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# Generic left-separated spaces and calibers

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**Abstract**

In this paper we use a natural forcing to construct a left-separated topology on an arbitrary cardinal $\kappa$. The resulting left-separated space $X_\kappa$ is also 0-dimensional $T_2$, hereditarily Lindelöf, and countably tight. Moreover if $\kappa$ is regular then $d(X_\kappa)=\kappa$, hence $\kappa$ is not a caliber of $X_\kappa$, while all other uncountable regular cardinals are. This implies that some results of [A.V. Archangelskiĭ, Topology Appl. 104(2000) 13–16] and [I. Juhász, Z. Szentmiklóssy, Topology Appl. 119 (2002) 315–324] are, consistently, sharp.

We also prove it is consistent that for every countable set $A$ of uncountable regular cardinals there is a hereditarily Lindelöf $T_3$ space $X$ such that $\varrho=\mathrm{cf}(\varrho)>\omega$ is a caliber of $X$ exactly if $\varrho\notin A$.
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## 1. Introduction

Let us start by recalling that a regular cardinal $\varrho$ is said to be a caliber of a topological space $X$ (in symbols: $\varrho\in\mathrm{Cal}(X)$) if among any $\varrho$ open subsets of $X$ there are always $\varrho$ many with non-empty intersection. Note that in this paper we restrict the notion of caliber to regular cardinals, although the definition does make sense for singular cardinals as well. Note also that $\varrho\in\mathrm{Cal}(X)$ implies that $X$ has no cellular family of size $\varrho$. Hence, as any infinite $T_2$ space has an infinite cellular family, for all spaces of interest we have $\mathrm{Cal}(X)\subset\mathbb{R}$, where $\mathbb{R}$ denotes the class of all uncountable regular cardinals.

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It is trivial to see that if $\varrho = \mathrm{cf}(\varrho) > d(X)$ then $\varrho \in \mathrm{Cal}(X)$, moreover Šanin proved in [9] that, for any fixed $\varrho$, the property of spaces $\varrho \in \mathrm{Cal}(X)$ is fully productive. Consequently, for any cardinal $\kappa$ we have $\mathrm{Cal}(2^\kappa) = \mathrm{Cal}([0,1]^\kappa) = \mathbb{R}$, showing that the converse of the above relation between density and calibers is not valid. More precisely, no bound for the density of $X$ can be deduced from the fact that $X$ satisfies the condition $\mathrm{Cal}(X) = \mathbb{R}$ that we also call Šhanin's condition, even for very nice (e.g., compact Hausdorff) spaces $X$.

Such a converse, however, is valid if $X$ is a compact $T_2$ space of countable tightness, as was shown by Šapirovskiĭ in [8], see also [3, 3.25]. Indeed, in this case $\varrho \in \mathrm{Cal}(X)$ implies $d(X) < \varrho$ or, equivalently,

$$\mathrm{Cal}(X) = \left[d(X)^+, \infty\right),$$

where the interval on the right-hand side (just like in PCF theory) denotes an interval of regular cardinals.

More recently, in [1], Archangelskiĭ proved that if $X$ is Lindelöf $T_3$ and countably tight and $\omega_1 \in \mathrm{Cal}(X)$ then $d(X) \leqslant 2^\omega$. In [6] both Šapirovskiĭ's and Archangelskiĭ's results were strengthened and generalized, moreover, under CH, in the second result the conclusion $d(X) \leqslant 2^\omega = \omega_1$ was improved to $d(X) = \omega$. Of course, this immediately led us to the question if the use of CH here is essential.

In the present note we give an affirmative answer to this question, in fact we show that Archangelskiĭ's result is sharp for arbitrarily large values of the continuum $2^\omega$, even for hereditarily Lindelöf (in short HL) $T_3$ spaces of countable tightness. The examples showing this will be obtained by forcing generic left-separated 0-dimensional spaces in a natural way. Our methods will then be used to also solve some other problems raised in [6]. Moreover, we shall also prove the consistency of the statement that for any countable subset $A$ of $\mathbb{R}$ there is a countably tight HL $T_3$ space $X$ such that

$$\mathrm{Cal}(X) = \mathbb{R} \setminus A.$$

This is in sharp contrast with the compact case.

We do not know if there are similar consistency results for uncountable $A \subset \mathbb{R}$ and the following intriguing question also remains open: Is it provable in ZFC that a countably tight (hereditarily) Lindelöf $T_3$ space $X$ satisfying Šanin's condition $\mathrm{Cal}(X) = \mathbb{R}$ is separable?

