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ON UNIQUENESS OF PRIME MODELS

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Abstract. We prove there are theories (stable or countable) for which over every A there is a prime model but it is not necessarily unique. We also give a simplified proof of the uniqueness theorem for countable stable theories.

Let us call a model M of a (complete, first-order) theory T prime if it can be elementarily embedded into any other model of T. Vaught [V] (see e.g. [CK]) proved the uniqueness and gives necessary and sufficient conditions for the existence of a prime model for *countable* complete T. Morley generalized the notion to "prime over A" where A is a subset of some model of T (here the embeddings are the identity on A) and proved their existence for $\aleph_0$-stable T. Sacks inquired into their uniqueness. More exactly he conjectured (i) they are unique if T is $\aleph_0$-stable and even (ii) they are unique if for every A, the isolated types in $S(A)$ are dense. These questions appear in a preliminary version of the list of questions of [CK]. Shelah [Sh 3] proved their uniqueness for $\aleph_0$-stable T. In [Sh 1, IV, 5.6] it was proved that if T is countable and stable, and over every A there is a prime model, then it is unique (more details in [Sh 1]). We show here that we cannot omit the countability nor the stability assumption in this theorem, and give a simpler proof of the uniqueness, and finish by remarking a further result. In fact we prove more general results.

As we do not want to require from the reader knowledge of [Sh 1], (mainly Chapter IV) we list below all relevant information. For understanding the negative result (i.e. the counterexamples), the reader should read definitions B, C, E and the easy claims D, F (whose proof can be easily reconstructed). For understanding the uniqueness result the reader should believe G, H, I are true. For credits and proofs see [Sh 1] or [Sh 2].

REMARK. The most interesting case below is $\boldsymbol{P}^t_{\aleph_0}$ when we get the usual notions.

A. *Notation.* For notational simplicity, for a fixed T, let $\mathfrak{C} = \mathfrak{C}_T$ be a fixed $\bar{\kappa}$-saturated model of T, and M, N be elementary submodels of $\mathfrak{C}$ of cardinality $< \bar{\kappa}$, A, B, C subsets of $\mathfrak{C}$ of cardinality $< \bar{\kappa}$, a, b, c elements of $\mathfrak{C}$, and $\bar{a}$, $\bar{b}$, $\bar{c}$ finite sequences of them. We write $a \in M$ instead of $a \in |M|$ and $\bar{a} \in A$ instead of Range $(\bar{a}) \subseteq A$. Let tp$(\bar{a}, A)$ be the type $\bar{a}$ realizes over A.

B. DEFINITION. (1) $S^m(A)$ is the set of complete m-types over A realized in $\mathfrak{C}$. Let $S(A) = S^1(A)$.

(2) $p \in S^m(A)$ is $\boldsymbol{P}^t_\lambda$-isolated if there is $q \subseteq p$, $|q| < \lambda$, such that $q \vdash p$ (i.e. every sequence realizing q realizes p). If q is over $B \subseteq A$, $|B| < \lambda$, we say p is $\boldsymbol{P}^t_\lambda$-isolated over B.

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(3) $p \in S^m(A)$ is $\boldsymbol{P}^s_\lambda$-isolated if there is $B \subseteq A$, $|B| < \lambda$, and $p \upharpoonright B \vdash q$ (where $p \upharpoonright B = \{\varphi(\bar{x}, \bar{a}) \in p : \bar{a} \in B\}$). We say p is $\boldsymbol{P}^s_\lambda$-isolated over B.

C. Definition. (1) $\langle a_i : i < \alpha \rangle$ is a $\boldsymbol{P}^x_\lambda$-construction over A, if for each i, tp(b_i, $A \cup \{b_j : j < i\}$) is $\boldsymbol{P}^x_\lambda$-isolated. The construction is exemplified by $\langle B_i : i < \alpha \rangle$ if tp(b_i, $A \cup \{b_j : j < i\}$) is $\boldsymbol{P}^x_\lambda$-isolated over B_i.

