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## MORE ON ENTANGLED ORDERS

OFER SHAFIR AND SAHARON SHELAH

**Introduction.** This paper grew as a continuation of [Sh462] but in the present form it can serve as a motivation for it as well. We deal with the same notions, all defined in 1.1, and use just one simple lemma from there whose statement and proof we repeat as 2.1. Originally entangledness was introduced, in [BoSh210] for example, in order to get narrow boolean algebras and examples of the nonmultiplicativity of c.c.-ness. These applications became marginal when other methods were found and successfully applied (especially Todorčević walks) but after the pcf constructions which made their debut in [Sh-g] and were continued in [Sh462] it seems that this notion gained independence.

Generally we aim at characterizing the existence of strong and weak entangled orders in cardinal arithmetic terms. In [Sh462, §6] necessary conditions were shown for strong entangledness which in a previous version was erroneously proved to be equivalent to plain entangledness. In §1 we give a forcing counterexample to this equivalence and in §2 we get those results for entangledness (certainly the most interesting case). A new construction of an entangled order ends this section. In §3 we get weaker results for positively entangledness, especially when supplemented with the existence of a separating point (Definition 2.2). An antipodal case is defined in 3.10 and completely characterized in 3.11. Lastly we outline in 3.12 a forcing example showing that these two subcases of positive entangledness comprise no dichotomy. The work was done during the fall of 1994 and the winter of 1995. The second author proved Theorems 1.2, 2.14, the result that is mentioned in Remark 2.11 and what appears in this version as Theorem 2.10(a) with the further assumption  $\text{den}(I)^\theta < \mu$ . The first author is responsible for waving off this assumption (actually by showing that it holds in the general case), for Theorems 2.12 and 2.13 in Section 2 and for the work which is presented in Section 3.

Our only notational idiosyncrasy is obeying the Jerusalem convention — the stronger forcing condition is the greater one. We also abuse the logic by writing the assumption “There is a such-and-such order” as “ $I$  is such-and-such”. We thank Shani Ben David for the beautiful typing.

### §1. Entangledness is not strong entangledness.

**DEFINITION 1.1.** (a) A linear order  $(I, <)$  is called  $(\mu, \sigma)$ -entangled if for any matrix of distinct elements from it  $\langle t_i^\varepsilon \mid i < \mu, \varepsilon < \sigma_1 \rangle$  ( $\sigma_1 < \sigma$ ) and  $u \subset \sigma_1$  there are  $\alpha < \beta < \mu$  satisfying  $\forall \varepsilon < \sigma_1 (t_\alpha^\varepsilon < t_\beta^\varepsilon \leftrightarrow \varepsilon \in u)$ .

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(b) A linear order  $(I, <)$  is called  $(\mu, \sigma)$ -strongly entangled if for any matrix  $\langle t_i^\varepsilon \mid i < \mu, \varepsilon < \sigma_1 \rangle$  ( $\sigma_1 < \sigma$ ) and  $u \subset \sigma$ , s.t.  $\forall \alpha < \mu \forall \varepsilon_0 \in u \forall \varepsilon_1 \in \sigma_1 \setminus u (t_\alpha^{\varepsilon_0} \neq t_\alpha^{\varepsilon_1})$  there are  $\alpha < \beta < \mu$  satisfying  $\forall \varepsilon < \sigma_1 (t_\alpha^\varepsilon \leq t_\beta^\varepsilon \leftrightarrow \varepsilon \in u)$ .

(c) A linear order is called  $(\mu, \sigma)$  positively [positively\*] entangled if for every  $\sigma_1 < \sigma$  and any matrix  $\langle t_\alpha^\varepsilon \mid \varepsilon < \sigma_1, \alpha < \mu \rangle$  s.t.  $\forall \varepsilon < \sigma_1 \forall \alpha, \beta < \mu (t_\alpha^\varepsilon \neq t_\beta^\varepsilon)$  and  $u \in \{\emptyset, \sigma_1\}$  there are  $\alpha < \beta$  [ $\alpha \neq \beta$ ] satisfying  $\forall \varepsilon < \sigma_1 (t_\alpha^\varepsilon < t_\beta^\varepsilon \leftrightarrow \varepsilon \in u)$ .

(d) The phrase “ $I$  is  $(\mu, \sigma)$  entangled with minimal  $\mu$ ” stands for “ $I$  is  $(\mu, \sigma)$  entangled but not  $(\mu', \sigma)$  entangled for any  $\mu' < \mu$ ”.

**THEOREM 1.2.** *For any cardinals  $\lambda = \lambda^{<\lambda} > \theta$ ,  $\text{cf} \mu = \kappa > \lambda$  there is a cardinal preserving forcing adding a  $(\mu, \theta^+)$ -entangled order with minimal  $\mu$ . In particular, it is not  $(\mu, \theta^+)$  strongly entangled.*

**PROOF.** Fix  $\langle \mu_i \mid i < \text{cf} \mu \rangle$  increasing to  $\mu$  and define  $\mathbb{P} = \{p \mid \text{dom } p \in [\mu]^{<\lambda}, \text{ for some } \alpha < \lambda \text{ ran } p \subseteq {}^2, p \text{ is 1-1}, \forall \alpha (2\alpha \in \text{dom } p \leftrightarrow 2\alpha + 1 \in \text{dom } p) \forall \alpha, \beta \in [\mu_i, \mu_i + \mu_i) (p(2\alpha) < p(2\beta) \leftrightarrow p(2\alpha + 1) < p(2\beta + 1))\}$  where  $<$  is the lexicographic order.  $p \leq q$  iff  $\text{dom } p \subset \text{dom } q$  and  $\forall \alpha \in \text{dom } p (p(\alpha) \triangleleft q(\alpha))$ . Easily  $\mathbb{P}$  is  $\lambda$ -closed. In order to see that it is also  $\lambda^+$ -c.c. (hence cardinals preserving) note that  $\forall \alpha < \lambda \forall p_0, p_1, p'_0, p'_1 \in {}^2 (p_0 \triangleleft p'_0 \wedge p_1 \triangleleft p'_1 \wedge p_0 \neq p_1) \rightarrow (p_0 < p_1 \leftrightarrow p'_0 < p'_1)$ , so that if  $\langle p_\alpha \mid \alpha < \lambda^+ \rangle$  are from  $\mathbb{P}$  wlog  $\{\text{dom } p_\alpha \mid \alpha < \lambda^+\}$  is a  $\Delta$  system and for some  $p^* \in \mathbb{P} \forall \alpha < \lambda^+ \forall i \in \text{dom } p_\alpha (p_\alpha(i) = p^*(\text{otp } \text{dom } p_\alpha \cap i))$ . Now define for  $\alpha < \beta < \lambda^+$

$$q(x) = \begin{cases} p_\alpha(x) \wedge 0 & x \in \text{dom } p_\alpha \\ p_\beta(x) \wedge 1 & x \in \text{dom } p_\beta \setminus \text{dom } p_\alpha \end{cases}$$

