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$L_{\infty\omega}$-free algebras

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In Memory of Evelyn Nelson

Abstract. In this paper the study of which varieties, in a countable similarity type, have non-free $L_{\infty\omega}$-free (or equivalently $\aleph_1$-free) algebras is completed. It was previously known that if a variety satisfies a property known as the construction principle then there are such algebras. If a variety does not satisfy the construction principle then either every $L_{\infty\omega}$-free algebra is free or for every infinite cardinal κ, there is a κ^+-free algebra of cardinality κ^+ which is not free. Under the set theoretic assumption $V = L$, for any variety $\mathcal{V}$ in a countable similarity type, either the class of free algebras is definable in $L_{\omega_1\omega}$ or it is not definable in any $L_{\infty\kappa}$.

0. Introduction

The question we will consider in this paper can be approached from many different points of view. The simplest form of the question is "which varieties possess an algebra which is $\aleph_1$-free but not free." By $\aleph_1$-free we mean that "most" countable subalgebras are free. In this paper any variety we consider will be in a countable similarity type, i.e. have at most countably many operations and constant symbols. There are various possible meanings for "most." In the case of a Schreier variety (i.e. one in which every subalgebra of a free algebra is free) "most" is equivalent to "all." So a group or an abelian group is $\aleph_1$-free if and only if every countable subgroup is free in the appropriate variety. However in a variety, such as the variety of all semigroups, even a free semigroup would not be $\aleph_1$-free if we demanded that all countable subsemigroups were free. For an infinite cardinal κ, there are several possible definitions of κ-free. (All the definitions are equivalent for $\aleph_1$.) The results in this paper and in [EM] do not depend on which definition is adopted since the existence results satisfy the most stringent of the definitions and the non-existence results require only the weakest definition. In this paper we will only need a definition of κ-free for successor

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cardinals, so we will only give a definition in this case. For more information on possible definitions see [S1], [HO] or [EM]. (Some set-theoretic complications are involved with the definition given in [S1] but we will ignore them here.)

DEFINITION. Suppose $\mathcal{V}$ is a variety. An algebra A is said to be *κ^+-free* if there is a *closed and unbounded family* $\mathcal{X} \subseteq \mathcal{P}_{\kappa^+}(A)$ consisting of free subalgebras; that is $\mathcal{X}$ is closed under the union of increasing chains of cardinality at most κ and any subset of A of cardinality κ is contained in an element of $\mathcal{X}$. Here $\mathcal{P}_{\kappa^+}(A)$ is the set of subsets of A of cardinality at most κ.

A more general version of the first question is "in a variety $\mathcal{V}$, for which cardinals κ is there a κ-free algebra of cardinality κ which is not free?" By Shelah's compactness theorem for singular cardinals ([S1] or [Ho]), any such cardinal must be regular.

From the point of view of logic, a particularly interesting class of algebras are the $L_{\infty\kappa}$-free algebras, i.e. algebras which are $L_{\infty\kappa}$-equivalent to a free algebra (in the variety we are considering). The infinitary language $L_{\infty\kappa}$ is formed from the atomic formulas by allowing negation, arbitrary conjunctions, and quantification over strings of fewer than κ variables. The language $L_{\omega_1\omega}$ is formed similarly except that only conjunctions of countable sets and quantification of finite sets are allowed. The existence of non-free $L_{\infty\kappa}$-free algebras for arbitrarily large κ is connected to the question "in which varieties is the class of free algebras definable in some $L_{\infty\kappa}$?" The conjectured answer is that for a variety the class of free algebras is definable in some $L_{\infty\kappa}$ if and only if it is definable in $L_{\omega_1\omega}$. This conjecture cannot be taken at face value. For example the class of free abelian groups is definable in $L_{\infty\kappa}$ if and only if any abelian group which is κ-free is in fact free. (This sharp result depends on [S5]. The weaker result that the class of free abelian groups is definable in $L_{\infty\infty}$ if and only if for some κ, κ-free implies free follows from a downward Lowenheim–Skolem argument [M1]) Whether or not such a cardinal exists depends on the underlying set theory. If κ is a strongly compact cardinal then any κ-free abelian group is free. On the other hand in L, there is no cardinal κ, such that every κ-free abelian group is free. There is always an abelian group of cardinality $\aleph_1$ which is not free but is $L_{\infty\omega_1}$-equivalent to a free abelian group.

It is possible to give algebraic characterizations of being $L_{\infty\kappa}$-free. Here, we will only note a few facts. An algebra is $L_{\infty\omega}$-free if and only if it is $L_{\omega_1\omega}$-free if and only if it is $\aleph_1$-free. If an algebra is κ^+-free then it is $L_{\infty\kappa}$-free. In this paper we shall refer to $L_{\omega_1\omega}$-free algebras or $L_{\infty\omega}$-free algebras, rather than the synonymous $\aleph_1$-free or countably free. This choice of terminology reflects our bias to the logical motivation, but we hope the reader will substitute whichever term is favoured.

In [EM] the significance of the existence of non-free $L_{\infty\kappa}$-free algebras of cardinality κ is studied. The major result of [EM] concerning these algebras is as follows.

THEOREM 0.1 [EM]. *Let $\mathscr{V}$ be a variety in a countable similarity type.*

(a) *If for some infinite cardinal κ, there is an $L_{\infty\kappa}$-free (i.e. $L_{\infty\kappa}$-equivalent to a free algebra) algebra of cardinality κ which is not free, then there is an $L_{\infty\omega_1}$-free algebra of cardinality $\aleph_1$ which is not free.*
(b) *If $V = L$ and there is an $L_{\infty\omega_1}$-free algebra of cardinality $\aleph_1$ which is not free, then for every regular non-weakly compact cardinal κ there is an $L_{\infty\kappa}$-free κ-free algebra of cardinality κ which is not free.*

In parts (a) and (b) of the theorem above, we can show that there are $2^{\aleph_1}$ and 2^{κ} such algebras. In [EM], the existence of a non-free $L_{\infty\omega_1}$-free algebra of cardinality $\aleph_1$ in a variety $\mathscr{V}$ is shown to be equivalent to the construction principle which is a statement about countable free algebras. A variety $\mathscr{V}$ satisfies the construction principle if:

There are countable free algebras $H \subseteq K \subseteq L$ so that

(i) $\{h_n : n < \omega\}$ is a free basis of H for every finite subset $S \subseteq \omega$, the algebra generated by $\{h_n : n \in S\}$ is a free factor with a free complement of L;
(ii) $L = K * F(\omega)$ and H is not a free factor of L.

