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ON THE EXISTENCE OF REGULAR TYPES

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The main results in the paper are the following.

Theorem A. *Suppose that T is superstable and $M \subset N$ are distinct models of T^{eq}. Then there is a $c \in N \setminus M$ such that $t(c/M)$ is regular.*

For $M \subset N$ two models we say that $M \subset_{na} N$ if for all $a \in M$ and $\theta(x, a)$ such that $\theta(M, a) \neq \theta(N, a)$, there is a $b \in \theta(M, a) \setminus \mathrm{acl}(a)$.

Theorem B. *Suppose that T is superstable, $M \subset_{na} N$ are models of T^{eq}, and p is a regular type non-orthogonal to $t(N/M)$. Then there is a $c \in N$ such that $t(c/M)$ is regular and non-orthogonal to p. Furthermore, there is a formula $\theta \in t(c/M)$ such that $a \in \theta(N)$ and $t(a/M) \not\perp p \Rightarrow t(a/M)$ is regular.*

We use these results to obtain 'good' tree decompositions of models in (possibly uncountable) superstable theories with NDOP. See Definition 5.1 for the undefined terms.

Theorem C. *Suppose that T is superstable with* NDOP *and $M \models T^{eq}$. Then every $\subset_{na}$-decomposition inside M extends to a $\subset_{na}$-decomposition of M. Furthermore, if $\langle N_\eta, a_\eta \colon \eta \in I\rangle$ is any $\subset_{na}$-decomposition of M, then M is minimal over $\bigcup N_\eta$ and for all $\eta \in I$, M is dominated by $\bigcup N_\eta$ over N_η.*

Using some stable group theory we show that when $\mathrm{Th}(M)$ is superstable with NDOP and $\langle N_\eta \colon \eta \in I\rangle$ is a tree decomposition of M, then M is constructible over $\cup N_\eta$ with respect to a very strong isolation relation (Section 6).

0. Introduction

Regular types play a crucial role in most theorems concerning the number of models of a superstable theory. They are central to the classification of countable first-order theories in [11]. Theorems A and B help to close the gap between superstable and totally transcendental theories (the results are well known for totally transcendental theories [10, §D]).

It is unfortunate that this paper was not written several years ago. The results in [11, XII and XIII] would have been easier to prove with the theorems in this paper. Without them there are significant technical difficulties in obtaining good tree decompositions of superstable theories with NDOP. This distracts us from

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the real difference between superstable and totally transcendental theories: the existence of prime models over sets. So, the ambitious reader could obtain a smoother proof of the Main Gap by incorporating this paper into [11].

Looking for more important applications, since we never assume that the theory is countable in this paper, it may play an active part in the classification of uncountable theories. Outside of the context of the Main Gap we still see Theorems A and B as important technical results about superstable theories. We would be very surprised if they did not play an important role in other topics, e.g., Vaught's conjecture for superstable theories.

The experienced reader who has seen other attempts to prove (or circumvent) Theorem A (see, in particular, [13, §1]) may be surprised at how straightforward the proof is. The reason is the availability of 'nice' non-trivial regular pairs. This is definitely the main ingredient in the proof. The rest of the proof is an excercise in manipulating p-simple and p-semi-regular types. (This is carried out in Section 3.) The niceness of non-trivial regular pairs was defined and proved by Shelah for superstable theories with NDOP (Theorem 2.2). This, of course, is all that is needed for Theorem C. Subsequently, Hrushovski eliminated the NDOP assumption (Theorem 2.1). This is a major advancement in working with general superstable theories.

Section 4, culminating in a proof of Theorem B, is largely a continuation of Section 3. The main lemma was already proved in Section 3. In Section 4 we focus on the $\subset_{na}$ relation through a couple of technical lemmas. Combining these with Section 3 leads quickly to Theorem B.

We will say very little about Section 5 in this introduction as it involves many terms with lengthy definitions. The basic idea is to obtain a very nice tree decomposition of a model of a superstable theory with NDOP. We refer the reader to the introduction of that section for a detailed discussion.

The main result in the last section is that for T superstable with NDOP, a minimal model over an independent tree of models is j-constructible over the tree. Admittedly this is a rather minor result but its proof may be of major importance. It is the prototype of combining Hrushovski's theorems about the existence of definable group actions with notions of isolation. We feel that other results will come from the methods developed here.

All of the theorems proved in this paper are due to Shelah. The organization of the material and the proofs as detailed here are by Buechler (as are any remaining errors). Buechler wishes to thank Bradd Hart for many helpful conversations on the topics contained herein, and Udi Hrushovski for his comments on a preliminary version of this paper. Shelah would like to thank Steven Buechler for so much improving the paper.

1. Background and some definitions

We assume that the reader has a working knowledge of stability theory as expounded in [10] and [11]. Our notation will largely follow [10]. We also assume that the reader knows the main definitions and theorems in [12] or [5], but not necessarily the proofs. In keeping with [10] the term 'regular' will only be applied to stationary types. The generalized RK-order on types, written $\lhd$ is defined in [10, C.13]. For stationary p and q in a stable theory $p \lhd q$ iff $p \leqslant_s q$, where the latter is as defined in [11, V, 2.1(2)]. We use $p \Box q$ to denote $p \lhd q$ and $q \lhd p$ (where Makkai uses $p \bowtie q$). For p a stationary type, $p^{(n)}$ denotes the type (over $\mathrm{dom}(p)$) of an independent sequence of realizations of p of length n. $R^\infty(-)$ denotes ∞-rank, i.e., the rank $R(-, L, \infty)$.

Throughout the paper we will tacitly work in T^{eq}. When we say that M is a model, we mean that $M = N^{\mathrm{eq}}$ for N some model of the original theory. $\mathrm{acl}(-)$ is algebraic closure as computed in T^{eq}.

For p a stationary type we call c a *restricted canonical base of p* if $\mathrm{acl}(\mathrm{Cb}(p)) = \mathrm{acl}(c)$. In a superstable theory there is always a restricted canonical base in $\mathrm{Cb}(p)$.

Definition. We say that q (a possibly incomplete type) is *hereditarily orthogonal* to p if $q' \supset q \Rightarrow q' \perp p$.

For A, B and C sets we say that A is *dominated by B over C*, written $A \lhd B(C)$, if for all d, $d \mathop{\downarrow\!\!\!\downarrow}_C B \Rightarrow d \mathop{\downarrow\!\!\!\downarrow}_C A$. As is easily verified, domination is transitive.

If $\mathbf{F}^x_\lambda$ is an isolation relation as discussed in [11, IV] we shorten $\mathbf{F}^x_{\aleph_0}$-isolated to x-isolated, etc.

Definition. For p a stationary type and $\varphi \in p$ we say that (p, φ) is a *regular pair* (RP) if for all q, $\varphi \in q$ and $R^\infty(q) < R^\infty(\varphi) \Rightarrow q \perp p$.

Lemma 1.1. (a) *Suppose that T is superstable, p is stationary and $\varphi \in p$ is such that $\varphi \in q$, and q forks over* $\mathrm{dom}(\varphi) \Rightarrow q \perp p$. *Then there is a $\varphi' \vdash \varphi$ over* $\mathrm{dom}(\varphi)$ *and a p' such that (p', φ') is an RP and $p' \Box p$.*

(b) *If (p, φ) is an RP, $\varphi \in q$ and $q \not\perp p$, then (q, φ) is an RP.*

(c) *If (p, φ) is an RP, $\varphi \in q$ and $q \perp p$, then q is hereditarily orthogonal to p.*

Proof. Easy.

Caveat. Shelah defines (p, φ) to be a regular pair if it satisfies the hypothesis on φ and p in Lemma 1.1(a). The added condition on rank in our definition is usually required in working with regular pairs. Because of this and Lemma 1.1 we choose to add it to the definition.

Abusing the terminology (badly) we will say that p is an RP if there is a φ such that (p, φ) is an RP.

Definition. Let (p, φ) be an RP, $\mathrm{dom}(p) = A$. We say that (p, φ) is *nice* if whenever a realizes p and b is such that $a \not\mathop{\downarrow\!\!\!\downarrow}_A b$, there is a formula $\theta(x, y) \in t(ab/A)$ such that $\vDash \theta(a', b') \Rightarrow \mathrm{stp}(a'/b'A)$ is hereditarily orthogonal to p.

Definition. (a) We say that a formula ψ is *α-remote* if every $q \in S(\mathfrak{C})$ containing ψ is non-orthogonal to a type of *∞-rank* $< \alpha$.

(b) Let p be an RP. We say that ψ is *p-remote* if ψ is $R^\infty(p)$-remote and hereditarily orthogonal to p.

(c) Let (p, φ) be an RP with $A = \mathrm{dom}(p)$. We say that (p, φ) is *f-nice* if for any a realizing p and b such that $a \not\downarrow_A b$, there is a formula $\psi(x, y) \in t(ab/A)$ such that for all b', $\psi(x, b') \vdash \varphi$ and $\psi(x, b')$ is p-remote.

(The 'f' in f-nice is, of course, for 'forte'.) It is clear that f-nice $\Rightarrow$ nice. With an example (unpublished) Hrushovski has shown that the converse is false. In Hrushovski's terminology a type is α-remote if it is $R^\infty(<\alpha)$-analyzable, where $R^\infty(<\alpha)$ is the class of formulas of ∞-rank $< \alpha$.

It is helpful to observe that if ψ is α-remote, $\psi \in q \in S(\mathfrak{C})$ and r is a regular type non-orthogonal to q, then r is non-orthogonal to a regular type of ∞-rank $< \alpha$, which contains ψ. Also, if (p, φ) is an RP and $\vdash \psi \to \varphi$, then ψ is p-remote iff it is $R^\infty(p)$-remote. This is the usual way in which p-remote formulas arise.

With the notion of a nice RP we are trying to approximate that "forking with a realization of p is definable". (This property would be defined by replacing "$\psi(x, b')$ is hereditarily orthogonal to p" by "$\psi(x, b')$ forks over A".) In what follows it will be obvious why such a definability of forking would be desirable. In the next section we will see that non-trivial regular pairs are nice.

Lemma 1.2. *Suppose that p and p' are RPs, p is parallel to p' and* $\mathrm{dom}(p)$ *and* $\mathrm{dom}(p')$ *are both algebraically closed. Then p is f-nice (nice) iff p' is f-nice (nice).*

Proof. The proof is the same for nice and f-nice. Let $A = \mathrm{dom}(p)$. It suffices to consider the case when $A \subset B = \mathrm{dom}(p')$. It is immediate that if p is nice then so is p'. Suppose that p' is nice, a realizes p and $a \not\downarrow_A b$. W.l.o.g., $ab \downarrow_A B$. Now apply the niceness of p' and the definability over A of $t(ab/B)$ to find the desired formula in $t(ab/A)$.

Definition. Let p be regular, q a non-algebraic strong type over A. We say that q is *p-semi-regular* if there are: a realizing q, $M \supset A$ an a-model with $a \downarrow_A M$ and $c_0, \ldots, c_n$ such that

(i) $p_i = t(c_i/M)$ is conjugate to p' and non-orthogonal to p, $p' \parallel p$,

(ii) $a \in \mathrm{acl}(M\bar{c})$,

(iii) if $A_i \subset M$ is such that p_i is based on A_i and $r \supset p_i \mid A_i$ forks over A_i, then $q \perp r$.

Part (iii) in the definition is rather important. It is this which allows us to conclude that if q is p-semi-regular, then for some n, $q \,\Box\, p^{(n)}$ (see [11, V, 4.1]).

This definition differs slightly from Shelah's in that he only requires $p_i \square p$ in (i). The present definition elucidates the fact that the p_i's witnessing p-semi-regularity must be chosen carefully. That is, q p-semi-regular and $p' \square p$ does not imply that q is p'-semi-regular. It is important to bear this in mind. The main existence lemma for semi-regular types is the following.

Lemma 1.3 [11, V, 4.12]. *Suppose that T is superstable, p is regular and* $\mathrm{stp}(a/A)$ *is such that $p \not\perp \mathrm{stp}(a/A)$ and if $q \supset p$ is a forking extension, then $q \perp \mathrm{stp}(a/A)$. Then there is a $c \in \mathrm{acl}(aA)$ such that $\mathrm{stp}(c/A)$ is p-semi-regular.*

The next concept was originally defined and investigated in [11, V, §4], but its first major use was by Hrushovski in his beautiful work [7]. We repeat the definitions here.

Definition. (a) Let p be regular, q a strong type. We say that q is *p-simple* if there are: an a-model $M \supset \mathrm{dom}(p) \cup \mathrm{dom}(q)$, a realizing $q \mid M$ and I a Morley sequence in $p \mid M$ such that $t(a/MI)$ is hereditarily orthogonal to p.

(b) For q a p-simple strong type, the *p-weight of q*, $w_p(q)$, is the cardinality of a minimal I witnessing the p-simplicity as in (a).

Clearly, a p-semi-regular type is p-simple. Unlike p-semi-regularity, p-simplicity is invariant under $\square$. That is, if q is p-simple and $p' \square p$, then q is p'-simple. Also, if $\bar{b} = b_0 \cdots b_n$ is such that $\mathrm{stp}(b_i/A)$ is p-simple for all $i \leq n$, then $\mathrm{stp}(\bar{b}/A)$ is p-simple. (Another property not shared by p-semi-regular types.) Each of these uses the fact that if $\mathrm{stp}(\bar{a}/A\bar{b})$ and $\mathrm{stp}(\bar{b}/A)$ are p-simple, then $\mathrm{stp}(\bar{a}/A)$ is p-simple. Furthermore,

$$w_p(\bar{a}\bar{b}/A) = w_p(\bar{a}/A\bar{b}) + w_p(\bar{b}/A).$$

Lemma 1.4. *Let (p, φ) be a nice RP. Suppose that A is algebraically closed, $t(b/A)$ is p-semi-regular and $w_p(b/Aa) = n$. Then there is a formula $\theta(x, y) \in t(ba/A)$ such that $\models \theta(b', a') \Rightarrow \theta(x, a')$ is p-simple and $w_p(b'/Aa') \leq n$.*

Proof. Let $M \supset A$ be a large saturated model, $ba \perp_A M$. By the definability of types and the fact that p is non-orthogonal to A it suffices to find such a formula θ over M. W.l.o.g., there are conjugates of (p, φ), (p_i, φ_i), $i \leq n$, $p_i \in S(M)$, $p_i \not\perp p$, and c_i realizing p_i such that $b \in \mathrm{acl}(M\bar{c})$. W.l.o.g., $w_p(\bar{c}/Mb) = 0$. An easy computation yields: $w_p(\bar{c}/aM) = w_p(b/aM)$. Since each (p_i, φ_i) is nice it is easy to find a formula $\sigma(\bar{x}, y) \in t(\bar{c}a/M)$ such that $\models \sigma(\bar{c}', a') \Rightarrow \models \varphi_i(c_i')$ and $w_p(\bar{c}'/a'M) \leq n$. Let $\theta \in t(ba/M)$ be such that $\models \theta(b', a') \Rightarrow$ there is a $\bar{c}'$, $\models \sigma(\bar{c}', a')$ and $b' \in \mathrm{acl}(\bar{c}', M)$. Since each φ_i is p-simple, $\theta(x, a')$ is p-simple. The p-weight computation is easy, completing the proof. $\square$

Note that the p-simplicity of $\theta(x, a')$ does not require the niceness of p.