and this element from  $\mathbb{P}$  satisfies  $p_\alpha, p_\beta < q$ . Any  $\mathbb{P}$ -generic set induces  $A = \langle e_\alpha \mid \alpha < \mu \rangle \subset {}^\lambda 2$  which are distinct and satisfy  $\forall \alpha, \beta \in [\mu_i, \mu_i + \mu_i) (e_{2\alpha} < e_{2\beta} \leftrightarrow e_{2\alpha+1} < e_{2\beta+1})$ . Again,  $A$  is ordered lexicographically. This shows that  $(A, <)$  is not  $(\mu_1, 2)$ -entangled for all  $\mu_1 < \mu$ . Suppose by contradiction that  $A$  is not  $(\mu, \lambda)$ -entangled. In that case there is  $p \in \mathbb{P}$  and  $p \Vdash \langle t_i^\varepsilon \mid i < \mu, \varepsilon < \lambda_1 \rangle$ ,  $\lambda_1 < \lambda$ ,  $u = \{2p \mid p < \lambda_1\}$  is a counterexample”. For  $i < \mu$  pick  $p < p_i$  and  $\langle \alpha(\varepsilon, i) \mid \varepsilon < \lambda_1 \rangle \subset \text{dom } p_i$  such that  $p_i \Vdash \bigwedge_{\varepsilon < \lambda_1} t_i^\varepsilon = e_{\alpha(\varepsilon, i)}$ . Wlog for some  $p^* \in \mathbb{P} \forall i < \mu \forall j \in \text{dom } p_i (p_i(j) = p^*(\text{otp } \text{dom } p_i \cap j))$ .

We can assume also that for every  $i < \mu \{e_{\alpha(\varepsilon, i)} \mid \varepsilon < \lambda_1\} \subset \text{dom } p_i$  and that  $\langle \text{otp } \text{dom } p_i \cap \alpha(\varepsilon, i) \mid \varepsilon < \lambda_1 \rangle$  does not depend on  $i$  (here we use the inequality  $\kappa > \lambda$ ). By the  $\lambda^+$ -c.c. we have two comparable elements, call them  $p_i, p_j$ . Now define

$$q(\gamma) = \begin{cases} p_i(\gamma) \wedge 0 & \gamma = \alpha(i, 2p) \\ p_i(\gamma) \wedge 1 & \gamma = \alpha(i, 2p + 1) \\ p_j(\gamma) \wedge 1 & \gamma = \alpha(j, 2p) \\ p_j(\gamma) \wedge 0 & \gamma = \alpha(j, 2p + 1) \\ p_i(\gamma) \wedge 0 & \gamma \in \text{dom } p_i \text{ not as above} \\ p_j(\gamma) \wedge 0 & \gamma \in \text{dom } p_j \text{ not as above} \end{cases}$$

But  $p < q \Vdash \langle t_i^\varepsilon \rangle$  is not a counterexample by looking at  $i, j$ , a contradiction.

For the second claim of the theorem apply [Sh462, 6.24(2)]. +

**§2. Positive results on entangled orders.** First we quote the useful lemma [Sh462, 1.2(4)].

LEMMA 2.1. *If  $I$  is  $(\mu, 2)$  entangled then the density of  $I$  is smaller than  $\mu$ .*

PROOF. Otherwise define inductively a sequence of intervals  $\langle (a_\alpha^0, a_\alpha^1) \mid \alpha < \mu \rangle$  s.t.  $(a_\alpha^0, a_\alpha^1)$  exemplifies the nondensity of  $\{a_\beta^0, a_\beta^1 \mid \beta < \alpha\}$  in  $I$ , i.e., disjoint to this set. Now the matrix  $\{a_\alpha^i \mid i < 2, \alpha < \mu\}$  contradicts the entangledness with respect to  $u = \{0\}$ .  $\dashv$

DEFINITION 2.2. For a linear order  $I$  and  $x, y \in I$  we define  $\langle x, y \rangle := (x, y)_I \cup (y, x)_I$ . We call the point  $x$   $\mu$ -separative if  $|\{y \in I \mid y < x\}|, |\{y \in I \mid y > x\}| \geq \mu$ . Let  $f(x) = \min_{y \in I \setminus \{x\}} |\langle x, y \rangle|$ .

The following theorem states a basic property of entangled orders.

THEOREM 2.3. *If  $\lambda = \theta^+$  is infinite and  $(I, <)$  is  $(\mu, \lambda)$ -entangled linear order with minimal  $\mu$  then  $|\{x \in I \mid f(x) < \mu\}| < \mu$ .*

PROOF. Suppose first  $\mu$  is limit. We assume that  $|I| = \mu$  — if  $|I| < \mu$  the conclusion is trivial, and if  $|I| > \mu$  we take any subset of  $I$  of cardinality  $\mu$ . Fix a strictly increasing sequence of successor cardinals converging to  $\mu$ ,  $\langle \mu_\alpha \mid \alpha < \text{cf} \mu \rangle$ . Define on  $I$  the equivalence relation  $xEy \leftrightarrow |\langle x, y \rangle| < \mu$ . We look for disjoint intervals  $\langle I_\alpha \mid \alpha < \text{cf} \mu \rangle$  satisfying  $|I_\alpha| \geq \mu_\alpha$ . If the conclusion of the theorem fails then the union of the equivalent classes which have more than one element is of power  $\mu$ . If there are  $\mu$  many such classes choose  $\langle a_\beta^0, a_\beta^1 \mid \beta < \mu \rangle$  with no repetitions s.t.  $\forall \beta, \alpha < \mu \forall i, j \in \{0, 1\} (a_\alpha^i E a_\beta^j \leftrightarrow \alpha = \beta)$  which contradicts  $(\mu, 2)$ -entangledness with respect to  $u = \{0\}$ . So there are less than  $\mu$  classes. If any equivalence class has power smaller than  $\mu$  then choose by induction for  $I_\alpha$  any sufficiently large but yet unchosen class. Otherwise fix one class  $J$ ,  $|J| = \mu$ . Pick  $x \in J$  and wlog  $|\{y \in I \mid y > x\}| = \mu$  (inversion of the order does not affect the entangledness). Choose inductively  $\langle x_\alpha \mid \alpha < \text{cf} \mu \rangle$  where  $x_\alpha$  will be taken as any element above the previous ones and if  $\alpha = \beta + 1$  for some  $\beta$  then  $|\langle x_\beta, x_\alpha \rangle| > \mu_\beta$ . Set  $I_\alpha = (x_\alpha, x_{\alpha+1})_I$ . Next choose counterexamples for  $(\mu_\alpha, \lambda)$ -entangledness  $\langle t_i^\varepsilon \mid \varepsilon < \theta, i \in [\sum_{\beta < \alpha} \mu_\beta, \mu_\alpha] \rangle$ ,  $u = \langle 2\varepsilon \mid \varepsilon < \theta \rangle$ . For any  $\alpha < \text{cf} \mu$  choose different elements  $\langle t_i^\varepsilon \mid \varepsilon \in \{\theta, \theta + 1\}, i \in [\sum_{\beta < \alpha} \mu_\beta, \mu_\alpha] \rangle$  in  $I_\alpha \setminus \langle t_i^\varepsilon \mid \varepsilon < \theta, i \in [\sum_{\beta < \alpha} \mu_\beta, \mu_\alpha] \rangle$  (this is possible as by the entangledness  $\theta < 2^\theta \leq \mu$ ). Wlog all the  $t_i^\varepsilon$  above are with no repetitions. This contradicts the  $(\mu, \lambda)$ -entangledness with respect to  $u' = u \cup \{\theta\}$ . For a successor cardinal the proof is simpler since we may disregard the counterexamples.  $\dashv$

DEFINITION 2.4. For a linear order  $I$   $\text{c.c.}(I)$  is the first cardinality in which there is no family of disjoint nonempty open intervals. We define  $\text{h.c.c.}(I) = \min\{\text{c.c.}(J) \mid J \in [I]^{|I|}\}$ .

