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Mathematical Logic

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Forcing axiom failure for any $\lambda > \aleph_1$

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Annotated Content

§1. A forcing axiom for $\lambda > \aleph_0$ fail

David Aspero asks on the possibility of, see Definition below, the forcing axiom $FA(\mathfrak{K}, \aleph_2)$ for the case $\mathfrak{K}$ = the class of forcing notions preserving stationarily of subsets of $\aleph_1$ and of $\aleph_2$. We answer negatively for any regular $\lambda > \aleph_1$ (even demanding adding no new sequence of ordinals of length $< \lambda$), see 1.16 below)

1.1. Definition.

1) Let $FA(\mathfrak{K}, \lambda)$, the λ-forcing axiom for $\mathfrak{K}$ mean that $\mathfrak{K}$ is a family of forcing notions and for any $\mathbb{P} \in \mathfrak{K}$ and dense open sets $\mathscr{I}_i \subseteq \mathbb{P}$ for $i < \lambda$ there is a directed $G \subseteq \mathbb{P}$ meeting every $\mathscr{I}_i$.

2) If $\mathfrak{K} = \{\mathbb{P}\}$ we may write $\mathbb{P}$ instead of $\mathfrak{K}$.

1.2. Definition. *Let λ be regular uncountable. We define a forcing notion $\mathbb{P} = \mathbb{P}^2_\lambda$ as follows:*

(A) if $p \in \mathbb{P}$ iff $p = (\alpha, \bar{S}, \bar{W}) = (\alpha^p, \bar{S}^p, \bar{C}^p)$ satisfying

(a) $\alpha < \lambda$

(b) $\bar{S}^p = \langle S_\beta : \beta \leq \alpha \rangle = \langle S^p_\beta : \beta \leq \alpha \rangle$

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(c) $\bar{C}^p = \langle C_\beta : \beta \le \alpha \rangle = \langle C^p_\beta : \beta \le \alpha \rangle$
such that
(d) S_β *is a stationary subset of* λ *consisting of limit ordinals*
(e) C_β *is a closed subset of* β
(f) if $\beta \le \alpha$ *is a limit ordinal then* C_β *is a closed unbounded subset of* β
(g) if $\gamma \in C_\beta$ *then* $C_\gamma = \gamma \cap C_\beta$
(h) $C_\beta \cap S_\beta = \emptyset$
(i) for every $\beta \le \alpha$ *and* $\gamma \in C_\beta$ *we have* $S_\gamma = S_\beta$
(B) order: natural $p \le q$ *iff* $\alpha^p \le \alpha^q$, $\bar{S}^p = \bar{S}^q \restriction (\alpha^p + 1)$ *and* $\bar{C}^p = \bar{C}^q \restriction (\alpha^p + 1)$.

1.3. Observation

1) $\mathbb{P}^2_\lambda$ is a (non empty) forcing notion of cardinality 2^λ.
2) $\mathscr{I}_i = \{p \in \mathbb{P}^2_\lambda : \alpha^p \ge i\}$ is dense open for any $i < \lambda$.

Proof. 1) Obvious.

2) Given $p \in \mathbb{P}^2_\lambda$ if $\alpha^p \ge i$ we are done. So assume $\alpha^p < i$ and for $\gamma \in (\alpha^p, i]$ let S^q_γ be S^* for any stationary subset S^* of $\{\delta < \lambda : \delta > i$ a limit ordinal$\}$ which does not belong to $\{S^p_\beta : \beta \le \alpha^p\}$ and let $C^q_\gamma = \{j : \alpha^p < j < \gamma\}$ and $q = (i, \bar{S}^{p\frown}\langle S^q_\gamma : \gamma \in (\alpha^p, i]\rangle, \bar{C}^{p\frown}\langle C^q_\gamma : \gamma \in (\alpha^p, i]\rangle)$. It is easy to check that $p \le q \in \mathbb{P}^2_\lambda$ and $q \in \mathscr{I}_i$. □

1.4. Claim. *Let* $\lambda = \mathrm{cf}(\lambda)$ *be regular uncountable and* $\mathbb{P} = \mathbb{P}^2_\lambda$. *For any stationary* $S \subseteq \lambda$ *and* $\mathbb{P}^2_\lambda$*-name* $\underset{\sim}{f}$ *of a function from* $\gamma^* \le \lambda$ *to the ordinals or just to* $\mathbf{V}$ *and* $p \in \mathbb{P}$ *there are* q, δ *such that:*

⊠*(i)* $p \le q \in \mathbb{P}$
(ii) $\alpha^q = \delta + 1$
(iii) $\delta \in S$ *if* $\gamma^* = \lambda$
(iv) q *forces a value to* $\underset{\sim}{f} \restriction (\delta \cap \gamma^*)$
(v) if $\beta < \delta \cap \gamma^*$ *and* $\Vdash_{\mathbb{P}}$ "Rang$(\underset{\sim}{f}) \subseteq \lambda$" *then* $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\beta) < \delta$".

Proof. Without loss of generality S is a set of limit ordinals. We prove this by induction on γ^*, so without loss of generality $\gamma^* = |\gamma^*|$ and without loss of generality $\gamma^* < \lambda \Rightarrow \gamma^* = \mathrm{cf}(\gamma^*)$, but if $\gamma^* < \lambda$ the set S is immaterial so without loss of generality

⊛ $\gamma^* < \lambda$ & $\delta \in S \Rightarrow \mathrm{cf}(\delta) \ge \gamma^*$.

