i \mid i < \delta \rangle$ is increasing continuous, each A_i is algebraically closed, and for all $i < \delta$ and all $\beta < \delta$,*

$$M_\beta \mathop{\smile\!\!\!\!\!\smile}\nolimits_{M_\beta \cap A_i} A_i.$$

Let i^* be a successor less than δ, $\beta^* < \delta$, β^* a successor, and let $p \in S(A_{i^*} \cap M_{\beta^*})$ (or even a type over $A_{i^*} \cap M_{\beta^*}$ in fewer than λ variables). Let

$$p' \in S((A_{i^*} \cap M_{\beta^*}) \cup M_{\beta^* - 1})$$

such that p' does not fork over p. Then there exists an $a \in M_{\beta^*}$ such that a realizes p', $A \mathop{\smile\!\!\!\!\!\smile}\nolimits_{M_{\beta^*} \cap A_{i^*}} a$, and if $A'_i = A_i \cup \{a\}$ for $i \geq i^*$ and $A'_i = A_i$ for $i < i^*$, then for all $\beta < \delta$ and all $i < \delta$,

$$M_\beta \mathop{\smile\!\!\!\!\!\smile}\nolimits_{M_\beta \cap A'_i} A'_i.$$

Proof. Let $B \subseteq M_{\beta^*}$ such that $|B| < \lambda$, $A^*_i \cap M_{\beta^*} \subseteq B$, and

$$M_{\beta^*} \mathop{\smile\!\!\!\!\!\smile}\nolimits_{M_{\beta^* - 1} B} A.$$

Let $a \in M_{\beta^*}$ such that a realizes p and $a \mathop{\smile\!\!\!\!\!\smile}\nolimits_{A^*_i \cap M_{\beta^*}} B \cup M_{\beta^* - 1}$. Since

$$M_{\beta^*} \mathop{\smile\!\!\!\!\!\smile}\nolimits_{M_{\beta^* - 1} \cup B} A,$$

we have $a \mathop{\smile\!\!\!\!\!\smile}\nolimits_{M_{\beta^* - 1} \cup B} A$, which implies that

$$a \mathop{\smile\!\!\!\!\!\smile}\nolimits_{A^*_i \cap M_{\beta^*}} M_{\beta^* - 1} \cup A.$$

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Since for all $i \geqslant i^*$, $a \Downarrow_{A_i} M_{\beta^*-1} \cup A$, we have for all $\gamma < \beta^*$,

$$a \Downarrow_{A_i} M_\gamma \cup A,$$

which implies that

$$a \cup A_i \Downarrow_{A_i \cap M_\gamma} M_\gamma.$$

Since $a \subseteq M_{\beta^*}$, we also have for all $\gamma \geqslant \beta^*$,

$$a \cup A_i \Downarrow_{(a \cup A_i) \cap M_\gamma} M_\gamma.$$

LEMMA 2.10. *Let* $\bar{M} \in K^s_\delta$. *Let* $A \subseteq M^*$ *such that* $|A| < \lambda$ *and* $A = \bigcup_{i<\delta} A_i$ *where* $\langle A_i \mid i < \delta \rangle$ *is increasing continuous, each* A_i *is algebraically closed and for all* $i < \delta$ *and all* $\beta < \delta$,

$$M_\beta \Downarrow_{M_\beta \cap A_i} A_i.$$

Let $i^* < \delta$, $\beta^* < \delta$, *where* β^* *and* i^* *are successors, and let* $p \in S(A_i \cap M_\beta)$. *Let* $p' \in S((A_i \cap M_{\beta^*}) \cup M_{\beta^*-1})$ *such that* p' *does not fork over* p. *Let* $f \in \mathrm{Aut}(A)$ *such that for all* $i < \delta$, $f[A_i] = A_i$. *Then there exist* $\{a_i \mid i \in \mathbb{Z}\} \subseteq M^*$ *and an extension* f' *of* f *with domain* $A \cup \{a_i \mid i \in \mathbb{Z}\}$ *such that* a_0 *realizes* p', $a_0 \in M_{\beta^*}$, *and for all* $i \in \mathbb{Z}$, $f'(a_i) = a_{i+1}$ *and if* $A'_i = A_i \cup \{a_i \mid i \in \mathbb{Z}\}$ *for* $i \geqslant i^*$ *and* $A'_i = A_i$ *for* $i < i^*$, *then for all* $\beta < \delta$,

$$M_\beta \Downarrow_{M_\beta \cap A'_i} A'_i, \quad A_\delta \Downarrow_{A_i} A'_i,$$

and

$$M_{\beta-1} \cup (M_\beta \cap A) \Downarrow_{M_{\beta-1} \cup (M_\beta \cap A_i)} M_\beta \cap A'_i.$$

