On countable theories with models - homogeneous models only

1. Theorem : Suppose every model of T of power λ is model homogeneous and $\lambda > |T|$ and T is countable. Then $\lambda \geq \lambda(T)$ (= the first cardinality in which T is stable), T is superstable, unidimensional and every model of T of power $\mu > \lambda(T)$ is model homogeneous.

Proof : We known that (see [Sh 2]):

(*) if M_1, M_2 are models homogeneous model of T of power λ and $\{N/\approx : N \prec M_1, \|N\| = |T|\} = \{N/\approx : N \prec M_2, \|N\| = |T|\}$ then $M_1 \approx M_2$.

By [Sh 1], Ch. VIII, 4.2. if T is not superstable T has non-isomorphic models of power λ which contradict (*).

Suppose T is stable but not unidimensional. By V. 2.10 T has an $\mathbf{F}^a_{|T|^+}$-saturated model M of cardinality $> (2^\lambda)^+$, with a maximal indiscernible set $\mathbf{I} \subseteq M$ of power $\kappa_r(T)$. Let $\bar{c} \in \mathbf{I}$ (so $\bar{c} \in M$), $\mathbf{I}_1 \overset{def}{=} \mathbf{I} - \{\bar{c}\}$ and let $N \prec M$ be $\mathbf{F}^a_{|T|^+}$-primary over $\bigcup \mathbf{I}$ and $N_1 \prec N$ be $\mathbf{F}^a_{\kappa_r(T)}$-primary over $\bigcup \mathbf{I}_1$. By [Sh 1] IV 4.18 N_1 is isomorphic to N by an isomorphism f mapping $\mathbf{I}_1$ onto $\mathbf{I}$, and clearly M omit $Av(\mathbf{I}, \bigcup \mathbf{I})$, N_1 omit $Av(\mathbf{I}_1, \bigcup \mathbf{I}_1)$. So clearly we cannot extend f to an elementary mapping f^* from N into M (as then $f^*(\bar{c})$ realizes $Av(\mathbf{I}, \bigcup \mathbf{I})$). So if $\lambda > \|N\|$ we can finish (see below). Let N^* be such that $N^* \prec M, \|N^*\| = |T|$ and:

$$(N^*, N_1 \cap N^*, f \restriction (N^* \cap N_1), \mathbf{I} \cap N^*, \bar{c}) \prec (M, N_1, f, \mathbf{I}, \bar{c})$$

and $\mathbf{I} \subseteq N^*$ (possible as $|\mathbf{I}| = \kappa_r(T) = \aleph_0 = |T|$). Again $f \restriction (N^* \cap N_1)$ cannot be extended to an elementary mapping f^* from $N^* \cap N_1$ into M (as then $f^*(\bar{c})$ realizes $Av(\mathbf{I}, \bigcup \mathbf{I})$). So M is not $|T|^+$-model homogeneous and $\|M\| \geq \lambda$, so we can find $M^+, N^* \prec M^+ \prec M$, $\|M^+\| = \lambda$ and M^+ contradicts a hypothesis . So T is unidimensional.

The proof that $\lambda > \lambda(T)$ is similar (M will be $\mathbf{F}^a_{\kappa_r(T)}$-prime over $\mathbf{I}$, $|\mathbf{I}| = \kappa_r(T)$, so $\|M\| = \lambda(T)$, $\mathbf{I}$ maximal in M).

If T has the otop, we get contradiction as in the case of unsuperstable T. T cannot have the dop as it is unidimensional.

Now note:

2. Lemma: If T is superstable unidimensional (or even just with no $M,\varphi,\bar{a}$ such that $\aleph_0 \le |\varphi(M,\bar{a})| < \|M\|$) and with the $(<\infty,\lambda>)$-existence property, then (see [Sh 1] Ch. XI §2), $(\mathbf{F}^t_{\aleph_0}, \subseteq_t)$ satisfies Ax. A4-6, B1-6, C1$^+$, C2, C3$^+$, D2^{-1}, E1^1, E2$^+$ $\lambda(\mathbf{T}^t_{\aleph_0}, \subseteq_t) \le |T|$.