Let χ be large enough (e.g. $\chi = (\beth_3(\lambda))^+$), $<^*_\chi$ is a well ordering of $\mathscr{H}(\chi)$ and choose $\bar{N} = \langle N_i : i < \lambda \rangle$ such that

⊚(a) $N_i \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ is increasing continuous
(b) $\lambda, p, \underset{\sim}{f}, S$ belongs to N_i hence $\mathbb{P} \in N_i$
(c) $\|N_i\| < \lambda$
(d) $N_i \cap \lambda \in \lambda$
(e) $\langle N_j : j \le i \rangle$ belong to N_{i+1}; hence $i \subseteq N_i$ so $\lambda \subseteq \cup\{N_i : i < \lambda\}$.

Let $\delta_i = N_i \cap \lambda$, and let $i(*) = \mathrm{Min}\{i : i < \lambda$ is a limit ordinal and $\delta_i \in S\}$, it is well defined as $\langle \delta_i : i < \lambda \rangle$ is strictly increasing continuous hence $\{\delta_i : i < \lambda\}$ is a club of λ; so by $\circledast$ we know that $\gamma^* < \lambda \Rightarrow \mathrm{cf}(i(*)) = \mathrm{cf}(\delta_{i(*)}) \geq \gamma^*$. Let α_i^* be δ_i for $i \leq i(*)$ a limit ordinal and be $\delta_i + 1$ for $i < i(*)$ a non limit ordinal. Now by induction on $i \leq i(*)$ choose p_i^- and if $i < i(*)$ also p_i and prove on them the following:

(∗)(i) $p_i, p_i^- \in \mathbb{P} \cap N_{i+1}$
(ii) p_i is increasing
(iii) $\alpha^{p_i} > \alpha_i^*$ (and $\delta_{i+1} > \alpha^{p_i}$ follows from $p \in \mathbb{P} \cap N_{i+1}$)
(iv) $S_{\alpha_i^*}^{p_i} = S$ and $C_{\alpha_i^*}^{p_i} = \{\alpha_j^* : j < i\}$
(v) p_i^- is the $<_\chi^*$-first q satisfying:
$q \in \mathbb{P}$
$j < i \Rightarrow p_j \leq q$
$\alpha^q > \delta_i$
$S_{\alpha_i^*}^q = S$ and
$C_{\alpha_i^*}^q = \{\alpha_j^* : j < i\}$
(vii) p_i is the $<_\chi^*$-first q such that:
$q \in \mathbb{P}$
$p_i^- \leq q$
q forces a value to $\underset{\sim}{f}(i)$ if $\gamma^* < \lambda$
q forces a value to $\underset{\sim}{f} \restriction \delta_i$ if $\gamma^* = \lambda$.

There is no problem to carry the definition, recalling the inductive hypothesis on γ^* and noting that $\langle (p_j^-, p_j) : j < i \rangle \in N_{i+1}$ by the " $<_\chi^*$-first" being used to make our choices as $\langle N_j : j \leq i \rangle \in N_{i+1}$ hence $\langle \delta_j : j \leq i \rangle \in N_{i+1}$ and also $\langle \alpha_j^* : j \leq i \rangle \in N_{i+1}$ (and $p, \underset{\sim}{f} \in N_0 \prec N_{i+1}$).

Now $p_{i(*)}^-$ is as required. □

1.5. Conclusion. Let $\lambda = \mathrm{cf}(\lambda) > \aleph_0$. Forcing with $\mathbb{P}_\lambda^2$ add no bounded subset of λ and preserve stationarity of subsets of λ (and add no new sequences of ordinals of length $< \lambda$).

Proof. Obvious from 1.4. □

1.6. Claim. *Let* $\lambda = \mathrm{cf}(\lambda) > \aleph_0$. *If* FA($\mathbb{P}_\lambda^2$), *(the forcing axiom for the forcing notion* $\mathbb{P}_\lambda^2$, λ *dense sets) holds, then there is a witness* $(\bar{S}, \bar{C})$ *to* λ *where*

1.7. Definition.

1) For λ regular uncountable, we say that $(\bar{S}, \bar{C})$ is a witness to λ or $(\bar{S}, \bar{C})$ is a λ-witness if:
(a) $\bar{S} = \langle S_\beta : \beta < \lambda \rangle$
(b) $\bar{C} = \langle C_\beta : \beta < \lambda \rangle$
(c) for every $\alpha < \lambda$, $(\alpha, \bar{S} \restriction (\alpha+1), \bar{C} \restriction (\alpha+1)) \in \mathbb{P}_\lambda^2$.
2) For $(\bar{S}, \bar{C})$ a witness for λ, let $F = F_{(\bar{S},\bar{C})}$ be the function $F : \lambda \to \lambda$ defined by

$$F(\alpha) = Min\{\beta : S_\alpha = S_\beta\}.$$

3) For $\beta < \lambda$ let $W_{(\bar{S},\bar{C})}^\beta = \{\alpha < \lambda : F_{(\bar{S},\bar{C})}(\alpha) = \beta\}$.