Proof. We define $\{a_i \mid i \in -n, \ldots, 0, \ldots, n\}$ by induction on n such that if $A'_i = \mathrm{acl}(A_i \cup \{a_i \mid i \in -n, \ldots, 0, \ldots, n\})$ for $i \geqslant i^*$ and $A'_i = A_i$ for $i < i^*$, then for all $i < \delta$ and all $\beta < \delta$,

$$M_\beta \Downarrow_{M_\beta \cap A'_i} A'_i, \quad A_\delta \Downarrow_{A_i} A'_i,$$

and

$$M_{\beta-1} \cup (M_\beta \cap A) \Downarrow_{M_{\beta-1} \cup (M_\beta \cap A)} M_\beta \cap A'_i,$$

and $f_n = f \cup \{(a_i, a_{i+1}) \mid -n \leqslant i < n\}$ is an elementary map. In addition we define a sequence of successor ordinals $\langle \beta_i \mid i \in \mathbb{Z} \rangle$ such that $\beta_i < \beta_j$ if $|i| < |j|$, and $\beta_n < \beta_{-n}$ such that

$$a_{n+1} \Downarrow_{M_{\beta_{n+1}} \cap A_{i^*}} M_{\beta_{n+1}-1} \cup A \cup \{a_{-n}, \ldots, a_0, \ldots, a_n\}$$

and

$$a_{-(n+1)} \Downarrow_{M_{\beta_{-(n+1)}} \cap A_{i^*}} M_{\beta_{-(n+1)}-1} \cup A \cup \{a_{-n}, \ldots, a_0, \ldots, a_n, a_{n+1}\}.$$

Define a_0 as in the previous lemma. Suppose that $\{a_{-n}, \ldots, a_0, \ldots, a_n\}$ and β_i for $-n \leqslant i \leqslant n$ have been defined satisfying the conditions. Let $C = \mathrm{acl}\, C$ such that for some $B \subseteq C$ with $|B| < \kappa_r(T)$, $\mathrm{acl}\, B = C$, $C \subseteq M_{\beta_{-n}} \cap A_{i^*}$ and

$$a_n \Downarrow_C A \cup \{a_{-n}, \ldots, a_0, \ldots, a_{n-1}\}.$$

Let $\beta_{n+1} > \beta_{-n}$ be a successor such that $f(C) \subseteq M_{\beta_{n+1}} \cap A_{i^*}$. Let $a_{n+1} \in M_{\beta_{n+1}}$ realize

$$f_n(tp(a_n / A \cup \{a_{-n}, \ldots, a_0, \ldots, a_{n-1}\}))$$

and in addition

$$a_{n+1} \Cup_{M_{\beta_{n+1}} \cap A_i^*} A \cup M_{\beta_{n+1}-1}.$$

Impose similar conditions on $a_{-(n+1)}$. Now as in the proof of the previous lemma, all the conditions of the induction hold.

LEMMA 2.11. *Let δ be an ordinal less than λ^+ such that $cf\delta \geq \aleph_1 + \kappa_r(T)$. Let $f \in \mathrm{Aut}_E(M^*)$ with $|E| < \lambda$. Let $\bar{M} \in K^s_\delta$. Then there exist $\bar{N}^1$, $\bar{N}^2 \in K^s_\delta$, $f_1 \in \mathrm{Aut}_E(\bar{N}^1)$, $f_2 \in \mathrm{Aut}_E(\bar{N}^2)$ with $E \subseteq N^1_0$, $E \subseteq N^2_0$ such that*

(1) $f = f_2 f_1$,

(2) *for all i, $\beta < \delta$ and all $l \in \{0, 1\}$,*

$$M_\beta \Cup_{M_\beta \cap N^l_i} N^l_i,$$

(3) *for all i, $\beta < \delta$ and all $l \in \{0, 1\}$,*

$$(N^l_{i+1} \cap M_{\beta+1}, c)_{c \in (N^l_{i+1} \cap M_\beta) \cup (N^l_i \cap M_{\beta+1})}$$

is saturated of cardinality λ,

(4) $(N^l_{i+1} \cap M_0, c)_{c \in N^l_i \cap M_0}$ *is saturated of cardinality λ.*

Proof. Without loss of generality we may assume that $E = \emptyset$. By induction on $\alpha < \lambda$ we build increasing continuous sequences $\langle A^\alpha_i |\ i \leq \delta \rangle$, $\langle B^\alpha_i |\ i \leq \delta \rangle$, $\langle f^\alpha_1 |\ \alpha < \lambda \rangle$, $\langle f^\alpha_2 |\ \alpha < \lambda \rangle$ such that

(1) $M^* = \bigcup_{\alpha<\lambda} A^\alpha_\delta = \bigcup_{\alpha<\lambda} B^\alpha_\delta$,

(2) $N^1_i = \bigcup_{\alpha<\lambda} A^\alpha_i$, $N^2_i = \bigcup_{\alpha<\lambda} B^\alpha_i$,

(3) $f^\alpha_1 \in \mathrm{Aut}(A^\alpha_\delta)$ such that $f^\alpha_1[A^\alpha_i] = A^\alpha_i$,

(4) $f^\alpha_2 \in \mathrm{Aut}(B^\alpha_\delta)$ such that $f^\alpha_2[B^\alpha_i] = B^\alpha_i$,