LEMMA 2.5. *If  $\lambda = \theta^+$  and  $I$  is  $(\mu, \lambda)$ -entangled linear order with minimal  $\mu$  then for any  $\{\sigma_i \mid i < \theta\} \subset \text{h.c.c.}(I)$  we have  $\prod_{i < \theta} \sigma_i < \text{cf} \mu$ .*

PROOF. Assume not. After throwing away less than  $\mu$  points of  $I$  we ensure that  $\forall x \in I (f(x) = \lambda)$  (by Lemma 2.3). Suppose the theorem fails for  $\{\sigma_i \mid i < \theta\}$ . Choose for every  $i < \theta$  a collection of disjoint intervals  $\{I_\alpha^i \mid \alpha < \sigma_i\}$  and distinct

functions in  $\Pi_{i < \theta} \sigma_i$ ,  $\langle f_\alpha \mid \alpha < \text{cf} \mu \rangle$ . Fix counterexamples  $t_i^\varepsilon, u$  and cardinals  $\langle \mu_\alpha \mid \alpha < \text{cf} \mu \rangle$  as above. For any  $\varepsilon < \theta$  choose  $\langle t_i^{\theta+2\varepsilon}, t_i^{\theta+2\varepsilon+1} \mid i \in \left[ \sum_{\beta < \alpha} \mu_\beta, \mu_\alpha \right) \rangle$  different elements from  $I_{f_\alpha(\varepsilon)}^\varepsilon \setminus \langle t_i^\varepsilon \mid \varepsilon < \theta, i \in \left[ \sum_{\beta < \alpha} \mu_\beta, \mu_\alpha \right) \rangle$  (remember that  $|I_\alpha^i| = \mu$ ). Wlog all the  $t_i^\varepsilon$  are with no repetitions and so  $\langle t_i^\varepsilon \mid \varepsilon < 2\theta, i < \mu \rangle$  contradicts the  $(\mu, \lambda)$  entangledness with respect to  $u' = u \cup \langle \theta + 2\varepsilon \mid \varepsilon < \theta \rangle$ .  $\dashv$

LEMMA 2.6. *If  $\lambda = \theta^+$  and  $I$  is  $(\mu, \lambda)$  entangled linear order with minimal  $\mu$  then  $\kappa = \text{h.c.c.}(I)$  satisfies  $\kappa^\theta \leq \text{cf} \mu$ .*

PROOF. Choose  $\langle \sigma_i \mid i < \text{cf} \kappa \rangle$  unbounded in  $\kappa$ . If  $\text{cf} \kappa \leq \theta$  then  $\kappa^\theta = \Pi_{i < \text{cf} \kappa} \sigma_i < \text{cf} \mu$  by Lemma 2.5. Otherwise by the same lemma  $\forall \sigma < \kappa (\sigma^\theta < \text{cf} \mu)$  so that  $\kappa^\theta = \kappa \cdot \sum_{i < \text{cf} \kappa} \sigma_i^\theta \leq \text{cf} \mu$  (remember that  $\kappa \leq \text{cf} \mu$ ).  $\dashv$

The next corollary strengthens [Sh462, 6.17(a)] where  $I$  was assumed to be  $(\mu, \lambda)$  strongly entangled and we got only  $\forall \theta < \lambda (2^\theta < \mu)$ .

COROLLARY 2.7. *If  $I$  is  $(\mu, \lambda)$  entangled linear order with density  $\chi$  then  $\forall \theta < \lambda (\chi^\theta < \mu)$ .*

PROOF. Fix  $\theta < \lambda$ . Wlog  $\lambda = \theta^+$  and  $\mu$  is minimal for which  $I$  is  $(\mu, \lambda)$  entangled. Let  $\kappa = \text{h.c.c.}(I)$ . We know that  $\kappa \in \{\chi, \chi^+\}$ . By Lemma 2.6 we have to consider only the case  $\kappa^\theta = \mu$ . By the proof above it follows that  $\text{cf} \kappa > \theta$  and  $\mu = \kappa \cdot \sum_{i < \text{cf} \kappa} \sigma_i^\theta$  (we keep the same notation) so that  $\kappa = \mu = \text{cf} \mu$ .  $\chi < \mu = \kappa$  holds by 2.1 and we can use Lemma 2.5 to get the desired conclusion.  $\dashv$

REMARK 2.8. The case  $\chi < \kappa = \mu (= \chi^+)$  follows) occurs for example in the construction from [BoSh210] if we assume CH (here  $\chi = \aleph_0$  and  $\mu = \aleph_1$ ).

CONCLUSION 2.9. *If  $I$  is  $(\mu, \lambda)$  entangled,  $\mu$  is regular,  $\theta < \lambda$  and  $\langle t_\varepsilon^i \mid i < \mu, \varepsilon < \theta \rangle$  is a matrix of different elements from  $I$  then for  $A \in [\mu]^\mu$  and a sequence of mutually disjoint intervals  $\langle I_\varepsilon \mid \varepsilon < \theta \rangle \forall i \in A \forall \varepsilon < \theta (t_\varepsilon^i \in I_\varepsilon)$ .*

PROOF. This is immediate from Corollary 2.7 and [Sh462, 1.2(3)].  $\dashv$

THEOREM 2.10. (a) *If  $I$  is a  $(\mu, \lambda)$  entangled order with minimal  $\mu$  and  $\lambda = \theta^+$  then  $\lambda < \text{h.c.c.}(I) \leq \text{cf} \mu$ . (b) *If  $I$  is a  $(\mu, \lambda)$  entangled order with minimal  $\mu$  and  $\text{cf} \mu \neq \text{cf} \lambda < \lambda$  then its density  $\chi$  satisfies  $\lambda < \chi$ .**

PROOF. (a) Assume the contrary. By 2.6  $\text{h.c.c.}(I) \leq \text{cf} \mu$  so here  $\lambda \geq \text{h.c.c.}(I)$ . For any  $x \in I$  choose a strictly increasing sequence converging to it with minimal (hence a regular cardinal) length  $\langle a_\alpha^x \mid \alpha < r(x) \rangle$ . By the assumption  $\forall x (r(x) < \lambda)$ . As  $|\text{rang} r| \leq |\theta \cup \{\theta\}| = \theta < \lambda \leq \text{cf} \mu$  (by Lemma 3.1) for some  $a = \langle x_i \mid i < \mu \rangle \subset I$  and some  $\sigma < \lambda \forall x \in A (r(x) = \sigma)$ . Wlog  $\forall x \in I (f(x) = \mu)$ . Define  $\langle t_\varepsilon^i \mid i < \mu, \varepsilon < \sigma \rangle$  by induction on  $i$ : for any  $\varepsilon < \sigma$  choose  $t_\varepsilon^i \in (a_{\varepsilon^{N_i}}, a_{\varepsilon^{N_i+1}}^{N_i})$  different from previously chosen  $t$ 's. This contradicts the  $(\mu, \lambda)$  entangledness with respect to  $u = \langle 2\varepsilon \mid \varepsilon < \sigma \rangle$ . (b) Assume not. Since  $\text{cf} \mu \neq \text{cf} \lambda$  for some  $\theta^+ < \lambda$   $I$  is  $(\mu, \theta^+)$  entangled with minimal  $\mu$  hence we get the conclusion of Theorem 2.3. Now in the proof of (a)  $r$  is into  $\lambda$  since  $\text{h.c.c.}(I) \leq \chi^+$  and we can ensure only its boundedness on a large  $A \subset I$ . Now take  $t_\varepsilon^i$  to be in  $(a_{\varepsilon_1^{N_i}}, a_{\varepsilon_1^{N_i+1}}^{N_i})$  where  $\varepsilon_1$  is  $\varepsilon$  modulo  $r(x_i)$ .  $\dashv$

REMARK 2.11. Note that the conclusion of 2.10(a) ( $\lambda < \text{cf} \mu$ ) is tight in view of Theorem 1.2. For inaccessible  $\lambda$  we have a forcing example of a  $(\mu, \lambda)$  entangled order with minimal  $\mu$  and  $\text{cf} \mu = \lambda$ .