Our notation and terminology follows [2,3] in topology and [7] in forcing.

## 2. Generic left-separated spaces

Let $\nu$ be an arbitrary limit ordinal and consider the suborder $P_\nu$ of the Cohen order $\mathrm{Fn}(\nu^2, 2)$ that consists of those $p \in \mathrm{Fn}(\nu^2, 2)$ which satisfy conditions (i) and (ii) below:

(i) if $\langle\alpha,\alpha\rangle \in D(p)$ then $p(\alpha,\alpha) = 1$;
(ii) if $p(\alpha,\beta) = 1$ then $\alpha \leqslant \beta$.

Clearly, $P_\nu$ is a complete suborder of $\mathrm{Fn}(\nu^2, 2)$, hence it is CCC and thus preserves cardinals and cofinalities.

It is straight–forward to check that for any pair $\langle\alpha,\beta\rangle\in\nu^2$ the set

$$D_{\alpha,\beta}=\{p\in P_\nu:\ \langle\alpha,\beta\rangle\in D(p)\}$$

is dense in $P_\nu$, consequently if $G\subset P_\nu$ is $P_\nu$-generic over $V$ then

$$F=\bigcup G:\nu^2\to 2,$$

i.e., $F$ defines a directed graph on $\nu$ by $F(\alpha,\beta)=1$ meaning that an edge goes from $\alpha$ to $\beta$.

Now, in $V[G]$, for any $\alpha\in\nu$ and $i\in 2$ let

$$U_{\alpha,i}=\{\beta\in\nu:\ F(\alpha,\beta)=i\},$$

and $\tau_G$ be the (0-dimensional) topology on $\nu$ generated by the subbase

$$\mathcal{S}_G=\{U_{\alpha,i}:\ \alpha\in\nu,\ i\in 2\}.$$

In other words, $\tau_G$ is the graph topology on $\nu$ determined by the directed graph $F$ in the sense of [4] or [5].

For all $\alpha\in\nu$ the minimal element of $U_{\alpha,1}$ is $\alpha$ and this shows that $\tau_G$ is left-separated in its natural well-ordering. This immediately implies that $\tau_G$ is $T_2$ and thus, by 0-dimensionality, also $T_3$.

All finite intersections of the elements of $\mathcal{S}_G$ form a base $\mathcal{B}_G$ of $\tau_G$. A typical element of $\mathcal{B}_G$ is of the form

$$[\varepsilon]=\bigcap\{U_{\alpha,\varepsilon(\alpha)}:\ \alpha\in D(\varepsilon)\},$$

where $\varepsilon\in\mathrm{Fn}(\nu,2)$.

All this was easy. Let us now turn to the less obvious properties of the topology $\tau_G$.

**Lemma 2.1.** *$\tau_G$ is* HL.

**Proof.** Assume, indirectly, that $p\in P_\nu$ forces that $\langle[\dot\varepsilon_i]:\ i\in\omega_1\rangle$ are right-separating neighbourhoods of the points $\langle\dot x_i:\ i\in\omega_1\rangle$ in $\nu$, where WLOG we may assume that $i<j$ implies $\dot x_i<\dot x_j$. Then for every $i\in\omega_1$ there are $p_i\in P_\nu$, $\xi_i\in\nu$, and $\eta_i\in\mathrm{Fn}(\nu,2)$ such that $p_i\leqslant p$ and $p_i\Vdash\dot x_i=\xi_i$ and $\dot\varepsilon_i=\eta_i$. We may also assume that $D(p_i)=a_i^2$ for some $a_i\in[\nu]^{<\omega}$, moreover, $\xi_i\in a_i$ and $D(\eta_i)\subset a_i$. By a standard $\Delta$-system and counting argument we can find $i,j\in\omega_1$ with $i<j$ such that

(a) $p_i\restriction(a_i\cap a_j)^2=p_j\restriction(a_i\cap a_j)^2$, i.e., $p_i$ and $p_j$ are compatible as functions;
(b) $\eta_i\restriction a_i\cap a_j=\eta_j\restriction a_i\cap a_j$;
(c) $\xi_i\in a_i\setminus a_j$, $\xi_j\in a_j\setminus a_i$, and $\xi_i<\xi_j$.