(5) $f[A^\alpha_i] = A^\alpha_i$, $f[B^\alpha_i] = B^\alpha_i$,

(6) $|A^\alpha_\delta| < |\alpha|^+ + \kappa_r(T) + \aleph_1$,

(7) $|B^\alpha_\delta| < |\alpha|^+ + \kappa_r(T) + \aleph_1$,

(8) $A^\alpha_\delta = B^\alpha_\delta$,

(9) $f^2_\alpha f^1_\alpha = f \restriction A^\alpha_\delta$,

(10) for all $\beta < \delta$, all $i < \delta$, and all $\alpha < \lambda$,

$$M_\beta \Cup_{M_\beta \cap A^\alpha_i} A^\alpha_i,$$

(11) for all $\beta < \delta$, all $i < \delta$, and all $\alpha < \lambda$,

$$M_\beta \Cup_{M_\beta \cap B^\alpha_i} B^\alpha_i,$$

(12) for all i, $\beta < \delta$ and all $l \in \{0, 1\}$,

$$(N^l_{i+1} \cap M_{\beta+1}, c)_{c \in (N^l_{i+1} \cap M_\beta) \cup (N^l_i \cap M_{\beta+1})}$$

is saturated of cardinality λ,

(13) $(N^l_{i+1} \cap M_0, c)_{c \in N^l_i \cap M_0}$ is saturated of cardinality λ,

(14) for all $i < \delta$ and all $\alpha < \lambda$,

$$A^\alpha_\delta \Cup_{A^\alpha_i} A^{\alpha+1}_i,$$

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(15) for all $i<\delta$ and all $\alpha<\lambda$,

$$B^\alpha_\delta \Cup_{B^\alpha_i} B^{\alpha+1}_i,$$

(16) for all $\beta<\delta$, all $i<\delta$, and all $\alpha<\lambda$,

$$M_\beta \cup (M_{\beta+1}\cap A^\alpha_\delta) \Cup_{M_\beta\cup(M_{\beta+1}\cap A^\alpha_i)} M_{\beta+1}\cap A^{\alpha+1}_i,$$

(17) for all $\beta<\delta$, all $i<\delta$, and all $\alpha<\lambda$,

$$M_\beta \cup (M_{\beta+1}\cap B^\alpha_\delta) \Cup_{M_\beta\cup(M_{\beta+1}\cap B^\alpha_i)} M_{\beta+1}\cap B^{\alpha+1}_i.$$

At limit stages we take unions. Let α be even. Let $M^*=\langle m_\alpha|\ \alpha<\lambda\rangle$. In the induction we define $\langle p_\alpha|\ \alpha$ is even and $\alpha<\lambda\rangle$ such that each $p_\alpha \in S((M_{\beta+1}\cap A^\alpha_{i+1})\cup M_\beta)$ for some $i,\beta<\delta$ and such that for all $i<\delta$, all $\beta<\delta$, all $A\subseteq M^*$ such that $|A|<\kappa_r(T)$, and all $p\in S(\operatorname{acl} A)$, there exist λ elements $p_\alpha\in\langle p_\alpha|\ \alpha<\lambda\rangle$ such that $p_\alpha\in S((M_{\beta+1}\cap A^\alpha_{i+1})\cup M_\beta)$, p_α is a non-forking extension of p, p_α is realized in $A^{\alpha+1}_{i+1}\cap M_{\beta+1}$, and for all $j\leq i$, $A^\alpha_j=A^{\alpha+1}_j$. By the proof of Lemma 2.6, this insures that (12) is true and (13) holds for $l=1$ when we finish our construction. So let i^*, $\beta^*<\delta$ such that $p_\alpha\in S((M_{\beta^*+1}\cap A^\alpha_{i^*+1})\cup M_{\beta^*})$. By Lemma 2.10 we can find extensions $(A^\alpha_i)'$ of A^α_i with $(A^\alpha_i)'=A^\alpha_i$ for $i\leq i^*$, and an extension f_1' of f_1 such that $f_1'[(A^\alpha_i)']=(A^\alpha_i)'$, p_α is realized in $M_{\beta^*+1}\cap(A^\alpha_{i^*+1})'$, and for all $\beta<\delta$ and all $i<\delta$,

$$M_{\beta-1}\cup(M_\beta\cap A^\alpha_\delta)\Cup_{M_{\beta-1}\cup(M_\beta\cap A^\alpha_i)} M_\beta\cap(A^\alpha_i)',$$

$$A^\alpha_\delta\Cup_{A^\alpha_i}(A^\alpha_i)' \quad\text{and}\quad M_\beta\Cup_{M_\beta\cap(A^\alpha_i)'}(A^\alpha_i)'.$$

Let F_1' be an extension of f_1' to an automorphism of M^*. By iterating ω times the procedure in the proof of Lemma 2.8, we can find $D\subset M^*$ such that $|D|<\kappa_r(T)+\omega_1$, if m is the least element of $\langle m_\alpha|\ \alpha<\lambda\rangle$ then $m\in D$, D is closed under f, f^{-1}, F_1', $(F_1')^{-1}$, and for some i^{**}, $i^{***}<\delta$ if $A^{\alpha+1}_i=(A^\alpha_i)'\cup D$ for $i\geq i^{**}$, and $(A^\alpha_i)'$ for $i<i^{**}$, and if $B^{\alpha+1}_i=B^\alpha_i\cup D$ for $i\geq i^{***}$ and $B^{\alpha+1}_i=B^\alpha_i$ for $i<i^{***}$, then

$$M_\beta\cup(M_\beta\cap A^\alpha_\delta)\Cup_{M_\beta\cup(M_{\beta+1}\cap A^\alpha_i)} M_{\beta+1}\cap A^{\alpha+1}_i,$$

$$A^\alpha_\delta\Cup_{A^\alpha_i}A^{\alpha+1}_i \quad\text{and}\quad M_\beta\Cup_{M_\beta\cap A^{\alpha+1}_i}A^{\alpha+1}_i,$$

and

$$M_\beta\cup(M_{\beta+1}\cap B^\alpha_\delta)\Cup_{M_\beta\cup(M_{\beta+1}\cap B^\alpha_i)} M_{\beta+1}\cap B^{\alpha+1}_i,$$

$$B^\alpha_\delta\Cup_{B^\alpha_i}B^{\alpha+1}_i \quad\text{and}\quad M_\beta\Cup_{M_\beta\cap B^{\alpha+1}_i}B^{\alpha+1}_i.$$