Here $*$ denotes free product and $F(\kappa)$ denotes the free algebra on κ generators. (Although free products may not exist between arbitrary elements of the variety, the coproduct of any two $\aleph_1$-free algebras is always a free product.) We say an algebra $A \subseteq B$ is a *free factor* if there is an algebra C so that $B = A * C$. The algebra C is referred to as a *complement* to A.

This theorem leaves open the question of what happens in varieties in which every $L_{\infty\omega_1}$-free algebra of cardinality $\aleph_1$ is free. It is possible for a variety $\mathscr{V}$ not to satisfy the construction principle and for the class of free algebras to still not be definable in any $L_{\infty\kappa}$. For example, and this is an example which should be kept in mind, consider the variety of abelian groups of exponent 6. This is the variety of abelian groups which satisfies the law "$6x = 0$." Any member of this variety is of the form, $C(2)^{(\mu)} \oplus C(3)^{(\lambda)}$, the direct sum of μ copies of the cyclic group of order 2 and λ copies of the cyclic group of order 3. In this variety a group is free if $\mu = \lambda$ (i.e. it is a direct sum of cyclic groups of order 6). For an infinite cardinal κ, $C(2)^{(\mu)} \oplus C(3)^{(\lambda)}$ is $L_{\infty\kappa}$-free if either $\mu = \lambda$ or $\kappa \leq \mu, \lambda$. In particular, $C(2)^{(\aleph_0)} \oplus$

$C(3)^{(\aleph_1)}$ is $\aleph_1$-free but not free. This example leads to the hope that we can show in a variety with no non-free $L_{\infty\omega_1}$-free algebra of cardinality $\aleph_1$ that every $L_{\infty\omega}$-free algebra is determined by finitely many cardinal invariants which can be varied at will and that the free algebras are definable in some $L_{\infty\kappa}$ if and only if there is exactly one invariant if and only if the free algebras are definable in $L_{\omega_1\omega}$. There is some evidence in [EM] that this conjecture is true.

Before stating a theorem from [EM], which we shall use extensively we will review some facts about $L_{\infty\omega}$. Suppose $A \subseteq B$, then A is an $L_{\omega_1\omega}$-elementary substructure of B, denoted $A <_{\omega_1\omega} B$, if for every $L_{\omega_1\omega}$-formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in A$, $A \models \varphi(a_1, \ldots, a_n)$ if and only if $B \models \varphi(a_1, \ldots, a_n)$. If A and B are free algebras there is a simple criterion for being an $L_{\omega_1\omega}$-elementary substructure. Suppose $A \subseteq B$ are free algebras. A necessary and sufficient condition for $A <_{\omega_1\omega} B$ is that any finite subset of A can be extended to a free basis of A if and only if it can be extended to a free basis of B. We will tacitly use this fact repeatedly.

THEOREM 0.2 [EM]. *Suppose that $\mathscr{V}$ is a variety in a countable similarity type which has no $L_{\infty\omega_1}$-free algebra of cardinality $\aleph_1$ which is not free, i.e. $\mathscr{V}$ does not satisfy the construction principle.*

(a) *If $A <_{\omega_1\omega} B$ and A, B are $L_{\omega_1\omega}$-free then there is a free algebra $C \subseteq B * F(|B|)$ so that $A * C = B * F(|B|)$.*
(b) *If A is $L_{\omega_1\omega}$-free then $A * F(|A|)$ and $*_{|A|} A$ are free.*

These results suggest that an $L_{\omega_1\omega}$-free algebra is a free algebra which is lacking some elements.

In a variety which does not satisfy the construction principle, there is an obvious strategy for creating $L_{\infty\kappa}$-free non-free algebras for arbitrarily large cardinals κ. One could begin with an $L_{\omega_1\omega}$-free algebra A which is not free. Inside A, choose a free $L_{\omega_1\omega}$-elementary substructure B so that B is maximal (in the sense that there is no $a \in A \setminus B$ so that a generates a free algebra and $\langle B \cup a \rangle = B * \{a\}$ and $\{B \cup a\rangle$ is an $L_{\omega_1\omega}$-elementary substructure of A). Next for κ^+ greater than the cardinality of A, let C be the free product of κ copies of A amalgamated over B. Finally we would like to assert that $C * F(\kappa)$ is $L_{\infty\kappa}$-free but not free. There are various problems with this sketch. Why for example should C be $L_{\omega_1\omega}$-free? With some modifications, this sketch can be made to work. In order to complete the analysis we will use the notions and machinery Shelah developed in [S3] and [S4] for analyzing excellent classes.

Excellent classes play the role in the study of $L_{\omega_1\omega}$ that ω-stable classes play in the study of the first order theories. In an excellent class it is possible to build prime models over various sets and it is just this ability which will let us find

$L_{\omega_1\omega}$-free algebras in which some dimension is not maximal. A shortened version of the present paper might be "read [EM], [S3] and [S4] and notice that they are related." Essentially [S3] and [S4] provide a mechanism for analyzing excellent classes and the results of [EM] show the relevance of [S3] and [S4]. The analysis shows (for the varieties which do not satisfy the construction principle) that either every $L_{\omega_1\omega}$-free algebra is free or there are at least two dimensions which can be varied independently. If the latter possibility holds then for every κ there is a κ^+-free algebra of cardinality κ^+ which is not free. Putting all the results together we get a division of varieties into three classes. The first division is between those classes which do satisfy the construction principle and those which do not. The varieties which do not satisfy the construction principle are in turn divided into two classes, the ones whose free algebras have one dimension and those with at least two dimensions. In section three, we will see that there are at most a finite number of dimensions.