Let us then define $q:(a_i\cup a_j)^2\to 2$ in such a way that (1) $q\supset p_i\cup p_j$, moreover (2)

$$q(\alpha,\xi_j)=p_i(\alpha,\xi_i)$$

if $\alpha\in D(\eta_i)\setminus a_j$ and $\alpha\leqslant\xi_i<\xi_j$, and finally (3) $q(\alpha,\beta)=0$ for every other pair $\langle\alpha,\beta\rangle\in(a_i\cup a_j)^2\setminus(a_i^2\cup a_j^2)$, not covered by cases (1) and (2). It is easy to see that $q\in P_\nu$ because it satisfies (i) and (ii). Moreover, $q(\alpha,\xi_j)=p_i(\alpha,\xi_i)$ holds for every $\alpha\in D(\eta_i)$:

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for $\alpha \in D(\eta_i) \cap a_j$ this follows from (b) and for $\alpha \in D(\eta_i) \setminus a_j$ this follows from (2) and (3). Consequently

$$q \Vdash \dot{x}_j = \xi_j \in [\eta_i] = [\dot{\varepsilon}_i],$$

contradicting that $p \Vdash \dot{x}_j \notin [\dot{\varepsilon}_i]$. □

Next we show that $\tau_G$ is countably tight.

**Lemma 2.2.** $\tau_G$ *has countable tightness.*

**Proof.** Let us assume that for a $P_\nu$-name $\dot{A}$ and some ordinal $\xi$ we have a condition $p \in P_\nu$ which forces $\xi \in \dot{A}'$, i.e., that $\xi$ is an accumulation point of $\dot{A}$. Since $\tau_G$ is left separated, we may also assume that $p \Vdash \xi < \dot{A}$, i.e., $p$ forces that every element of $\dot{A}$ is bigger than $\xi$. It can also be assumed that $\langle \xi, \xi \rangle \in D(p)$.

Let $\lambda$ be a large enough regular cardinal such that $H(\lambda)$ contains "everything in sight", e.g., $P_\nu$, $\dot{A} \in H(\lambda)$, etc. Fix a countable elementary submodel $N$ of $\langle H(\lambda), \in \rangle$ such that $\nu, \xi, p, \dot{A} \in N$. Clearly, we shall be done if we can prove the following claim.

**Claim.** $p \Vdash \xi \in (N \cap \dot{A})'$.

To see this, consider any $\varepsilon \in \mathrm{Fn}(\nu, 2)$ and let $q \leqslant p$ be an arbitrary extension of $p$ in $P_\nu$ such that $D(q) = a^2$ with $a \in [\nu]^{<\omega}$, $D(\varepsilon) \subset a$, and $q \Vdash \xi \in [\varepsilon]$, i.e., $q(\alpha, \xi) = \varepsilon(\alpha)$ for every $\alpha \in D(\varepsilon)$.

Then $q_N = q \cap N \in N$ is an extension of $p$ hence $q_N \Vdash |[\varepsilon_N] \cap \dot{A}| \geqslant \omega$, where $\varepsilon_N = \varepsilon \cap N$. But we also have $q_N \in N$, hence $N \prec H(\lambda)$ implies that there is an extension $r \leqslant q_N$ with $r \in N$ and an ordinal $x \in N \setminus a$ such that $D(r) = b^2$, $x \in b$, and

$$r \Vdash x \in N \cap \dot{A} \cap [\varepsilon_N].$$

Clearly $q \restriction (a \cap N)^2 = r \restriction (a \cap N)^2 = q_N$ and $D(q) \cap D(r) = (a \cap N)^2$, hence $q$ and $r$ are compatible as functions. We can thus define $q^* \supset q \cup r$ with the following additional stipulation: $q^*(\alpha, x) = \varepsilon(\alpha)$ whenever $\alpha \in D(\varepsilon) \setminus N$. Note that neither $q$ nor $r$ is defined for a pair $\langle \alpha, x \rangle$ of this form because $x \notin a \cap N$ and $\alpha \notin N$. Also, $q^* \in P_\nu$ because (i) holds trivially and (ii) holds because if $\alpha \in D(\varepsilon) \setminus N$ and $\varepsilon(\alpha) = 1$ then by $q(\alpha, \xi) = \varepsilon(\alpha) = 1$ we have $\alpha \leqslant \xi < x$. Finally, if $\alpha \in D(\varepsilon) \cap N$ then we have $q^*(\alpha, x) = r(\alpha, x) = \varepsilon_N(\alpha) = \varepsilon(\alpha)$ because $r \Vdash x \in [\varepsilon_N]$, consequently

$$q^* \Vdash x \in N \cap \dot{A} \cap [\varepsilon] \neq \emptyset.$$

This completes the proof of the claim and thus of Lemma 2.2. □

Note that Lemmas 2.1 and 2.2 immediately yield us that $\langle \nu, \tau_G \rangle$ is a countably tight $L$ space if $\nu \geqslant \omega_1$.