(2) B is $\boldsymbol{P}^x_\lambda$-constructible over A if $B = A \cup \{b_i : i < \alpha\}$ for some b_i, α where $\langle b_i : i < \alpha \rangle$ is a $\boldsymbol{P}^x_\lambda$-construction over A.

(3) M is $\boldsymbol{P}^x_\lambda$-saturated if for every $p \in S(A)$, $A \subseteq M$, p $\boldsymbol{P}^x_\lambda$-isolated, p is realized in M. (So $\boldsymbol{P}^s_\lambda$-saturation is λ-saturation; and $\boldsymbol{P}^t_\lambda$-saturation is λ-compactness provided that T has the $\boldsymbol{P}^t_\lambda$-extension property (see E), which is the interesting case.)

(4) M is $\boldsymbol{P}^x_\lambda$-prime over A if $A \subseteq M$, M is $\boldsymbol{P}^x_\lambda$-saturated and for every $\boldsymbol{P}^x_\lambda$-saturated N, $A \subseteq N$, there is an elementary embedding of M into N over A.

(5) M is $\boldsymbol{P}^x_\lambda$-primary over A if $|M|$ is $\boldsymbol{P}^x_\lambda$-constructible over A and is $\boldsymbol{P}^x_\lambda$-saturated.

D. *Claim.* (1) Every $\boldsymbol{P}^x_\lambda$-primary model over A is $\boldsymbol{P}^x_\lambda$-prime over A.

(2) Over every A there is a $\boldsymbol{P}^x_\lambda$-primary (hence prime) model *if* T satisfies the $\boldsymbol{P}^x_\lambda$-extension property, defined below.

(3) If A is $\boldsymbol{P}^x_\lambda$-constructible over B, $\bar{a} \in A$ then tp($\bar{a}$, B) is $\boldsymbol{P}^x_\lambda$-isolated, provided that λ is regular (or cf $\lambda \geq \kappa(T)$, see H).

(4) If $\langle b_j : j < \alpha \rangle$, is a $\boldsymbol{P}^x_\lambda$-construction over A, exemplified by $\langle B_i : i < \alpha \rangle$, we can change the order of the b's (and it will still be a $\boldsymbol{P}^x_\lambda$-construction) if $b_j \in B_i$ implies b_j precedes b_i in the new order.

(5) If C is $\boldsymbol{P}^x_\lambda$-constructible over A and $A \subseteq B \subseteq C$, with $|B - A| <$ cf λ *then* tp($\bar{c}$, B) is $\boldsymbol{P}^x_\lambda$-isolated for any $\bar{c} \in C$, provided that λ is regular or cf $\lambda \geq \kappa(T)$.

E. Definition. (1) T satisfies the $\boldsymbol{P}^t_\lambda$-extension property if every 1-type p over a set A, $|p| < \lambda$, has a $\boldsymbol{P}^t_\lambda$-isolated extension in $S(A)$.

(2) T satisfies the $\boldsymbol{P}^s_\lambda$-extension property if every 1-type p over a set B, $B \subseteq A$, $|B| < \lambda$ has a $\boldsymbol{P}^s_\lambda$-isolated extension $S(A)$.

F. *Claim.* If $\langle b_i : i < \alpha \rangle$ is a $\boldsymbol{P}^x_\lambda$-construction over A, $C \subseteq A \cup \{b_i : i < \alpha\}$, $|C| \leq \kappa$, $\lambda \leq \kappa$ then by reordering the b_i's we can assume $C \subseteq A \cup \{b_i : i < \kappa\}$ (see D(4)).

G. Theorem. *Every two $\boldsymbol{P}^x_\lambda$-primary models over A are isomorphic over A provided that λ is regular* (*or* cf $\lambda \geq \kappa(T)$, *see* H) (Ressayre, see [Sh 1, IV, 3.9]).