In this paper, we will explain the notions from [S3] and [S4]. In our particular case we will be able to verify the properties from [S3] and [S4] directly rather than merely quoting those papers. Assuming the results of [EM], we will give a self-contained and relatively simple proof of our main theorem.

THEOREM 11. *Suppose $\mathscr{V}$ is a variety in a counable similarity type which does not satify the construction principle. Either every $L_{\omega_1\omega}$-free algebra is free or for every infinite cardinal κ there is a non-free algebra of cardinality κ^+ which is κ^+-free.*

To obtain the more detailed results in section 3, we rely more heavily on the machinery. A subsidiary goal of this paper is to explain the notions involved with excellent classes in terms of a concrete and important example. We have tried to make the paper accessible to anyone with a rudimentary knowledge of model theory and universal algebra.

1. Starting the interpretation

We fix $\mathscr{V}$ a variety in a countable language which every $L_{\infty\omega_1}$-free algebra of cardinality ω_1 is free. We let $\mathscr{K}$ be the class of algebras which are $L_{\omega_1\omega}$-equivalent to $F(\omega)$. Suppose M is an element of $\mathscr{K}$. For $\bar{a} = (a_1, \ldots, a_n) \in M$, $B \subseteq M$, the type of $\bar{a}$ over B, denoted $\mathrm{tp}(\bar{a}, B)$ is $\{\varphi(x_1, \ldots, x_n, b_1, \ldots, b_m) : \varphi$ is an $L_{\omega_1\omega}$-formula, $b_1, \ldots, b_m \in B$, and $M \models \varphi[a_1, \ldots, a_n, b_1, \ldots, b_m]\}$. There are only countably many different types over the empty set, so any type over a finite set is equivalent to a single formula.

The next two propositions are true without the assumption on $\mathcal{V}$, and are known.

PROPOSITION 1. *Suppose* $M \in \mathcal{K}$ *and* F *is a free algebra then* $M \prec_{\infty\omega} M * F$.

Proof. Easy.

PROPOSITION 2. *Suppose* F *is a free algebra and* X *is a free basis of* F. *If* $\bar{c}$ *realizes the same* ($L_{\omega_1\omega}$-)*type as a finite subset of* X, *then* $\bar{c}$ *can be extended to a free basis of* F.

Proof. Easy (but vital).

Let $\text{Bas}(x_1, \ldots, x_n)$ be the ($L_{\omega_1\omega}$-)formula which says that $x_1, \ldots, x_n$ are distinct and they satisfy the same $L_{\omega_1\omega}$-type as a subset of a free basis of a free algebra. By the previous lemma, in a free algebra $\text{Bas}(x_1, \ldots, x_n)$ expresses the statement "$x_1, \ldots, x_n$ are distinct and can be extended to a free basis." We will let $\text{Bas}(x)$ denote the (complete) type it implies.

In the study of a complete first order theory there is a non-essential but useful simplification, namely we can assume that we are working inside a large saturated model. In general, when we study models of an $L_{\omega_1\omega}$-theory we cannot assume that we are working inside a large model. However, by virtue of the following theorem, we can assume that all our models and sets are subsets of some large free algebra.

THEOREM 3. $\mathcal{K}$ *is an amalgamation class*; *i.e. if* $M \prec_{\omega_1\omega} N$ *and* $M \prec_{\omega_1\omega} N'$ *then there is a free algebra* $G \in \mathcal{K}$ *and* $L_{\omega_1\omega}$-*elementary embeddings of* f, f' *of* N, N' *into* G *which agree on* M *such that the range* $(f) \cap$ *range* $(f') = f(M)$.

Proof. By taking the free product of N, N' with free algebras we can assume (by 0.2(b)) that $N = M * F$ and $N' = M * F'$. The algebra $G = M * F * F'$, is as desired.

From now on we will assume that all our sets are subsets of some large free algebra, $\mathfrak{C}$. By "$\models$" we will mean satisfaction in this free algebra. This assumption allows us to give a simpler approach than that in [S3] and [S4]. In particular we do not have to introduce the auxiliary class $\mathcal{K}^+$ nor do we have to make the assumption that sets satisfy atomic types. By model we shall mean an $L_{\omega_1\omega}$-elementary submodel of the large free algebra. We shall always use M and N to denotes models. By a free algebra we will mean a model which is also a free algebra.

The class $\mathcal{K}$ is said to be $\aleph_0$-*stable* if for every countable $M \in \mathcal{K}$ (there is only one model up to isomorphism) $S_*(M) = \{\text{tp}(\bar{a}, M) : \bar{a} \in \mathfrak{C}\}$ is countable. We warn the reader that being $\aleph_0$-stable is not the decisive property that it is in the case of first-order theories.

PROPOSITION 4. *The class $\mathcal{K}$ is $\aleph_0$-stable.*

Proof. Suppose we have countable models $M \subseteq N$. Let F be a countable rank free algebra. By Theorem 0.2, $N * F$ is isomorphic over M to $M * F$. So any type over M which is realized in N is realized in $M * F$.

By the same method we can show (this follows from the general theory, as well) that for any model M of cardinality κ there are only κ types over M.

DEFINITION. Suppose p is a type over a finite set B. The type p is *stationary* if there is a countable model $M \supseteq B$ and $\bar{a}$ so that $\text{tp}(\bar{a}, B) = p$, and for any formula φ with parameters from B if $\bar{c}, \bar{c}' \in M$ and $\text{tp}(\bar{c}, B) = \text{tp}(\bar{c}', B)$ then $\models \varphi(\bar{a}, \bar{c})$ if and only if $\models \varphi(\bar{a}, \bar{c}')$. (Alternatively, we say that $\text{tp}(\bar{a}, M)$ *does not split over B*.)