Proof of 1.6. Let $\mathcal{I}_i = \{p \in \mathbb{P}^2_\lambda : \alpha^p \geq i\}$, by 1.3(2) this is a dense open subset of $\mathbb{P}^2_\lambda$, hence by the assumption there is a directed $G \subseteq \mathbb{P}^2_\lambda$ such that $i < \lambda \Rightarrow \mathcal{I}_i \cap G \neq \emptyset$. Define $S_\alpha = S^p_\alpha$, $C_\alpha = C^p_\alpha$ for every $p \in G$ such that $\alpha^p \geq \alpha$.

Now check. □

1.8. Observation. Let $(\bar{S}, \bar{C})$ be a witness for λ and $F = F_{(\bar{S},\bar{C})}$.

1) If $\alpha < \lambda$ then $F(\alpha) \leq \alpha$.
2) If $\alpha < \lambda$ is limit then $F(\alpha) < \alpha$.
3) If $\alpha < \lambda$ then $\alpha \in W^{F(\alpha)}_{(\bar{S},\bar{C})}$.
4) If $\alpha < \lambda$ and $i = F(\alpha)$ and $\beta \in C_\alpha$ then $\beta S_\alpha = S_i$.

Proof. Easy (remember that each S_α is a set of limit ordinals $< \lambda$ and that for limit $\alpha \leq \alpha^p$, $p \in \mathbb{P}^2_\lambda$ we have $\alpha = \sup(C_\alpha)$). □

1.9. Claim. *Assume $(\bar{S}, \bar{C})$ is a λ-witness and $S^* \subseteq \lambda$ satisfies $\delta \in S^* \Rightarrow \mathrm{cf}(\delta) \geq \theta > \aleph_0$ and $F_{(\bar{S},\bar{C})} \restriction S^*$ is constant and S^* is stationary. Then there is a club E^* of λ such that: $(\bar{S}, \bar{C}, S^*, E^*)$ is a strong (λ, θ)-witness, where*

1.10. Definition

1) We say that $\mathbf{p} = (\bar{S}, \bar{C}, S^*, E^*)$ is a strong λ-witness if
(a) $(\bar{S}, \bar{C})$ is a λ-witness
(b) $S^* \subseteq \lambda$ is a set of limit ordinals and is a stationary subset of λ
(c) E^* is a club of λ
(d) for every club E of λ, for stationarily many $\delta \in S^*$ we have

$$\delta = \sup\{\alpha \in C_\delta : \alpha < Suc^1_{C_\delta}(\alpha, E^*) \in E\}$$

where
$(*)$(i) $Suc^0_{C_\delta}(\alpha) = Min(C_\delta \backslash (\alpha + 1))$,
(ii) $Suc^1_{C_\delta}(\alpha, E^*) = \sup(E^* \cap Suc^0_{C_\delta}(\alpha))$.

2) We say $(\bar{S}, \bar{C}, S^*, E^*)$ is a strong (λ, θ)-witness if in addition
(e) $\delta \in S^* \Rightarrow cf(\delta) \geq \theta$.

3) For $(\bar{S}, \bar{C}, S^*, E^*)$ a strong λ-witness we let $\bar{C}' = \langle C'_\delta : \delta \in S^* \cap acc(E^*)\rangle$, $C'_\delta = C_\delta \cup \{Suc^1_{C_\delta}(\alpha, E^*) : \alpha \in C_\delta\}$; if $\mathbf{p} = (\bar{S}, \bar{C}, S^*, E^*)$ we write $\bar{C}' = \bar{C}'_{\mathbf{p}}$ and $\bar{S}_{\mathbf{p}} = \bar{S}$, $\bar{C}_{\mathbf{p}} = \bar{C}$, $S^*_{\mathbf{p}} = S^*$, $E^*_{\mathbf{p}} = E^*$. We call $(\bar{S}, \bar{C}, S^*, E^*, \bar{C}')$ an expanded strong λ-witness (or (λ, θ)-witness).

1.11. Observation. In Definition 1.10(3) for $\delta \in S^* \cap \mathrm{acc}(E^*)$ we have:

⊛ C'_δ is a club of δ, $\mathrm{Min}(C'_\delta) \geq \sup(E^* \cap \mathrm{Min}(C_\delta))$ and if $\gamma_1 < \gamma_2$ are successive members of C_δ then $C'_\delta \cap (\gamma_1, \gamma_2)$ has at most one member (which necessarily is $\sup(E^* \cap \gamma_2)$) hence $\mathrm{acc}(C'_\delta) = \mathrm{acc}(C_\delta)$ and $\alpha \in C_\delta \wedge \alpha < \mathrm{Suc}^1_{C_\delta}(\alpha) \Rightarrow \alpha \notin C'_\delta \backslash C_\delta$ and $\mathrm{acc}(C_\delta) = \mathrm{acc}(C'_\delta)$.

Proof of 1.9. As in [Sh:g, III], but let us elaborate, so assume toward contradiction that for no club E^* of λ is $(\bar{S}, \bar{C}, S^*, E^*)$ a strong (λ, θ)-witness. We choose by induction on n sets E^*_n, E_n, A_n such that:

(a) E_n^*, E_n are clubs of λ
(b) $E_0^* = \lambda$
(c) E_n is a club of λ such that the following set is not stationary (in λ)

$$A_n = \{\delta \in S^* : \delta \in \text{acc}(E_n^*) \text{ and } \delta = \sup\{\alpha \in C_\delta : \alpha < \text{Suc}^1_{C_\delta}(\alpha, E_n^*) \in E_n\}\}$$

(d) E_{n+1}^* is a club of λ included in $\text{acc}(E_n^* \cap E_n)$ and disjoint to A_n.