Next we give a weakening of Conclusion 2.9 which is valid also for singular  $\mu$ .

**THEOREM 2.12.** *If  $I$  is  $(\mu, \lambda)$ -entangled then for any  $\theta < \lambda$  and for any matrix  $\langle t_i^\varepsilon \mid \varepsilon < \theta, i < \mu \rangle$  of distinct elements there is a sequence of disjoint intervals  $\langle I_\varepsilon \mid \varepsilon < \theta \rangle$  such that all the  $\alpha$ 's and  $\beta$ 's in the definition of entangledness can be chosen to satisfy  $\forall \varepsilon < \theta (t_\alpha^\varepsilon, t_\beta^\varepsilon \in I_\varepsilon)$ .*

**PROOF.** Suppose the theorem fails for  $I$  with density  $\chi$ . By Conclusion 2.9  $\mu$  is singular. Let  $\langle \mu_i \mid i < \text{cf}\mu \rangle$  be a strictly increasing sequence of successor cardinals and  $\langle t_i^\varepsilon \mid \varepsilon < \theta, i < \mu \rangle$  a counterexample to the theorem. As  $\chi^0 < \mu \text{ wlog } \mu_0 > \chi^0$  and by induction on  $i$  we can choose  $\langle I_\varepsilon^i \mid \varepsilon < \theta, i < \text{cf}\mu \rangle$  such that  $\langle I_\varepsilon^i \mid \varepsilon < \theta \rangle$  are disjoint for all  $i < \text{cf}\mu$ , for  $i < \text{cf}\mu \left\{ v \in \left[ \sum_{\alpha < i} \mu_\alpha, \mu_i \right] \mid \forall \varepsilon < \theta (t_v^\varepsilon \in I_\varepsilon^i) \right\} \mid = \mu_i$  — wlog this set is  $\left[ \sum_{\alpha < i} \mu_\alpha, \mu_i \right)$  — and  $\forall i < \text{cf}\mu \exists j(i) < \text{cf}\mu \forall j > j(i) \exists \varepsilon < \theta \left( I_\varepsilon^i \cap \left\{ t_v^\varepsilon \mid v \in \left[ \sum_{\alpha < j} \mu_\alpha, \mu_j \right] \right\} \right) < \mu_j$  hence wlog  $\forall i, j < \text{cf}\mu (i \neq j \rightarrow \exists \varepsilon < \theta (I_\varepsilon^i \cap I_\varepsilon^j = \emptyset))$ . As  $\langle t_i^\varepsilon \mid i < \mu, \varepsilon < \theta \rangle$  is a counterexample, for any  $i < \text{cf}\mu$  there is  $u_i \subset \theta$  such that  $\forall \alpha, \beta \in \left[ \sum_{\alpha < i} \mu_\alpha, \mu_i \right) \exists \varepsilon < \theta (t_\alpha^\varepsilon < t_\beta^\varepsilon \leftrightarrow \varepsilon \notin u_i)$ . By a previous lemma  $2^{\theta} < \text{cf}\mu$  so wlog the  $u_i$ 's are the same  $u$ . Now  $\langle s_i^\varepsilon \mid i < \mu, \varepsilon < 3\theta \rangle$  defined by  $s_i^\varepsilon = t_{3i}^\varepsilon$ ,  $s_i^{\theta+\varepsilon} = t_{3i+1}^\varepsilon$ ,  $s_i^{2\theta+\varepsilon} = t_{3i+2}^\varepsilon$  ( $i < \mu, \varepsilon < \theta$ ) contradicts the  $(\mu, \lambda)$ -entangledness with respect to  $u' = u \cup [\theta, 2\theta)$ .  $\dashv$

**THEOREM 2.13.** *If  $I$  is  $(\mu, \theta^+)$ -entangled with minimal  $\mu$  then there are two  $\theta^+$ -closed  $\mu$ -c.c. posets whose product is not  $\mu$ -c.c.*

**PROOF.** Let  $\langle x_\alpha \mid \alpha < \mu \rangle$  be distinct elements of  $I$ . Denote by  $\prec$  the partial order on  $E = \{(x_{2\alpha}, x_{2\alpha+1}) \mid \alpha < \mu\}$  which is the product of  $\prec_I$  with itself. Let  $A = \{a \in [E]^{\leq \theta} \mid a \text{ is } \prec\text{-chain}\}$  and  $B = \{a \in [E]^{\leq \theta} \mid \neg \exists x, y \in a (x \prec y)\}$ .  $A$  and  $B$  are  $\theta^+$ -closed when ordered by inclusion and  $A \times B$  is not  $\mu$ -c.c. since  $\{((x_{2\alpha}, x_{2\alpha+1}), (x_{2\alpha}, x_{2\alpha+1})) \mid \alpha < \mu\}$  is an antichain in it. If  $\langle a_\alpha \mid \alpha < \mu \rangle \subseteq [E]^{\leq \theta}$  then look at any matrix  $\langle t_i^\varepsilon \mid i < \mu, \varepsilon < \theta \rangle$  satisfying  $\forall \alpha < \mu \{ (t_{2\varepsilon}^\alpha, t_{2\varepsilon+1}^\alpha) \mid \varepsilon < \theta \} \supset a_\alpha$  and apply Theorem 2.11 with respect to  $u = \emptyset$  to see that it is not an  $A$ -antichain and with respect to  $u = \{2\beta \mid \beta < \theta\}$  to see that it is not a  $B$ -antichain. This proves the theorem.  $\dashv$

By previous theorems the existence of a  $(\lambda^+, \lambda)$  entangled order implies that  $\lambda^{<\lambda} = \lambda$ . Below we give sufficient conditions.