Similar results hold for α odd. Let $f^{\alpha+1}_1=F_1\restriction A^{\alpha+1}_i$ and $f^{\alpha+1}_2=f(f^{\alpha+1}_1)^{-1}$.

3. *The proof of the small index property*

Definition 3.1. Let δ be a limit ordinal and let $\bar N\in K^s_\delta$. Then $f\in\operatorname{Aut}^*(\bar N)$ if and only if $f\in\operatorname{Aut}(M^*)$ and for some $n\in\omega$, $f[N_\alpha]=N_\alpha$ for every α such that $n\leq\alpha\leq\delta$. In addition $\operatorname{Aut}^*_A(\bar N)=\{f\in\operatorname{Aut}^*(\bar N)|\ f\restriction A=\mathrm{id}_A\}$.

Definition 3.2. Let δ be a limit ordinal and let $\bar N\in K^s_\delta$. Let $B\subseteq N_0$ as in the above definition. If for every $f\in\operatorname{Aut}(M^*)$,

$$(f\in\operatorname{Aut}^*(\bar N)\wedge f\restriction B=\mathrm{id}_B)\Rightarrow f\in G,$$

then we define

$$E = \{C \subseteq B \mid f \in \mathrm{Aut}^*(\bar{N}) \wedge f \restriction C = \mathrm{id}_C \Rightarrow f \in G\}.$$

LEMMA 3.3. *Let δ be a limit ordinal and let $\bar{N} \in K^s_\delta$. Let $B \subseteq N_0$ such that $(N_0, c)_{c \in B}$ is saturated. Let $C = \mathrm{acl}\, C$, $C \subseteq B$, and g be an elementary map with $\mathrm{dom}\, g = B$ and $g \restriction C = \mathrm{id}_C$. Let $(N_0, c)_{c \in B \cup g[B]}$ be saturated, and*

$$B \mathbin{\cup\!\!\!\cup}_C g(B).$$

Then the following are equivalent:

(1) *$C \in E$;*

(2) *all extensions of g in $\mathrm{Aut}^*(\bar{N})$ are in G;*

(3) *some extension of g in $\mathrm{Aut}^*(\bar{N})$ is in G.*

Proof. (1)$\Rightarrow$(2) This is trivial.

(2)$\Rightarrow$(3) We just need to prove that g has some extension in $\mathrm{Aut}^*(\bar{N})$. But this follows easily by the saturation for every $j < \delta$ of $(N_{j+1}, c)_{c \in N_j}$.

(3)$\Rightarrow$(1) Let $f \in \mathrm{Aut}^*(\bar{N})$ such that $f \restriction C = \mathrm{id}_C$. Let $n \in \omega$ and $g^* \in \mathrm{Aut}^*(\bar{N})$ such that $g^* \supseteq g$, $f, g^* \in \mathrm{Aut}(\bar{N} \restriction [n, \delta))$, and $g^* \in G$. Let $B' \subseteq N_{n+1}$ such that $B' \mathbin{\cup\!\!\!\cup}_C N_n$ and $tp(B'/C) = tp(B/C)$. Let $g_1 \in \mathrm{Aut}(\bar{N} \restriction [n+2, \delta))$ such that g_1 maps $g(B)$ onto B' and $g_1 \restriction B = \mathrm{id}_B$. Since $g_1 \restriction B = \mathrm{id}_B$, we have $g_1 \in G$. Let $g_2 = g_1 g^* (g_1)^{-1}$. Again $g_2 \in G$, $g_2 \restriction C = \mathrm{id}_C$, and $g_2[B] = B'$. As $B' \mathbin{\cup\!\!\!\cup}_C N_n$, $f \in \mathrm{Aut}(\bar{N} \restriction [n, \delta))$ and $f \restriction C = \mathrm{id}_C$, clearly

$$f(B') \mathbin{\cup\!\!\!\cup}_C N_n.$$

Therefore there exists $g_3 \in \mathrm{Aut}(\bar{N} \restriction [n+2, \delta))$ such that $g_3 \restriction B' = f \restriction B'$ and $g_3 \restriction N_n = \mathrm{id}_{N_n}$, whence $g_3 \in G$. Now $(g_3)^{-1} f \restriction B' = \mathrm{id}_{B'}$, so $(g_2)^{-1}(g_3)^{-1} f g_2 = \mathrm{id}_B$ and hence $(g_2)^{-1}(g_3)^{-1} f g_2 \in G$. But this implies that $f \in G$.