The main lemmas we use on p-simple types (quoted below) are basic facts 4) and 5) in [7, §3]. Below T is superstable.

Lemma 1.5 [11, V, 4.20]. *Suppose that p is regular and non-orthogonal to* $\mathrm{stp}(a/A)$. *Then there is a* $b \in \mathrm{acl}(Aa)$ *such that* $\mathrm{stp}(b/A)$ *is p-simple and non-orthogonal to p.*

Lemma 1.6. *Suppose that* $\mathrm{stp}(a/A)$ *is p-simple. Then there is a* $d \in \mathrm{dcl}(aA)$ *such that* $\mathrm{stp}(a/Ad) \square p^{(n)}$, *for some n, and* $w_p(d/A) = 0$.

Proof. This proof is easier than the one in [7] because we are assuming that T is superstable. First, in Claim 1, we establish a useful basic fact about p-simple types.

An elementary fact about regular types we will be using is that if r is any type and $s \not\perp r$ is regular, then there is a regular $r' \supset r$ such that $r' \square s$.

Claim 1. *If q is a regular type non-orthogonal to* $\mathrm{stp}(a/A)$, *then $q \square p$ or there is a $q' \square q$ such that q' is hereditarily orthogonal to p.*

Since $\mathrm{stp}(a/A)$ is p-simple there is an a-model $M \supset A$, $a \downarrow_A M$, and I a Morley sequence in $p \mid M$ such that $w_p(a/MI) = 0$. There is a $b \not\downarrow_M a$ such that $t(b/M) \square q$. Assuming that $q \perp p$ we must have $b \downarrow_M I$. Thus, $\mathrm{stp}(b/MI) \square q$ and this strong type is non-orthogonal to $\mathrm{stp}(a/MI)$. By the fact mentioned above there is an $r \supset \mathrm{stp}(a/MI)$ such that $r \square \mathrm{stp}(b/MI)$. Thus, r is hereditarily orthogonal to p and $r \square q$, proving the claim.

Turning our attention to the lemma, since extensions of p-simple types are p-simple we may use induction on $R^{\infty}(a/A)$ to see that it suffices to prove

Claim 2. *If we do not have* $\mathrm{stp}(a/A) \square p^{(n)}$, *for some n, there is a* $d \in \mathrm{dcl}(aA) \setminus \mathrm{acl}(A)$ *such that* $w_p(d/A) = 0$.

Let q be a regular type non-orthogonal to $\mathrm{stp}(a/A)$ which is orthogonal to p. By Claim 1 we may assume that q is hereditarily orthogonal to p. We further pick q to have least ∞-rank under these requirements. We want to apply Lemma 1.3 so consider q', a forking extension of q. If q' is non-orthogonal to $\mathrm{stp}(a/A)$, then using the above fact on regular types we can find a regular $q'' \supset q'$, $q'' \not\perp \mathrm{stp}(a/A)$. Since q'' is hereditarily orthogonal to p and $R^{\infty}(q'') < R^{\infty}(q)$ we contradict the minimality assumption on $R^{\infty}(q)$. Thus, every forking extension of q is orthogonal to $\mathrm{stp}(a/A)$. By Lemma 1.3 there is $d_0 \in \mathrm{acl}(aA)$ such that $\mathrm{stp}(d_0/A)$ is q-semi-regular. Clearly, this strong type is hereditarily orthogonal to p, as is the element d of $\mathfrak{C}^{\mathrm{eq}}$ naming $\{d' : d \equiv d_0(aA)\}$. Since $d \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A)$, we have proved the lemma. $\square$

We say that a formula ψ is *p-simple* if every $q \in S(\mathfrak{C})$ containing ψ is p-simple. Notice that if (p, φ) is an RP, then φ is p-simple. We leave the proof of the following to the reader.

Lemma 1.7. *Suppose that p is an RP, $A = \mathrm{acl}(A)$ and $q \in S(A)$ is p-semi-regular. Then there is a $\psi \in q$ which is p-simple.*

Definition. Suppose that p is regular, $M \subset N$ are models and θ is a formula over M such that θ is p-simple and $b \in \theta(N)$, $t(b/M) \not\perp p \Rightarrow t(b/M) \square p$. *Then,* if $b \in \theta(N)$ and $t(b/M) \not\perp p$ we say that $(t(b/M), \theta)$ is a *regular pair relative to N.*

2. Notes of depth zero

Initially, Theorems A and B were proved only for theories with NDOP. The reason being that the main technical lemma, Theorem 2.1, was proved only with this additional assumption. This first result was due to Shelah and is stated here as Theorem 2.2. Later, while working on a different problem, Udi Hrushovski found a rather straightforward proof of Theorem 2.1 as it is stated (see [8]). We will furnish a proof of Theorem 2.2 since we would like the paper to be largely self-contained for theories with NDOP.

Theorem 2.1. *Suppose that T is superstable, A is algebraically closed and $t(a/A)$ is a non-trivial RP. Then for some $a' \in \mathrm{dcl}(aA) \setminus A$, $t(a'/A)$ is an f-nice RP.*

Theorem 2.2. *Let T be superstable with* NDOP, *p an RP of depth* 0 *with* dom(p) *algebraically closed. Then p is f-nice.*

We prove the theorem with a series of lemmas. By Lemma 1.2 it suffices to consider the case when $\mathrm{dom}(p) = M$ is an a-model. Let $\varphi \in p$ be such that (p, φ) is an RP. Suppose that a realizes p and b is such that $a \not\downarrow_M b$.

Let $\psi(x, b) \in t(a/bM)$ be such that $\psi(x, b) \vdash \varphi$ and $\psi(x, b)$ forks over M.

Let $M^* \supset M$ be a large saturated model containing b. Let $M^+ = M \cup \{e \in M^*: \vDash\varphi(e)$ and $R^\infty(e/M) < R^\infty(\varphi)\}$. Let I be a basis for p in M^*, $N_0 \subset M^*$ an a-prime model over $I \cup M$, $N_1 \subset M^*$ a-prime over M^+, and $N \subset M^*$ a-prime over $N_0 \cup N_1$. Since $N_0 \lhd I(M)$ and $N_1 \lhd M^+(M)$, $N_0 \downarrow_M N_1$.

Lemma 2.3. (a) $p(N) = p(M^*)$.

(b) *There is no infinite set of indiscernibles over $M^+ \cup I$ in $p(N)$.*

Proof. (a) Suppose, towards a contradiction, that there is a $b \in p(M^*) \setminus N$. By NDOP we know that $t(b/N)$ is non-orthogonal to N_0 or N_1. By a slight refinement to [10, D.11(v)] we can find a $b' \in N[b] \subset M^*$ such that $t(b'/N)$ is regular,

$b' \perp\!\!\!\perp_{N_0} N$ or $b' \perp\!\!\!\perp_{N_1} N$, and b' realizes p_0, a type parallel to p over some finite subset of M. If $R^\infty(b'/M) < R^\infty(p_0)$, $b' \in M^+$, a contradiction. Thus, $t(b'/M) = p$. Since $t(M^+/M)$ is hereditarily orthogonal to p, $b' \perp\!\!\!\perp_M N_1$. If $N \perp\!\!\!\perp_{N_1} b'$, then $N \perp\!\!\!\perp_M b'$. But N_0 contains a basis for p in M^*, so $b' \not\perp\!\!\!\perp_M N_0$. Thus, $N \not\perp\!\!\!\perp_{N_1} b'$ and we may conclude that $N \perp\!\!\!\perp_{N_0} b'$. Since p has depth 0 and $N_0 = M[I]$, $t(b'/N_0) \not\perp M$. Thus, we can find a $b'' \in M^*$ realizing p_0 such that $b' \not\perp\!\!\!\perp_N b''$ and $b'' \perp\!\!\!\perp_M N$. b'' cannot realize $p \mid N$ since $N \supset I$, so $t(b''/M) \neq p$. Thus, $b'' \in M^+$, contradicting that $b'' \notin N$. This proves (a).

(b) By NDOP we know that N is minimal over $I \cup M^+$ among a-models. It is easy to show that if $J \subset N$ is an infinite set of indiscernibles over $I \cup M^+$ and b realizes $\mathrm{Av}(J/N)$, then $N \cup b$ is a-atomic over $I \cup M^+$. This contradicts the minimality of N, proving (b). $\square$

Actually, (b) implies

(1) *there is no infinite set of indiscernibles over M^+ in $\psi(N, b) \cap p(N)$.*

(This is because $\psi(x, b)$ is hereditarily orthogonal to p. There is a finite $I_0 \subset I$ such that $I \setminus I_0 \perp\!\!\!\perp_{MI_0b} \psi(N, b) \cap p(N)$. If $J \subset \psi(N, b) \cap p(N)$ is an infinite set of indiscernibles over M^+, then all but finitely much of J is indiscernible over $M^+ \cup I_0 \cup b$, hence, indiscernible over $M^+ \cup I$.) So, Lemma 2.3(a) implies

(2) *there is no infinite set of indiscernibles over M^+ in $\psi(M^*, b) \cap p(M^*)$.*

Let $\Gamma = \{\theta\colon \theta$ is a formula over M, $\theta \vdash \varphi$ and $R^\infty(\theta) < R^\infty(\varphi)\}$.
Of course, we may as well assume that $M^* = \mathfrak{C}$, yielding

Corollary 2.4. *There is no infinite set of indiscernibles over M^+ in $\psi(\mathfrak{C}, b) \cap p(\mathfrak{C})$.*

So, the hereditary orthogonality of $\psi(x, b)$ to p allowed us to eliminate I in Lemma 2.3(b). The importance of Corollary 2.4 is that we can now define a theory expressing that there *is* a set of indiscernibles over M^+ in $\psi(\mathfrak{C}, b) \cap p(\mathfrak{C})$, and apply compactness. This gives us the desired formula in $t(b/M)$. The details are as follows.

Add to the language constant symbols for the elements of $M \cup b$ and a unary predicate symbol I. Let $T' \supset T$ be the theory expressing

(i) the elementary diagram of $M \cup b$,
(ii) $I \subset \psi(\mathfrak{C}, b) \cap p(\mathfrak{C})$, and
(iii) I is infinite and indiscernible over $M^+ = \bigcup\{\theta(\mathfrak{C})\colon \theta \in \Gamma\}$.

By Corollary 2.4, T' is inconsistent. Thus, there are: $\psi'(x, y) \in t(ab/M)$, $\psi'(x, y) \vdash \psi(x, y)$, $\theta \in \Gamma$ and $c \in M$, such that for all b'

(3) *there is no infinite set of indiscernibles over $c \cup \theta(\mathfrak{C})$ in $\psi'(\mathfrak{C}, b')$.*

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W.l.o.g., c contains the parameters from M in θ and ψ'. We now have to check that ψ' is the desired formula. Notice that $R^\infty(\theta)<R^\infty(p)$ and θ is hereditarily orthogonal to p. Pick an arbitrary b' and suppose that $q\in S(\mathfrak{C})$ contains $\psi'(x, b')$. By (3), $q \not\perp \theta$. Since $\varphi\in q$ and $\theta\vdash\varphi$, we must have $p\perp q$. It follows immediately that $\psi'(x, b')$ is p-remote. Thus, p is f-nice.

Hrushovski has another proof of Theorem 2.2 which does not use that T has NDOP, only that the depth of p is 0. But, of course, we need NDOP to know that the non-trivial RPs have depth 0. In applications, however, we usually need the niceness of all non-trivial RPs, and we need NDOP to know that all of these have depth 0.

Combining the main theorem in [3] with [4, Lemma 2.5] gives: Suppose that T is superstable, $U(p)=1$, p is non-trivial and $\mathrm{dom}(p)=A$ is algebraically closed. Then there is a $\varphi\in p$ such that $\varphi\in q\in S(A)$ non-algebraic$\Rightarrow q\not\perp p$. The next result can be seen as a generalization of this. Indeed, the proofs are very similar. This proposition will not be used in the rest of the paper.

Proposition 2.5. *Suppose that T is superstable with* NDOP, *p is a non-trivial RP and $A=\mathrm{dom}(p)$ is algebraically closed. Then there is a $\varphi'\in p$ such that (p, φ') is an RP and for all $q\in S(\mathfrak{C})$, $\varphi'\in q\Rightarrow q\not\perp p$ or q is p-remote.*

Proof. W.l.o.g., $A=M$ is an a-model. Let $\varphi\in p$ be such that (p, φ) is an RP. By the non-triviality of p there is $\{a_0, a_1, a_2\}\subset p(\mathfrak{C})$ pairwise M-independent, but not M-independent. T having NDOP implies that the depth of p is 0 (see [12, 5.10]), so by Theorem 2.2, (p, φ) is f-nice. Thus, there is a formula $\psi(x_0, x_1, x_2)\in t(a_0a_1a_2/M)$ such that: if $\psi_0(x, \bar y)$ is obtained from ψ by some permutation of the variables, then

(4) *for all $\bar b$, $\psi_0(x, \bar b)\vdash\varphi$ and $\psi_0(x, \bar b)$ is p-remote.*

Let $\psi'(x, y)=\exists z\,\psi(x, y, z)$. Notice that $\psi'(x, a_1)$ does not fork over M. By the definability of types there is a formula φ' over M such that for all $r\in S(M)$, $\varphi'\in r$ iff the non-forking extension of r over a_1M contains $\psi'(x, a_1)$. Of course, $\varphi'\vdash\varphi$ and $\varphi'\in p$, so (p, φ') is an RP.

Now suppose that $\models\varphi'(b)$, $b\notin M$ and $b\perp\!\!\!\perp_M a_1$. If $t(b/M)\not\perp p$, we are done, so suppose that $t(b/M)\perp p$. As $\models\varphi'(b)$, $t(b/M)$ is hereditarily orthogonal to p, so we need to show that $t(b/M)$ is non-orthogonal to a set of ∞-rank$<\alpha=R^\infty(p)$. Let c satisfy $\psi(b, a_1, z)$.

Claim. $c \not\perp\!\!\!\perp_M a_1$.

Suppose, towards a contradiction, that $c\perp\!\!\!\perp_M a_1$. Suppose first that $t(c/M)\perp p$. Since $b, c\in\varphi(\mathfrak{C})$, $t(bc/M)$ is hereditarily orthogonal to p. Thus, $a_1\perp\!\!\!\perp_M bc$, contradicting (4). So, $t(c/M)\not\perp p$, hence $t(c/M)\square p$. Since $t(c/a_1bM)$ is hereditarily orthogonal to p, $c\not\perp\!\!\!\perp_{a_1M} b$. Thus, $t(b/M)$ is non-orthogonal to p, contradiction.