Our next lemma is the main result about calibers of $\tau_G$. In fact, for some applications to be given later, we formulate a slightly stronger result about calibers of initial segments of $\nu$ as subspaces of $\langle \nu, \tau_G \rangle$. So for $\alpha \leqslant \nu$ we let $X_\alpha$ denote the subspace of $\langle \nu, \tau_G \rangle$ on $\alpha$. Note that for any $\beta \in \nu \setminus \alpha$ we have $U_{\beta,1} \cap \alpha = \emptyset$, consequently for any $\varepsilon \in \mathrm{Fn}(\nu, 2)$ we

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have either $[\varepsilon] \cap \alpha = \emptyset$ or $[\varepsilon] \cap \alpha = [\varepsilon \restriction \alpha] \cap \alpha$. Therefore the trace of the base $\mathcal{B}_G$ on $\alpha$ can be written as

$$\mathcal{B}_G \restriction \alpha = \big\{[\varepsilon] \cap \alpha\colon\ \varepsilon \in \mathrm{Fn}(\alpha, 2)\big\} \cup \{\emptyset\}.$$

**Lemma 2.3.** *If $\alpha \leqslant \nu$ is any limit ordinal and $\varrho$ is an uncountable regular cardinal with $\varrho < \mathrm{cf}(\alpha)$ then $\varrho \in \mathrm{Cal}(X_\alpha)$. Moreover, we also have $d(X_\alpha) = \mathrm{cf}(\alpha)$ and so $\mathrm{Cal}(X_\alpha) = \mathbb{R} \setminus \{\mathrm{cf}(\alpha)\}$.*

**Proof.** By the above remark, to see the first part it clearly suffices to show that whenever $p \Vdash \{\dot{\varepsilon}_i\colon i \in \varrho\} \subset \mathrm{Fn}(\alpha, 2)$ then for some $\xi \in \alpha$ there is a $q \leqslant p$ such that

$$q \Vdash \big|\big\{i \in \varrho\colon\ \xi \in [\dot{\varepsilon}_i]\big\}\big| = \varrho.$$

To see this, first we find for each $i \in \varrho$ an $\eta_i \in \mathrm{Fn}(\alpha, 2)$ and an extension $p_i \leqslant p$ such that $p_i \Vdash \dot{\varepsilon}_i = \eta_i$. We may also assume that $D(p_i) = a_i^2$ for some $a_i \in [\nu]^{<\omega}$ and $D(\eta_i) \subset a_i$ for all $i \in \varrho$. But then

$$\Big|\bigcup\{a_i\colon\ i \in \varrho\}\Big| \leqslant \varrho < \mathrm{cf}(\alpha),$$

hence (the trace of) this union is bounded in $\alpha$. Consequently, there is an ordinal $\xi < \alpha$ with $a_i \cap \alpha < \xi$ for all $i \in \varrho$. Now extend each $p_i$ to a condition $q_i \in P_\nu$ such that $q_i(\alpha, \xi) = \eta_i(\alpha)$ for all $\alpha \in D(\eta_i)$. This is clearly possible because

$$D(\eta_i) \subset a_i \cap \alpha < \xi.$$

Note that then $q_i \Vdash \xi \in [\eta_i] = [\dot{\varepsilon}_i]$.

Since $P_\nu$ is CCC and $q_i \leqslant p$ for all $i \in \varrho$, there is a condition $q \in P_\nu$ with $q \leqslant p$ such that

$$q \Vdash \big|\big\{i \in \varrho\colon\ q_i \in \dot{G}\big\}\big| = \varrho,$$

hence clearly

$$q \Vdash \big|\big\{i \in \varrho\colon\ \xi \in [\dot{\varepsilon}_i]\big\}\big| = \varrho,$$

which was to be shown.

To see that $d(X_\alpha) = \mathrm{cf}(\alpha)$ first note that $d(X_\alpha) \geqslant \mathrm{cf}(\alpha)$ is trivial because $X_\alpha$ is left-separated in its natural ordering. On the other hand, if $S \subset \alpha$ is any cofinal subset of $\alpha$ in the ground model $V$ then $S$ will be dense in $X_\alpha$. Indeed, it is again sufficient to show that $S \cap [\varepsilon] \neq \emptyset$ for every $\varepsilon \in \mathrm{Fn}(\alpha, 2)$, and this follows by a straight–forward density argument. Consequently we have $d(X_\alpha) \leqslant \mathrm{cf}(\alpha)$, hence $d(X_\alpha) = \mathrm{cf}(\alpha)$.