H. Theorem. *There is a regular cardinal $\kappa = \kappa_r(T)$ such that for $\lambda \geq 2^{|T|}$, $\lambda = \lambda^{<\kappa}$ $(= \sum_{\mu<\kappa} \lambda^\mu)$ iff T is stable in λ (i.e. $|A| \leq \lambda \Rightarrow |S(A)| \leq \lambda$). T is called stable if $\kappa < \infty$ and then $\kappa \leq |T|^+$.*

I. Theorem. *For stable T we can define a notion "p forks over A" (for a type p) such that*

(1) *for every $p \in S(A)$ there is $B \subseteq A$, $|B| < \kappa$, over which p does not fork.*

(2) *If $p \subseteq q$, $B \subseteq C$, q does not fork over B, then p does not fork over C.*

(3) *If p forks over A, then some finite subset of p does.*

(4) *If $p \in S^m(C)$, $A \subseteq B \subseteq C$, p does not fork over B and $p \upharpoonright B$ does not fork over A then p does not fork over A.*

(5) *If $B \subseteq C$,* tp($\bar{a}$, $C \cup \bar{b}$) *does not fork over B iff* tp($\bar{b}$, $C \cup \bar{a}$) *does not fork over B.*

(6) *If* tp($\bar{a}$, A) *does not fork over $B \subseteq A$, and* tp($\bar{b}$, $A \cup \bar{a}$) *does not fork over B, then* tp($\bar{a} \wedge \bar{b}$, A) *does not fork over B.*

(7) *If* tp($\bar{a}$, A) *is* $\boldsymbol{P}^t_\lambda$*-isolated and does not fork over* $B \subseteq A$, *then* tp($\bar{a}$, B) *is* $\boldsymbol{P}^t_\lambda$*-isolated.*

1. DEFINITION. For any linearly ordered I, let L^I be the (first-order) language with a two-place relation E_s for each $s \in I$ (we shall write xE_sy instead of $E_s(x, y)$). Let T^I be the theory in L^I: with the sentences saying for $s, t \in I$

(i) each E_s is an equivalence relation,

(ii) each E_s-equivalence class is infinite, and E_s has infinitely many equivalence classes,

(iii) for $s < t$, E_t refines E_s; moreover, each E_s-equivalence class is divided by E_t to infinitely many equivalence classes.

2. *Claim.* T^I is a complete theory with elimination of quantifiers, it is stable and $\kappa_r(T^I)$ is the first regular κ such that in I there is no increasing sequence of length κ.

PROOF. Easy (for $\kappa_r(T^I)$ use H). (This is [Sh 1, Ex. II, 2.3] not proved there.)

From here until conclusion 7 we deal with theories T^I. We first show that (under certain assumptions on I) T has $\boldsymbol{P}^s_\lambda$ ($\boldsymbol{P}^t_\lambda$) prime models over every A.

3. LEMMA. (1) *Let* $B \subseteq A$, $|B| < \mu$, *p a* 1*-type over* B. *Then p has a* $\boldsymbol{P}^s_\mu$*-isolated extension in* $S(A)$. *In more detail, for each* q, $p \subseteq q \in S(B)$, *either q is realized in A or q has a unique extension in* $S(A)$.

(2) *T satisfies the extension property for* $\boldsymbol{P}^s_\mu$, *hence over every A a* $\boldsymbol{P}^s_\mu$*-prime model exists* (*see* D(1), D(2), E(2)).

PROOF. (1) Clearly it suffices to prove the second phrase. Suppose $q \in S(B)$ has two extensions in $S(A)$. We will show it is realized in A. Let $J = \{s \in I$: for some $b \in B$, $(xE_sb) \in q\}$, so clearly J is an initial segment of I. By an assumption above, for some formula φ with parameters from A, $q \cup \{\varphi^{\boldsymbol{t}}\}$ is consistent (for $\boldsymbol{t} = 0, 1$ where $\varphi^0 = \varphi, \varphi^1 = \neg\varphi$). As by 2, T^I has elimination of quantifiers, we can assume φ is quantifier-free, hence we can assume φ is atomic. If $\varphi = (x = a)$ then a realizes q, as desired, so we can assume for some $s \in I$, $a \in A$, $\varphi = xE_sa$. Clearly $s \notin J$ (otherwise for some b, $xE_sb \in q$, and then if $\models bE_sa$ then $q \vdash xE_sa$, and if $\models bE_sa$ then $q \vdash \neg xE_sa$, both cases contradicting the choice of φ). We now prove that a realizes q; as T^I has elimination of quantifiers, it suffices to prove that $xE_tb \in q \Leftrightarrow \models aE_tb$ for every $b \in B$, $t \in I$ (clearly for equality this holds).