So, we know that $c \not\downarrow_M a_1$, hence, $R^\infty(c/a_1M) < \alpha$. If $b \not\downarrow_{a_1M} c$, then $t(b/M)$ is non-orthogonal to a set of ∞-rank $< \alpha$. So, we suppose that $b \downarrow_M ca_1$. Since $\models \psi(b, a_1, c)$ and $\psi(x, a_1, c)$ is p-remote, $\mathrm{stp}(b/a_1cM)$, hence, $t(b/M)$, is non-orthogonal to a set of ∞-rank $< \alpha$. This proves the proposition. $\square$

3. Existence of regular types

The main goal of this section is to prove Theorem A. The major proposition, however, will play an essential role in the proof of Theorem B. What we try to do here is to show that if there is a $c \in N \setminus M$ with $t(c/M)$ p-semi-regular, then there is such a c with $t(c/M) \square p$. That is, reduce the problem of finding a regular type non-orthogonal to p, to finding a p-semi-regular. We fall slightly short of this goal, Proposition 3.2 being the actual result. However, Theorem A follows easily from this proposition using Lemma 1.3 on the existence of semi-regular types. Throughout the section we work in a superstable theory.

Handling trivial types becomes rather easy because of the following. This is [12, 5.11(5)], but we repeat the proof here.

Proposition 3.1. *Suppose that* $\mathrm{stp}(a/A)$ *is non-orthogonal to the trivial regular type* p, *and if* $q \supset p$ *is a forking extension, then* $q \perp \mathrm{stp}(a/A)$. *Then there is a* $c \in \mathrm{acl}(Aa)$ *such that* $\mathrm{stp}(c/A)$ *is regular and non-orthogonal to* p.

Proof. W.l.o.g., $A = \emptyset$. By Lemma 1.3 there is an $a' \in \mathrm{acl}(a)$ such that $\mathrm{stp}(a')$ is p-semi-regular. Certainly, if we prove the lemma for a' we will have it for a. Thus, we may assume that $\mathrm{stp}(a)$ is p-semi-regular. We may also assume that $\mathrm{dom}(p) = B$ is finite, $a \downarrow B$, and there is a b realizing p such that $a \not\downarrow_B b$.

Let $J = \{b_iB_i\colon i < \omega\}$ be a Morley sequence in $\mathrm{stp}(bB/a)$ with $b_0B_0 = bB$, $B' = \bigcup \{B_i\colon i < \omega\}$. Let c be a restricted canonical base of $\mathrm{stp}(bB/a)$. We have $c \in \mathrm{acl}(a) \setminus \mathrm{acl}(\emptyset)$, $c \downarrow B'$ and $c \in \mathrm{acl}(\bigcup J)$. The first condition implies that $\mathrm{stp}(c)$ is p-semi-regular.

Claim. $w_p(c) = 1$.

Supose that $w_p(c) > 1$. We work towards contradicting the triviality of $p_0 = t(b_0/B') = p \mid B'$. Notice that $b_i \downarrow_{B_i} B'$ and $c \not\downarrow_{B'} b_i$. Since $c \in \mathrm{acl}(\bigcup J)$, the only way that $w_p(c)$ can be >1 is if $\{b_i\colon i < \omega\}$ is pairwise B'-independent, but not B'-independent. By the triviality of p_0 (and its conjugates: $t(b_i/B')$) there are d_i realizing p_0 with $d_i \not\downarrow_{B'} b_i$ for $i = 1, 2$ (see [12]). It follows that $\{b_0, d_1, d_2\}$ is pairwise B'-independent, but not B'-independent. This contradicts the triviality of p to prove the claim.

Since $\mathrm{stp}(c)$ is p-semi-regular with p-weight, hence weight $= 1$, it is regular by [11, V, 4.8]. $\square$

The next result is the main technical lemma in this section and the next. It is here where Theorem 2.1 is used.

Proposition 3.2. *Suppose that $N \supset M$ are models and there is an $a \in N \backslash M$ such that $t(a/M)$ is p-semi-regular, for p some RP.*

(i) *If p is trivial, then there is a $c \in \operatorname{acl}(aM)$ such that $t(c/M) \square p$.*

(ii) *Suppose that p is non-trivial. Then there is a formula ψ over M such that ψ is p-simple of p-weight 1, and there is a $c \in \psi(N)$ with $w_p(c/M) = 1$.*

Proof. (i) follows immediately from Proposition 3.1, so, we assume that p is non-trivial. Hence, w.l.o.g., p is a nice RP by Theorem 2.1. (There is a $p' \square p$ which is a nice RP and an $a' \in \operatorname{acl}(aM) \backslash M$ with $t(a'/M)$ p'-semi-regular. Thus, since p-simple and p'-simple are the same, we may replace a and p by a' and p'.)

Let $r = t(a/M)$. By Lemma 1.7 there is a p-simple formula in r. The problem is that this formula may have p-weight >1. What we will do is to take an extension of r of p-weight 1, find a formula in the type witnessing this, and use Lemma 1.4 to 'pull' this formula into M in a way which guarantees that it is satisfied in $N \backslash M$.

Let $r' \supset r$ be a regular type non-orthogonal to p such that $\operatorname{dom}(r') = M \cup d$, where $d \downarrow_M N$. Let c realize r'. By the niceness of p and Lemma 1.4 there is a formula $\sigma(x, y) \in t(cd/M)$ such that for all d', $\sigma(x, d')$ is p-simple of p-weight ≤ 1.

Since $t(a/M) \not\perp r'$ and $d \not\downarrow_M a$ we may choose d large enough so that there is a c' realizing r' with $a \not\downarrow_{dM} c'$. Let $\sigma'(x, w, z) \in t(ac'd/M)$ be such that

(1) $\qquad \models \sigma'(a, f, e) \;\Rightarrow\; a \not\downarrow_M fe$ and $\models \sigma(f, e)$.

Since $a \downarrow_M d$ we can find a $d' \in M$ with $\models \exists w\, \sigma'(a, w, d')$. Let $c'' \in \sigma(a, N, d')$. Since $a \not\downarrow_M c''$, $c'' \in N \backslash M$ and $t(c''/M) \not\perp p$ (since $t(a/M)$ is p-semi-regular). Observing that $\models \sigma(c'', d')$ and $\sigma(x, d')$ is p-simple of p-weight 1 completes the proof. $\square$

We can now prove Theorem A rather easily. Let $M \subset N$ be as hypothesized. Let p be a regular type non-orthogonal to $t(N/M)$ of least ∞-rank under this assumption. Applying Lemma 1.3 we get an $a \in N \backslash M$ such that $t(a/M)$ is p-semi-regular. If p is trivial, we can apply Proposition 3.2(i) to finish the proof. So, suppose that p is non-trivial and c is as in Proposition 3.2(ii). By the minimal rank assumption on p we can again apply Lemma 1.3 to get a $c' \in \operatorname{acl}(cM)$ such that $t(c'/M)$ is p-semi-regular. $w_p(c'/M) \leq w_p(c/M) = 1$. Thus, by [11, V, 4.8], $t(c'/M)$ is regular. This proves Theorem A.

4. Refining the method to get Theorem B

The advantages of Theorem B over Theorem A are fairly clear. Not only do we get control over which regular type is realized in the difference, but assuming that $M \subset_{na} \mathfrak{C}$, we can choose it to be an RP. As usual, throughout the section T is superstable.

We will want to apply Proposition 3.2. This requires that we have an $a \in N \setminus M$ such that $t(a/M)$ is p-semi-regular. In obtaining Theorem A we got such an a by a minimal rank assumption on p. In this section we use the $\subset_{\text{na}}$ assumption to obtain the next two technical lemmas.

Lemma 4.1. *Suppose that $M \subset_{\text{na}} N$, $c \in \theta(N) \setminus M$, and there is a $d \in \operatorname{acl}(cM) \setminus M$. Further suppose that there are d', c' and B such that: $d'c' \equiv dc$ (M), $\{cd, d', B\}$ is M-independent, and $c' \not\downarrow_{dd'BM} c$. Then there is an $e \in \theta(N) \setminus M$ such that $R^\infty(e/M) < R^\infty(c/M)$ and $c \not\downarrow_{dM} e$.*

Proof. W.l.o.g., $R^\infty(c/M) = R^\infty(\theta)$, θ is over $\emptyset$ and $d \in \operatorname{acl}(c)$. Let $\psi(x, y, z) \in t(c'd'B/cdM)$ be such that whenever $\models \psi(c'', d'', b')$,

(1) $\models \theta(c'')$,
(2) $d'' \in \operatorname{acl}(c'')$,
(3) $c \not\downarrow_{dM} c''d''b''$.

By assumption we know that $\exists x\, \psi(x, y, z)$ (which is a formula over cdM) does not fork over M. Since $B \downarrow_M d'cd$ there is a $b' \in M$ such that $\models \exists x\, \psi(x, d', b')$. Now we use that $d' \downarrow_M cd$, $d' \equiv d(M)$ and $M \subset_{\text{na}} N$ to find a $d'' \in M$ such that

(4) $\models \exists x\, \psi(x, d'', b')$, and
(5) $d'' \notin \operatorname{acl}(\emptyset)$.

Let e satisfy $\psi(x, d''b')$ in N. We have $e \not\downarrow_{dM} c$ by (3) (so, $e \notin M$) and $\models \theta(e)$. Since $d'' \in \operatorname{acl}(e) \setminus \operatorname{acl}(\emptyset)$ (by (2) and (5)), $d'' \not\downarrow e$. Thus, $R^\infty(e/M) \leq R^\infty(e/d'') < R^\infty(\theta) = R^\infty(c/M)$. This proves the lemma. $\square$

Lemma 4.2. *Suppose that $M \subset_{\text{na}} N$, p is regular and $c \in N \setminus M$ is such that*

(i) $t(c/M)$ is p-simple, non-orthogonal to p, and $R^\infty(c/M)$ is minimal under this assumption, or

(ii) $\theta \in t(c/M)$ is a p-simple formula, $t(c/M) \not\perp p$, and $R^\infty(c/M)$ is minimal in $\{R^\infty(c'/M) \colon c' \in \theta(N) \text{ and } t(c'/M) \not\perp p\}$.

Then, for some n, $t(c/M) \square p^{(n)}$.

Proof. We will prove the lemma under the assumptions in (i), stating at the end the alteration needed to handle (ii).

First note that by Lemma 1.5 and the elementary properties of ∞-rank,

(6) *if $a \in N \setminus M$ and $t(a/M) \not\perp p$, then there is an $a' \in N \setminus M$ such that $t(a'/M)$ is p-simple, non-orthogonal to p, and $R^\infty(a'/M) \leq R^\infty(a/M)$.*

By Lemma 1.6 there is a $d \in \operatorname{acl}(cM)$ such that $t(c/dM) \square p^{(n)}$ for some n and $w_p(d/M) = 0$. We assume, towards a contradiction, that $d \notin M$.

Claim. *There are $c'd' \equiv cd$ (M) and $M_0 \supset M$ such that $\{d', cd, M_0\}$ is M-independent and $c' \not\downarrow_{M_0dd'} c$.*

Let $M_0 \supset M$ be an a-model such that $c \downarrow_M M_0$. W.l.o.g., $\mathrm{dom}(p) = M_0$. By the definition of p-simple there is I, an M_0-independent sequence of realizations of p such that $w_p(c/M_0I) = 0$ and $w_p(I/M_0c) = 0$. Find $c'd' \stackrel{s}{=} cd$ (M_0I), $c'd' \downarrow_{M_0I} cd$. Since $w_p(d/M_0) = 0$, $d \downarrow_M IM_0$. Thus, $\{d', cd, M_0\}$ is M-independent by the transitivity of independence. Since $w_p(I/cM_0) = 0$ and $w_p(c'/IM_0) = 0$, $w_p(c'/cM_0) = 0$. Thus, $c' \not\downarrow_{dd'M_0} c$. This proves the claim.

Now we can apply Lemma 4.1 to obtain an $e \in N \setminus M$ such that $e \not\downarrow_{dM} c$ and $R^\infty(e/M) < R^\infty(c/M)$. The forking condition implies that $w_p(c/edM) < w_p(c/M)$, so $t(e/M) \not\perp p$. Using (6) we contradict the minimal rank assumption on $t(c/M)$ in (i). This implies that $d \in M$, finishing the proof.

To obtain the lemma under the assumption in (ii), simply observe that in applying Lemma 4.1 we may require that $\models \theta(e)$, and there is no need to apply (6). $\square$

Corollary 4.3. *Suppose that $M \subset_{\mathrm{na}} N$, p is regular and non-orthogonal to $t(N/M)$. Then there is an $a \in N \setminus M$ such that $t(a/M)$ is p-semi-regular.*

Proof. This follows immediately from Lemma 1.5, 4.2(i) and 1.3. $\square$

We can now prove Theorem B when p is non-trivial. Applying Corollary 4.3 and Proposition 3.2 we obtain a formula ψ over M which is p-simple of p-weight 1, and a $c \in \psi(N)$ with $w_p(c/M) = 1$. Pick a ψ which has least ∞-rank under all of these assumptions. By Lemma 4.2(ii) and the fact that $w_p(\psi) = 1$,

(7) *if $c' \in \psi(N)$ and $t(c'/M) \not\perp p$ then $t(c'/M) \square p$.*

This proves Theorem B for non-trivial p.

Now we move on to the case when p is trivial. By Corollary 4.3 we have a $c \in N \setminus M$ such that $t(c/M)$ is p-semi-regular. W.l.o.g., p is an RP. By Lemma 1.7 there is a $\psi_0 \in t(c/M)$ which is p-simple. Pick ψ of least ∞-rank among the p-simple formulas, θ, over M such that there is an $a \in \theta(N)$ with $t(a/M) \not\perp p$.

Claim. *$a \in \psi(N)$ and $t(a/M) \not\perp p \Rightarrow t(a/M) \square p$.*

By Lemma 4.2(ii), $b \in \psi(N)$ and $t(b/M) \not\perp p \Rightarrow t(b/M) \square p^{(n)}$ for some n. Thus, it suffices to show that $w_p(a/M) \leq 1$. Suppose, towards a contradiction, that $t(a/M) \square p^{(n)}$ for $n > 1$. By Proposition 3.1 there is a $b \in \mathrm{acl}(aM)$ such that $w_p(a/bM) = n - 1$. Iterating this procedure, using Lemma 1.6 if necessary, we obtain a $b \in \mathrm{acl}(aM)$ such that $t(a/bM) \square p$. Suppose that $b' \equiv b$ (M) and $b' \downarrow_M ab$. Since $t(a/bM)$ is trivial and non-orthogonal to M there is an a' such that $a'b' \equiv ab$ (M) and $a \not\downarrow_{bb'M} a'$. This proves the claim.

Theorem B follows immediately for trivial p.

Hrushovski has an example showing that Theorem B is false without the $M \subset_{\mathrm{na}} N$ assumption. We call a model *na-correct* if $N \subset_{\mathrm{na}} \mathfrak{C}$. Notice that N is na-correct if for all $a \in N$ and non-algebraic $\theta(x, a)$, $\theta(N, a) \not\subset \mathrm{acl}(a)$.

Corollary 4.4. *Suppose that M is na-correct and p is a regular type non-orthogonal to M. Then there is a $q \in S(M)$ such that $p \not\perp q$ and q is an RP.*

Theorem B also yields the following 'three model lemma'.

Theorem 4.5. *Suppose that $M_0 \subset M_1 \subset M_2$, $M_0 \subset_{\mathrm{na}} M_2$, and $a \in M_2$ is such that $t(a/M_1)$ is regular and non-orthogonal to M_0. Then there is a $c \in M_2$, $c \not\perp_{M_1} a$. $c \perp_{M_0} M_1$ and $t(c/M_0)$ is an RP relative to M_2.*

Proof. Most of the work goes into proving

Claim. $t(a/M_1) \not\perp t(M_2/M_0)$.