**THEOREM 2.14.** *If  $\lambda^{<\lambda} = \lambda > \beth_w$  and  $2^\lambda = \lambda^+$  then there is a  $(\lambda^+, \lambda)$  entangled order (also strongly as  $\lambda = \lambda^{<\lambda}$ ).*

**PROOF.** Fix an enumeration of all the triples  $(\gamma, \bar{\eta}, \varepsilon)$  where  $\varepsilon, \gamma < \lambda$  and  $\bar{\eta} = \langle \eta^\alpha \mid \alpha < \gamma \rangle \subset {}^\varepsilon \lambda$  is a sequence of different functions,  $\langle (\gamma_\alpha, \bar{\eta}_\alpha, \varepsilon_\alpha) \mid \alpha < \lambda \rangle$  (remember that  $\lambda^{<\lambda} = \lambda$ ). By [Sh460, 3.5]  $\lambda^{<\lambda} = \lambda > \beth_w$  implies that there are  $\lambda$  disjoint stationary subsets of  $\lambda \langle S_\alpha \mid \alpha < \lambda \rangle$  s.t. for each  $\alpha < \lambda$   $D\ell(S_\alpha)$  holds. We remind the reader that  $D\ell(S_\alpha)$  is a weakening of diamond and here we use the following form of it: there is a sequence  $\langle P_\beta \mid \beta \in S_\alpha \rangle$  s.t.  $P_\beta$  is a family of less than  $\lambda$  sequences of length  $\gamma_\alpha$  of functions from  ${}^\alpha \lambda$  and given any sequence of length  $\gamma_\alpha$  of functions from  ${}^\lambda \lambda$ ,  $\langle f_i \mid i < \gamma_\alpha \rangle$ , for stationary many  $\beta \in S_\alpha \langle f_i \upharpoonright \beta \mid i < \gamma_\alpha \rangle \in P_\beta$ . Since  $2^\lambda = \lambda^+$  there is a cofinal and increasing sequence of functions  $\langle f_\alpha \mid \alpha < \lambda^+ \rangle$  in  $({}^\lambda \lambda, <^*)$  where  $<^*$  means eventual

dominance. Now set  $A = \{f \in {}^\alpha\lambda \mid \exists \beta, \delta < \lambda (2^\delta < \gamma_\beta \wedge \alpha \in S_\beta \wedge \eta_\beta^{2^\delta} \triangleleft f)\}$  and define  $I = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$  and  $f <_I g$  iff  $f \upharpoonright \alpha \in A \leftrightarrow f(\alpha) < g(\alpha)$  where  $\alpha = \min\{\beta < \lambda \mid f(\beta) \neq g(\beta)\}$ . To prove that  $I$  is as required let  $\gamma < \lambda$ ,  $u \subset \gamma$  and  $\langle f_{\alpha_v^\beta} \mid \beta < \lambda^+, v < \gamma \rangle$  be as in Definition 1.1. To simplify the notation we write  $f_v^\beta$  for  $f_{\alpha_v^\beta}$ . Wlog  $\langle \alpha_v^\beta \mid \beta < \lambda^+ \rangle$  is increasing for all  $v < \gamma$ , (for this replace the set of indices by an inductively chosen sequence of length  $\lambda^+$ )  $\gamma$  is an infinite cardinal and  $u = \langle 2\alpha \mid \alpha < \gamma \rangle$ . For every  $\beta < \lambda^+$  there is  $\varepsilon(\beta) < \lambda$  s.t.  $\langle f_v^\beta \upharpoonright \varepsilon(\beta) \mid v < \gamma \rangle$  are distinct so that on  $B \in [\lambda^+]^{\lambda^+}$  all  $\varepsilon(\beta)$  are equal to some  $\varepsilon^*$  and all  $\langle f_v^\beta \upharpoonright \varepsilon(\beta) \mid v < \gamma \rangle$  are the same, to be denoted by  $\bar{\eta}^*$ . Let  $\beta$  be s.t.  $(\gamma, \bar{\eta}^*, \varepsilon^*) = (\gamma_\beta, \bar{\eta}_\beta, \varepsilon_\beta)$ . If some  $\varepsilon_0 < \lambda$  and  $\bar{\eta} = \langle \bar{\eta}_\alpha \mid \alpha < \gamma \rangle \subset {}^{\varepsilon_0}\lambda$  satisfy that for all  $i < \lambda^+$  and  $\delta < \lambda$  there is  $\zeta \in B$  s.t.  $\bar{\eta} = \langle f_v^\zeta \upharpoonright \varepsilon_0 \mid r < \gamma \rangle$  and  $\min\{f_v^\zeta(\varepsilon_0) \mid r < \gamma\} > \delta$  then we are clearly done (take such  $\zeta$  with respect to  $(0, 0)$  then such  $\zeta'$  with respect to  $(\zeta, \sup\{f_v^\zeta(\varepsilon, 0) \mid v < \gamma\})$ ). Otherwise for every  $\bar{\eta}$  as above there are witnesses for its failure,  $i(\bar{\eta})$  and  $\delta(\bar{\eta})$ . Since  $\lambda^{<\lambda} = \lambda$  the supremum of  $i(\bar{\eta})$  over all relevant  $\bar{\eta}$  is less than  $\lambda'$ , denote it by  $i^*$ . Define  $\delta: S_\beta \rightarrow \lambda$  by  $\delta(\alpha) = \sup\{\delta(\bar{\eta}) \mid \bar{\eta} \in P_\alpha\} < \lambda$  and using the cofinality of the  $f_\alpha$ 's find  $\zeta \in B \setminus i^*$  for which  $\delta <^* f_\zeta \upharpoonright S_\beta$ . Now using  $D\ell(S_\beta)$  there is  $\alpha \in S_\beta$  s.t.  $\langle f_v^\zeta \upharpoonright \alpha \mid v < \gamma \rangle \in P_\alpha$ , moreover we can get  $\alpha > \sup \min\{\varepsilon \in S_\beta \mid \delta(\varepsilon) > f_v^\zeta(\varepsilon)\}$  so  $\min f_v^\zeta(\alpha) > \delta(\alpha) \geq \delta(\langle f_v^\zeta \upharpoonright \alpha \rangle)$ , a contradiction.  $\dashv$

REMARK 2.15. Notice that for  $\lambda$  as in the theorem the construction in [BoSh210] gives only a  $(\lambda^+, \aleph_0)$  entangled order. However, their proof gives also a  $(\aleph_1, \aleph_0)$  entangled order and that is done assuming only  $\text{cf}2^{\aleph_0} = \aleph_1$ . Remember that under  $MA + 2^{\aleph_0} > \aleph_1$ , there is no such an order at all.

**§3. Results on positively entangled orders.**

THEOREM 3.1. *If  $I$  is a  $(\mu, \lambda)$  [positively\*] [positively] entangled linear order with minimal  $\mu$  then  $\text{cf}\mu \geq \text{cf}\lambda$ .*

PROOF. Suppose not. We deal with positive entangledness (the other cases are similar). Fix  $\langle \mu_\alpha \mid \alpha < \text{cf}\mu \rangle$  increasing to  $\mu$  and  $\langle \lambda_\alpha \mid \alpha < \text{cf}\mu \rangle$  s.t. for every  $\alpha < \text{cf}\mu$   $\lambda_\alpha < \lambda$  and  $I$  is not  $(\mu_\alpha, \lambda_\alpha^+)$  positively entangled and counterexamples  $\langle t_i^\varepsilon \mid i < \mu_\alpha, \varepsilon \in [\sum_{j<i} \lambda_j, \sum_{j \leq i} \lambda_j] \rangle$ , wlog all with respect to  $u = \emptyset$  (here  $\sum$  stands for ordinal summation). In each row  $\varepsilon$  choose fillers  $\langle t_i^\varepsilon \mid \mu_\alpha \leq i < \mu \rangle$  different from  $\langle t_i^\varepsilon \mid i < \mu_\alpha \rangle$ . As  $\sum_{i < \text{cf}\mu} \lambda_i < \lambda$  this contradicts the  $(\mu, \lambda)$ -positively entangledness with respect to  $u = \emptyset$ .  $\dashv$