For $t \leq s$ this follows by the consistency of $q \cup \{xE_sa\}$. (If c realizes this type then $xE_tb \in q \Leftrightarrow \models cE_tb \Leftrightarrow \models aE_tb$—the last arrow as $\models cE_sa$.) So let $t >$ s, hence $t \notin J$, hence for each $b \in B$, $xE_tb \notin q$ (by the definition of J, as $s \notin J$ and J is an initial segment of I), so we have to prove $\models \neg aE_tb$. For this, it siffices to prove $\models \neg\, aE_sb$ (as $s \leq t$), but otherwise $q \cup \{xE_sa\}$ would not be consistent.

(2) Immediate.

4. DEFINITION. We say the linearly ordered set I is μ-proper if whenever $J \subseteq I$, $|J| < \mu$ then for some $J_0 \subseteq I$, $J < J_0$, $|J_0| < \mu$ and for no $t \in I$, $J < t < J_0$ (for $\mu = \aleph_0$ this just says each element of I has an immediate successor).

5. LEMMA. (1) *Suppose I is* $\boldsymbol{\mu}$*-proper. Any* 1*-type p over* A, $|p| < \mu$ *has a* $\boldsymbol{P}^t_\mu$*-isolated extension in* $S(A)$, *i.e. there is a* 1*-type q over* A, $p \subseteq q$, $|q| < \mu$, *and q has a unique extension in* $S(A)$.

(2) T^I *satisfies the extension property for* $\boldsymbol{P}^t_\mu$, *hence over every A a* $\boldsymbol{P}^t_\mu$*-primary model exists* (*see* D(1), D(2), E(1)).

PROOF. (1) As T^I has elimination of quantifiers, we can assume p consists of quantifier-free formulas, indeed atomic and negation of atomic formulas. Let $B = \{b \in A;\, b$ appears in $p\}$ and $J_0 = \{s \in I$: for some b, $xE_s b \in p\}$, so $|B|$, $|J_0| < \mu$; hence by the μ-properness of I, there is $J_1 \subseteq I, |J_1| < \mu$, $J_0 < J_1$ but for no $t \in I$, $J_0 < t < J_1$. By inessential changes we can assume $s \in J_0 \cup J_1$ whenever E_s appears in p, and $p \subseteq q$ where q has the form

$$\{xE_s b_s\colon s \in J_0\} \cup \{\neg xE_s b\colon s \in J_1,\ b \in B\} \cup \{x \neq b\colon b \in B\}.$$

We can also assume no element of A realizes q. Now we prove q has a unique extension in $S(A)$. By the assumption above for each $a \in A$, $q \vdash x \neq a$. Now consider an atomic formula $\varphi = xE_t a$; if for some $s \in J_0$, $t \leq s$, then $q \vdash \varphi$ when $\models aE_t b_s$ and $q \vdash \neg\varphi$ when $\models \neg\, aE_t b_s$. So assume there is no such s, hence by J_1's choice for some $s \in J_1$, $s \leq t$. If for some $s(0) \in J_0$, $\models \neg aE_{s(0)} b_{s(0)}$, then $p \vdash \neg\varphi$, so assume there is no such $s(0)$. As $a \in A$ does not realize q, for some $b \in B$, and $s(1) \in J_1$, $\models aE_{s(1)} b$, but $\neg xE_{s(1)} b \in q$ hence $q \vdash \neg\varphi$.