THEOREM 3.4. *Let $|T| < \lambda$. Let $\bar{M} \in K^s_\delta$. Let $G \subseteq \mathrm{Aut}^*(M)$. If*

$$f \in \mathrm{Aut}^*_{M_0}(\bar{M}) \;\Rightarrow\; f \in G$$

but for no $C \subseteq M_0$ with $|C| < \lambda$ does

$$f \in \mathrm{Aut}^*_C(\bar{M}) \;\Rightarrow\; f \in G,$$

then

$$[\mathrm{Aut}(M^*) : G] > \lambda.$$

Proof. Suppose that the theorem does not hold. Let $\langle h_i \mid i < \lambda \rangle$ be a list of the representatives of the left G cosets of $\mathrm{Aut}(\bar{M} \restriction [1, \delta))$ possibly with repetition. Let $\lambda = \bigcup_{\zeta < cf\lambda} \lambda_\zeta$ with $\langle \lambda_\zeta \mid \zeta < cf\lambda \rangle$ increasing continuous and $|T| \leqslant |\lambda_0| \leqslant |\lambda_\zeta| < \lambda$. Let $M_0 = \bigcup_{\zeta < cf\lambda} M^0_\zeta$ and $M_1 = \bigcup_{\zeta < cf\lambda} M^1_\zeta$ with each being a continuous chain such that $|M^i_\zeta| \leqslant |\lambda_\zeta|$.

Now we define by induction on $\zeta < cf\lambda$, $N_{0,\zeta}$, $N_{1,\zeta}$, f_ζ, B_ζ, and $h_{j,\zeta}$ for $j < \lambda_\zeta$ such that:

(1) f_ζ is an automorphism of $N_{1,\zeta}$;

(2) $\langle f_\zeta \mid \zeta < cf\lambda \rangle$ is increasing continuous;

(3) if $j < \lambda_\zeta$ and there is an $h \in \mathrm{Aut}(\bar{M} \restriction [1, \delta))$ such that

(a) h extends f_ζ,
(b) $hG = h_jG$,
then $h_{j,\zeta}$ satisfies (a) and (b);

(4) B_ζ is a subset of $N_{1,\zeta}$ of cardinality at most $|\lambda_\zeta|$;
(5) $M^1_\zeta \subseteq B_\zeta$;
(6) $N_{0,\zeta} \subseteq B_{\zeta+1}$ and $B_{\zeta+1}$ is closed under $h_{j,\varepsilon}$ and $h^{-1}_{j,\varepsilon}$ for $j < \lambda_\varepsilon$ and $\varepsilon \leqslant \zeta$;
(7) $f^{-1}_{\zeta+1}(B_{\zeta+1}) \Cup_{N_{0,\zeta}} N_{0,\zeta+1}$;
(8) $N_{1,\zeta} \Cup_{N_{0,\zeta}} M_0$;
(9) $M_1 = \bigcup_{\zeta < cf\lambda} N_{1,\zeta}$ and $M_0 = \bigcup_{\zeta < cf\lambda} N_{0,\zeta}$;
(10) $|N_{0,\zeta}| \leqslant |\lambda_\zeta|$;
(11) $(N_{1,\zeta+1}, c)_{c \in N_{1,\zeta}}$ is saturated of cardinality λ;
(12) $(M_1, c)_{c \in M_0 \cup N_{1,\zeta}}$ is saturated of cardinality λ.

For $\zeta = 0$ let B_0 be empty, let $N_{0,0}$ be a submodel of M_0 of cardinality $|\lambda_0|$, let $N_{1,0}$ be a saturated submodel of M_1 of cardinality λ such that $N_{1,0} \Cup_{N_{0,0}} M_0$ and let $f_\zeta = \mathrm{id}_{N_{1,0}}$. At limit stages take unions. If $\zeta = \varepsilon + 1$, let B_ζ be as in (4), (5), (6). Let $N_{0,\zeta} \subseteq M_0$ such that $B_\zeta \Cup_{N_{0,\zeta}} M_0$, $N_{0,\varepsilon} \subseteq N_{0,\zeta}$, $M^0_\zeta \subseteq N_{0,\zeta}$, $|N_{0,\zeta}| \leqslant \lambda_\zeta$. Let $N_{1,\zeta} \subseteq M_1$ such that $B_\zeta \subseteq N_{1,\zeta}$, $N_{1,\zeta} \Cup_{N_{0,\zeta}} M_0$, $(N_{1,\zeta}, c)_{c \in N_{1,\varepsilon}}$ is saturated of cardinality λ, and $(M_1, c)_{c \in M_0 \cup N_{1,\zeta}}$ is saturated of cardinality λ. Let f_ζ be an extension of $f_\varepsilon \restriction N_{1,\varepsilon}$ to an automorphism of $N_{1,\zeta}$ so that

$$f^{-1}_\zeta(B_\zeta) \Cup_{N_{1,\varepsilon}} N_{0,\zeta}.$$

Since $N_{0,\zeta} \Cup_{N_{0,\varepsilon}} N_{1,\varepsilon}$, we have $f^{-1}_\zeta(B_\zeta) \Cup_{N_{0,\varepsilon}} N_{0,\zeta}$. Let f be an extension of $\bigcup_{\zeta < cf\lambda} f_\zeta$ to an element of $\mathrm{Aut}(\bar{M} \restriction [1, \delta))$. We have defined f so that

(1) (by non-forking calculus) for all $\zeta < cf\lambda$ and all $j < \lambda_\zeta$,

$$f^{-1}h_{j,\zeta}(M_0) \Cup_{N_{0,\zeta}} M_0,$$

(2) $f^{-1}h_{j,\zeta} \restriction N_{0,\zeta} = \mathrm{id}$.