The most important property that stationary types enjoy is that they have a unique definable extension to any set. Suppose p is a stationary type over B, and M and $\bar{a}$ are as in the definition. If $B \subseteq C$ then define the *stationarization* of p to C to be $\{\varphi(\bar{x}, \bar{c})$: there is $\bar{e} \in M$ so that $\text{tp}(\bar{c}, B) = \text{tp}(\bar{e}, B)$ and $\models \varphi(\bar{a}, \bar{e})\}$. If p is a type over C and $B \subseteq C$, define $p \restriction B = \{\varphi(\bar{x}, \bar{b}) \in p : \bar{b} \in B\}$.

There may be types which are not stationary but the following theorem proves that there are enough stationary types. We will give a proof of this theorem which works in our setting. A more general proof of the theorem can also be given using the notion of rank.

THEOREM 5. *If p is a type over a model M then there is some finite $B \subseteq M$ so that $p \restriction B$ is stationary and p is the stationarization of $p \restriction B$ to M.*

Proof. We can assume that M is countable. (This assumption can be justified in any one of several ways. However, we will only need the countable case in the proof of the main theorem.) Choose $\bar{a}$ which realizes p. Let F be a free algebra of countable rank so that $\bar{a} \in M * F$. Take X a free basis for M. Choose B a finite subset of X so that $\bar{a}$ is in the subalgebra generated by $B \cup F$. Suppose, $\text{tp}(\bar{c}, B) = \text{tp}(\bar{c}', B)$. Choose a finite set D so that $B \cup D$ can be extended to a free basis of M and $\bar{c} \in \langle B \cup D \rangle$. Since $\text{tp}(\bar{c}, B) = \text{tp}(\bar{c}', B)$ we can choose D' so that $B \cup D'$ can be extended to a free basis of M and some bijection from $B \cup D$ to $B \cup D'$ which is the identity on B extends to an isomorphism of $\langle B \cup D \rangle$ with

$\langle B \cup D \rangle$ which takes $\bar{c}$ to $\bar{c}'$. So there is an automorphism of M which fixes B and takes $\bar{c}$ to $\bar{c}'$. Extending this automorphism to $M * F$ we have shown that $\operatorname{tp}(\bar{a}, \bar{c}) = \operatorname{tp}(\bar{a}, \bar{c}')$.

From the proof of this theorem we can understand exactly what the stationary types are. Consider formulas, $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, of the following form:

$$\exists z_1, \ldots, z_k \operatorname{Bas}(y_1, \ldots, y_m, z_1, \ldots, z_k) \bigwedge_{1<i<n} x_i = w_i(y_1, \ldots, y_m, z_1, \ldots, z_k)$$

where w_i is a term in the language of the algebra. The proof above shows that if $\models \operatorname{Bas}(b_1, \ldots, b_m)$ then $\varphi(x_1, \ldots, x_n, b_1, \ldots, b_m)$ implies a (complete) stationary type. Also any stationary type is essentially of this form. More exactly, any type over a model is the stationarization of a type implied by such a formula to the model. In particular, $\operatorname{Bas}(x)$ implies a stationary type. If M is any model then the stationarization of $\operatorname{Bas}(x)$ to M is the type implied by the atomic diagram of M together with $\{\operatorname{Bas}(x, \bar{b}) : \bar{b} \in M \text{ and } \models \operatorname{Bas}(\bar{b})\}$.

Because of Theorem 5, we can liberalize our definition of stationary type and stationarization. Suppose p is a type over any set C (finite or infinite). We say that p is *stationary* if there is a finite set $B \subseteq C$ so that $p \upharpoonright B$ is stationary and p is the stationarization of $p \upharpoonright B$. If $C \subseteq D$, then the stationarization of p to D is the stationarization of $p \upharpoonright B$ to D. It is an easy exercise to show that the stationarization of p to D is well defined. In terms of these revised definitions, Theorem 5 says that any type over a model is stationary.

A key notion is that of the dimension of a stationary type.

DEFINITION. Suppose that p is a stationary type over B. The *dimension* of p for (A_1, A_2, A_3) is the least cardinal κ, such that for some $C \subseteq A_2$, $|C| = \kappa$ there is a stationary type q over $A_1 \cup C$ so that q and p have the same stationarization to $A_1 \cup A_2$ and q is not realized in A_3. We assume that $B \subseteq A_1 \cup A_2$. We denote the dimension of p for (A_1, A_2, A_3) by $\operatorname{Dim}_p(A_1, A_2, A_3)$.

In the most interesting case where M is a model $\operatorname{Dim}_p(B, M, M) \geq \kappa$ if and only if there is an indiscernible sequence of length κ in M based on p (i.e. there is a sequence $(a_i : i < \kappa)$ of elements of M such that a_j realizes the stationarization of p to $\{a_i : i < j\} \cup B$. (Theorem 1.4 [S3].)

DEFINITION. A model M is (λ-) *full* if for all stationary p, $\operatorname{Dim}_p(\emptyset, M, M) \geq |M|$ ($\geq \lambda$). We say M is weakly (λ-) full over A if $A \subseteq M$ and for all stationary p, $\operatorname{Dim}_p(A, M, M) \geq |M|$ ($\geq \lambda$).

Note that for a model M the least cardinal λ such that M is not λ-full must be a successor cardinal.

The model theoretic notions of full and λ-full correspond to notions of free and λ^+-free.

THEOREM 6. (a) *A model M is free if and only if it is full.* (b) *If a model is λ-full then it is λ^+-free.* (c) *A model M is (λ-) full if and only if* $\mathrm{Dim}_{\mathrm{Bas}(x)}(\emptyset, M, M) = |M|$ $(\geq\lambda)$.