So for any atomic formula φ with parameters from A, $q \vdash \varphi$ or $q \vdash \neg\varphi$. As T^I has elimination of quantifiers, this proves q has a unique extension in $S(A)$.

(2) Immediate.

6. LEMMA. *Suppose $x = t$, I is μ-proper or $x = s$. Suppose also $s(i)$ $(i < \lambda)$ is a strictly increasing sequence in I, λ regular, $\lambda > \mu = \mathrm{cf}\,\mu$ or $\lambda > \mu^+$. Then the $\boldsymbol{P}_\mu^x$-prime model (over $\emptyset$) is not unique.*

PROOF. By D(1), D(2) and 5.2 there is a $\boldsymbol{P}_\mu^x$-prime model (over $\emptyset$) M, $|M| = \{a_i\colon i < \alpha_0\}$, $\mathrm{tp}(a_i, A_i)$ is $\boldsymbol{P}_\mu^x$-isolated where $A_i = \{a_j\colon j < i\}$. Now we prove the following:

$(*)$ if $b_i \in |M|$ for $i < \lambda$, and for each $\alpha < \lambda$, for all large enough $i < \lambda$, the b_i's are $E_{s(\alpha)}$-equivalent *then* for some $b \in M$, for each $\alpha < \lambda$ for all large enough $i < \lambda$, $\models bE_{s(\alpha)} b_i$.

Suppose such b does not exist, hence we can assume the b_i's are distinct.

By F we can assume that $b_i \in A_\lambda$ for each $i < \lambda$. Let for $\alpha < \lambda$, $\zeta(\alpha) < \lambda$ be such that $\zeta(\alpha) \leq i < \lambda \Rightarrow b_i E_{s(\alpha)} b_{\zeta(\alpha)}$ and $\alpha < \zeta(\alpha)$. As we assume appropriate $b \in |M|$ does not exist, for each $B \subseteq |M|$, $|B| < \lambda$, there is $\xi_0 < \lambda$, such that $\xi_0 \leq \xi < \lambda$, $b \in B \Rightarrow\ \models \neg bE_{s(\xi_0)} b_\xi$ (as λ is regular). Hence it is easy to find $\delta < \lambda$, $\mathrm{cf}\,\delta \geq \mu$ such that:

(i) $b_j \in A_\delta$ iff $j < \delta$ iff $\zeta(j) < \delta$,

(ii) for each $i < \delta$ for some $\xi_i < \delta$, for every ξ, $\xi_i \leq \xi < \delta$ and $b \in A_i$, $\models \neg bE_{s(\xi_i)} b_\xi$.

By D(3), $\mathrm{tp}(b_\delta, A_\delta)$ is $\boldsymbol{P}_\mu^x$-isolated, hence for some $i < \delta$, $\mathrm{tp}(b_\delta, A_i) \vdash \mathrm{tp}(b_\delta, A_\delta)$, but $b_{\zeta(\xi_i)}$ realizes the former but not the latter, contradiction.

Now we shall exhibit another $\boldsymbol{P}_\mu^x$-prime model, not isomorphic to M. Trivially every elementary submodel N of M which is $\boldsymbol{P}_\mu^x$-saturated is a $\boldsymbol{P}_\mu^x$-prime model.

So choose $a_0 \in M$, and let N be the submodel of M with universe $\{b \in |M|$; for some $i < \lambda$, $\models \neg bE_{s(i)} a_0\}$. We leave to the reader the proof that N is an elementary submodel of M and N is $\boldsymbol{P}_\mu^x$-saturated hence is $\boldsymbol{P}_\mu^x$-prime. Let us prove N is not isomorphic to M by showing N fails to satisfy $(*)$. For each $i < \lambda$ there is $b_i \in |M|$ such that $\models b_i E_{s(i)} a_0 \wedge \neg b_i E_{s(i+1)} a_0$, so $b_i \in N$. By the definition of N, no $b \in N$ as required in $(*)$ exists.

7. *Conclusion.* Let μ be an infinite cardinality, and λ be μ^+ when μ is regular, and μ^{++} otherwise.