For $n = 0$, E_n^* is defined by clause (b).

If E_n^* is defined, choose E_n as in clause (c), possible by our assumption toward contradiction, also $A_n \subseteq S^*$ is defined and not stationary. So obviously E_{n+1}^* as required in clause (d) exists.

So $E^* =: \cap\{E_n^* : n < \omega\}$ is a club of λ and let $\alpha(*)$ be the constant value of $F_{(\bar{S},\bar{C})} \restriction S^*$, exists by an assumption of the claim. Recall that $S_{\alpha(*)}$ is a stationary subset of λ, so clearly $E^{**} =: \{\delta \in E^* : \delta = \sup(\delta \cap E^* \cap S_{\alpha(*)})\}$ is a club of λ. As S^* is a stationary subset of λ, we can choose $\delta^* \in S^* \cap E^{**}$. For each $n < \omega$ we have $\delta^* \in S^* \cap E^{**} \subseteq E^{**} \subseteq E^* \subseteq E_{n+1}^*$ hence $\delta^* \notin A_n$ hence $\beta_n^* = \sup\{\beta \in C_{\delta^*} : \beta < \text{Suc}^1_{C_{\delta^*}}(\beta, E_n^*) \in E_n\}$ is $< \delta^*$ but $\in C_{\delta^*}$. But $\delta^* \in S^*$ so $\text{cf}(\delta^*) \geq \theta > \aleph_0$, hence $\beta^* = \sup\{\beta_n^*, \text{Min}(C_{\delta^*}) : n < \omega\}$ is $< \delta^*$ but $\geq \text{Min}(C_{\delta^*})$ and it belongs to C_{δ^*}. As $\delta^* \in E^{**}$, we know that $\delta^* = \sup(\delta^* \cap E^* \cap S_{\alpha(*)})$ hence there is $\gamma^* \in E^* \cap S_{\alpha(*)} \cap (\text{Suc}^0_{C_{\delta^*}}(\beta^*), \delta^*)$. But $\delta^* \in S^* \subseteq S_{\alpha(*)}$ recalling by the choice of $\alpha(*)$ above $F_{(\bar{S},\bar{C})}(\delta^*) = \alpha(*)$ hence by Claim 1.8(4), i.e., Definition 1.2(1), clause (A)(h) and Definition 1.7(1) we have $C_{\delta^*} \cap S_{\alpha(*)} = \emptyset$ hence $\gamma^* \notin C_{\delta^*}$. But $\delta^* > \gamma^* > \beta^* \geq \text{Min}(C_{\delta^*})$ and C_{δ^*} is a closed subset of δ^* hence $\zeta^* = \max(C_{\delta^*} \cap \gamma^*)$ is well defined and so, recalling $\beta^* \in C_{\delta^*}$ we have

$$(\forall n < \omega)(\beta_n^* \leq \beta^* < \text{Suc}^0_{C_{\delta^*}}(\beta^*) \leq \zeta^* \in C_{\delta^*}).$$

Let $\xi^* = \text{Suc}^0_{C_{\delta^*}}(\zeta^*)$ so clearly $\gamma^* \in (\zeta^*, \xi^*)$. Now for every n we have $\sup(\xi^* \cap E_n^*) \in [\gamma^*, \xi^*]$ as $\gamma^* \in E^* \cap S_{\alpha(*)} \subseteq E^* \subseteq E_n^*$.

So recalling $\zeta^* < \gamma^*$ clearly $\zeta^* < \sup(\xi^* \cap E_n^*)$; if also $\sup(\xi^* \cap E_n^*) \in E_n$ then recalling $\xi^* = \text{Suc}^0_{C_{\delta^*}}(\zeta^*)$, $\text{Suc}^1_{C_{\delta^*}}(\zeta^*, E_n^*) \equiv \sup(\xi^* \cap E_n^*)$ we have $\zeta^* \leq \beta_n^*$ (see its choice and see the choice of β_n^* above), but this contradicts $\zeta^* \geq \text{Suc}^0_{C_{\delta^*}}(\beta^*) > \beta^* \geq \beta_n^*$ and the definition of A_n (see clause (c) of $(*)$), contradiction. So necessarily $\sup(\xi^* \cap E_n^*)$ does not belong to E_n hence does not belong to E_{n+1}^*, hence $\sup(\xi^* \cap E_n^*) > \sup(\xi^* \cap E_{n+1}^*)$.

So $\langle \sup(\xi^* \cap E_n^*) : n < \omega \rangle$ is a strictly decreasing sequence of ordinals, contradiction. □

1.12. Definition. *Assume*

$(*)_1$ *$(\bar{S}, \bar{C}, S^*, E^*, \bar{C}')$ is an expanded strong λ-witness so $\bar{C}' = \langle C'_\delta : \delta \in S^* \rangle$, $C'_\delta = C_\delta \cup \{Suc^1_{C_\delta}(\alpha, E^*) : \alpha \in C_\delta\}$ or just*

$(*)_2$ *$S^* \subseteq \lambda$ is a stationary set of limit ordinals, $\bar{C}' = \langle C'_\delta : \delta \in S^* \rangle$, C'_δ is a club of δ, E^* a club of λ.*

We define a forcing notion $\mathbb{P} = \mathbb{P}_{\bar{C}'}$

(A) $c \in \mathbb{P}$ *iff*
  (a) c *is a closed bounded subset of* λ
  (b) if $\delta \in S^* \cap c$ *then* $\{\alpha \in C'_\delta : Suc^0_{C'_\delta}(\alpha) \in c\}$ *is bounded in* δ

Let $\alpha^c = \sup(c)$.