Proof. (a) Assume first that M is free. Let λ be the cardinality of M. Choose X a free basis for M. Suppose p is a stationary type over a finite subset B of M. Let C be any subset of M of cardinality less than λ and q a stationary type over C which has the same stationarization to M as p. Choose Y a subset of X of cardinality less than λ so that C is contained in the subalgebra generated by Y. Since q can be realized in $\langle Y\rangle * F(\omega)$ and $|X\setminus Y|\geq\aleph_0$, q can be realized in M.

Let M be an uncountable full model and let p be the stationary type generated by Bas (x). Enumerate the elements of M as $\{a_i : i<\lambda\}$. We define an increasing continuous sequence M_i $(i<\lambda)$ of submodels of M. The sequence will have the following properties. Each M_i will be a free algebra of cardinality $|i|+\aleph_0$. For all i, $a_i\in M_{i+1}$. For all i, M_i is a free factor with a free complement of M_{i+1}. Let M_0 be any countable submodel. Suppose M_i has been defined. Let N be a submodel of cardinality $|i|+\aleph_0$ which contains M_i and a_i. Choose $(b_j : j<|i|+\aleph_0)$ a sequence so that b_j realizes the stationarization of p to $N\cup\{b_l : l<j\}$. Let M_{i+1} be the subalgebra generated by $N\cup\{b_j : j<|i|+\aleph_0\}$. It is not hard to see that $M_{i+1}=N*F$, where F is the free algebra generated by $\{b_j : j<|i|+\aleph_0\}$. By Theorem 0.2, M_{i+1} is free and M_i is a free factor with a free complementary factor of M_{i+1}. At limit ordinals i, we let $M_i=\bigcup_{j<i}M_j$. Since $M=\bigcup_{i<\kappa}M_i$, M is free.

(b) Suppose M is λ-full. Then $\mathscr{C}=\{N : |N|=\lambda \text{ and } \mathrm{Dim}_{\mathrm{Bas}(x)}(\emptyset, N, N)=\lambda\}$ is closed and unbounded as a subset of $\mathscr{P}_{\lambda^+}(M)$. Since every element of $\mathscr{C}$ is free, M is λ^+-free. (It is also possible to let $\mathscr{C}$ be the collection of full submodles of cardinality λ.)

(c) This assertion has been proved in the course of proving the other parts.

If the cardinality of M is at least λ, then the converse of (b) holds as well. Given the synonyms for free and λ^+-free in terms of full and λ-full we our task is to prove that if there is a model which is not full then for every successor cardinal λ there is a model of cardinality λ which is λ-full but not full.

2. Excellence

In our discussion of excellent classes we will be able to avoid talking about an important concept, namely that of good sets. Because we will avoid good sets we will not actually give a definition of excellent classes but will only give a proof that our class is excellent. The reader who is content with the main theorem and not interested in the results of section 3, can ignore good sets and excellent classes.

DEFINITION. A triple (M_0, A_1, A_2) is in *stable amalgamation* if M_0 is a model, $M_0 \subseteq A_1, A_2$ and for any $\bar{b} \in A_1$, $\operatorname{tp}(\bar{b}, A_2)$ is the stationarization of $\operatorname{tp}(\bar{b}, M_0)$.

Intuitively, (M_0, A_1, A_2) are in stable amalgamation if the subalgebra generated by $A_1 \cup A_2$ is the free product of A_1 and A_2 amalgamated over M.

PROPOSITION 7. *Suppose models (M_0, M_1, M_2) are in stable amalgamation, M_0 is countable and M_1, M_2 are weakly full over M_0. The subalgebra generated by $M_1 \cup M_2$ is a submodel.*

Proof. By Theorem 0.2 and Theorem 6, $M_i = M_0 * F_i$ for some free algebra F_i, where $i = 1, 2$. Choose X a free basis of M_0 and Y_i a free basis of F_i. Since Y_1 realizes the stationarization to M_2 of its type over M_0, $\langle M_1 \cup M_2 \rangle$ is freely generated by $X \cup Y_1 \cup Y_2$. So it is a submodel.

THEOREM 8. *Suppose models (M_0, M_1, M_2) are in stable amalgamation and M_0 is countable. The subalgebra generated by $M_1 \cup M_2$ is a model.*

Proof. The proof is by induction on the complexity of formulas. It is enough to show that if $\bar{b} \in M_1 \cup M_2$ and $\models \exists x \varphi(x, \bar{b})$ then there is $a \in \langle M_1 \cup M_2 \rangle$ so that $\models \varphi(a, \bar{b})$. By Proposition 7, if F is a free algebra of large enough rank then $\langle M_1 \cup M_2 \rangle * F$ is a submodel. So there is a term w, elements $c_1, \ldots, c_n$ of a free basis of F and $\bar{d} \in M_1 \cup M_2$ so that $\models \varphi(w(c_1, \ldots, c_n, \bar{d}), \bar{b})$. Let F_1 be a free algebra of large rank and consider $\langle M_1 \cup M_2 \rangle * F_1 * F$. Choose $X \cup Y$ a free basis of $\langle M_1 \cup M_2 \rangle * F_1$ so that X is a free basis of M_0. Choose a finite subset Z of $X \cup Y$ so that $\bar{d}, \bar{b}$ are in the algebra generated by Z. Choose $a_1, \ldots, a_n \in X \setminus Z$. Let $a = w(a_1, \ldots, a_n, \bar{d})$. There is an automorphism of $\langle M_1 \cup M_2 \rangle * F_1 * F$ which fixes $\bar{d}, \bar{b}$ and takes a_i to c_i. Hence $\models \varphi(a, \bar{b})$.

The assumption that M_0 is countable is not needed. The theorem without this assumption can be deduced from Theorem 8 by a downward Löwenheim–Skolem argument. There is another argument which is more typical of classification theory. Rather than taking a free basis for M_0 and adding F_1, we can choose an infinite sequence $(a_n : n \in \omega)$ in M_0 based on $\operatorname{Bas}(x)$. It is possible, by standard arguments, to find an infinite subsequence based on the stationarization of $\operatorname{Bas}(x)$ to $\bar{b}$. We have chosen our approach so that we can give a self contained proof of our main theorem.