LEMMA 3.2. *If a  $(\mu, \lambda)$  positively\* entangled linear order  $I$  has a  $\mu$ -separative point then  $\forall \theta < \lambda (2^\theta < \mu)$ .*

PROOF. Let  $x$  be such a point and suppose by contradiction  $\theta < \lambda$ ,  $2^\theta \geq \mu$ . Fix distinct functions  $\langle f_\alpha \mid \alpha < \mu \rangle \subset {}^\theta 2$ . Define  $\langle t_i^\varepsilon \mid \varepsilon < \theta, i < \mu \rangle$  inductively on  $i$ : choose any  $x_0 < x < x_1$  different from previously chosen  $t$ 's and put  $t_i^{2^{\varepsilon+\ell}} = x_\ell$  for  $\ell \in \{0, 1\}$  if  $f_i(\varepsilon) = 0$  and  $t_i^{2^{\varepsilon+\ell}} = x_{1-\ell}$  else. This contradicts the  $(\mu, \lambda)$ -positively\* entangledness.  $\dashv$

COROLLARY 3.3. (a) *If  $I$  is  $(\mu, \lambda)$  positively\* entangled then  $\chi = \text{den}I \geq \lambda$ .* (b) *If  $I$  is  $(\mu, \lambda)$  positively\* entangled then it is not  $(\lambda, 2)$  entangled.*

PROOF. Assume  $I$  is a counterexample for (a).  $\text{Wlog } \mu$  is the minimal cardinal s.t.  $I$  is  $(\mu, \chi^+)$  positively\* entangled. If there is no  $\mu$ -separating point in  $I$  we can define inductively a monotone sequence in  $I$  of length  $\text{cf}\mu$  which is greater than  $\chi$  by Theorem 3.1, a contradiction. If there is a  $\mu$ -separating point then by Lemma 2.3  $2^\lambda < \mu$ , a contradiction. (b) follows from (a) and Lemma 2.1 [Sh462, 1.2(4)].  $\dashv$

THEOREM 3.4. *If  $I$  is  $(\mu, \lambda)$  positively entangled then  $\forall \theta < \lambda (2^\theta < \mu)$ .*

PROOF. Suppose this fails for some  $\theta$ .  $\text{Wlog } \lambda = \theta^+$ . In view of Lemma 3.2 we can assume that  $I$  has no  $\mu$ -separating point. It follows that  $\text{cf}\mu < \mu$ . For any  $\mu_1 < \mu$  there is a  $\mu_1$ -separating point, otherwise  $\text{wlog } \forall x \in I |\{y \in I \mid y < x\}| < \mu_1$ , so we can define an increasing sequence of length  $\mu_1 + 1$  and pick the last element of it. By Lemma 3.2  $I$  is not  $(\mu_1, \lambda)$  positively\* entangled for any  $\mu_1 < \mu$ . But now if  $\langle \mu_\alpha \mid \alpha < \text{cf}\mu \rangle$  are increasing to  $\mu$ ,  $\langle \langle t_i^\varepsilon \mid \varepsilon < \theta, i \in [\mu_\alpha, \mu_{\alpha+1}] \rangle \mid \alpha < \text{cf}\mu \rangle$  are counterexamples for  $(\mu_\alpha, \lambda)$  positively\* entangledness and  $\langle I_\alpha \mid \alpha < \text{cf}\mu \rangle$  is an inductively chosen monotone sequence of intervals s.t.  $|I_\alpha| \geq \mu_\alpha$  (here we use the nonexistence of a  $\mu$ -separating point) then pick for every  $\alpha < \text{cf}\mu$  different  $\langle t_i^\theta \mid i \in [\mu_\alpha, \mu_{\alpha+1}] \rangle$  from  $I_\alpha$  to contradict the  $(\mu, \lambda)$  positively entangledness with  $\langle t_i^\varepsilon \mid \varepsilon \leq \theta, i < \mu \rangle$ .  $\dashv$

THEOREM 3.5. *If  $I$  is  $(\mu, \lambda)$  positively entangled with minimal  $\mu$  which has a  $\mu$ -separative point and  $\lambda = \theta^+$  then  $2^\theta < \text{cf}\mu$ . In particular  $\lambda \leq \text{cf}\mu$ .*

PROOF. Let  $x \in I$  be  $\mu$ -separating and assume that  $2^\theta \geq \text{cf}\mu$ . Fix distinct  $\langle f_\alpha \mid \alpha < \text{cf}\mu \rangle \subset {}^\theta 2$  and choose  $\langle t_i^\varepsilon \mid \varepsilon < \theta, i \in [\mu_\alpha, \mu_{\alpha+1}] \rangle$  counterexamples for  $(\mu_\alpha, \lambda)$  positively entangledness, wlog all with respect to  $u = \emptyset$ . For every  $\varepsilon < \theta$  choose by induction on  $\alpha x_0 < x < x_1$  different from previously chosen elements and put  $t_\alpha^{\theta+\varepsilon} = x_\ell$  for  $\ell \in \{0, 1\}$  if  $f_\beta(\varepsilon) = 0$  and  $t_\alpha^{\theta+\varepsilon} = x_{1-\ell}$  else (here  $\beta$  is s.t.  $\alpha \in [\mu_\beta, \mu_{\beta+1}]$ ).  $\langle t_\alpha^\varepsilon \mid \alpha < \mu \varepsilon < \theta + \theta \rangle$  contradicts the  $(\mu, \lambda)$  positively entangledness.  $\dashv$

Notice that below one cannot wave off the assumption  $\text{cf}\mu \neq \text{cf}\lambda$  (see Remark 2.11).

COROLLARY 3.6. *If  $I$  is  $(\mu, \lambda)$  positively entangled with minimal  $\mu$  which has a  $\mu$ -separative point and  $\text{cf}\mu \neq \text{cf}\lambda$  then  $\forall \theta < \lambda (2^\theta < \text{cf}\mu)$  and  $\lambda < \text{cf}\mu$ .*

PROOF. As  $\text{cf}\mu \neq \text{cf}\lambda$  there is  $\theta_1 < \lambda$  such that  $I$  is  $(\mu, \theta^+)$  entangled with minimal  $\mu$  for every  $\theta_1 \leq \theta < \lambda$  so we can use Theorem 3.5. Note that the possibility  $\lambda = \text{cf}\mu$  is excluded by the assumption.  $\dashv$

DEFINITION 3.7. A linear order  $I$  is called hereditarily separative if every  $A \in [I]^{|I|}$  has a  $|I|$ -separative point. The assumption below ( $\lambda$  is singular strong limit  $\Rightarrow pp\lambda =^+ 2^\lambda$ ) is not known to be independent of ZFC. See [Sh-g].