Proof: Suppose M is not model homogeneous, $\|M\| > \lambda(T)$ and we shall get a contradiction thus finishing. Clearly M is not saturated, hence by [Sh 1] IX 1.8 T is not $\aleph_0$ stable, [condition (3) fail hence (6) with λ,μ there standing for λ, $|T|$ here, now M is easily not μ^+-model homogeneous.] So $\lambda(T) = 2^{\aleph_0}$. As M is not model homogeneous there are $\mu < \|M\|$, $M_0 \prec M_1 \prec M$, $M^0 \prec M$ f an isomorphism from M_0 onto M^0 which cannot be extended to an elementary embedding of M_1 into M and $\|M_1\| \le \mu$. By [Sh 1] 2.6(2), and Lemma 2, M_0 has an $(\mathbf{F}^t_{\aleph_0}, \subseteq)$-decomposition $\langle N_\eta : \eta \in \{<>, <i>: i < \alpha_0\}\rangle$, N_η countable, and it can be extended to an $(\mathbf{F}^t_{\aleph_0}, \subseteq)$-decomposition $\langle N_\eta ; \eta \in \{<>, <i>: i < \alpha_1\}\rangle$ of M_1, $\|N_\eta\| = \aleph_0$. Clearly $\langle N^+_\eta : \eta \in \{<>, <i>: i < \alpha^0\}\rangle$ is an $(\mathbf{F}^t_{\aleph_0}, \subseteq)$-decomposition of M^0 when $N^+_\eta = f(N^+_\eta)$ and it can be extended to an $(\mathbf{F}^t_{\aleph_0}, \subseteq)$-decomposition of M : $\langle N^+_\eta : \eta \in \{<>, <i>, : i < \beta\rangle\}$, $\|N^+_\eta\| = \aleph_0$.

We can define by induction on $\alpha, \alpha_0 \le \alpha < \alpha_1$ a model $M_{0,\alpha}$: $M_{0,0} = M_0$, $M_{0,\alpha+1}$ is prime over $M_{0,\alpha} \cup N_{<\alpha>}$ (it exists as T has the $(<\infty,2)$-existence property), and for limit δ, $M_{0,\delta} = \bigcup_{\alpha_0 \le \alpha < \delta} M_{0,\alpha}$. We then can try to define by induction on $\alpha, \alpha_0 \le \alpha < \alpha_1$, an elementary embedding f_α of $M_{0,\alpha}$ into M, f_α extending f and f_β when $\alpha_0 \le \beta < \alpha$. If f_{α_1}, is defined this contradicts the choice of M_0, M_1, M^0, f. So for some α, f_α is defined but $f_{\alpha+1}$ is not, and (by renaming) W.l.o.g. $\alpha = \alpha_0$, $\alpha_1 = \alpha_0 + 1$. By the $(<\infty,2)$-existence property there is no N^1 realizing $f(tp_*(N_{<\alpha_0>}, N_{<>}))$, which is independent over $(M^0, f(N_{<>}))$.

Choose $N \prec M, \|N\| = |T|$, $|N_{<>}| \cup |N^+_{<>}| \cup |N_{<\alpha_0>}| \subseteq N$, f maps $N \cap M_0$ onto $N \cap M^1$, $tp_*(N, M_0)$ does not fork over $N \cap M_0$, $tp_*(N, M^0)$ does not fork over $M^0 \cap N, N \cap M_0 \prec M_0$. As $\|M\| > 2^{\aleph_0} + \mu$ w.l.o.g. $tp(N^+_{\alpha_0+i}, N^+_{<>})$ is constant for $i < \mu^+$ and $\{N^+_{\alpha_0+i} : i < \mu^+\}$ is independent over $(M_1 \cup M^0 \cup N, N^+_{<>})$.

If $\lambda \le \mu^+$ let M_λ be prime over $N \cup \bigcup_{i<\lambda} N^+_{\alpha_0+i}$ (exists by the $(<\infty,2)$-existence property). So w.l.o.g. $M_\lambda \prec M$, and we shall show that M_λ is not $|T|^+$-model homogeneous, thus getting a contradiction hence every model of T of power $> \lambda(T)$ is model homogeneous thus finishing the proof . The non $|T|^+$-model homogeneity of M_λ is exemplified by $M_0 \cap N, M_1 \cap N$, and $f \restriction (M_0 \cap N)$. For this it suffices to prove that $f(tp_*(N_{<\alpha_0>}, M_0 \cap N))$ is not realized in M_λ, so suppose N^+ realizes it, $N^+ \prec M_\lambda$. So $N^+ \prec M$. Easily $tp_*(N^+, M_1 \cup M^0 \cup N)$ does not fork over N, (as M_λ is atomic over $N \cup \bigcup_{i<\lambda} N^+_{\alpha_0+i}$) and we have chosen N such that $tp_*(N, M^0)$ does not fork over $M^0 \cap N$ so by III 0.1, $tp_*(N^+ \cup N, M^0)$ does not fork over $M^0 \cap N$, so $tp_*(N^+, M^0)$ does not fork over $M^0 \cap N$. So N^+ realizes over M^0 the stationarization of $f(tp_*(N_{<\alpha_0>}, N \cap M_0))$ so we can show that it realizes $f(tp_*(N_{<\alpha_0>}, M_0))$, contradiction.

If $\lambda \ge \mu^+$ we can "lengthen" $\{N^+_{\alpha_0+1} : i < \mu^+\}$ and the proof is similar.

References

[Sh1]

S. Shelah, Classification Theory, completed for countable. North Holland Publ. Co.

[Sh2]

---------, Classification of non elementary classes II, Abstract Elementary classes, here.