(B) *order:* $c_1 \le c_2$ *iff* c_1 *is an initial segment of* c_2.

1.13. Claim. *Let* $\mathbb{P} = \mathbb{P}_{\bar{C}'}$ *be as in Definition 1.12.*

1) $\mathbb{P}$ *is a (non empty) forcing notion.*
2) For $i < \lambda$ *the set* $\mathcal{I}_i = \{c \in \mathbb{P} : i < \sup(c)\}$ *is dense open.*

Proof. 1) Trivial.
2) If $c \in \mathbb{P}$, $i < \lambda$ and $c \notin \mathcal{I}_i$ then let $c_2 = c \cup \{i+1\}$, clearly $(c_2 \backslash c) \cap S^* = \emptyset$ as S^* is a set of limit ordinals hence $c_2 \in \mathbb{P}$ and obviously $c \le c_2 \in \mathcal{I}_i$. □

1.14. Claim. *Assume* $\mathbf{p} = (\bar{S}, \bar{C}, S^*, E^*, \bar{C}')$ *is an expanded strong* λ*-witness.*
Forcing with $\mathbb{P} = \mathbb{P}_{\bar{C}'}$ *add no new bounded subsets of* λ*, no new sequence of ordinals of length* $< \lambda$ *and preserve stationarity of subsets of* λ*.*

Proof. Assume $p \in \mathbb{P}$, $\gamma^* \le \lambda$ and $\underset{\sim}{f}$ is a $\mathbb{P}$-name of a function from γ^* to the ordinals or just to $\mathbf{V}$ and $S \subseteq \lambda$ is stationary and we shall prove that there are q, δ satisfying (the parallel of) ⊠ of 1.4, i.e.,

⊠(i) $p \le q \in \mathbb{P}$
(ii) $\alpha^q = \delta + 1$
(iii) $\delta \in S$ if $\gamma^* = \lambda$
(iv) q forces a value to $\underset{\sim}{f} \restriction (\delta \cap \gamma^*)$
(v) if $\beta < \delta \cap \gamma^*$ and $\Vdash$ "$\underset{\sim}{f} : \gamma^* \to \lambda$" then $q \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\beta) < \delta$".

This is clearly enough for all the desired consequences. We prove this by induction on γ^*, so without loss of generality $\gamma^* = |\gamma^*|$ and without loss of generality $\gamma^* < \lambda \Rightarrow \gamma^* = \mathrm{cf}(\gamma^*)$, but if $\gamma^* < \lambda$ then S is immaterial so without loss of generality $\gamma^* < \lambda \;\&\; \delta \in S \Rightarrow \mathrm{cf}(\delta) \ge \gamma^*$. Also we can shrink S as long as it is a stationary subset of λ and recall that $F_{(\bar{S},\bar{C})}$ is regressive on limit ordinals (see Observation 1.8(2)) so without loss of generality $F_{(\bar{S},\bar{C})} \restriction S$ is constantly say $\alpha(*)$.

Let χ be large enough and choose $\bar{N} = \langle N_i : i < \lambda\rangle$ such that

⊚(a) $N_i \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ is increasing continuous
(b) $\lambda, p, \underset{\sim}{f}, S$ belongs to N_i hence $\mathbb{P} \in N_i$
(c) $\|N_i\| < \lambda$
(d) $N_i \cap \lambda \in \lambda$
(e) $\langle N_j : j \le i\rangle$ belong to N_{i+1} (hence $i \subseteq N_i$, so $\lambda \subseteq \cup\{N_i : i < \lambda\}$)
(f) $N_{i+1} \cap \lambda \in S_{\alpha(*)}$ and $N_0 \cap \lambda \in S_{\alpha(*)}$.

Let $\delta_i = N_i \cap \lambda$ and $i(*) = \text{Min}\{i : i < \lambda$ is a limit ordinal and $\delta_i \in S\}$, it is well defined as $\langle \delta_i : i < \lambda \rangle$ is (strictly increasing continuous) hence $\{\delta_i : i < \lambda\}$ is a club of λ, hence $\gamma^* < \lambda \rightarrow \text{cf}(i(*)) = \text{cf}(\delta_{i(*)}) \geq \gamma^*$.

Let[1] $W =: \{i \leq i(*)\colon i > 0$ and if $i < i(*)$ and $j < i$ then $C'_{\delta_{i(*)}} \cap \delta_{i+1} \nsubseteq \delta_{j+1}\}$. Clearly $W \cap i(*)$ is a closed subset of $i(*)$ and as $\delta_{i(*)} = \sup(C_{\delta_{i(*)}})$, also $W \cap i(*)$ is unbounded in $i(*)$. Also as by 1.11 we have $(\alpha \in \text{ acc } C'_{\delta_{i(*)}}) \Rightarrow \alpha \in C_{\delta_{i(*)}} \Rightarrow C'_\alpha = C'_{\delta_{i(*)}} \cap \alpha$ clearly

$(*)$ if $i \in W$ then $\langle N_j : j \in W \cap (i+1) \rangle \in N_{i+1}$.