There is a stable theory T of cardinality λ such that for $x = t, s$, over every A there is a $\boldsymbol{P}_\mu^x$-prime model; but even for $A = \emptyset$ it is not unique.

PROOF. Just choose T^I with $I = \lambda$.

8. DEFINITION. (1) Let M^0 be the following model: $|M^0| = \boldsymbol{Z} \cup \{h: h: \boldsymbol{Z} \to \boldsymbol{Z}$ such that for all small enough n, $h(n) = 0\}$ ($\boldsymbol{Z}$ the integers).

$P = \boldsymbol{Z}$,

$Q = |M_0| - P$,

$<$ the natural order on $\boldsymbol{Z}$,

$E = \{\langle h_1, h_2, n\rangle: h_1, h_2 \in Q, n \in P$, and for every integer $m \leq n$, $h_1(m) = h_2(m)\}$,

F a two-place function from Q to P,

$F(h_1, h_2) = \min\{n \in \boldsymbol{Z}: h_1(n+1) \neq h_2(n+1)\}$ when $h_1 \neq h_2$, and $F(h, h) = 0$,

S a one-place function from P to P—the successor,

S^{-1} the inverse of S.

(2) Let T^0 be the (first-order) theory of M^0.

9. *Claim.* T^0 is a countable complete theory which has elimination of quantifiers.

10. LEMMA. *Over each A (for T^0) there is a $\boldsymbol{P}_{\aleph_0}^t$-prime model over A. It is not unique if $A \subseteq P$, A has order-type $\boldsymbol{Z} \times \omega_1$. Similarly for $\boldsymbol{P}_\lambda^t$.*

12.[2] *Conclusion.* There is a countable complete theory for which over every A there is a $\boldsymbol{P}_{\aleph_0}^t$-prime model, but it is not always unique.[3]

Now we prove the uniqueness theorem.

By G, there is a unique $\boldsymbol{P}_\lambda^t$-primary model over each A, hence it suffices to prove

13. LEMMA. *Suppose C is $\boldsymbol{P}_\lambda^t$-constructible over A, $A \subseteq B \subseteq C$, T stable, then B is $\boldsymbol{P}_\lambda^t$-constructible over A provided that at least one of the following conditions hold:*

(a) *$\lambda^+ \geq \kappa(T)$, λ regular,*

(b) *(cf λ) $\geq \kappa(T)$,*

(c) *for some λ_0, $\lambda_0^+ \geq \kappa(T)$ and the conclusion of the lemma holds whenever $|C - A| \leq \lambda_0$.*

PROOF. The proof proceeds by induction on $|C - A|$.

If $|C - A| \leq$ cf λ, the result is immediate using any enumeration of $|B - A|$ by D(3) and D(5). Now in cases (a) and (b) let $\lambda_0 =$ cf λ and in case (c) take λ_0 satisfying case (c). In any case we have $\lambda_0^+ \geq \kappa(T)$ and the conclusion holds if $|C - A| = \mu$ is $\leq \lambda_0$. Thus we can assume $\mu > \lambda_0$. Now we find C_i ($i < \mu$) such that

(i) C_i is increasing continuous and $C = \bigcup_{i<\mu} C_i$,

(ii) $|C_i| \leq |i| + \lambda_0$,

(iii) for each $\bar{c} \in C_i$, tp($\bar{c}$, B) does not fork over $B \cap C_i$ and tp($\bar{c}$, A) does not fork over $A \cap C_i$,

(iv) C_{i+1} is $\boldsymbol{P}_\lambda^x$-constructible over $A \cup C_i$.

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[2]This is a complement to 14 (= [Sh 1, IV, 5.6]); by it there is no countable stable example. So now we describe a countable unstable example, which is, essentially, an encoding of the uncountable example.