THEOREM 3.8. *If  $I$  is hereditarily separative  $(\mu, \lambda)$ -positively entangled with minimal  $\mu$ ,  $\text{cf}\mu \neq \text{cf}\lambda$  and ( $\lambda$  is singular strong limit  $\Rightarrow pp\lambda =^+ 2^\lambda$ ) then  $\lambda^{<\lambda} < \text{cf}\mu$ .*

PROOF. If  $\lambda$  is not strong limit then for some  $\theta_1 < \lambda$  we have  $\lambda \leq 2^{\theta_1}$  and by Theorem 3.5  $\forall \theta < \lambda (\lambda^\theta \leq 2^{\theta+\theta_1} < \text{cf}\mu)$ . If  $\lambda$  is inaccessible  $\lambda^{<\lambda} = \lambda$  so we can apply Corollary 3.6. We are left with the case  $\lambda$  is strong limit singular,  $pp\lambda =^+ 2^\lambda$ . Fix  $\theta < \lambda$ . By the trivial direction of [Sh410, 3.7] there are functions  $\langle f_\alpha \mid \alpha < \lambda^\theta \rangle \subset {}^\theta \lambda$  s.t.  $\forall \alpha < \beta < \lambda^\theta \exists \varepsilon < \theta (f_\alpha < f_\beta(\varepsilon))$ . Assume that  $\lambda^\theta \geq \text{cf}\mu$ . If  $A$  is an equivalence class of the equivalence relation  $xEy \leftrightarrow |(x, y)_I| < \mu$  and



is of cardinality  $\mu$  then pick any  $x \in A$ .  $\text{Wlog } |\{y \in A : y > x\}| = \mu$ . Since  $I$  is hereditarily separative  $\{y \in A \mid y > x\}$  has  $\mu$ -separative point, call it  $z$ . In particular  $|(x, z)_I| = \mu$  so  $x \not\perp z$ , a contradiction. We conclude that any equivalence class of  $E$  is of size less than  $\mu$  which implies that there are at least  $\text{cf}\mu$  many such classes. By Corollary 3.6  $\lambda < \text{cf}\mu$  and as  $\lambda$  is strong limit  $(2^\theta)^+ < \lambda$ . Choosing any  $(2^\theta)^+$  distinct equivalence classes of  $E$  they inherit the order  $I$  since they are convex subsets of it so by the Erdős-Rado theorem  $\theta$  many from them form a monotone sequence, call it  $\langle J_\alpha \mid \alpha < \theta \rangle$ . Replacing it by  $\langle J'_\alpha \mid \alpha < \theta \rangle$  where  $J'_\alpha = \text{convex}(J_{2\alpha} \cup J_{2\alpha+1})$  we ensure also  $\forall \alpha (|J'_\alpha| = \mu)$ , (this is as  $J'_\alpha$  contains an interval between two nonequivalent points). Of course, this can be done for any  $\tau < \lambda$  instead of  $\theta$ . Starting from any such, wlog, increasing sequence  $\langle J_\alpha \mid \alpha < \text{cf}\lambda \rangle$  (remember that  $\text{cf}\lambda < \lambda$ ) we fix a strictly monotone sequence of cardinals converging to  $\lambda$ ,  $\langle \lambda_\alpha \mid \alpha < \text{cf}\lambda \rangle$ . Any  $J_\alpha$  is also hereditarily separative so it contains by the same argument monotone sequence of length  $\lambda_\alpha$  of intervals of power  $\mu \langle J'_\beta \mid \beta < \lambda_\alpha \rangle$ . If in one  $J_\alpha$  there is no increasing sequence of length  $\lambda_\alpha$  then starting from decreasing intervals  $\langle J_{\alpha'} \mid \alpha < \text{cf}\mu \rangle$  inside this  $J_\alpha$  we can take all the sequences decreasing. Otherwise we take them all increasing. Concatenating them yields a monotone sequence of intervals  $\langle I_\alpha \mid \alpha < \lambda \rangle$ ,  $\forall \alpha (|I_\alpha| = \mu)$ . Now choose  $\langle \mu_\alpha \mid \alpha < \text{cf}\mu \rangle$   $\langle t_\varepsilon^\alpha \mid \alpha < \mu, \varepsilon < \theta \rangle$  as in the proof of Theorem 3.5. For all  $\varepsilon < \theta$  choose by induction on  $\alpha$   $t_{\theta+\varepsilon}^\alpha \in I_{f_\beta(\varepsilon)} \setminus \{t_{\theta+\varepsilon}^\gamma \mid \gamma < \alpha\}$  where  $\alpha \in [\mu_\beta, \mu_{\beta+1})$ . This is always possible because  $\forall \alpha (|I_\alpha| = \mu)$ . Now check that  $\langle t_\varepsilon^\alpha \mid \alpha < \mu, \varepsilon < \theta + \theta \rangle$  contradicts the  $(\mu, \lambda)$ -positively entangledness. We conclude that  $\forall \theta < \lambda (\lambda^\theta < \text{cf}\mu)$ . As  $\lambda < \text{cf}\mu$  this gives the desired inequality.  $\dashv$

Compare the following with Theorem 2.10(b).

**COROLLARY 3.9.** *If  $I$  is  $(\mu, \lambda)$  positively entangled hereditarily separative linear order with minimal  $\mu$  and with density  $\chi$ ,  $\text{cf}\mu \neq \text{cf}\lambda < \lambda$  and  $(\lambda$  is strong limit singular  $\rightarrow \text{pp}\lambda =^+ 2^\lambda)$  then  $\chi > \lambda$ .*

**PROOF.** Assume that  $I$  is a counterexample and deduce by Corollary 3.3(a) that  $\chi = \lambda$ . Fix  $A \in [I]^\lambda$  dense in  $I$ . For every  $x \in I$  find a well ordered sequence of elements from  $A$  converging to  $x$  of minimal length  $\langle \alpha_\alpha^x \mid \alpha < r(x) \rangle$ . By minimality  $r(x)$  is always a regular cardinal hence smaller than  $\lambda$ . By Theorem 3.8  $\lambda^{<\lambda} < \text{cf}\mu$  so there are two distinct points in  $I$  with the same sequences, a clear contradiction.  $\dashv$

Below we deal with a typical example of orders  $I$  that (usually) have no  $|I|$ -separative points.

**DEFINITION 3.10.** If  $\mu$  is a singular cardinal then a linear order  $I$  is called “of type  $s_\mu$ ” if it contains for some (equivalently any) sequence of cardinals converging to  $\mu$   $\langle \mu_\alpha \mid \alpha < \text{cf}\mu \rangle$  an isomorphic copy of  $\bigcup_{\alpha < \text{cf}\mu} \{\mu_\alpha\} \times \mu_\alpha$  ordered by  $(\alpha, \beta) < (\alpha_1, \beta_1)$  iff  $\alpha < \alpha_1$  or  $\alpha = \alpha_1$  and  $\beta > \beta_1$ . We say that “ $s_\mu$  is  $(\mu, \lambda)$  positively entangled” if some (equivalently any) order of type  $s_\mu$  has this property.

**THEOREM 3.11.**  $s_\mu$  is  $(\mu, \theta^+)$ -positively entangled iff  $\theta < \text{cf}\mu$  and  $(\text{cf}\mu)^\theta < \mu$ .