THEOREM 9. *Suppose M_i $(i \in I)$ is a family of $L_{\omega_1\omega}$-free algebras each of which contains M, a countable $L_{\omega_1\omega}$-elementary substructure. The free product of $M_i(i \in I)$ amalgamated over M is $L_{\omega_1\omega}$-free.*

Proof. Choose $M_i' \supseteq M_i$ so that $M_i' = M * F_i$. The free product of M_i $(i \in I)$ amalgamated over M is contained in $M * *_{i\in I} F_i$. So for all i, $(M, M_i, \bigcup_{i\neq j} M_j)$ is in stable amalgamation. A repeated application of Theorem 8 finishes the proof.

LEMMA 10. *Suppose $M \subseteq N$ are countable models such that M is a proper subset of N, the stationarization of* Bas (x) *to M is not realized in N, and for any tupe p over M either* $\mathrm{Dim}_p(M, N, N) = 0$ *or* $\mathrm{Dim}_p(M, N, N) = \aleph_0$. *For any cardinal κ let $N(\kappa)$ denote the free product of κ copies of N amalgamated over M. Then for any uncountable cardinal κ, $N(\kappa^+) * F(\kappa)$ is κ^+-free and not free.*

Proof. By Theorem 9, $N(\kappa^+)$ is $L_{\omega_1\omega}$-free. Enumerate the copies of N as N_i $(i < \kappa^+)$. We first show that the stationarization of Bas (x) to M is not realized in $N(\kappa^+)$. Suppose not and that $w(\bar{a}, \bar{b}_1, \ldots, \bar{b}_n)$ realizes this type where $\bar{a} \in M$ and $\bar{b}_i \in N_i \setminus M$, and n is minimal. Note that by the choice of N, n is at least 2. Let $p = \mathrm{tp}(\bar{b}_2, M)$. Since the type of $\bar{b}_2$ over $M \cup \bar{b}_1$ is the stationarization of p and $\dim_p(M, N, N) = \aleph_0$, there is $\bar{b}_2' \in N_1$ so that $\mathrm{tp}(\bar{b}_2, M \cup \bar{b}_1) = \mathrm{tp}(\bar{b}_2', M \cup \bar{b}_1)$. Restating this last equation, we have $\mathrm{tp}(\bar{b}_2 \cup \bar{b}_1, M) = \mathrm{tp}(\bar{b}_2' \cup \bar{b}_1, M)$. Since $\mathrm{tp}(\bar{b}_2 \cup \bar{b}_1, M \cup \bigcup_{2<i} \bar{b}_i)$ and $\mathrm{tp}(\bar{b}_2' \cup \bar{b}_1, M \cup \bigcup_{2<i} \bar{b}_i)$ are both the stationarization of the same type, $w(\bar{a}, \bar{b}_1, \bar{b}_2', \ldots, \bar{b}_n)$ realizes the stationarization of Bas (x) to M. This contradicts the minimality of n.

We have already seen in Theorem 6 that $N(\kappa^+) * F(\kappa)$ is κ^+-free. It remains to show that it is not full. Let q be the stationarization of Bas (x) to $M * F(\kappa)$. Assume that $w(\bar{a}, \bar{f})$ realizes q where $\bar{a} \in N(\kappa^+)$ and $\bar{f} \in F(\kappa)$. Choose a finite set $B \subseteq M$ so that $\mathrm{tp}(\bar{a}, M)$ is the stationarization of $\mathrm{tp}(\bar{a}, B)$. Since $(M, N(\kappa^+), M * F(\kappa))$ are in stable amalgamation, $\mathrm{tp}(\bar{a}, M \cup \bar{f})$ is the stationarization of $\mathrm{tp}(\bar{a}, B)$. Choose $\bar{b} \in M$ so that $\mathrm{tp}(\bar{b}, B) = \mathrm{tp}(\bar{f}, B)$. Then $w(\bar{a}, \bar{b})$ realizes q, so in particular it realizes the stationarization of Bas (x) to M. This is a contradiction.

The method used in the proof above will be familiar to the reader acquainted with classification theory.

THEOREM 11. *Suppose $\mathcal{V}$ is a variety in a countable similarity type which does not satisfy the construction principle. Either every $L_{\omega_1\omega}$-free algebra is free or for every infinite cardinal κ there is a non-free algebra of cardinality κ^+ which is κ^+-free.*

Proof. Assume that there is an $L_{\omega_1\omega}$-free algebra which is not free. By the previous lemma, it is enough to show that there are models $M \subseteq N$ satisfying the hypothesis of Lemma 10. Choose N_1 an $L_{\omega_1\omega}$-free algebra which is not full. Suppose N_1 is κ-full but not κ^+-full. Choose $M_1 \subseteq N_1$ so that $\text{Dim}_{\text{Bas}(x)}(M_1, N_1, N_1) = 0$ and for all stationary types p over M_1 either $\text{Dim}_p(M_1, N_1, N_1) = 0$ or $\text{Dim}_p(M_1, N_1, N_1) > \kappa$. To see that M_1 exists, first choose M_2 of cardinality κ so that $\text{Dim}_{\text{Bas}(x)}(M_2, N_1, N_1) = 0$. As was observed after the proof of Proposition 4, there are only κ types over any model of cardinality κ. So we can define an increasing chain of models $(M_n : n > 1)$ of cardinality κ so that for every type p over M_n if $\text{Dim}_p(M_n, N_1, N_1) \leq \kappa$, than $\text{Dim}_p(M_{n+1}, N_1, N_1) = 0$. Let $M_1 = \bigcup_{1<n} M_n$.

We now want to reduce $M_1 \subseteq N_1$ to countable models. There are various ways to do this. The simplest way is to choose a cardinal λ greater than $|N|$ and let C be an elementary substructure of $(H(\lambda), \epsilon)$ such that M_1, N_1 and $F(\omega)$ belong to C. Here $H(\lambda)$ denotes the sets of hereditary cardinality less than λ. Then $M = M_1 \cap C$ and $N = N_1 \cap C$ are as desired.