Also note that

$(**)$ if $i < i(*)$ is nonlimit, then $\delta_i > \sup(C_{\delta_{i(*)}} \cap \delta_i)$ hence $\delta_i > \sup(C'_{\delta_{i(*)}} \cap \delta_i)$. [Why? By 1.8(4) as $\delta_{i(*)} \in S \subseteq S_{\alpha(*)}$ recalling the choice of $\alpha(*)$ clearly $C_{\delta_{i(*)}} \cap S_{\alpha(*)} = \emptyset$ but by clause $\circledcirc(f)$ we have $\delta_i \in S_{\alpha(*)}$ so $\delta_i \notin C_{\delta^*}$. But $C_{\delta_{i(*)}}$ is a closed subset of $\delta_{i(*)}$ hence $\delta_i > \sup(C_{\delta_{i(*)}} \cap \delta_i)$, and $C'_{\delta_{i(*)}} \cap \delta_i \setminus \sup(C_{\delta_{i(*)}} \cap \delta_i)$ has at most two members (see 1.11) so $C'_{\delta_{i(*)}} \cap \delta_i$ is a bounded subset of δ_i so we are done.]

Now by induction on $i \in W$ we choose p_i, p_i^- and prove on them the following:

$(*)$(i) $p_i, p_i^- \in \mathbb{P} \cap N_{i+1}$

(ii) p_i is increasing (in $\mathbb{P}$)

(iii) $\max(p_i) > \delta_i$ (of course $\delta_{i+1} > \max(p_i)$ as $p_i \in \mathbb{P} \cap N_{i+1}$)

(iv) $p_i^- = p \cup \{\sup(\delta_i \cup (C'_{\delta_{i(*)}} \cap \delta_{i+1})) + 1\}$ if $i = \text{Min}(W)$

(v) if $0 < i = \sup(W \cap i)$ and $\gamma_i = \max(C'_{\delta_{i(*)}} \cap \delta_{i+1})$ so $\delta_i \leq \gamma_i < \delta_{i+1}$ then $p_i^- = \cup\{p_j : j \in W \cap i\} \cup \{\delta_i, \gamma_i + 1\}$

(vi) if $j < i$ are in W then $p_j \leq p_i^- \leq p_i$

(vii) $i \in W, i < i(*)$ and $j = \text{Max}(W \cap i)$ so $j < i$ and $\gamma_i = \max(\{\delta_i\} \cup (C'_{\delta^*} \cap \delta_{i+1}))$ so $\delta_i \leq \gamma_i < \delta_{i+1}$ then $p_i^- = p_j \cup \{\gamma_i + 1\}$

(viii) p_i is the $<^*_\chi$-first $q \in \mathbb{P}$ satisfying

(α) $p_i^- \leq q \in \mathbb{P}$

(β) if $\gamma^* < \lambda$ then q forces a value to $\underset{\sim}{f}(\text{otp}(\{j < i : j \in W$ and $\text{otp}(j \cap W)$ is a successor ordinal$\})$

(γ) if $\gamma^* = \lambda$ then q forces a value to $\underset{\sim}{f} \restriction \delta_i$

(ix) $p_i^- \setminus \bigcup_{j<i} p_j$ and $p_i \setminus p_\cdot^-$ are disjoint to $C'_{\delta_{i(*)}} \setminus \text{acc}(C'_{\delta_{i(*)}})$, which include the set $\{\text{Suc}^1_{C_{\delta_{i(*)}}}(\alpha, E^*) : \alpha \in C_{\delta_{i(*)}}$ and $\alpha < \text{Suc}^1_{C_{\delta_{i(*)}}}(\alpha)\}$.

Note that clause (ix) follows from the rest; we now carry the induction.

Case 1. $i = \text{Min}(W)$.

Choose p_i^- just to fulfill clauses (iv), note that $\delta_i \leq \gamma_i < \delta_{i+1}$ as $i \in W \cap i(*)$ and then choose p_i to fulfill clause (viii).

[1] if $\text{cf}(\delta_{i(*)}) > \aleph_0$ then $W = \{i < i(*) : \delta_i \in C_{\delta_{i(*)}}\} \cup \{\delta_{i(*)}\}$ is O.K.

Case 2. $i = \text{Min}(W\setminus(j+1))$ and $j \in W$.

Choose p_i^- by clauses (vii) and then p_i by clause (viii).

Case 3. $0 < i = \sup(W \cap i)$.

A major point is $\langle p_j : j < i\rangle \in N_{i+1}$, this holds as $\langle p_j^-, p_j, j \in i \cap W\rangle$ is definable from $\bar{N} \restriction \delta_i, \underset{\sim}{f}, p, C'_{\delta_{i(*)}} \cap N_{i+1}$ all of which belong to N_{i+1} and $N_{i+1} \prec (\mathscr{H}(\chi), \in, <^*_\chi)$.

Let p_i^- be defined by clause (v), note that $\delta_i \leq \gamma_i < \delta_{i+1}$ as $i \in W$ and $p_i^- \in \mathbb{P}$ as:

(α) $(\forall j < i)[p_j \in \mathbb{P}]$ and

(β) $\delta_i = \sup(\cup\{\delta_j : j < i \text{ and } j \in W\})$. [Why? As $\delta_i < \max(p_j) < \delta_{i+1}$ by clause (iii)] and

(γ) $\alpha \in p_i^- \cap S^* \Rightarrow \sup(p_i^- \cap C'_\alpha \setminus \text{acc}(C'_\delta)) < \alpha$. [Why? If $\alpha < \delta_i$ then for some $j \in i \cap W$ we have $\alpha < \delta_j$ so p_j is an initial segment of p_i^- hence $\sup(p_i^- \cap C'_\alpha) = \sup(p_j \cap C'_\alpha) < \alpha$. If $\alpha = \delta_i$ we can assume $\alpha \in S^*$ but clearly $\alpha = \delta_i \in C'_{\delta_{i(*)}}$ by the definition of W and the assumption of case 3; so by $(\bar{S}, \bar{C})$ being a λ-witness, $C'_{\delta_i} = C'_{\delta_{i(*)}} \cap \delta_i$ so by clause (ix) the demand (in (γ)) hold.]