By Lemma 3.3 none of the $f^{-1}h_{j,\zeta}$ are in G, a contradiction as for some $j < \lambda$, $fG = h_jG$, so for some ζ, $j < \lambda_\zeta$, $h_jG = h_{j,\zeta}G = fG$.

LEMMA 3.5. *Let $|T| < \lambda$. Let $cf\delta \geqslant \kappa_r(T) + \aleph_1$. Suppose $[\mathrm{Aut}(M^*) : G] \leqslant \lambda$ and assume that for no $A \subseteq M^*$ with $|A| < \lambda$ is $\mathrm{Aut}_A(M^*) \subseteq G$. Then for some $\bar{N} \in K^s_\delta$,*

$$\bigwedge_{\alpha < \delta} \mathrm{Aut}^*_{N_\alpha}(\bar{N}) \nsubseteq G.$$

Proof. Suppose that the theorem does not hold. Let $\bar{M} \in K^s_\delta$. Then there exists an $\alpha < \delta$ such that $\mathrm{Aut}^*_{M_\alpha}(\bar{M}) \subseteq G$. Without loss of generality we may assume that $\alpha = 0$. By Lemma 3.4 there exists $E \subseteq M_0$ such that $|E| < \lambda$ and $\mathrm{Aut}_E(\bar{M}) \subseteq G$. Let $f \in \mathrm{Aut}_E(M^*) \backslash G$. By Lemma 2.11 we can find $\bar{N}^1$, $\bar{N}^2 \in K^s_\delta$ and automorphisms $f_1 \in \mathrm{Aut}_E(\bar{N}^1)$ and $f_2 \in \mathrm{Aut}_E(\bar{N}^2)$ such that

(1) $E \subset N^1_0$, $E \subset N^2_0$,
(2) $f = f_2 f_1$,
(3) $f_1 \restriction E = f_2 \restriction E = \mathrm{id}_E$,
(4) for all $\alpha, \beta < \delta$,

(a) $N^1_\alpha \Cup_{N^1_\alpha \cap M_\beta} M_\beta$,

(b) $N^2_\alpha \Cup_{N^2_\alpha \cap M_\beta} M_\beta$,

(c) $(N^1_{\alpha+1} \cap M_{\beta+1}, c)_{c \in (N^1_{\alpha+1} \cap M_\beta) \cup (N^1_\alpha \cap M_{\beta+1})}$ is saturated of cardinality λ,

(d) $(N^2_{\alpha+1} \cap M_{\beta+1}, c)_{c \in (N^2_{\alpha+1} \cap M_\beta) \cup (N^2_\alpha \cap M_{\beta+1})}$ is saturated of cardinality λ,

(e) $(N^1_{\alpha+1} \cap M_0)_{c \in N^1_\alpha \cap M_0}$ is saturated of cardinality λ,

(f) $(N^2_{\alpha+1} \cap M_0)_{c \in N^2_\alpha \cap M_0}$ is saturated of cardinality λ.

Since $f \notin G$, can assume without loss of generality that $f_1 \notin G$. Also, since we suppose the theorem untrue, we can assume there is an $F \subseteq N^1_0$ such that $(N^1_0, c)_{c \in F}$ is saturated and $\mathrm{Aut}_F(\bar{N}^1) \subseteq G$. By Lemma 3.4 we can assume that $|F| < \lambda$ and, without loss of generality, that $E \subseteq F$. Let for $\alpha < \delta$,

$$F_\alpha = F \cap M_\alpha.$$

By Lemma 3.6 we can find a sequence $\langle F'_\alpha | \; \alpha < \delta \rangle$ such that for each α, $F_\alpha \subseteq F'_\alpha$ with $|F'_\alpha| < \lambda$ and, for each $\beta < \alpha$, $F'_\alpha \cap M_\beta = F'_\beta$, and if $F' = \bigcup_{\alpha<\delta} F'_\alpha$ then

$$M_\alpha \cap N^1_0 \Cup_{F'_\alpha} F'.$$

We define by induction on $\alpha < \delta$ a map g_α, an automorphism of $M_\alpha \cap N^1_0$, such that

(1) for all β, $\alpha < \delta$, if $\beta < \alpha$ then $g_\beta \subseteq g_\alpha$,

(2) if α is a limit then $g_\alpha = \bigcup_{\beta<\alpha} g_\beta$,

(3) $g_\alpha(F'_\alpha) \Cup_E F'_\alpha$,

(4) $g_\alpha \restriction E = \mathrm{id}_E$.