The reader who is familiar with [S3] and [S4] will realize that we have proved more than is needed to show the class $\mathcal{K}$ is excellent. In the following lemma I is a finite set partially ordered by $<_I$ such that any two elements have a lower bound (i.e. I is a meet semilattice). Let 0 denote the least element of I.

LEMMA 12. *Suppose I is a finite meet semilattice and $(M_i : i \in I)$ a system of countable models and $<$ an ordering of I satisfying the following*:

(i) *If $i <_I j$, then $i < j$,*
(ii) *If $i <_I j$, then $M_i \subseteq M_j$;*
Let A_i denote $\bigcup_{j <_I i} M_j$.
(iii) *For any i and $\bar{a} \in M_i$, if* $\text{tp}(\bar{a}, A_i)$ *is stationary then* $\text{tp}(\bar{a}, \bigcup_{j<i} M_j)$ *is the stationarization of* $\text{tp}(\bar{a}, A_i)$.
(iv) *For any i, M_i is weakly full over A_i.*
Then $\bigcup_{i \in I} M_i$ generates a model.

Proof. The proof is by induction on the cardinality of I. Of course, for $|I| = 1$, there is nothing to prove. Notice that the induction hypothesis implies that for all $i \neq 0$, A_i generates a model. So for $i \neq 0$, every type over A_i is stationary and indeed A_i can be replaced by the model is generates. The induction step follows from applying Proposition 7.

Rather than give the definition of being excellent which would involve more definitions we will just point out that being excellent is implied for an $\aleph_0$-stable class by Lemma 12 by specializing I to range over partially ordered sets of the form $\mathcal{P}^-(n)$. Here $\mathcal{P}^-(n)$ is the collection of subsets of $\{0, \ldots, n-1\}$ of cardinality at most $n-1$, ordered by containment.

THEOREM 13. *Suppose $\mathcal{V}$ is a variety in a countable similarity type which does not satisfy the construction principle. Let $\mathcal{K}$ be the class of algebras $L_{\omega_1\omega}$-equivalent to the free algebra of rank ω. The class $\mathcal{K}$ is excellent.*

Originally our proof of Theorem 11 relied on facts about excellent classes. By Theorem 5.9 of [S4], if $\mathcal{K}$ is an excellent class then either $\mathcal{K}$ has (up-to-isomorphism) exactly one model in each cardinal or in every uncountable cardinality there are at least two non-isomorphic models. In the latter case it is possible to show for any infinite cardinal κ that there is a model of cardinality κ^+ which is κ-full but not full (see [Ha] or [GH]). To deduce Theorem 11 from these facts, all that is needed is the interpretation of "full" and "κ-full" in terms of "free" and "κ^+-free."

3. Dimensions

It is possible to obtain a deeper understanding of the structure of $L_{\infty\omega}$-free algebras by using the machinery of (strongly) regular types. The theory of regular types for excellent classes is analogous to that for $\aleph_0$-stable first-order theories as found in [S2] *V*. The theory is worked out in detail in [Ha] and will be presented in [GH]. Since any type over a finite set is implied by a formula there is no difference in the setting of excellent classes between regular and strongly regular types. Here we will content ourselves with a definition of regular types and a statement of enough of their properties so that the major theorem can be understood. For the remainder of this section fix a countable model M_0.

DEFINITION. Suppose p is a stationary non-algebraic type over a set $B \supseteq M_0$. (A type is *algebraic* if it has only a finite number of realizations in any model.) We say p is *regular* if for any model $N \supseteq B$ if q extends p then either q is the stationarization of p to N or if a realizes the stationarization of p to N and b realizes q then a realizes the stationarization of p to $N \cup \{b\}$. (In considering regular types it is enough to look at types in only one variable.) Suppose B and C contain M_0. Two stationary types p and q based on B and C respectively are said to be *orthogonal* if for any model M containing B, C whenever a and b realize the stationarizations of p and q to M then a realizes the stationarization of p to $M \cup \{b\}$, and vice versa (although this is implied).

For regular types, based on sets containing M_0 being non-orthogonal is an equivalence relation. We shall be concerned with the finite dimensional case.

DEFINITION. An excellent class $\mathcal{K}$ is *finite dimensional* if there is a natural number n so that any set of mutually orthogonal regular types based on on sets containing M_0 has cardinality at most n. The minimum such n is called the *dimension* of $\mathcal{K}$.

If $\mathcal{K}$ is finite dimensional, p is a regular type based on a set containing M_0 and M is a model containing M_0 then there is a regular type q over M so that q is not orthogonal to p. In particular any regular type over a set containing M_0 is non-orthogonal to a regular type over M_0. As well, if p and q are non-orthogonal regular types over a model M then $\mathrm{Dim}_p(\emptyset, M, M) = \mathrm{Dim}_q(\emptyset, M, M)$. So the dimension of a regular type depends only on its equivalence class (under non-orthogonality). Note that the $\mathrm{Dim}_q(\emptyset, M, M)$ is at least $\aleph_0$. The basic facts we will use is given in the following theorem.

THEOREM 14. *Suppose $\mathcal{K}$ is a finite dimensional excellent class and $p_1, \ldots, p_n$ enumerates a maximal set of mutually orthogonal regular types over M_0.*

(a) *If $M, N \supseteq M_0$ are models then $M \cong N$ if $\mathrm{Dim}_{p_i}(\emptyset, M, M) = \mathrm{Dim}_{p_i}(\emptyset, N, N)$ for all $1 \le i \le n$.*
(b) *For any collection of infinite cardinals $\lambda_1, \ldots, \lambda_n$ there is a model M so that $\mathrm{Dim}_{p_i}(\emptyset, M, M) = \lambda_i$ for all $1 \leqslant i \leqslant n$.*

To apply this theorem, we use the following proposition.