Let p be an RP non-orthogonal to $t(a/M_1)$ with $\mathrm{dom}(p) = A_0 \perp_{M_0} M_2$. Assuming that A_0 is sufficiently large we can find a finite B, $B \perp_{M_1} aA_0$, and a c_0 realizing $p \mid M_1BA_0$ such that $c_0 \not\perp_{M_1BA_0} a$. Let $I = \{c_iA_i : i < \omega\}$ be a Morley sequence in $\mathrm{stp}(c_0A_0/aBM_1)$, c a restricted canonical base of this strong type. Thus, $c \in \mathrm{acl}(aBM_1) \cap \mathrm{acl}(\bigcup I)$ and, since $I \perp_{M_1} B$, $c \not\perp_{BM_1} a$. Let $\theta_i \in t(c_i/A_i)$ witness that this type is an RP. Since $B \perp_{M_1} \{a\} \cup \{A_i : i < \omega\}$ we can 'pull B into M_1' to find a

$$c' \in \mathrm{acl}(aM_1) \cap \mathrm{acl}\Big(\bigcup_{i<\omega} \theta_i(\mathfrak{C}) \cup A_i\Big)$$

such that $c' \not\perp_{M_1} a$. Since $\{A_i : i < \omega\} \cup M_2$ is M_0-independent, $c' \perp_{M_0} \{A_i : i < \omega\}$. Since each θ_i is p-simple, $t(c'/M_0)$ is p-simple.

$$a \underset{M_1}{\not\perp} c' \;\Rightarrow\; w_p(c'/aM_1) < w_p(c'/M_1) \leqslant w_p(c'/M_0).$$

It follows that $t(aM_1/M_0) \not\perp p$, proving the claim.

Now, by Theorem B there is a formula θ over M_0 such that θ is p-simple and $b \in \theta(M_2) \setminus M_0$, $t(b/M_0) \not\perp p \Rightarrow t(b/M_0) \square p$. It is an easy exercise to show that there is a $b \in \theta(M_2)$ such that $b \not\perp_{M_1} a$. It follows that $b \perp_{M_0} M_1$, proving the theorem. $\square$

5. Obtaining tree decompositions and extending the logic

The primary purpose of this section is to obtain a 'good' tree decomposition of each model of a superstable theory with NDOP. One possible form of 'good' is the following notion.

Definition 5.1. We say that $\langle N_\eta, a_\eta : \eta \in I\rangle$ is a $\subset_{\mathrm{na}}$*-decomposition inside* N if

(a) $\langle N_\eta : \eta \in I\rangle$ is an independent tree of models with $|N_\eta| = |T|$ for all $\eta \in I$.
(b) $N_\eta \subset_{\mathrm{na}} N$ and $a_\eta \in N_\eta$.

(c) $t(a_\eta/N_{\eta^-})$ is regular (when $\mathrm{lh}(\eta)>0$) and orthogonal to $N_{\eta^{--}}$ (when $\mathrm{lh}(\eta)>1$),

(d) $N_\eta \lhd a_\eta\ (N_{\eta^-})$ when $\mathrm{lh}(\eta)>0$.

We say that $\langle N_\eta, a_\eta : \eta \in I\rangle$ is a $\subset_{\mathrm{na}}$-*decomposition of* N if it is a maximal $\subset_{\mathrm{na}}$-decomposition inside N.

Notice that the only restriction on $a_\emptyset$ is that it is an element of $N_\emptyset$. It appears only because its exclusion would be a notational headache.

We will prove Theorem C stated in the abstract. If we removed (d) from the definition the results in the previous section alone would yield the theorem. Obtaining the theorem as stated will require a significant amount of work. The second topic of the section, extending the logic, will be left until later.

The key proposition is the following. Theorem C will be easy given this. Notice that we do not need NDOP here.

Proposition 5.1. *Suppose that T is superstable, N and M are models of T with $N \subset_{\mathrm{na}} M$ and $N \subset A \subset M$. Then there is a model $N' \supset A$, $N' \subset_{\mathrm{na}} M$ $N' \lhd A\ (N)$ and $|N'| = |A| + |T|$.*

We separate the proof into two lemmas. In our first lemma we show that the relation "cA is not dominated by A over M" can be witnessed in a particular canonical way.

Lemma 5.2. *Suppose that T is superstable, $M \models T$, $N \subset_{\mathrm{na}} M$, $N \subset A \subset M$ and $c \in M$ is such that* $\mathrm{stp}(c/A)$ *is p-semi-regular, p some RP, and cA is not dominated by A over N. Then there is an $e \in M$ such that*

(i) *$t(e/N)$ is an RP relative to M,*

(ii) *$e \downarrow_N A$, and*

(iii) *$e \not\downarrow_A c$.*

Proof. Let b be such that $b \downarrow_N A$ and $b \not\downarrow_A c$. There is a $b' \in \mathrm{acl}(bN)$ such that $t(b'/N)$ is p-simple and for all $B \supset N$ with $b \downarrow_N B$ and all e with $\mathrm{stp}(e/B)$ p-semi-regular, $b \downarrow_{b'B} e$. (Take b' to be the P-internal part of b, where P is the class of p-semi-regular types. See [6, Proposition 5, p. 16], replacing $\emptyset$ by N and $\mathrm{int}_P(A)$ by $\mathrm{int}_P(A/N)$.) Thus, $b' \downarrow_N A$ and $b' \not\downarrow_A c$, so we may assume that $t(b/N)$ is p-simple.

Since $\mathrm{stp}(c/A) \square p^{(n)}$, for some n, $\mathrm{stp}(b/A)$ is p-simple and $c \not\downarrow_A b$, we have $w_p(c/bA) < w_p(c/A)$ by [7, Fact 2, p. 139]. Thus, $w_p(b/cA) < w_p(b/A) = w_p(b/N)$, implying that $p \not\perp t(cA/N)$. A fortiori, $p \not\perp t(M/N)$. By Theorem B there is a $q \in S(N)$ such that q is an RP relative to M non-orthogonal to p and realized in M.

There is a $B \supset N$, $B \downarrow_N Acb$, and an $\bar f = \langle f_0, \ldots, f_n\rangle$ realizing $q^{(n+1)} \mid B \cup A$ such that $w_p(b/\bar f B) = 0$. Since $b \not\downarrow_{BA} c$ and $\mathrm{stp}(c/AB)$ is p-semi-regular, $\bar f \not\downarrow_{BA} c$.

Let θ witness that q is an RP relative to M. Since $B \perp_N Ac$ and N is a model

(1) *there is an* $\bar{e} \subset \theta(M)$ *such that* $\bar{e} \not\perp_A c$.

Using that stp(c/A) is p-semi-regular we find an $\bar{e}'e_k \subset \bar{e}$ such that

(2) $f \in \bar{e}'e_k \Rightarrow t(f/N) \square p$, $\bar{e}' \perp_N Ac$ *and* $e_k \not\perp_{A\bar{e}'} c$.

By an argument similar to the one used in obtaining (1) we find a formula $\tau(x)$ over Ac such that $\models \tau(e) \Rightarrow \models \theta(e)$ and $e \not\perp_A c$. For any such $e \in M$ we must have $t(e/A) \not\perp p$, so $t(e/N) \square p$ and $e \perp_N A$. This proves the lemma. □

The next lemma is the key to the entire section. The proof we give here is due to Hrushovski. It has the advantages over Buechler's proof of being 1/5 as long and correct. The proof reads most smoothly if we use the following notation due to Hrushovski. Let p be a strong type over A in the variable x, $\varphi(x, y)$ a formula over A. We let $(d_p x)\,\varphi(x, y)$ be the formula in y such that for all b, $\models (d_p x)\,\varphi(x, b)$ iff $\varphi(x, b) \in p \mid Ab$. We read $(d_p x)\ \varphi(x, y)$ as "for generic x realizing p, $\varphi(x, y)$ holds".

Lemma 5.3. *Suppose that T is superstable, $N \subset_{na} M$ are models of T, $N \subset A \subset M$ and $\theta(x, b)$ is a formula over A such that $\theta(M, b) \not\subset \mathrm{acl}(b)$. Then there is a $c \in \theta(M, b) \setminus \mathrm{acl}(b)$ such that $cA \lhd A\ (N)$.*

Proof. Let $\theta_1(x, b_1)$ be a formula over A of least ∞-rank such that $\theta_1(x, b_1) \vdash \theta(x, b)$ and $\theta_1(M, b_1) \not\subset \mathrm{acl}(b)$. Let $c \in \theta_1(M, b_1)$. We will show that $cA \lhd A\ (N)$.

Suppose, towards a contradiction, that cA is not dominated by A over N. By Lemma 5.2 there is an $e \in M$ such that $e \not\perp_A c$ and $e \perp_N A$. Let $a \in A$ be such that $e \not\perp_N ca$ and $b_1 \subset a$, $\theta_2(x, a)$ a formula in $t(c/aN)$ implying $\theta_1(x, b_1)$. Let $\varphi(x, y, z) \in t(cae/N)$ be such that $\varphi(ca, z)$ forks over N and $\varphi(x, y, z) \vdash \theta_2(x, y)$. Let $r = t(a/N)$. Define an equivalence relation by: $e' \sim e''$ iff $(d_r y)\,\forall x(\varphi(x, y, e') \leftrightarrow \varphi(x, y, e''))$. Clearly, $e/{\sim} \notin N$, since otherwise we would have $c \in N$.

Choose a finite $D \subset N$ such that $cae \perp_D N$. As $N \subset_{na} M$ there exists an $e' \in N$ such that $e'/{\sim} \notin \mathrm{acl}(D)$ and $\models \exists x\,\varphi(x, a, e')$. Let $q = \mathrm{stp}(e'/D)$ and $\sigma(x, y) = (d_q z)\varphi(x, y, z)$.

Claim 1. $(d_r y)\,\exists x\,[\theta_2(x, y) \wedge \neg(\varphi(x, y, e') \leftrightarrow \sigma(x, y))]$.

Suppose that this is not the case. Then $(d_r y)\,\forall x\,[\varphi(x, y, e') \leftrightarrow \theta_2(x, y) \wedge \sigma(x, y)]$. If e'' realizes q, $(d_r y)\,\forall x\,[\varphi(x, y, e'') \leftrightarrow \theta_2(x, y) \wedge \sigma(x, y)]$, so $(d_r y)\,\forall x\,[\varphi(x, y, e') \leftrightarrow \varphi(x, y, e'')]$. Thus, $e' \sim e''$. Since we can pick e'' so that $e'' \perp_D e'$, we conclude that $e'/{\sim} \in \mathrm{acl}(D)$. This contradiction proves the claim.

Thus, as a realizes r and $a \perp_D e'$, $\models \exists x\,[\theta_2(x, a) \wedge \neg(\varphi(x, a, e') \leftrightarrow \sigma(x, a))]$. Let c' witness this quantifier. Then, $\models \theta_2(c', a)$.

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Claim 2. $c' \not\downarrow_{Da} e'$.

Suppose that $\models\varphi(c', a, e')$. Then $\models\neg\sigma(c', a)$, so by the definition of σ, we must have $c'a \not\downarrow_D e'$. If $\models\sigma(c', a)$, then $\models\neg\varphi(c', a, e')$, so again by the definition of σ, $c'a \not\downarrow_D e'$. Since $a \downarrow_D e'$, the claim holds.

Since $e' \not\downarrow_{Da} c'$, $c' \notin \mathrm{acl}(a)$. $\models\theta_2(c', a)$ and $e' \in N \Rightarrow R^{\infty}(c'/aN) < R^{\infty}(c'/aD) = R^{\infty}(\theta_2(x, a))$. This contradicts the definition of θ_2, proving the lemma. □

Proposition 5.1 now follows easily from these lemmas. Lemma 5.5 below will virtually complete the proof of Theorem C. The next lemma is used there and may hold some interest in its own right. Recall that we write $A \subset_t B$ when $A \subset B$ and for all formulas $\theta(x)$ over A such that $\models\theta(b)$ for some $b \in B$, $\models\theta(a)$ for some $a \in A$. (Makkai writes $<_{TV}$ for $\subset_t$.)

Lemma 5.4. *Suppose that $N \subset_{na} M_0$ are models, $M_0 \supset B \supset A \supset N$, $M_0 \supset C \supset A \supset N$, $C \downarrow_A B$, $A \subset_t B$ and $C \lhd A\ (N)$. Then $C \cup B \lhd B\ (N)$.*

Proof. The key to this proof is [11, XI, 1.4] which we quote as

(3) *Given any set D, $p \in S(D)$ and $\psi(x, y)$ over D there is a formula $\theta(y)$ over D such that for all b, $p \cup \{\psi(x, b)\}$ does not fork over D iff $\models\theta(b)$.*

This fact is used to establish the following strengthening of the hypothesis: $A \subset_t B$.

Claim 1. *For all formulas $\psi(y)$ over C such that $\models\psi(b)$ for some $b \in B$, there is a $c \in C$ such that $\models\psi(c)$.*

Write $\psi(y)$ as $\psi(a, y)$ where $a \in C$ and $\psi(x, y)$ is over A. Applying (3) to $p = t(a/A)$ and ψ we obtain a formula $\theta(y) \in t(b/A)$ such that for all b', $p \cup \{\psi(x, b')\}$ does not fork over A iff $\models\theta(b')$. Since $A \subset_t B$ there is a $b' \in A$ such that $\models\theta(b')$. This implies that $p \cup \{\psi(x, b')\}$ does not fork over A, hence $\psi(x, b') \in p$. That is, $\models\psi(a, b')$, proving the claim.

Turning our attention to the lemma, suppose that $C \cup B$ is not dominated by B over N. Then we can find a sequence $ca \in C$ such that $aB \lhd B\ (N)$, $\mathrm{stp}(c/aA)$ is p-semi-regular (for some regular p), and caB is not dominated by B over N. By Lemma 5.2 there is an $e \in M_0$ such that $t(e/N)\,\square\, p$ is an RP relative to M_0, $e \downarrow_N aB$ and $e \not\downarrow_{aB} c$. Since $C \downarrow_A B$, $c \not\downarrow_{aA} eb$ for some $b \in B$. Let $\varphi \in t(e/N)$ witness that this type is an RP relative to M_0. Let $\tau(z, y) \in t(eb/caA)$ be such that $\models\tau(e', b') \Rightarrow c \not\downarrow_{aA} e'b'$; $\psi(y)$ the formula $\exists z\ (\varphi(z) \wedge \tau(z, y))$.