[3]In Lemma 5.1, for $\mu = \aleph_0$ it suffices to assume jumps are dense in I (i.e. for $s < t$ in I there are s_0, t_0, $s \leq s_0 < t_0 \leq t$, t_0 the successor of s_0), and similary for any λ. Applying the idea to 9–12 (using $2 \times \boldsymbol{Q}$ instead $\boldsymbol{Z}$) we get in 12 an $\aleph_0$-categorical theory.

Each C_i is constructed in ω-steps using I(1) to guarantee (iii) and D(4) to guarantee (iv). Now we apply our induction hypothesis to $A \cup C_i$, $A \cup C_i \cup (B \cap C_{i+1})$, $A \cup C_{i+1}$ (for A, B, C resp.). So let us define α_i and b_j $(\alpha_i \leq j < \alpha_{i+1})$ such that $B \cap C_{i+1} - (A \cup C_i) = \{b_j: \alpha_i \leq j < \alpha_{i+1}\}$, and $\text{tp}(b_j, A \cup C_i \cup \{b_\gamma: \alpha_i \leq \gamma < j\})$ is $\boldsymbol{P}_\lambda^t$-isolated. If we show this type does not fork over $A \cup \{b_\gamma: \gamma < j\}$ we can conclude by I(7) that $\text{tp}(b_j, A \cup \{(b_\gamma: \gamma < j\})$ is $\boldsymbol{P}_\lambda^t$-constructible and then the theorem. To this end consider $\bar{c} \in C_i$; by I(2) as $\text{tp}(\bar{c}, B)$ does not fork over $B \cap C_i$, also $\text{tp}(\bar{c}, A \cup (B \cap C_{i+1}))$ does not fork over $B \cap C_i$. This implies (again by I(2)) that $\text{tp}(\bar{c}, A \cup \{b_\gamma: \gamma \leq j\})$ does not fork over $A \cup \{b_\gamma: \gamma < j\}$ so by I(5), $\text{tp}(b_j, A \cup \bar{c} \cup \{b_\gamma: \gamma < j\})$ does not fork over $A \cup \{b_\gamma: \gamma < j\}$. Now by I(3), $\text{tp}(b_j, A \cup C_i \cup \{b_\gamma: \gamma < j\})$ does not fork over $A \cup \{b_\gamma: \gamma < j\}$ whence by I(7), $\text{tp}(b_j, A \cup \{b_\gamma: \gamma < j\})$ is $\boldsymbol{P}_\lambda^t$-isolated. So $\{b_\gamma: \gamma < \bigcup_{i<\lambda} \alpha_i\}$ demonstrate B is $\boldsymbol{P}_\lambda^t$-constructible over A.

14. *Conclusion.* Suppose the conditions on T, λ, λ_0 from 13 are satisfied and over A there is $\boldsymbol{P}_\lambda^t$-primary model M (i,e. $|M|$ is $\boldsymbol{P}_\lambda^t$-constructible over A and $\boldsymbol{P}_\lambda^t$-saturated). Then this is the unique $\boldsymbol{P}_\lambda^t$-prime model over A. Note that for a countable stable T, we get uniqueness for $\boldsymbol{P}_\lambda^t$-prime models by (a) and H.

PROOF. By D(1), M is $\boldsymbol{P}_\lambda^t$-prime over A. If N is too then w.l.o.g. $N \prec M$ (as being $\boldsymbol{P}_\lambda^t$-prime over A, N can be elementarily embedded into M over A), so 13, by N is $\boldsymbol{P}_\lambda^t$-constructible over A, hence by G, N, M are isomorphic over A.

CONCLUDING REMARK. For those acquainted with [Sh 1] we remark (the number shows which place in this paper is relevant)

2. *Claim.* For T^I, every formula which is almost over A is equivalent to a formula over A.

5. LEMMA. *T^I satisfies axioms* XI 1, XII 1 for $\boldsymbol{P}_\mu^a$.

6. LEMMA. *The lemma holds also for $x = a$.*

13. LEMMA. *The lemma holds for $\boldsymbol{P}_\lambda^a$-construction too.*

14. *Conclusion.* The conclusion holds for $\boldsymbol{P}_\lambda^a$ too.

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