PROPOSITION 15. *Let $\mathcal{K}$ be the class of $L_{\omega_1\omega}$-free algebras for some variety in which the construction principle fails. Then $\mathcal{K}$ is finite dimensional.*

Proof. It is enough to show that there is a stationary type such that no non-algebraic type is orthogonal to it. (This claim is standard, cf. [S2] V or [Ha].) Let p be the type generated by the formula $\mathrm{Bas}(x)$. Suppose M is a model and q is a non-algebraic type over M. If F is a free algebra of rank $|M|$, then q is realized in $M * F$. So there is a finite subset X of a free basis of F and a in the subalgebra generated by $M \cup X$ so that a satisfies q. Since $\mathrm{tp}(a, M \cup X)$ is algebraic, it is not the stationarization of q to $M \cup X$. Let $X = \{b_1, \ldots, b_n\}$. Since b_k realizes the stationarization of p to $M \cup \{b_j : j < i\}$, q is not orthogonal to p.

There is some subtlety in the application of Theorem 14. Since all the countable $L_{\omega_1\omega}$-free algebras are actually free, any $L_{\omega_1\omega}$-free algebra N contains a copy of M_0. Fixing the copy of M_0, the dimensions of the isomorphic copies of $p_1, \ldots, p_n$ determine the isomorphism type of N relative to the choice of M_0. (Note that this is different than isomorphism over the choice of the copy of M_0.)

However a different choice of the copy of M_0 might give different choices for the copies of $p_1, \ldots, p_n$, in the sense that we may have permuted the equivalent classes of regular types over N. But any such permutation is induced by an automorphism of M_0. So in general we have a finite set of equivalence classes of regular types over a countable model M_0 and a permutation group acting on the equivalence classes induced by the automorphism group of M_0. Thus the isomorphism type of any model is determined by a finite set of dimensions together with a permutation group acting on these dimensions.

In the case of abelian groups of exponent 6, or more generally modules over a left-perfect ring, the notion of dimensions works out just as it should. That is for abelian groups of exponent 6, the regular types are $x \neq 0 \wedge 2x = 0$ and $x \neq 0 \wedge 3x = 0$. For each of these types the dimension of the type corresponds to the number of copies of $C(2)$ and $C(3)$ represented in a direct sum decomposition of the group.

4. Conclusion

We summarize the preceding results and those of [EM] in a series of theorems.

THEOREM 16. *For any variety of algebras $\mathcal{V}$ exactly one of the following possibilities holds.*

(i) *There is an $L_{\infty\omega_1}$-free algebra of cardinality $\aleph_1$ which is not free.*
(i)′ *Assume $V = L$. For every regular non-weakly compact cardinal κ, there is a non-free $L_{\infty\kappa}$-free algebra.*
(ii) *Every $L_{\infty\omega_1}$-free algebra of cardinality $\aleph_1$ is free. For every cardinal κ there is an $L_{\infty\kappa}$-free algebra of cardinality κ^+ which is not free.*
(iii) *Every $L_{\infty\omega}$-free algebra is free.*

The theorem above can be proved using the results of [EM] and the results in the sections before section 3. Using the results of section 3, it is possible to give cardinality equivalents to the possibilities.

THEOREM 17. *Let the possibilities be as in Theorem* 16. *Then*

(i) *is equivalent to there are $2^{\aleph_1}$ pairwise non-isomorphic $\aleph_1$-free algebras of cardinality $\aleph_1$.*
(ii) *is equivalent to there is more than* 1 *but finitely many pairwise non-isomorphic $\aleph_1$-free algebras of cardinality $\aleph_1$.*
(iii) *is equivalent to any $\aleph_1$-free algebra is free.*

Although the division of varieties into three classes is most naturally expressed in terms of uncountable algebras, it is interesting to note that each of (i), (ii) and (iii) above are equivalent to a condition on countable free algebras.

Given this division of varieties into three classes, it is natural to ask into which class interesting varieties fall. In retrospect, this question has a long history. Higman's construction in [Hi] of a countably free group which is not free uses the construction principle for the variety of all groups. For varieties of modules and almost all varieties of groups, it is known which class they belong to. The answer to the weaker question of which varieties have $\aleph_1$-free algebras is known for all varieties of modules and varieties of groups. Information on modules and some miscellaneous varieties can be found in [EM] and information on groups can be found in [M2]. We hope that those who work on other varieties of algebras will find the question of which classes their varieties belong to interesting in the context of their work.

REFERENCES

[EM] P. EKLOF, and A. MEKLER, *Categoricity results for $L_{\infty\kappa}$-free algebras,* Annals of Pure and Applied Logic *37* (1988) 81–99.
[GH] R. GROSSBERG and B. HART, *Classification of excellent classes,* J. Symbolic Logic (to appear)
[Ha] B. HART, Some results in classification theory, Ph.D. Thesis, McGill, 1987.
[Hi] G. HIGMAN, *Almost free groups,* Proc. London Math. Soc. *1* (1951), 231–248.
[Ho] W. HODGES, For singular, λ, λ-free implies free, Algebra Universalis *12* (1981), 205–220.
[M1] A. MEKLER, Applications of logic to group theory, Ph.D. thesis, Stanford, 1976.
[M2] A. MEKLER, Almost free groups in varieties, J. Algebra. (to appear)
[S1] S. SHELAH, *A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals,* Israel J. Math. *21* (1975), 319–340.
[S2] S. SHELAH, Classification Theory, North Holland 1978.
[S3] S. SHELAH, *Classification theory for non-elementary classes I: the number of uncountable models of $\psi \in L_{\omega_1\omega}$. Part A.* Israel J. Math. *46* (1983) 211–240.
[S4] S. SHELAH, *Classification theory for non-elementary classes I: the number of uncountable models of $\psi \in L_{\omega_1\omega}$. Part B.* Israel J. Math. *46* (1983) 241–273.
[S5] S. SHELAH, *Incompactness in Regular Cardinals,* Notre Dame J. Formal Logic *26* (1985) 195–228.

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