By Claim 1 there is a $b' \in A$ such that $\models\psi(b')$. Let $e' \in M_0$ satisfy $\varphi(z) \wedge \tau(z, b')$. Thus, $e'b' \not\downarrow_{aA} c$, which is the same as $e' \not\downarrow_{aA} c$ since $b' \in A$. Since $\mathrm{stp}(c/aA)$ is p-semi-regular, $\mathrm{stp}(e'/aA) \not\perp p$. As $\models\varphi(e')$ and $(t(e/N), \varphi)$ is an RP relative to M_0 which is non-orthogonal to p, $e' \downarrow_N Aa$. But this contradicts that $C \lhd A\ (N)$, proving the lemma. □

Lemma 5.5. *Suppose that T is superstable with* NDOP, *$M \models T$ and $\langle M_\eta, a_\eta : \eta \in I\rangle$ is an $\subset_{na}$-decomposition of M. Then*

(i) *$M \lhd \bigcup M_\eta\ (M_\eta)$, for all $\eta \in I$.*

(ii) *M is a-atomic over $\bigcup M_\eta$.*

(iii) *If $p \in S(M)$, then $p \not\perp M_\eta$ for some η.*

Proof. (i) Suppose, towards a contradiction, that M is not dominated by $\bigcup M_\eta$ over M_η. It is easy to find an A, $\bigcup M_\eta \subset A \subset M$ and an $a \in M$ such that a is not dominated by A over M_η and $\mathrm{stp}(a/A)$ is semi-regular. Since $M_\eta \subset_{na} M$ we may apply Lemma 5.2 to find an $e \in M$ such that $t(e/M_\eta)$ is an RP relative to M and $e \downarrow_{M_\eta} A$. By Proposition 5.1 there is an $N \subset_{na} M$, $|N| = |T|$, $N \supset eM_\eta$ and $N \lhd e\,(M_\eta)$. Let $\nu < \eta$ be minimal such that $t(e/M_\eta) \not\perp M_\nu$. By Theorem 4.5 there is an $e' \in N$ such that $e' \downarrow_{M_\nu} M_\eta$ and $t(e'/M_\nu)$ is regular. Since $N \lhd e\,(M_\eta)$ and $e' \in N$, $e' \downarrow_{M_\eta} A$, so $e' \downarrow_{M_\nu} A$ by the transitivity of independence. Again by Proposition 5.1 there is a model $N' \subset_{na} M$ such that $N' \supset M_\nu e'$ and $N' \lhd e'\,(M_\nu)$. It follows easily that $\langle M_\eta, a_\eta : \eta \in I\rangle \cup \{\langle N', e'\rangle\}$ is a $\subset_{na}$-decomposition inside M, contradicting that $\langle M_\eta, a_\eta : \eta \in I\rangle$ is a maximal such decomposition. This proves (i).

Now find an independent tree of a-models, $\langle N_\eta : \eta \in I\rangle$, such that $N_\eta \supset M_\eta$, $\bigcup N_\eta \downarrow_{\bigcup M_\eta} M$ and $N_\eta \downarrow_{M_\eta} \bigcup M_\eta$, for $\eta \in I$.

Claim 1. *For all $\nu \in I$, $M \cup \bigcup N_\eta \lhd \bigcup N_\eta\ (N_\nu)$.*

We are going to apply Lemma 5.4 with N_ν as N, $\bigcup M_\eta \cup N_\nu$ as A, $M \cup N_\nu$ as C and $\bigcup N_\eta$ as B. Bringing in the heavy artillery, [11, XII, 2.3(3)] implies that $\bigcup M_\eta \cup N_\nu \subset_t \bigcup N_\eta$, taking care of one hypothesis in Lemma 5.4. Since N_ν is an a-model we can simply take $\mathfrak{C}$ as M_0. It remains only to establish

$$M \cup N_\nu \lhd \bigcup M_\eta \cup N_\nu\ (N_\nu). \tag{4}$$

Let $c \downarrow_{N_\nu} \bigcup M_\eta$ Since $N_\nu \downarrow_{M_\nu} \bigcup M_\eta$, $cN_\nu \downarrow_{M_\nu} \bigcup M_\eta$ by the Pairs Lemma. Since $M \lhd \bigcup M_\eta\,(M_\nu)$ (by (i) of this lemma) we conclude that $cN_\nu \downarrow_{M_\nu} M$. Hence $c \downarrow_{N_\nu} M$, proving (4).

Applying Lemma 5.4 in that manner described above gives the claim.

Claim 2. *M is a-atomic over $\bigcup N_\eta$.*

Find an a-prime model $N \supset \bigcup N_\eta$ such that N is a-atomic over $M \bigcup N_\eta$. Since $b \in N \Rightarrow \mathrm{stp}(b/M \cup \bigcup N_\eta)$ is a-isolated, and N_η is an a-model, it is easily shown that

$$N \lhd M \cup \bigcup N_\eta\ (N_\eta), \quad \text{for all } \eta. \tag{5}$$

By Claim 1 and the transitivity of domination $N \cup M \lhd \bigcup N_\eta\ (N_\eta)$. A fortiori, $NM \lhd N\,(N_\eta)$. Suppose that $M \not\subset N$. Since T has NDOP, $t(M/N) \not\perp N_\eta$ for some η. Thus, there is a c such that $c \not\downarrow_N M$ and $c \downarrow_{N_\eta} N$; directly contradicting the above. Thus, $M \subset N$, proving Claim 2.

Now (ii) follows from Claim 2 since $\bigcup N_\eta \downarrow_{\bigcup M_\eta} M$.

Proceeding to (iii), let $p \in S(M)$. By Claim 2 and NDOP, $p \not\perp N_\eta$ for some η. Since $N_\eta \downarrow_{M_\eta} M$, $p \not\perp M_\eta$. $\square$

Proof of Theorem C. That every $\subset_{\mathrm{na}}$-decomposition inside M extends to an $\subset_{\mathrm{na}}$-decomposition of M is proved like Lemma 5.5(i). By Lemma 5.5(i) it remains only to show that M is minimal over $\bigcup M_\eta$, for $\langle M_\eta, a_\eta : \eta \in I\rangle$ any $\subset_{\mathrm{na}}$-decomposition of M. Suppose, towards a contradiction, that there is a model M' with $M \supsetneqq M' \supset \bigcup M_\eta$. Let $p \in S(M')$ be a regular type realized in M. Since $\langle M_\eta, a_\eta : \eta \in I\rangle$ is also a $\subset_{\mathrm{na}}$-decomposition of M' we may apply Lemma 5.5(iii) to obtain an η such that $p \not\perp M_\eta$. By Theorem 4.5 there is an $a \in M$ such that $t(a/M') \square p$ and $a \downarrow_{M_\eta} M'$. This is easily shown to contradict that $\langle M_\eta, a_\eta : \eta \in I\rangle$ is a maximal $\subset_{\mathrm{na}}$-decomposition inside M. This completes the proof. $\square$

To finish this first topic let us summarize the additional information about na-correct models given by Proposition 5.1. Let K^{na} be the na-correct models of a superstable theorey T. First notice that $\subset = \subset_{\mathrm{na}}$ on K^{na}. Proposition 5.1 becomes: For $M, N \in K^{\mathrm{na}}$, $N \subset A \subset M$, implies that there is an $N' \in K^{\mathrm{na}}$, $A \subset N' \subset M$ and $N' \lhd A\,(N)$. Combining this with Corollary 4.4 gives a very well-behaved theory for K^{na}.

Our next topic makes a natural companion to this model theory of K^{na} although it may not seem so at first. Let T be a superstable theory with NDOP, $M \models T$. We expand M to M^+ by adding a predicate for every subset of $M^{<\omega}$ definable in the logic $L(\exists^{>|T|})$. Let T^+ be the first-order theory of M^+. We will show that T^+ is also superstable with NDOP and that the 'correct' models of T^+ can be decomposed as in Theorem C with respect to an inclusion relation that is natural for studying the $L(\exists^{>|T|})$-theory of M. Later we will mention how $\subset_{\mathrm{na}}$ and na-correct models fit into this subject.

First we need to give some additional notation and terminology. Let M be a model of an arbitrary theory T in the language L. Let Q denote the quantifier $\exists^{>|T|}$, $L(\mathrm{Q})$ the finitary logic in L with this quantifier [9]. We define expansions L^*, M^*, T^*, $L^\#$, $M^\#$ and $T^\#$ as follows. For every formula $\varphi(x, y) \in L$ let $R_{\mathrm{Q}x\,\varphi}(y)$ be a new predicate symbol in the obvious arity, $L^* = L \cup \{R_{\mathrm{Q}x\,\varphi} : \varphi \in L\}$. Expand M to the L^*-structure M^* by:

$$M^* \models R_{\mathrm{Q}x\,\varphi}(a) \quad \text{iff} \quad M \models \mathrm{Q}x\,\varphi(x, a), \quad \text{for all } a \in M.$$

Let $M^\# = (M^*)^{\mathrm{eq}}$, $L^\#$ the language of $M^\#$ and $T^\# = \mathrm{Th}(M^\#)$.

Let $L = L_0$, $M = M_0$, and for $i < \omega$, $M_{i+1} = (M_i)^\#$, $L_{i+1} = (L_i)^\#$ and $T_{i+1} = \mathrm{Th}(M_{i+1})$. M^+ is defined to be $\bigcup M_i$, $L^+ = \bigcup L_i$, $T^+ = \bigcup T_i$. Any T^+ obtained in this way is called an *$L(\mathrm{Q})$-expansion of T.*

Notice that M^+ is not simply obtained by adding a predicate for each $L(\mathrm{Q})$-definable subset of M. M^+ also contains imaginary elements for equivalence relations definable with the new predicates. Furthermore, M^+ has relations for the $L(\mathrm{Q})$-definable subsets of these new imaginary elements. So, M^+ is really an '$L(\mathrm{Q}, \mathrm{eq})$-expansion' of M, and we may identify T^+ with $(T^+)^{\mathrm{eq}}$.

For $\varphi(x, \bar{y})$ a formula in $L(\mathrm{Q})$ and $N \models T^+$ we write $N \models$ "$\mathrm{Q}x\, \varphi(x, \bar{a})$" if $N \models R_{\mathrm{Q}x\,\varphi}(\bar{a})$. For $\varphi(x, \bar{y}) \in L^+$, $\varphi = R_\psi$, we may write $N \models$ "$\mathrm{Q}x\, \varphi(x, \bar{a})$" for $N \models R_{\mathrm{Q}x\,\psi}(\bar{a})$. For A a set we say that a is in $\mathrm{acl}_{\mathrm{m}}(A)$, called the *small closure of A*, if there is a formula $\varphi(x)$ over A such that $\models \varphi(a)$ and $\models$"$\neg\mathrm{Q}x\, \varphi(x)$".

Of course, in considering T^+ as an approximation to an $L(\mathrm{Q})$-theory we are not interested in all models. In the next definition we isolate the relevant models and the corresponding inclusion relation.

Definition. Let T^+ be an $L(\mathrm{Q})$-expansion of some theory T.

(1) $N \models T^+$ is called *correct for T^+* if

(a) $N \models$ "$\neg\mathrm{Q}x\, \varphi(x, \bar{a})$" $\Rightarrow |\varphi(N, \bar{a})| \leq |T|$, for all $\varphi \in L^+$,

(b) $N \models$ "$\mathrm{Q}x\, \varphi(x, \bar{a})$" $\Rightarrow \varphi(N, \bar{a}) \not\subset \mathrm{acl}_{\mathrm{m}}(\bar{a})$, for all $\varphi \in L^+$.

(2) We call $N \models T^+$ *standard* if $N \models$ "$\mathrm{Q}x\, \varphi(x, \bar{a})$" iff $|\varphi(N, \bar{a})| > |T|$, for all $\varphi \in L^+$.

(3) For sets $A \subset B$ we write $A \subset_{\mathrm{m}} B$ if

(a) $\mathrm{acl}_{\mathrm{m}}(A) \cap B \subset A$, and

(b) whenever $\models \theta(b, a)$, $b \in \mathrm{acl}(B)$, $a \in \mathrm{acl}(A)$, there is a $b' \in \mathrm{acl}(A)$, $b' \notin \mathrm{acl}_{\mathrm{m}}(a)$ with $\models \theta(b', a)$, for every $\theta \in L^+$.

Observe that for all A and B, $A \subset_{\mathrm{m}} B \Rightarrow A \subset_{\mathrm{na}} B$.

There is one $L(\mathrm{Q})$-expansion of T which is of special interest. Let T^{na} be the $L(\mathrm{Q})$-expansion associated with the monster model. Notice that $\mathfrak{C} \models \mathrm{Q}x\, \varphi(x, a)$ iff $\varphi(\mathfrak{C}, a)$ is infinite, so in this case "$\mathrm{Q}x$" means "there are infinitely many x".

Our main result about T^+ is

Proposition 5.6. *Suppose that T is superstable with* NDOP *and T^+ is an $L(\mathrm{Q})$-expansion of T. Then T^+ is superstable with* NDOP.

Shelah's proof of this result centered on the tree decomposition of models enabled by NDOP. After hearing of the theorem Hrushovski suggested that the proposition may follow from Bouscaren's Theorem (Lemma 5.7). The details of this second proof of the proposition were worked out by Buechler and appear below. As Bouscaren's Theorem relies heavily on the tree decomposition of models it is likely that these two proofs are essentially the same.

Lemma 5.7. (i) [2, Main Theorem] *Suppose that T is superstable with* NDOP *and $N \subset M$ are models of T. Let (M, N) be the pair obtained by adding a predicate symbol for N. Then $T' = \mathrm{Th}(M, N)$ is superstable with* NDOP.

(ii) [2, Corollary 3.2] *If T, M and N are as above and p is a type in T' such that $p \vdash$ "$v \in N$", Then $U(p) \leq U(p \restriction L)$.*

Recall that T^+ is the limit of an ω-sequence of expansions T_i. We will use Lemma 5.7(i) repeatedly to show that each T_i is superstable with NDOP. Then we use (ii) of the lemma to see that T^+ is superstable with NDOP.

Terminology. There are many possible interpretations of the term "T^0 is interpretable in T^1". Below we will only be dealing with one rather simple case. Let φ be a formula without parameters in T^1. We say that T^0 *is interpretable in* T^1 *on* φ if there is L', a sublanguage of the language of T^1, such that $T^0 = \mathrm{Th}(\varphi(\mathfrak{C}) \restriction L')$ (after changing the language of T^0, if necessary).

Recall the definition of M^* and T^* above.

Lemma 5.8. *Suppose that T is superstable with* NDOP, $M \models T$, $T^* = \mathrm{Th}(M^*)$ *and p is some type in T^*. Then T^* is superstable with* NDOP *and $U(p) \leqslant$ the U-rank of $p \restriction L$ as computed in T.*

Proof. It is not hard to see that there is a model $N' \subset M^*$, $|N'| = |T|$, such that for all $\varphi(x, y) \in L$ and $a \in N'$, $N' \models R_{Qx\,\varphi}(a)$ iff $(M, N) \models (\exists x \notin N)\, \varphi(x, a)$, where $N = N' \restriction L$. This shows that T^* is interpretable in $T' = \mathrm{Th}(M, N)$ on $\mathbf{N}$, where $\mathbf{N}$ is the predicate symbol representing the submodel. It follows from [1] that not only T', but also T^* is superstable with NDOP.

Now let p be any type in T^*, (M_0, N_0) a saturated model of T'. W.l.o.g., $\mathrm{dom}(p) \subset N_0$.

Claim. *Suppose that T^0 is interpretable in the superstable theory T^1 and p is a complete type in T^0. Then there is p', a completion of p in T^1, such that $U(p) \leqslant U(p')$.*

This claim is proved by an easy induction on $U(p)$, which we leave to the reader.