So easily p_i^- is as required. If $i < i(*)$ we can choose p_i by clause (viii) using the induction hypothesis if $\gamma^* = \lambda$. So we have carried the definition and $p_{i(*)}^-$ is as required. □

1.15. Conclusion.

1) If $\mathbf{p} = (\bar{S}, \bar{C}, S^*, E^*)$ is a strong λ-witness and $\bar{C}' = \bar{C}'_{\mathbf{p}}$ and $\mathbb{P} = \mathbb{P}_{\bar{C}'}$, then FA($\mathbb{P}, \lambda$) fails.

2) In part (1), $\mathbb{P}_{\bar{C}'}$ is a forcing of cardinality $\leq 2^{<\lambda}$, add no new sequence of ordinals of length $< \lambda$ and preserve stationarity of subsets of any $\theta = \text{cf}(\theta) \in [\aleph_1, \lambda]$.

Proof. 1) Recall that by Claim 1.13(2), $\mathscr{I}_i$ is a dense open subset of $\mathbb{P}$. Now if $G \subseteq \mathbb{P}_{\bar{C}'}$ is directed not disjoint to $\mathscr{I}_i$ for $i < \lambda$, let $E = \cup\{p : p \in G\}$. By the definition of $\mathbb{P}_{\bar{C}'}$ and $\mathscr{I}_i$ clearly E is an unbounded subset of λ and by the definition of $\mathbb{P}_{\bar{C}'}$ and G being directed, $p \in G \Rightarrow E \cap (\max(p)+1) = p$ and (p is closed) hence E is a closed unbounded subset of λ. So E contradicts the definition of "$(\bar{S}, \bar{C}, \bar{S}^*, \bar{E}^*, \bar{C}')$ being a strong λ-witness".

2) Follows from 1.14 and direct checking. □

1.16. Conclusion. Let λ be regular $> \aleph_1$. Then there is a forcing notion $\mathbb{P}$ such that:

(α) $\mathbb{P}$ of cardinality $\leq 2^\lambda$

(β) forcing with $\mathbb{P}$ add no new sequences of ordinals of length $< \lambda$

(γ) forcing with $\mathbb{P}$ preserve stationarity of subsets of λ (and by clause (β) also of any $\theta = \text{cf}(\theta) \in [\aleph_1, \lambda)$)

(δ) FA($\mathbb{P}, \lambda$) fail.

Proof. We try $\mathbb{P}^2_\lambda$, it satisfies clause (α), (β), (γ) (see 1.3(1), 1.5, 1.6). If it satisfies also clause (δ) we are done otherwise by Claim 1.6 there is a λ-witness $(\bar{S}, \bar{C})$. Let $S^* \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) > \aleph_0\}$ be stationary, so by 1.9 for some club E^* of λ, the quadruple $\mathbf{p} = (\bar{C}, \bar{S}, S^*, E^*)$ is a strong λ-witness (see Definition 1.10), and let $\bar{C}' = \bar{C}'_{\mathbf{p}}$.

Now the forcing notion $\mathbb{P} = \mathbb{P}_{\bar{C}'}$ (see Definition 1.12) satisfies clauses (α), (β), (γ) by claims 1.15(2) and also clause (δ) by claim 1.15(1). So we are done. $\square$

§2. There are $\{\aleph_1\}$-semi-proper not proper forcing notion[2]

By [Sh:f, XII,§2], it was shown when no "remnant of large cardinal properties holds" (e.g. $\neg 0'_{\#}$) then every quite semi-proper forcing is proper, more fully UReg-semi-properness implies properness. This leaves the problem

(∗) is the statement (for every forcing notion $\mathbb{P}$, "$\mathbb{P}$ is proper" follows from $\mathbb{P}$ is "semi-proper, i.e., $\{\aleph_1\}$-semi proper") consistent or is the negation provable in ZFC.

David Asparo raises the question and we answer affirmatively: there are such forcing notions. So the iteration theorem for semi proper forcing notions in [Sh:f, X] is not covered by the one on proper forcing notions even if $0^\#$ does not exist.

2.1. Claim. *There is a forcing notion $\mathbb{P}$ of cardinality $2^{\aleph_2}$ which is not proper but is $\{\aleph_1\}$-semi proper. This follows from 2.2 using $\kappa = \aleph_2$.*

2.2. Claim. *Assume $\kappa = \mathrm{cf}(\kappa) > \aleph_1$, $\lambda = 2^\kappa$. Then there is $\mathbb{P}$ such that*

(a) $\mathbb{P}$ is a forcing notion of cardinality 2^κ
(b) if $\chi > \lambda$, $p \in \mathbb{P} \in N \prec (\mathcal{H}(\chi), \in)$, N countable, then there is $q \in \mathbb{P}$ above p such that $q \Vdash$ "$N \cap \kappa \lhd N[\underset{\sim}{G}_{\mathbb{P}}] \cap \kappa$" ($\lhd$ means initial segment); this gives $\mathbb{P}$ is $\{\aleph_1\}$-semi proper and more
(c) there is a stationary $\mathcal{S} \subseteq [\lambda]^{\aleph_0}$ such that $\Vdash_{\mathbb{P}}$ "$\mathcal{S}$ is not stationary"
(d) $\mathbb{P}$ is not proper.