Now, if $\varrho \in \mathbb{R}$ and $\varrho > d(X_\alpha) = \mathrm{cf}(\alpha)$ then $\varrho \in \mathrm{Cal}(X_\alpha)$, trivially. Finally, $\mathrm{cf}\,\alpha \notin \mathrm{Cal}(X_\alpha)$ is again obvious because $X_\alpha$ is left-separated. □

It is immediate from the above lemmas that if $\kappa$ is regular and $\kappa^\omega = \kappa$ then, in $V^{P_\kappa}$, we have $2^\omega = \kappa$ and the space $X_\kappa$ is HL, 0-dimensional $T_2$, countably tight with $d(X_\kappa) = \kappa = 2^\omega$, and $\mathrm{Cal}(X_\kappa) = \mathbb{R} \setminus \{\kappa\}$. In particular, this shows that Archangelskiĭ's result from [1] saying that a countably tight Lindelöf $T_3$ space $X$ with $\omega_1 \in \mathrm{Cal}(X)$ satisfies $d(X) \leqslant 2^\omega$ (or the more general Corollary 1.2 of [6] saying that for such a space $X$ with $\varrho^+ \in \mathrm{Cal}(X)$ we have $d(X) \leqslant 2^\varrho$) is, at least consistently, sharp.

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Clearly, in a Lindelöf space of countable tightness every free sequence is countable. Consequently, if we also have $\kappa > \omega_\omega$ then the space $X_\kappa$ establishes in addition that from Corollary 1.5 of [6] (saying that if $X$ is a countably tight $T_3$ space with no free sequence of length $\omega_\omega$ and satisfying $\{\omega_n\colon 0 < n < \omega\} \subset \mathrm{Cal}(X)$ then $X$ is separable provided that $\omega_\omega$ is strong limit) the assumption that $\omega_\omega$ be strong limit cannot be omitted.

With a little extra work we can deduce from our lemmas the following result showing that we have, again consistently, much more freedom in prescribing $\mathrm{Cal}(X)$ for Lindelöf (even HL) and countably tight $T_3$ spaces than in the case of compact and countably tight spaces.

**Theorem 2.4.** *Let $\kappa$ be any cardinal. Then, in $V^{P_\kappa}$, for every countable subset $A$ of $\mathbb{R} \cap \kappa$ there is a HL and countably tight 0-dimensional $T_2$, hence $T_3$, space $X$ such that $\mathrm{Cal}(X) = \mathbb{R} \setminus A$.*

**Proof.** For any $\varrho \in A$ let $X_\varrho$ be the subspace $\langle \varrho, \tau_G \restriction \varrho \rangle$ as in 2.3 and then let

$$X = \bigoplus \{X_\varrho\colon \varrho \in A\}$$

be the (disjoint) topological sum of these subspaces. Since $A$ is countable, it is obvious that $X$ is HL, countably tight, and 0-dimensional $T_2$. For any $\varrho \in A$ then $X_\varrho$ is a clopen subspace of $X$, hence, by Lemma 2.3, we have $\varrho \notin \mathrm{Cal}(X_\varrho)$, implying that $\varrho \notin \mathrm{Cal}(X)$ as well. On the other hand, if $\lambda \in \mathbb{R} \setminus A$ and $\mathcal{G}$ is a family of open sets in $X$ with $|\mathcal{G}| = \lambda$ then, again by the countability of $A$, there is a $\varrho \in A$ such that $|\{G \in \mathcal{G}\colon G \cap X_\varrho \neq \emptyset\}| = \lambda$, hence by Lemma 2.3 we have $\lambda \in \mathrm{Cal}(X_\varrho)$ which implies that also $\lambda \in \mathrm{Cal}(X)$. □

A natural question that we could not answer is if a similar result could be proved for uncountable sets $A$ of regular (uncountable) cardinals. Finally, our methods leave open the following very natural and interesting question formulated below.

**Problem 2.5.** Is it provable in ZFC that a Lindelöf $T_3$ space $X$ of countable tightness satisfying Šanin's condition $\mathrm{Cal}(X) = \mathbb{R}$ is separable?

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