By the claim there is a completion p' of p in T' with $U(p) \leqslant U(p')$. Since T^* is interpretable on $\mathbf{N}$, $p' \vdash v \in \mathbf{N}$. By Lemma 5.7(ii), $U(p') \leqslant U(p' \restriction L)$. Since $p' \restriction L = p \restriction L$, we have proved the lemma. □

Since $T^\# = (T^*)^{\mathrm{eq}}$ we conclude that $T^\#$ is also superstable with NDOP.

We can now complete the proof of Proposition 5.6. Let $M \models T$ be such that $T^+ = \mathrm{Th}(M^+)$. Let M_i, L_i and T_i be as defined above. We will prove that T^+ is superstable by showing that every type has ordinal U-rank. Let p be a complete type in T^+. Let j be minimal such that the variables in p range over sorts appearing in T_j. For $i \geqslant 0$ let $p_i = p \restriction L_{i+j}$.

We can associate with any such p the descending sequence of ordinals $\langle U(p_0), U(p_1), \ldots \rangle$ by Lemma 5.8. We leave it to the reader to see that there cannot be a chain $p = q_0 \supset q_1 \supset \cdots$ such that for all i, q_{i+1} forks over $\mathrm{dom}(q_i)$. Thus, T^+ is superstable.

Now we will show that T^+ has NDOP. Let $N_0 \subset N_1, N_2$ be a-models of T^+ such that $N_1 \downarrow_{N_0} N_2$; N an a-prime model over $N_1 \cup N_2$. Let $p \in S(N)$. We must show that $p \not\perp N_i$, for $i=1$ or 2. Let a realize p and M be an a-prime model over $N \cup \{a\}$. For $j<\omega$ let $N_i^j = N_i \restriction L_j$ $(i=1,2)$ and $N^j = N \restriction L_j$; all models of T_i, a superstable theory with NDOP. We may construct N so that N^j is a-prime over $N_2^j \cup N_1^j$ for $j<\omega$. For q any type in T^+ let q_i denote $q \restriction L_i$, with the convention that $q \restriction L_i = q \restriction L_j$ for $i<j$ and L_j the smallest language containing the sorts over which the variables in q range.

Let $c \in M \setminus N$ be such that $\lim_{i<\omega} U(t(c/N)_i) = \alpha$ is minimal. Let $q = t(c/N)$, and let $i<\omega$ be such that $U(q_j)=\alpha$ for $j \geq i$. Then, for all $c' \in M \setminus N$, $U(c'/N^i) \geq \alpha$ in T_i. Notice that q_i is regular. Since T_i has NDOP and N^i is a-prime over $N_2^i \cup N_1^i$ we must have q_i non-orthogonal to N_2^i or N_1^i. Suppose that $q_i \not\perp N_2^i$. By a standard argument there is an $r \in S(N_2^i)$ such that $r \not\perp q_i$, $U(r)=\alpha$ and r is regular. Thus, working in T_i, there is a $c_0 \in M^i \setminus N^i$ such that c_0 realizes $r \mid N$. Let $r^+ = t(c_0/N)$ in T^+. By the minimality condition on α, $U(r_j^+) = U(r_j^+ \restriction N_2) = \alpha$ for $j \geq i$. Thus, for each j, r_j^+ does not fork over N_2, so r^+ does not fork over N_2. Since r^+ is realized in $M = N[a]$, $p \not\perp N_2$. This proves that T^+ has NDOP.

With Proposition 5.6 in hand our next goal is to prove a decomposition theorem analogous to Theorem C for T^+ with respect to $\subset_{\mathrm{m}}$. The explicit definition and result are as follows.

Definition. Let T^+ be an $L(\mathrm{Q})$-expansion of some T, N a correct model of T^+. We say that $\langle N_\eta, a_\eta : \eta \in I \rangle$ is a *$\subset_{\mathrm{m}}$-decomposition inside N* if (a), (c) and (d) of Definition 5.1 hold and

(b′) $N_\eta \subset_{\mathrm{m}} N$ and $a_\eta \in N_\eta$.

'$\subset_{\mathrm{m}}$-decomposition of N' is defined in the obvious way.

Theorem 5.9. *Suppose that T is superstable with* NDOP, *T^+ is an $L(\mathrm{Q})$-expansion of T and $M \models T^+$ is standard. Then every $\subset_{\mathrm{m}}$-decomposition inside M extends to a $\subset_{\mathrm{m}}$-decomposition of M. Furthermore, if $\langle N_\eta, a_\eta : \eta \in I \rangle$ is any $\subset_{\mathrm{m}}$-decomposition of M, M is minimal over $\bigcup N_\eta$ and for all $\eta \in I$, $M \lhd \bigcup N_\eta\ (N_\eta)$.*

The passage from $\subset_{\mathrm{na}}$ to $\subset_{\mathrm{m}}$ requires the following additional lemmas.

Lemma 5.10. *Suppose that T^+ is a superstable $L(\mathrm{Q})$-expansion of T, $M \models T^+$, $N \subset_{\mathrm{m}} M$ and $N \subset A \subset M$. Let $\theta(x)$ be a formula over A such that $M \models$ "$\neg \mathrm{Q}x\, \theta(x)$". Then $\theta(M)A \lhd A\ (N)$.*

Proof. Suppose that the lemma is false. After adding some elements of $\theta(M)$ to A, if necessary, there is a $c_0 \in \theta(M)$ such that c_0 is not dominated by A over N. Repeatedly applying Lemma 1.3, perhaps adding more elements to A, we find a $c \in \mathrm{acl}(c_0A)$ such that $\mathrm{stp}(c/A)$ is semi-regular and c is not dominated by A over N. Furthermore, there is a formula $\theta' \in t(c/A)$ such that $M \models$ "$\neg \mathrm{Q}x\, \theta'(x)$". By

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Lemma 5.2 there is an $e \in M$, $e \downarrow_N A$, such that $c \not\downarrow_A e$. Let θ' be $\theta'(x, a)$ where $M \models$ "$\forall y\, (\neg Qx\, \theta'(x, y))$". Since $e \downarrow_N a$ we can find an $a' \in N$ and a $c' \in M$ such that $\models \theta'(c', a')$ and $e \not\downarrow_N c'$. Thus, $c' \in M \setminus N$. This contradicts that $M \models$ "$\neg Qx\, \theta'(x, a')$" and $N \subset_m M$. □

Lemma 5.11. *Suppose that T^+ is a superstable $L(Q)$-expansion of T, $M \models T^+$ is standard, $N \subset_m M$ is a model of T^+, $N \subset A \subset M$ with $|A| = |T|$, and $\theta(x, b)$ is a formula over A with $M \models$ "$Qx\, \theta(x)$". Then there is a $c \in \theta(M) \setminus \mathrm{acl}_m(b)$ such that $cA \lhd A\,(N)$.*

Proof. Notice that this a generalization of Lemma 5.3. We begin the proof as in that previous lemma, but this situation is complicated by the *lack* of a lemma saying: $e \notin \mathrm{acl}_m(D)$ and $c \in \mathrm{acl}_m(D) \Rightarrow e \downarrow_D c$.

Let $\theta_1(x, b_1)$ be a formula over A of least ∞-rank such that $\theta_1(x, b_1) \vdash \theta(x, b)$ and $\theta_1(M, b_1) \not\subseteq \mathrm{acl}_m(b)$. Let $c \in \theta_1(M, b_1) \setminus \mathrm{acl}_m((b)$. We will show that $cA \lhd A\,(N)$.

Suppose, towards a contradiction, that cA is not dominated by A over N. Now proceed to find e, a, $\theta_2(x, a) \in t(c/aN)$ and $\varphi(x, y, z) \in t(cae/N)$ exactly as in Lemma 5.3. Let $r = t(a/N)$ and define $\sim$ as before.

Choose a finite $D_0 \subset N$ such that $cae \downarrow_{D_0} N$. As $N \subset_m M$ there exists an $e' \in N$ such that $\models \exists x\, \varphi(x, a, e')$ and $e'/\!\sim\, \notin \mathrm{acl}_m(D_0)$. Pick $D \subset N$ finite, $D_0 \subset D$, and $e' \in N$ such that $\models \exists x\, \varphi(x, a, e')$, $e^* = e'/\!\sim\, \notin \mathrm{acl}_m(D)$ and $R^\infty(e^*/D)$ is minimal under these conditions. Let $\varphi_0(u) \in t(e^*/\mathrm{acl}(D))$ be such that $R^\infty(e^*/D) = R^\infty(\varphi_0)$ and $\varphi_0(u) \vdash \exists z\, (z/\!\sim\, = u \wedge (d_r y)\, \exists x\, \varphi(x, y, z))$. Notice that $M \models$ "$Qx\, \varphi_0(x)$". Furthermore, the minimal rank assumption implies that if $D' \supset D$, $N \supset D'$ and $f \in \varphi_0(M) \setminus \mathrm{acl}_m(D')$, then $R^\infty(f/D') = R^\infty(\varphi_0)$.

Sublemma 1. *Suppose that $f \in \varphi_0(M) \setminus \mathrm{acl}_m(D)$ and $f \downarrow_D a$. Then there is a $c' \in \theta_2(M, a)$ such that $f \not\downarrow_{aD} c'$.*

Proof. Much of the work here was done in 5.3. First we find an f' with $f'/\!\sim\, = f$, $\models (d_r y)\, \exists x\, \varphi(x, y, f')$ and $f' \downarrow_{fD} a$. Since $f' \downarrow_D a$, $\models \exists x\, \varphi(x, a, f')$. Let $q = \mathrm{stp}(f'/D)$ and $\sigma(x, y) = (d_q z)\, \varphi(x, y, z)$.

Claim 1. $(d_r y)\, \exists x\, [\theta_2(x, y) \wedge \neg(\varphi(x, y, f') \leftrightarrow \sigma(x, y))]$.

This is proved like Claim 1 in Lemma 5.3.

As a realizes r and $a \downarrow_D f'$,

$$\models \exists x\, [\theta_2(x, a) \wedge \neg(\varphi(x, a, f') \leftrightarrow \sigma(x, a))].$$

Claim 2. $\models \theta_2(c', a) \wedge \neg(\varphi(c', a, f') \leftrightarrow \sigma(c', a))$ *implies that* $c' \not\downarrow_{aD} f'$.

This is proved like Claim 2 in Lemma 5.3.

For the work ahead we must transfer our attention to f, so to what is borrowed from 5.3 we add

Claim 3. *For c' as in Claim 2, $c' \not\downarrow_{aD} f$.*

Choose an $f'' \stackrel{s}{=} f'$ (fD) with $f'' \downarrow_{fD} ac'$. Since $f \downarrow_D a$, $f'' \downarrow_D a$. The definition of $\sim$ then implies that $\vDash \forall x\, (\varphi(x, a, f') \leftrightarrow \varphi(x, a, f''))$. It follows that $\vDash \theta_2(c', a) \wedge \neg(\varphi(c', a, f'') \leftrightarrow \sigma(c', a))$. So, by Claim 2 and the fact that $f' \stackrel{s}{=} f''$ (aD), $c' \not\downarrow_{aD} f''$. The claim follows immediately since $f'' \downarrow_{fD} ac'$.

This proves the sublemma.

So, the e^* chosen above satisfies $e^* \not\downarrow_{aD} c'$. We will reach a contradiction rather quickly once we show that $c' \notin \mathrm{acl}_{\mathrm{m}}(a)$. In that previous lemma, where we used $\mathrm{acl}(-)$ instead of $\mathrm{acl}_{\mathrm{m}}(-)$, this was an immediate consequence of the forking. In this case a lot of work remains.

Sublemma 2. $c' \notin \mathrm{acl}_{\mathrm{m}}(a)$.

Proof. Assume that $c' \in \mathrm{acl}_{\mathrm{m}}(a)$. Eventually we will contradict the minimality assumption on $R^\infty(\varphi_0)$ defined above.

Claim 4. *There is an $e^\# \in M$ such that $e^\# \notin \mathrm{acl}_{\mathrm{m}}(aN)$, $\vDash \varphi_0(e^\#)$ and $e^\# \not\downarrow_N a$.*

There are two distinct cases involved in the proof.

Case 1: $e^* \in \mathrm{acl}_{\mathrm{m}}(Da)$. In this case we let $\psi \in t(e^*a/D)$ be such that $M \vDash$ "$\forall y\, (\neg \mathrm{Q}\, \psi(x, y))$", $\psi_0(x) = d_r y\, \psi(x, y)$, a formula over $\mathrm{acl}(D)$. Since $e^* \notin \mathrm{acl}_{\mathrm{m}}((D)$, $M \vDash$ "$\mathrm{Q}x\, (\psi_0(x) \wedge \varphi_0(x))$". The standardness of M and the fact that $|N| = |T|$ yields an $e^\# \in M$, $e^\# \notin \mathrm{acl}_{\mathrm{m}}((Na)$ and $\vDash \psi_0(e^\#) \wedge \varphi_0(e^\#)$. $e^\# \notin \mathrm{acl}_{\mathrm{m}}(Da) \Rightarrow \vDash \neg\psi(e^\#, a)$, so by the definition of ψ_0 we must have $e^\# \not\downarrow_D a$. The minimal rank assumption on φ_0 implies that $R^\infty(e^\#/N) = R^\infty(e^\#/D)$. Hence, $e^\# \not\downarrow_N a$, proving the claim in this case.

Case 2: $e^* \notin \mathrm{acl}_{\mathrm{m}}(aD)$. Since we have assumed that $c' \in \mathrm{acl}_{\mathrm{m}}(a)$ this implies that $e^* \notin \mathrm{acl}_{\mathrm{m}}(ac'D)$. Let $\psi(x, ac') \in t(e^*/\mathrm{acl}(D)ac')$ have ∞-rank $< R^\infty(e^*/D)$ and imply φ_0. Since $M \vDash$ "$\mathrm{Q}x\, \psi(x, ac')$" and M is standard there is an $e^\# \in \psi(M, ac')$ with $e^\# \notin \mathrm{acl}_{\mathrm{m}}(aN)$. By the minimality assumption on $R^\infty(\varphi_0)$, $R^\infty(e^\#/N) = R^\infty(e^\#/D)$. Thus, $e^\# \not\downarrow_N ac'$. If $e^\# \downarrow_N a$ we contradict $c' \in \mathrm{acl}_{\mathrm{m}}(a)$ by Lemma 5.10. Thus, $e^\# \not\downarrow_N a$ and $e^\# \notin \mathrm{acl}_{\mathrm{m}}(Na)$, proving the claim in this second case.

We will show that the conditions exhibited in Claim 4 contradict the minimality of $R^\infty(\varphi_0)$. To do this, however, we will not work with the elements shown, but must introduce a model. By Claim 4 and the standardness of M there are e_0 and $N' \supset Na$ such that $N' \subset_{\mathrm{m}} M$, $|N'| = |T|$, $e_0 \in M \setminus N'$, $\vDash \varphi_0(e_0)$ and $e_0 \not\downarrow_N N'$. Let $I = \{e_i : i < \omega\}$ be a Morley sequence in $t(e_0/N')$. We passed from $e^\#$, a and N to e_0, N' and N in order to obtain the following.