Let $\alpha = \beta + 1$ and suppose g_β has been defined. Let $X \subseteq M_\alpha \cap N^1_0$ such that $X^\frown g_\beta(F'_\beta) \equiv F'^\frown_\alpha F'_\beta$ by h_β an extension of $g_\beta \restriction F'_\beta$ and

$$X \Cup_{g_\beta(F'_\beta)} F'_\alpha \cup (M_\beta \cap N^1_0).$$

Let $g'_\alpha = g_\beta \cup h_\beta$. Since $X \Cup_{g_\beta(F'_\beta)} g_\beta(M_\beta \cap N^1_0)$ and $F'_\alpha \Cup_{F'_\beta} M_\beta \cap N^1_0$, g'_α is an elementary map. Now let g_α be an extension of g'_α to an automorphism of $M_\alpha \cap N^1_0$. Let $g' = \bigcup_{\alpha<\delta} g_\alpha$. Then g' is an automorphism of N^1_0 such that for every $\alpha < \delta$,

$$g'[M_\alpha \cap N^1_0] = [M_\alpha \cap N^1_0].$$

By the saturation and independence of the N^1_α, M_β we can find an extension g of g' such that $g \in \mathrm{Aut}(\bar{N}_1)$ and $g \in \mathrm{Aut}(\bar{M})$. This gives a contradiction since $g(F) \Cup_E F$ and $g \in \mathrm{Aut}(\bar{N}_1)$ implies $g \notin G$, but $g \in \mathrm{Aut}(\bar{M})$ and $g \restriction E = \mathrm{id}_E$ implies $g \in G$.

LEMMA 3.6. *Let $\bar{M} = \langle M_\beta | \; \beta \leq \delta \rangle \in K^s_\delta$. Let $F \subseteq M^*$ with $|F| < \lambda$. Then there exists a set F' such that $|F'| < \lambda$, $F \subseteq F'$, and for all $\beta < \delta$,*

$$(*) \qquad M_\beta \Cup_{F' \cap M_\beta} F'.$$

Proof. Let $w \subseteq F$ be finite. There are less than $\kappa_r(T)$ values of $\alpha < \delta$ such that $w \not\Cup_{M_\alpha} M_{\alpha+1}$. Let a_w be the set of such α. For each $\alpha \in a_w$ let $w_\alpha \subseteq M_\alpha$ such that $|w_\alpha| < \kappa_r(T)$, and $w \Cup_{w_\alpha} M_\alpha$. Let $w^1 = \bigcup_{\alpha \in a_w} w_\alpha$. Let $F^1 = \bigcup_{w \subset F,\ \mathrm{finite}} w^1$ and repeat this procedure ω times with F^n relating to F^{n+1} as F is related to F^1. Let $F' = \bigcup_{n \in \omega} F^n$. Then F' satisfies $(*)$.

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LEMMA 3.7. *Let Tr be a tree of infinite height. Let* $\alpha <$ height(Tr) *and let* $\eta \in Tr \restriction \text{level}(\alpha+1)$. *Let* $\langle M_\beta | \ \beta \leq \alpha \rangle$ *be an increasing chain of models such that for all* $\beta < \alpha$, $(M_{\beta+1}, c)_{c \in M_\beta}$ *is saturated. Let* $M_\alpha \subseteq N_0 \subseteq N_1 \subseteq N_2 \subseteq N_3$ *with* $(N_{i+1}, c)_{c \in N_i}$ *saturated for* $i \leq 2$. *Suppose* $\langle h_\beta | \ \beta \leq \alpha \rangle$ *are such that*

(1) $h_\beta = \text{id}_{M_\beta}$,

(2) $h_\beta[N_i] = N_i$ *for* $i \leq 3$,

(3) $h_\beta[M_\gamma] = M_\gamma$ *for* $\gamma \leq \alpha$.

For each $\nu \in Tr \restriction \text{level}(\alpha+1)$ *let* m_ν, l_ν *be automorphisms of* N_0. *Suppose* $g_\eta \in \text{Aut}(N_0)$ *such that for all* $\nu \in Tr \restriction \text{level}(\alpha+1)$,

$$g_\eta m_\eta (m_\nu)^{-1} (g_\eta)^{-1} = l_\eta (l_\nu)^{-1} h_{\gamma[\eta,\nu]}^{\eta(\gamma[\eta,\nu]) < \nu(\gamma[\eta,\nu])}.$$

Let m_ν^+, l_ν^+ *be extensions of* m_ν *and* l_ν *to automorphisms of* N_1 *for all* $\nu \in Tr \restriction \text{level}(\alpha+1)$. *Then there exists a* $g'_\eta \in \text{Aut}(N_3)$ *extending* g_η *and for all* $\nu \in Tr \restriction \text{level}(\alpha+1)$ *automorphisms of* N_3, m'_ν *and* l'_ν *extending* m_ν^+ *and* l_ν^+ *respectively such that*

$$g'_\eta m'_\eta (m'_\nu)^{-1} (g'_\eta)^{-1} = l'_\eta (l'_\nu)^{-1} h_{\gamma[\eta,\nu]}^{\eta(\gamma[\eta,\nu]) < \nu(\gamma[\eta,\nu])}.$$

Proof. This is similar to the proof of Lemma 1.8.