**PROOF.** Throughout the proof fix a sequence of successor cardinals  $\langle \mu_\alpha \mid \alpha < \text{cf}\mu \rangle$  strictly increasing to  $\mu$ . First assume  $(\text{cf}\mu)^\theta < \mu$  and  $\theta < \text{cf}\mu$ . Given any  $\langle t_\varepsilon^\alpha \mid \alpha > \mu, \varepsilon < \theta \rangle$  as in Definition 1.1(c) then, as  $(\text{cf}\mu)^\theta < \mu$ , there is  $A < \mu$  of cardinality  $(2^\theta)^+$  for which if  $\alpha, \beta \in A$  and  $\varepsilon < \theta$  then  $t_\varepsilon^\alpha$  and  $t_\varepsilon^\beta$  have the

same first coordinate. Now we can find  $\alpha, \beta \in A$  satisfying  $\forall \varepsilon < \theta (t_\varepsilon^\alpha > t_\varepsilon^\beta)$  and  $\alpha < \beta$ . Otherwise color  $[A]^2$  with  $f(\{\alpha, \beta\}) = \min\{\varepsilon < \theta \mid t_\varepsilon^\alpha < t_\varepsilon^\beta\}$  (here  $\alpha < \beta$ ) and using Erdős-Rado get a homogeneous set of size  $\theta$  giving rise to a decreasing sequence of ordinals of this length, a contradiction. To get the other condition observe that  $\bigcup_{\varepsilon < \theta} \{\alpha < \mu \mid t_\varepsilon^\theta > t_\varepsilon^\alpha\}$  is of cardinality less than  $\mu$  as it is a union of size less than  $\text{cf}\mu$  of initial segments of  $s_\mu$ , which is of order type  $\mu$ . For any  $\alpha$  in its complements we have  $\forall \varepsilon < \theta (t_\varepsilon^\theta < t_\varepsilon^\alpha)$ . We conclude that  $s_\mu$  is  $(\mu, \theta^+)$ -positively entangled.

Suppose  $(\text{cf}\mu)^\theta \geq \mu$ , hence there are distinct  $\langle f_\alpha \mid \alpha < \mu \rangle \subset {}^\theta(\text{cf}\mu)$ . Wlog  $\forall \alpha < \mu_\alpha (\min f_\alpha > \alpha)$ . For  $\varepsilon < \theta$   $\beta = \mu_\alpha + \gamma < \mu_{\alpha+1}$  define  $t_\varepsilon^\beta = (f_\beta(\varepsilon), \gamma) \in s_\mu$ . Now fix any  $\alpha < \text{cf}\mu$  and choose a partition of  $\mu_{\alpha+2}$  to  $\mu_{\alpha+1}$  unbounded sets  $\langle A_\delta \mid \delta < \mu_{\alpha+1} \rangle$ . For any  $\varepsilon < \theta$  look at the relation on  $\mu_{\alpha+1} \setminus \mu_\alpha$  defined by  $\beta <_\varepsilon \gamma \leftrightarrow f_\beta(\varepsilon) < f_\gamma(\varepsilon)$ .  $\prec_\varepsilon$  is a partial order with no infinite decreasing sequences so we can define a rank function  $g_\varepsilon$  into  $\mu_{\alpha+2}$  satisfying  $\beta \prec_\varepsilon \gamma \rightarrow g_\varepsilon(\beta) < g_\varepsilon(\gamma)$  by  $\prec_\varepsilon$ -recursion:  $g_\varepsilon(\beta) = \min A_\beta \setminus \sup\{g_\varepsilon(\gamma) \mid \gamma <_\varepsilon \beta\}$ . For  $\beta \in \mu_{\alpha+1} \setminus \mu_\alpha$  set  $t_{\theta+\varepsilon}^\beta = (\alpha + 2, g_\varepsilon(\beta))$ . By the construction the  $t$ 's are different in each  $\mu$ -row. If  $\beta < \gamma < \mu$  then either  $\exists \alpha < \text{cf}\mu (\mu_\alpha \leq \beta < \gamma < \mu_{\alpha+1})$  in this case since the  $f_\alpha$ 's are distinct there is  $\varepsilon < \theta$  for which  $f_\beta(\varepsilon) \neq f_\gamma(\varepsilon)$ ; or  $f_\beta(\varepsilon) < f_\gamma(\varepsilon)$  so  $t_\varepsilon^\beta < t_\varepsilon^\gamma$  or  $f_\beta(\varepsilon) > f_\gamma(\varepsilon)$  which implies  $\beta >_\varepsilon \gamma$ ,  $g_\varepsilon(\beta) > g_\varepsilon(\gamma)$  and  $t_{\theta+\varepsilon}^\beta < t_{\theta+\varepsilon}^\gamma$ . We summarize that  $\forall \beta < \gamma < \mu \exists \varepsilon < \theta + \theta (t_\varepsilon^\beta < t_\varepsilon^\gamma)$  which means that  $s_\mu$  is not  $(\mu, \theta^+)$ -positively entangled.

Finally we show that  $s_\mu$  cannot be  $(\mu, (\text{cf}\mu)^+)$ -positively entangled. For this partition  $\text{cf}\mu$  into  $\text{cf}\mu$  mutually disjoint stationary sets  $\langle A_\alpha \mid \alpha < \text{cf}\mu \rangle$  and enumerate their elements  $A_\alpha = \langle a_i^\alpha \mid i < \text{cf}\mu \rangle$ . Wlog  $\forall \alpha (a_0^\alpha > \alpha)$ . For any  $\varepsilon < \text{cf}\mu$   $\beta = \mu_\alpha + \gamma < \mu_{\alpha+1}$  set  $t_\varepsilon^\beta = (a_\varepsilon^\alpha, \gamma) \in s_\mu$ . These  $t$ 's are different in each  $\mu$ -row. Now if for some  $\beta < \gamma < \mu \forall \varepsilon < \text{cf}\mu (t_\varepsilon^\beta < t_\varepsilon^\gamma)$  holds then necessarily there are distinct  $\alpha, \bar{\alpha} < \text{cf}\mu$  s.t.  $\beta \in [\mu_\alpha, \mu_{\alpha+1})$ ,  $\gamma \in [\mu_{\bar{\alpha}}, \mu_{\bar{\alpha}+1})$ . The function  $f = \{(a_\varepsilon^{\bar{\alpha}}, a_\varepsilon^\alpha) \mid \varepsilon < \text{cf}\mu\}$  is a one to one regressive function with domain  $A_{\bar{\alpha}}$  which is stationary — a contradiction.  $\dashv$

By the above theorem one can see that Theorem 3.5 does not hold generally. (For any  $\theta$  take  $\mu = (2^\theta)^{+\theta^+}$ . Now  $s_\mu$  is  $(\mu, \theta^+)$  positively entangled but  $\text{cf}\mu = \theta^+ \leq 2^\theta$ ).

**THEOREM 3.12.** *There is a c.c.c. forcing adding a  $(\aleph_\omega, \aleph_0)$  positively entangled linear order of density  $\aleph_0$  (in particular not of type  $s_{\aleph_\omega}$ ) which has no  $\aleph_\omega$ -separative point.*

**PROOF.** Fix any  $n < \omega$  and define  $\mathbb{P} = \{f \text{ is a function, } \text{dom} f \in [n \times \aleph_\omega]^{<\omega}, \text{ran} f \subset 2^{<\omega}, \text{ if } \aleph_m \leq \alpha < \beta < \aleph_{m+1} \text{ are in } \text{dom} f \text{ then } \exists i < n ((i, \alpha), (i, \beta)) \in \text{dom} f \wedge f(i, \alpha) <_{\ell_x} f(i, \beta))\}$ . The order is  $f \leq g$  iff  $\text{dom} f \supseteq \text{dom} g$  and  $\forall x \in \text{dom} g (g(x) \triangleleft f(x))$ . If  $G$  is  $\mathbb{P}$  generic we define  $I = \bigcup_{m < \omega} m + \{x \in 2^\omega \mid \forall i < \omega \exists f \in G \exists y \in n \times [\aleph_m, \aleph_{m+1}) (f(y) = x \upharpoonright i)\}$  after identifying  $2^\omega$  with Cantor set. The rest is almost identical to the proof of Theorem 1.1.  $\dashv$

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