Claim 5. *For all finite $I_0 \subset I$ and $e \in I \setminus I_0$, $e \notin \mathrm{acl}_{\mathrm{m}}(I_0N')$.*

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Suppose that the claim fails. Then by the indiscernibility of I there is a $k<\omega$ such that $e_0 \in \mathrm{acl_m}(N'e_1\cdots e_k)$. Since $e_1\cdots e_k \downarrow_{N'} e_0$ we can find $e'_1,\ldots,e'_k \in N'$ with $e_0 \in \mathrm{acl_m}(N'e'_1\cdots e'_k)=\mathrm{acl_m}(N')$. This contradicts that $e_0\in M\backslash N'$ and $N' \subset_{\mathrm{m}} M$, proving the claim.

Let $a' \in \mathrm{Cb}(e_0/N')$ be such that $e_0 \downarrow_{a'} N'$, and $k<\omega$ such that $a' \in \mathrm{acl}(e_0\cdots e_k)$. By these remarks and Claim 5 there is a formula $\theta(x_0,\ldots,x_k,y)\in t(e_0\cdots e_k a'/\mathrm{acl}(D))$ such that

(i) $\theta(x_0,\ldots,x_k,y)\vdash \varphi_0(x_i),\quad i\leqslant k,$

(ii) $\forall x_0\cdots x_k y\,[\theta(x_0,\ldots,x_k,y)\rightarrow \exists^{<\omega}y\,\theta(x_0,\ldots,x_k,y)$

$$\wedge \bigwedge_{i\leqslant k} \mathrm{Q}x_i\cdots \mathrm{Q}x_k\,\theta(x_0,\ldots,x_k,y)].$$

The assumption that $e_0 \not\downarrow_N N'$ implies that $a'\in M\backslash N$. Since $N\subset_{\mathrm{m}} M$ there is an $a''\in N\backslash \mathrm{acl_m}(D)$ such that $\vDash \exists x_0\cdots x_k\,\theta(x_0,\ldots,x_k,a'')$. Using (ii) in the definition of θ, the standardness of M and $N\subset_{\mathrm{m}} M$, we find $e'_0,\ldots,e'_k\in N$ such that $\vDash\theta(e'_0,\ldots,e'_k,a'')$ and $e'_i\notin \mathrm{acl_m}(Da''e'_0\cdots e'_{i-1})$, for $i\leqslant k$. Letting $\bar e'$ be the sequence of e'_i's, we get $a''\in\mathrm{acl}(\bar e')\backslash\mathrm{acl}(D)$, so $\bar e' \not\downarrow_D a''$. By the Pairs Lemma there is an i such that $e'_i \not\downarrow_{De'_0\cdots e'_{i-1}} a''$. Thus, $R^\infty(e'_i/Da''e'_0\cdots e'_{i-1})< R^\infty(e'_i/D)=R^\infty(\varphi_0)$ and $e'_i\notin \mathrm{acl_m}(Da''e'_0\cdots e'_{i-1})$, so we finally reach a contradiction to the minimality of $R^\infty(\varphi_0)$.

This contradiction proves Sublemma 2. Thus, $e' \not\downarrow_{Da} c'$ and $c'\notin \mathrm{acl_m}(a)$. This contradicts the minimality assumption on $R^\infty(\varphi_2(x,a))$, proving that $cA \lhd A(N)$. This proves the lemma. □

Proposition 5.12. *Suppose that T^+ is a superstable $L(\mathrm{Q})$-expansion of T, $M\vDash T^+$ is standard, $N\subset_{\mathrm{m}} M$, $N\vDash T^+$, $N\subset A\subset M$ and $|A|=|T|$. Then there is an $N'\supset A$, such that $N'\subset_{\mathrm{m}} M$, $N'\lhd A\ (N)$ and $|N'|=|T|$.*

Proof. This is immediate by Lemmas 5.10 and 5.11. □

Theorem 5.9 is now proved as was Theorem C. We leave the details to the reader.

6. Improving the constructibility of N over $\bigcup N_\eta$

Using the results in the last section, especially Lemma 5.5, it is not difficult to show that if T is superstable with NDOP and $\langle N_\eta\colon \eta\in I\rangle$ is an independent tree of models with N minimal over $\bigcup N_\eta$, then N is l-atomic over $\bigcup N_\eta$ (see [11, IV, Definition 2.3]). In [11, XII, §1] we encountered j-isolation, a strengthening of l-isolation. It was shown that for countable superstable theories and all sets A there is a j-constructible model over A. j-isolation plays a central role in finding useful consequences of NOTOP (see [11, XII, §4]). Here we show that for N and

$\langle N_\eta: \eta \in I\rangle$ as above, N is j-constructible over $\bigcup N_\eta$. In fact, we obtain the constructibility of N over $\bigcup N_\eta$ with respect to an isolation relation significantly stronger than j-isolation, and we can find such an N inside any $M \supset \bigcup N_\eta$. Also, we do not need to assume that T is countable.

Our method of proving this last result may actually be of more interest than the fact itself. It depends highly on Hrushovski's theorem guaranteeing the existence of definable group actions (Proposition 6.4). We will show that for N and $\langle N_\eta: \eta \in I\rangle$ as above there is a construction $\langle c_\alpha: \alpha < |N|\rangle$ of N over $\bigcup N_\eta$ such that whenever $t(c_\alpha/C_\alpha \cup \bigcup N_\eta)$ is non-algebraic there is a definable group acting transitively on some formula in this type. We will be able to choose this formula of minimal ∞-rank over $C_\alpha \cup \bigcup N_\eta$. Using results in stable group theory (Proposition 6.3) we get $t(c_\alpha/C_\alpha \cup \bigcup N_\eta)$ isolated with respect to a notion stronger than j-isolation (see Definition 6.6(3), below).

First, we will discuss the stable group theory involved in the proof. After developing the terminology of definable group actions we will state results from [6] on 'generic types' which generalize well-known results about stable groups.

Definition 6.1 (1) We say that $\bar\varphi(\bar a)$ is an *$\bar a$-definable group action if* $\bar\varphi = \langle \varphi_0(x, \bar a), \varphi_1(x, y, z, \bar a), \varphi_2(x, \bar a), \varphi_3(x, y, z, \bar a)\rangle$ where

(i) $\bar\varphi \restriction 2 = \langle \varphi_0, \varphi_1\rangle$ is a definable group where $A_{\bar\varphi} = \varphi_0(\mathfrak{C})$ is the universe and $\varphi_1(\mathfrak{C})$ is the graph of the operation, and

(ii) φ_3 defines a group action of $A_{\bar\varphi}$ on $B_{\bar\varphi} = \varphi_2(\mathfrak{C})$.

(2) We say that the definable group action $\bar\varphi$ is *transitive* if $A_{\bar\varphi}$ acts transitively on $B_{\bar\varphi}$; i.e., for all $b, b' \in B_{\bar\varphi}$ there is a $g \in A_{\bar\varphi}$ with $\models \varphi_3(g, b, b')$.

We adopt the notation used above that for $\bar\varphi$ a definable group action, φ_2 defines the set $B_{\bar\varphi}$ acted on by $A_{\bar\varphi} = \varphi_0(\mathfrak{C})$. $\cdot$ denotes the group operation defined by φ_1 and $\circ$ denotes the action defined by φ_3.

Notation 6.2. Let $\bar\varphi(\bar a)$ be a definable group action.

(i) For $\theta = \theta(x, \bar y)$ let $\theta^{[\bar\varphi]}(x, z, \bar y) = \varphi(z \circ x, \bar y)$. For Δ a set of formulas $\Delta^{[\bar\varphi]} = \{\theta^{[\bar\varphi]}: \theta \in \Delta\}$. We call Δ $\bar\varphi$-invariant if for every Δ-formula $\theta(x, \bar b)$ and $c \in A_{\bar\varphi}$, $\theta(c \circ x, \bar b)$ is a Δ-formula.

(ii) We say that $p \in S(\mathrm{acl}(\bar a))$ is *generic* if $p \vdash \varphi_2$ and whenever $g \in A_{\bar\varphi}$ and b realizes p with $g \perp_{\bar a} b$, $g \perp_{\bar a} g \circ b$. If $A = \mathrm{dom}(q) \supset \bar a$ we say that q is generic if there is a generic p with $q = p \restriction A$.

(iii) We call a formula $\theta(x, \bar b)$ *$\bar\varphi$-small* if $\theta \vdash \varphi_2$ and θ is not contained in any generic.

Proposition 6.3 (Hrushovski). *Let $\bar\varphi(\bar a)$ be a transitive definable group action; Δ finite and invariant.*

(1) *Generic types exist and $A_{\bar\varphi}$ acts transitively on the set of all generics in $S(\mathrm{acl}(\bar a))$.*

(2) *Suppose that $\bar{a} \subset A$. Then $\Gamma_\Delta = \{q \restriction \Delta: q \in S(A)$ is generic$\}$ is finite. In fact, $r \in \Gamma_\Delta \Rightarrow R(r, \Delta, \aleph_0) = R(\varphi_2, \Delta, \aleph_0)$.*

(3) *Assume that T is superstable, $\bar{a} \subset A$, θ is a formula over A, $\theta \vdash \varphi_2$, and θ is not $\bar{\varphi}$-small. Then* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

(i) *θ has no extension over A which is $\bar{\varphi}$-small.*

(ii) *There is no $\theta' \vdash \theta$ over A with $R^\infty(\theta') < R^\infty(\varphi_2)$.*

(iii) *For every finite Δ', $|\{q \restriction \Delta': \theta \in q \in S(A)\}| < \aleph_0$.*

Proof. The proof is easily obtained from Facts 12 and 13 in [6, Ch. 1]. (1) is just Facts 12 and 13(a). (2) is 13(c), using its proof to obtain the second claim. For (3) we need the following additional information about generics when T is superstable.

Claim. *If p is generic, then $R^\infty(p) = R^\infty(\varphi_2)$.*

Let $q \in S(\mathrm{acl}(\bar{a}))$ be such that $\varphi_2 \in q$ and $R^\infty(q) = R^\infty(\varphi_2)$. We first show that q is generic as follows. Pick $g \in A_{\bar{\varphi}}$ and b realizing $q \mid g\, \mathrm{acl}(\bar{a})$. Since b and $g \circ b$ are equi-definable over $g\bar{a}$, $R^\infty(g \circ b / g\bar{a}) = R^\infty(b / g\bar{a}) = R^\infty(b/\bar{a}) = R^\infty(\varphi_2)$. Since $R^\infty(g \circ b / \bar{a}) \leq R^\infty(\varphi_2)$, we conclude that $g \circ b \downarrow_{\bar{a}} g$, proving that q is generic. It follows easily from 13(a) that all generics have the same ∞-rank, proving the claim.

Now the equivalence of (i) and (ii) is clear. (ii) $\Rightarrow$ (iii) follows from (2) and the fact that every finite Δ is contained in the finite invariant set $\Delta^{[\bar{\varphi}]}$. □

We turn now to deeper results. The first is just a more explicit statement of Proposition 2 on p. 35 of [6]. Our claim that φ_0 and φ_2 may be chosen to extend given formulas in p and q follows from the proof, as does the transitivity of the action.

Lemma 6.4. *Suppose that T is stable, $p, q \in S(A)$ are stationary and the following hold.*

(i) *$\cdot$ is an operation from $p(\mathfrak{C}) \times p(\mathfrak{C})$ to $p(\mathfrak{C})$, $\circ$ an operation from $p(\mathfrak{C}) \times q(\mathfrak{C})$ to $q(\mathfrak{C})$, and each of these is the restriction of a definable relation.*

(ii) *If a and b are A-independent realizations of p, then $a \cdot b$ realizes p and $\{a, b, a \cdot b\}$ is pairwise A-independent.*

(iii) *If a realizes p, b realizes q and $a \downarrow_A b$, then $a \circ b$ realizes q and $\{a, b, a \circ b\}$ is pairwise A-independent.*

(iv) *If $\{a, b, c\} \subset p(\mathfrak{C})$ is A-independent, then $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.*

(v) *If $a, b \in p(\mathfrak{C})$, $c \in q(\mathfrak{C})$ and $\{a, b, c\}$ is A-independent, then $a \circ (b \circ c) = (a \cdot b) \circ c$.*

Then for any $\varphi^0 \in p$ and $\varphi^2 \in q$ there is a transitive A-definable group action $\bar{\varphi}$ such that $\varphi_0 \vdash \varphi^0$, $\varphi_2 \vdash \varphi^2$, $\varphi_0 \in p$, $\varphi_2 \in q$ and $\cdot_{\bar{\varphi}}$, $\circ_{\bar{\varphi}}$ extend $\cdot$ and $\circ$, respectively.

Proposition 6.5. *Suppose that T is stable, $A = \mathrm{acl}(A)$, $q \in S(A)$, $t(c/A) \not\perp q$ and $c \downarrow_A q(\mathfrak{C})$. Then there is a $d \in \mathrm{dcl}(cA)\setminus A$ and a transitive A-definable group action $\bar\varphi$ with $d \in B_{\bar\varphi}$. Furthermore, for any $\psi \in t(d/A)$ we may require that $\varphi_2 \vdash \psi$.*

Proof. This is basically a combination of Proposition 6.4 and Theorem 2 in §3.4 of [6]. It is not hard to show that $t(c/A) \not\perp q \Rightarrow$ there is a $d \in \mathrm{dcl}(cA)\setminus A$ such that $t(d/A)$ is q-internal. The proof of the quoted result yields the hypotheses of Proposition 6.4. This proves the proposition. $\square$

We come now to the isolation relations relevant to this section.

Definition 6.6 For $x \in \{c1, j1, (c, j)\}$ we define $\mathbf{F}^x_\lambda$ as the set of (p, B) with $p \in S(A)$, $A \supset B$, $|B| < \lambda$, such that

(1) if $x = c1$, there is a $\varphi \in p \upharpoonright B$ such that $q \in S(A)$ and $\varphi \in q \Rightarrow R^\infty(q) = R^\infty(\varphi) < \infty$;

(2) if $x = j1$, there is a $\varphi \in p \upharpoonright B$ such that for all finite Δ, $|\{q \upharpoonright \Delta: \varphi \in q \in S(A)\}| < \aleph_0$;

(3) if $x = (c, j)$ there is a $\varphi \in p \upharpoonright B$ simultaneously witnessing that $(p, B) \in \mathbf{F}^{c1}_\lambda \cap \mathbf{F}^{j1}_\lambda$.

As usual, $\mathbf{F}^x_{\aleph_0}$-isolated is abbreviated x-isolated, etc.

Remark 6.7. Suppose that $p \in S(A)$ is $j1$-isolated, as witnessed by φ. It follows that not only is p $j1$-isolated, but every $q \in S(A)$ containing φ is $j1$-isolated.

The next lemma only plays a minor role in our treatment, but it lends insight into the strength of $j1$-isolation.