THEOREM 3.8. *Let* $|T| < \lambda$. *Let* M^* *be a saturated model of cardinality* λ, *and let* $G \subseteq \text{Aut}(M^*)$. *Suppose that for no* $A \subseteq M$ *with* $|A| < \lambda$ *is* $\text{Aut}_A(M^*) \subseteq G$. *Suppose Tr is a tree of height* κ, *where* κ *is a regular cardinal greater than or equal to* $\kappa_r(T) + \aleph_1$ *such that each level of Tr is of size at most* λ, *but Tr having more than* λ *branches. Then*

$$[\text{Aut}(M^*) : G] > \lambda.$$

Proof. Suppose that the theorem does not hold. Then by Lemma 3.5 there is an $\bar{N} \in K^s_{\lambda \times \kappa}$ such that

$$\bigwedge_{\alpha < \lambda \times \kappa} \text{Aut}^*_{N_\alpha}(\bar{N}) \nsubseteq G.$$

By thinning $\bar{N}$ if necessary we can assume for each $\alpha < \kappa$ that there exists an automorphism $h_\alpha \in \text{Aut}_{N_{\lambda\times\alpha}}(\bar{N})$ such that $h_\alpha \notin G$. By induction on $\alpha < \kappa$ for every $\eta \in Tr \restriction \text{level}\,\alpha$, we define automorphisms g_η, m_η, l_η of $N_{\lambda\times\alpha}$ such that if $\rho \neq \nu$ then $l_\rho \neq l_\nu$ and

$$g_\rho m_\rho (m_\nu)^{-1} (g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h_{\gamma[\rho,\nu]}^{\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])}.$$

At limit steps we take unions. If $\alpha = \beta + 1$, for each $i < \lambda$ we define for some $\eta_i \in Tr \restriction \text{level}\,\alpha$, $g_{\eta_i} \in \text{Aut}(N_{\lambda\times\beta+3i})$ such that for each $\eta \in Tr \restriction \text{level}\,\alpha$, $\eta = \eta_i$ a cofinal number of times in λ, and for every $\nu \in Tr \restriction \text{level}\,\alpha$, $m^i_\nu \neq l^i_\nu \in \text{Aut}(N_{\lambda\times\beta+3i})$ such that

$$g_{\eta_i} m^i_{\eta_i} (m^i_\nu)^{-1} (g_{\eta_i})^{-1} = l^i_{\eta_i} (l^i_\nu)^{-1} h_{\gamma[\eta_i,\nu]}^{\eta_i(\gamma[\eta_i,\nu]) < \nu(\gamma[\eta_i,\nu])}$$

The g_{η_i}, m^i_ν, l^i_ν are easily defined by induction on $i < \lambda$ using Lemma 3.7. Then if we let $g_\eta = \bigcup \{g_{\eta_i} | \ \eta_i = \eta\}$, $m_\eta = \bigcup_{i<\lambda} m^i_\eta$ and $l_\eta = \bigcup_{i<\lambda} l^i_\eta$, we have finished. Let Br be the set of branches of Tr of height κ. For $\rho \in Br$ let

$g_\rho = \bigcup\{g_\eta \mid \eta < \rho\}$, $m_\rho = \bigcup\{m_\eta \mid \eta < \rho\}$, and $l_\rho = \bigcup\{l_\eta \mid \eta < \rho\}$. If $\rho \neq \nu$, then $g_\rho \neq g_\nu$ since without loss of generality we may assume that $\rho(\gamma[\rho, \nu]) < \nu(\gamma[\rho, \nu])$ and

$$g_\rho m_\rho (m_\nu)^{-1} (g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h^{\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]}$$

and

$$g_\nu m_\nu (m_\rho)^{-1} (g_\nu)^{-1} = l_\nu (l_\rho)^{-1}$$

implies

$$g_\rho (g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} = l_\rho (l_\nu)^{-1} h^{\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]}.$$

So if $g_\rho = g_\nu$, this would imply that $h^{\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]} = \mathrm{id}_{M^*}$, a contradiction. If

$$[\mathrm{Aut}(M^*) : G] \leq \lambda,$$

then for some $\rho, \nu \in Br$ we must have $l_\rho (l_\nu)^{-1} \in G$ and $g_\rho (g_\nu)^{-1} \in G$, but then we get a contradiction as $g_\rho (g_\nu)^{-1} l_\rho (l_\nu)^{-1} g_\nu (g_\rho)^{-1} \in G$ and $l_\rho (l_\nu)^{-1} \in G$, but $h^{\rho(\gamma[\rho,\nu]) < \nu(\gamma[\rho,\nu])}_{\gamma[\rho,\nu]} \notin G$.

COROLLARY 3.9. *Let $G \subseteq \mathrm{Aut}(M^*)$. Suppose that for no $A \subseteq M$ with $|A| < \lambda$ is $\mathrm{Aut}_A(M^*) \subseteq G$. suppose $|T| < \lambda$ and M^* does not have the small index property. Then:*

(1) *there is no tree of height an uncountable regular cardinal κ with at most λ nodes, but more than λ branches;*

(2) *for some strong limit cardinal μ, $cf\mu = \aleph_0$ and $\mu < \lambda < 2^\mu$;*

(3) *T is superstable.*

Proof. (1) This follows from the previous theorem.

(2) This follows from (1) and [**4**, 6.3].

(3) If T is stable in λ, then $\lambda = \lambda^{<\kappa_r(T)}$, so if $\kappa_r(T) > \aleph_0$, we can let κ from the previous theorem be the least κ such that $\lambda < \lambda^\kappa$.

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