Lemma 6.8. *Suppose that T is stable, $M \vDash T$, $A \supset M$ and $t(a/A)$ is $j1$-isolated. Then $aA \lhd A$ (M).*

Proof. Suppose that $\varphi(x, b) \in t(a/A)$ witnesses that this type is $j1$-isolated. Let $\Sigma = \{q \in S(\mathfrak{C})$: $\varphi \in q\}$ and q does not fork over A. For each finite Δ let $\Sigma_\Delta = \{q \upharpoonright \Delta: q \in \Sigma\}$. $q \in \Sigma \Rightarrow R(q, \Delta, \aleph_0) = R(q \upharpoonright A, \Delta, \aleph_0)$, so Σ_Δ is finite. Using that each $r \in \Sigma_\Delta$ is definable over $\mathrm{acl}(A)$ we obtain a formula $\tau_\Delta(\bar y)$ over $\mathrm{acl}(A)$ such that for $\theta(x, \bar y) \in \Delta$ and all $\bar c$, $\vDash \tau_\Delta(\bar c)$ iff $\theta(x, \bar c) \in r$ for some $r \in \Sigma_\Delta$.

Now suppose that $\bar c \downarrow_M A$ and $\bar c \not\downarrow_A a$. Let $\theta(x, \bar c) \in t(a/\bar c A)$ be a formula that forks over A, τ_θ the formula associated to $\{\theta(x, \bar y)\}$ as above. Then $\vDash \exists x\, (\theta(x, \bar c) \wedge \varphi(x)) \wedge \neg\tau_\theta(\bar c)$. Since $\bar c \downarrow_M A$ and M is a model there is a $\bar c' \in M$, $\vDash \exists x\, (\theta(x, \bar c') \wedge \varphi(x)) \wedge \neg\tau_\theta(\bar c')$. Thus, $\theta(x, \bar c')$ is in some $r \in \Sigma_{\{\theta\}}$, contradicting that $\vDash \neg\tau_\theta(\bar c')$. This proves the lemma. $\square$

Definition 6.9. (1) We say that $t(a/A)$ is *gp-isolated* if $a \in \mathrm{acl}(A)$ or there is a transitive A-definable group action $\bar\varphi$ with $\varphi_2 \in t(a/A)$.

(2) We say that $t(a/A)$ is *(c, gp)-isolated* if $a \in \mathrm{acl}(A)$ or there is a transitive A-definable group action $\bar{\varphi}$ with $\varphi_2 \in t(a/A)$ and for $q \in S(A)$, $\varphi_2 \in q \Rightarrow R^\infty(q) = R^\infty(\varphi_2) < \infty$.

Notice that for a type to be (c, gp)-isolated it is not sufficient to simply be both $c1$- and gp-isolated. Requiring φ_2 to be the c1-isolating formula has the important consequence that $\mathrm{stp}(a/A)$ is generic.

For $x = c1$, $j1$, etc., the notion of an *x-construction over A* is defined in the obvious way.

In this section (c, gp)-isolation is used largely in conjunction with the next lemma, which is an immediate corollary to Proposition 6.3(3).

Lemma 6.10. *If $t(a/A)$ is (c, gp)-isolated, then it is (c, j)-isolated.*

To prove the existence of enough (c, gp)-isolated types we will need the following refined notion.

Definition 6.11. Let ψ be a formula over A. We say that $\langle c_0, \ldots, c_n \rangle$ is a *gp-construction over A into ψ* if

(i) $\langle c_0, \ldots, c_n \rangle$ is a gp-construction over A, $\models \psi(c_n)$ and $c_i \in \mathrm{acl}(c_n C_i A)$.

(ii) If $c_i \notin \mathrm{acl}(C_i A)$ there is a transitive $C_i A$-definable group action $\bar{\varphi}$ with $\varphi_2 \in t(c_i/C_i A)$ and $\models \varphi_2(e) \Rightarrow$ there is a $d \in \psi(\mathfrak{C})$ with $e \in \mathrm{acl}(dC_i A)$.

$\langle c_0, \ldots, c_n \rangle$ is a *(c, gp)-construction over A into ψ* if it is a gp-construction over A into ψ, and under (ii), φ_2 $c1$-isolates $t(c_i/C_i A)$.

The next lemma is central to the proofs of our main results. Its proof is straight forward so we leave it to the reader.

Lemma 6.12. *Suppose that $\langle c_0, \ldots, c_n \rangle$ is a gp-construction over A into ψ, where ψ is a formula over $B \subset A$. There is a formula $\varphi(x_0, \ldots, x_n, \bar{y}) \in t(c_0 \cdots c_n A/B)$ such that $\models \varphi(d, \ldots, d_n, \bar{a}) \Rightarrow \langle d_0, \ldots, d_n \rangle$ is a gp-construction over $\bar{a}b$ into ψ.*

Our main goal is the next result, which is proved with a series of lemmas.

Theorem 6.13. *Suppose that T is superstable with* NDOP, *$\langle N_\eta : \eta \in I \rangle$ is an independent tree of models and $M \supset \bigcup N_\eta$. Then there is a model $M' \subset M$, $M' \supset \bigcup N_\eta$ such that M' is (c, gp)-constructible over $\bigcup N_\eta$.*

Proposition 6.14. *Suppose that T is superstable with* NDOP, *$\langle N_\eta : \eta \in I \rangle$ is an independent tree of a-models, $A \supset \bigcup N_\eta$ and for all $\eta \in I$, $A \lhd \bigcup N_\eta$ (N_η). Then for all formulas ψ over A there is a gp-construction over A into ψ.*

Proof. The proof is by induction on $R^\infty(\psi)$. Suppose that the proposition is true for all ψ' and all A' with $R^\infty(\psi')<R^\infty(\psi)$. W.l.o.g., $R^\infty(\psi)=R^\infty(a/A)$ for all $a\in\psi(\mathfrak{C})$, $A=\mathrm{acl}(A)$, and ψ is non-algebraic.

As in Claim 2 of Lemma 5.5, A is a-atomic over $\bigcup N_\eta$. Let p be a regular type of least ∞-rank which is non-orthogonal to $\mathrm{stp}(a/A)$ for some $a\in\psi(\mathfrak{C})$. By NDOP, $p\not\perp N_\eta$ for some $\eta\in I$. Since N_η is an a-model we may assume that $\mathrm{dom}(p)=N_\eta$. Let $a\in\psi(\mathfrak{C})$ be such that $t(a/A)\not\perp p$.

Claim. $a \mathop{\smile\!\!\!\!\!\!\,|}_A p(\mathfrak{C})$.

Let I be a basis for $p\mid A$ in $\mathfrak{C}$. If $a \not\mathop{\smile\!\!\!\!\!\!\,|}_A I$ we can easily contradict that $t(a/A)$ is $c1$-isolated since $I\mathop{\smile\!\!\!\!\!\!\,|}_{N_\eta} A$ and N_η is an a-model. Thus, $a\mathop{\smile\!\!\!\!\!\!\,|}_A I$. If $b\in p(\mathfrak{C})$ and $B\supset IA$, $R^\infty(b/B)<R^\infty(p)$, so $\mathrm{stp}(a/A)\perp\mathrm{stp}(b/B)$, by the minimal rank assumption on p. This proves the claim.

Since A is algebraically closed, we may apply Proposition 6.5 to obtain $\bar{\varphi}$, an A-definable transitive group action with $\varphi_2\in t(d/A)$ for some $d\in\mathrm{dcl}(aA)\setminus A$. Furthermore, we may assume that $\vDash\varphi_2(e)\Rightarrow$ there is an $a'\in\psi(\mathfrak{C})$ with $e\in\mathrm{dcl}(a'A)$. Since $R^\infty(a/dA)<R^\infty(\psi)$, we can apply induction to obtain $\langle c_0,\dots,c_n\rangle$, a gp-construction over Ad into ψ with $d\in\mathrm{acl}(c_nA)$. Now observe that $\langle d,c_0,\dots,c_n\rangle$ is a gp-construction over A into ψ, proving the proposition. □

Proposition 6.15. *Suppose that T is superstable, $M\vDash T$, $M\supset A$ and for every formula ψ over A there is a gp-construction over A into ψ. Then for every ψ over A, in M there is a (c,gp)-construction over A into ψ.*

Proof. This, too, is proved by induction on $R^\infty(\psi)=\alpha$. Assume that the proposition is true for all A and all formulas of ∞-rank $<\alpha$, and $\alpha\neq 0$.

First suppose that there are: $a\in\psi(M)$, $\langle b_0,\dots,b_{m-1}\rangle$, a (c,gp)-construction over A with $b_i\in\mathrm{acl}(aB_iA)$, and a $c\in\mathrm{acl}(aAB_m)\setminus\mathrm{acl}(AB_m)$ such that $t(c/AB_m)$ is $c1$-isolated and $R^\infty(c/AB_m)<\alpha$. Pick $\theta(x,y)\in t(ac/AB_m)$ such that $\vDash\theta(de)\Rightarrow\vDash\psi(d)$, $e\in\mathrm{acl}(dAB_m)$, $b_i\in\mathrm{acl}(dAB_i)$ for $i<m$, and $R^\infty(e/AB_m)=R^\infty(c/AB_m)$. This is possible since $t(c/AB_m)$ is $c1$-isolated. Let $\varphi(y)=\exists x\,\theta(x,y)$. By induction there is $\langle c_0,\dots,c_n\rangle\in M$, a (c,gp)-construction over AB_m into φ. Since $\vDash\varphi(c_n)$ and $c\notin\mathrm{acl}(AB_m)$, $c_n\notin\mathrm{acl}(AB_m)$, so $R^\infty(\theta(x,c_n))<\alpha$. We may again apply induction to find $\langle d_0,\dots,d_l\rangle\in M$, a (c,gp)-construction over AB_mC_{n+1} into $\theta(x,c_n)$. Now check that $\langle b_0,\dots,b_{m-1},c_0,\dots,c_n,d_0,\dots,d_l\rangle$ is a (c,gp)-construction over A into ψ. (For $i<m$, $b_i\in\mathrm{acl}(d_lB_iA)$ since $\vDash\theta(d_l,c_n)$. For $i\leqslant n$, $c_i\in\mathrm{acl}(d_lC_iB_mA)$ since $c_i\in\mathrm{acl}(c_nC_iB_mA)$ and $c_n\in\mathrm{acl}(d_lB_mA)$.) Thus, we may assume

(1) *If $a\in\psi(M)$, $\langle b_0,\dots,b_{m-1}\rangle$ is a (c,gp)-construction with $b_i\in\mathrm{acl}(aB_iA)$ for all i, $c\in\mathrm{acl}(aAB_m)\setminus\mathrm{acl}(AB_m)$ and $t(c/AB_m)$ is $c1$-isolated, then $R^\infty(c/AB_m)\geqslant\alpha$.*

By hypothesis there is $\langle c_0, \ldots, c_n \rangle$, a gp-construction over A into ψ. Let $\varphi(x_0, \ldots, x_n) \in t(c_0 \cdots c_n/A)$ be as guaranteed by Lemma 6.12. Let $\langle d_0, \ldots, d_n \rangle \in M$ satisfy φ with $R^\infty(d_i/D_iA)$ minimal.

Claim. *$t(d_i/D_iA)$ is (c, gp)-isolated.*

We do this by induction on i. If $d_i \in \mathrm{acl}(D_iA)$ we are done, so suppose that $d_i \notin \mathrm{acl}(D_iA)$. By the definition of a gp-construction into ψ, $\models\psi(d_n)$, $d_i \in \mathrm{acl}(d_nD_iA)$ and there is $\bar{\varphi}$, a transitive D_iA-definable group action with $\varphi_2 \in t(d_i/D_iA)$. By the inductive hypothesis D_i is a (c, gp)-construction over A with $d_j \in \mathrm{acl}(d_nD_jA)$ for $j<i$. Thus, we can apply (1) to conclude that $R^\infty(d_i/D_iA) \geq \alpha$. By the definition of a gp-construction into ψ there is an $a' \in \psi(\mathfrak{C})$ with $e \in \mathrm{acl}(a'D_iA)$ for any $e \in \varphi_2(\mathfrak{C})$. Thus, $R^\infty(\varphi_2) \leq \alpha$. Since we chose $R^\infty(d_i/D_iA)$ minimal, we conclude that $R^\infty(e/D_iA) = R^\infty(\varphi_2)$ for all $e \in \varphi_2(\mathfrak{C})$. This proves the claim, hence the proposition. □

Lemma 6.16. *Suppose that T is superstable with* NDOP, *$\langle M_\eta : \eta \in I \rangle$ is an independent tree of models and $A \supset M_\eta$ is such that $A \lhd \bigcup M_\eta$ (M_η) for all $\eta \in I$. Then for all formulas ψ over A there is a gp-construction over A into ψ.*

Proof. Let $\langle N_\eta : \eta \in I \rangle$ be an independent tree of a-models with $N_\eta \supset M_\eta$, $N_\eta \perp_{M_\eta} A$ and $\bigcup N_\eta \perp_{\bigcup M_\eta} A$. By [11, XII, 2.3(3)] $\bigcup M_\eta \subset_t \bigcup N_\eta$.

Claim. *For all formulas $\theta(x, y)$ such that $\models\theta(a, b)$ for $a \in A$ and $b \in \bigcup N_\eta$, there is a $b' \in \bigcup M_\eta$ with $\models\theta(a, b')$.*

This is proved just like Claim 1 of Lemma 5.4.

As in Claim 1 of Lemma 5.5, $\bigcup N_\eta A \lhd \bigcup N_\eta$ (N_η) for all $\eta \in I$. By Proposition 6.14 there is $\langle c_0, \ldots, c_n \rangle$, a gp-construction over $A \cup \bigcup N_\eta$ into ψ. Let $\varphi(x_0, \ldots, x_n, y) \in t(c_0 \cdots c_n \cup N_\eta/A)$ be as guaranteed by Lemma 6.12. By the claim there is an $a' \in \bigcup M_\eta$ such that $\models\exists x_0 \cdots x_n\, \varphi(x_0, \ldots, x_n, a')$. Witnessing these existential quantifiers gives the desired gp-construction into ψ. □

Proof of Theorem 6.13. This is obtained by iterated application of Propositions 6.14 and 6.15, Lemma 6.16 and the following corollary of Lemma 6.8:

If $\langle M_\eta : \eta \in I \rangle$ is an independent tree of models and A is (c, gp)-constructible over $\bigcup M_\eta$, then $\bigcup M_\eta A \lhd \bigcup M_\eta$ (M_η) for all $\eta \in I$.

Corollary 6.17. *Suppose that T is superstable with* NDOP, *$\langle N_\eta : \eta \in I \rangle$ is an independent tree of models and $M \supset \bigcup N_\eta$. Then there is a model $M' \subset M$, $M' \supset \bigcup N_\eta$ such that M' is (c, j)-constructible over $\bigcup N_\eta$.*

Proof. Combine Theorem 6.13 and Lemma 6.10. □

We have certainly not exhausted the consequences of the ideas in this section. As an example of an additional application we offer the following.

Corollary 6.18. *Suppose that T is superstable and non-multidimensional, $M, N \models T$ with $N \subset A \subset M$. Then there is a model $M' \subset M$, $A \subset M'$ and M' is dominated by A over N.*

Proof. We only outline the proof. We will find an M' which is (c, gp)-constructible over A. The corollary then follows from Lemmas 6.8 and 6.10. Combining various results in this section the problem reduces to showing that for N' an a-model with $N' \downarrow_N A$, there is a gp-constructible model over $N' \cup A$. This is proved as in Lemma 6.14 using that every regular type is non-orthogonal to an RP over N'. □

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