Proof. We give many details.

Stage A. Preliminaries.

Let $M^* = (\lambda, F_{n,m})_{n,m<\omega}$, with $F_{n,m}$ an $(n+1)$-place function, be such that for every $n < \omega$ and n-place function f from κ to κ there is $m < \omega$ such that $(\forall i_1, \ldots, i_n < \kappa)(\exists \alpha < \kappa)[f(i_1, \ldots, i_n) = F_{n,m}(\alpha, i_1, \ldots, i_n)]$.

Let S_1, S_2 be disjoint stationary subsets of κ of cofinality $\aleph_0$ (i.e. $\delta \in S_1 \cup S_2 \Rightarrow \mathrm{cf}(\delta) = \aleph_0$). Let

$$\mathcal{S} = \Big\{ a \in [\lambda]^{\aleph_0} : \text{ for some } b \in [\lambda]^{\aleph_0} \text{ we have } \begin{array}{ll} (\alpha) & a \subseteq b \text{ are closed under } F_{n,m} \text{ for } n, m < \omega, \\ (\beta) & \sup(a \cap \kappa) \in S_1, \sup(b \cap \kappa) \in S_2 \\ (\gamma) & (a \cap \kappa) \lhd (b \cap \kappa) \ (\lhd \text{ is being an initial segment}) \end{array} \Big\}$$

[2] done 2001/8/8

$$\mathbb{P} = \mathbb{P}_{\mathscr{S}} = \big\{\bar{a} : \ \bar{a} = \langle a_i : i \le \alpha\rangle \text{ is an increasing continuous sequence of members of } [\lambda]^{\aleph_0}\backslash\mathscr{S} \text{ of length } \alpha < \omega_1\big\}$$

Clearly clause (a) of 2.2 holds.

Stage B. $\mathscr{S}$ is a stationary subset of $[\lambda]^{\aleph_0}$. Why? Let N^* be a model with universe λ and countable vocabulary, it is enough to find $a \in \mathscr{S}$ such that $N^* \restriction a \prec N$. Without loss of generality N^* has Skolem functions and N^* expands M^*. Choose for $\alpha < \kappa, N_\alpha \prec N^*, \|N_\alpha\| < \kappa, \beta < \alpha \Rightarrow N_\beta \subseteq N_\alpha, \alpha \subseteq N_\alpha, N_\alpha$ increasing continuous. So $C =: \{\delta < \kappa : \delta$ a limit ordinal and $N_\delta \cap \kappa = \delta\}$ is a club of κ. Choose $\delta_1 < \delta_2$ from C such that $\delta_1 \in S_1, \delta_2 \in S_2$. Choose a countable $c_1 \subseteq \delta_1$ unbounded in δ_1, and a countable $c_2 \subseteq \delta_2$ unbounded in δ_2.

Choose a countable $M \prec N_{\delta_2}$ such that $M \cap N_{\delta_1} \prec N_{\delta_1}$ and $c_1 \cup c_2 \subseteq \delta$. Let $a = M \cap N_{\delta_1}, b = M \cap N_{\delta_2}$. As N^* expands M^*, clearly a, b are closed under the functions of M^*. Also $c_1 \subseteq M \cap \delta_1 = M \cap (N_{\delta_1} \cap \kappa) = a \cap \kappa \subseteq N_{\delta_1} \cap \kappa = \delta_1$ hence $\delta_1 = \sup(c_1) \le \sup(a \cap \kappa) \le \delta_1$ so $\sup(a \cap \kappa) = \delta_1$. Similarly $\sup(b \cap \kappa) = \delta_2$. Lastly, obviously $a \cap \kappa \triangleleft b \cap \kappa$ so b witnesses $a \in \mathscr{S}$, as required.

Stage C. $\Vdash_{\mathbb{P}}$ "$\mathscr{S}$ is not stationary". Why? Define $\underset{\sim}{a}^*_\alpha = \{a_\alpha : \bar{a} \in \underset{\sim}{G}_{\mathbb{P}}, \ell g(\bar{a}) > \alpha\}$.

Clearly

$(*)_0$ $\mathbb{P} \ne \emptyset$. [Why? Trivial.]

$(*)_1$ for $\alpha < \omega_1, \mathscr{I}^1_\alpha = \{\bar{a} \in \mathbb{P} : \ell g(\bar{a}) > \alpha\}$ is a dense open subset of $\mathbb{P}$. [Why? If $\langle a_i : i \le j\rangle \in \mathbb{P}, j < \gamma < \omega_1$ we let $a_i =: a_j$ for $i \in (j, \gamma]$ and then $\langle a_i : i \le j\rangle \le_{\mathbb{P}} \langle a_i : i \le \gamma\rangle$.]
Also

$(*)_2$ for $\beta < \lambda, \mathscr{I}^2_\beta = \{\bar{a} \in \mathbb{P} : \beta \in a_\alpha$ for some $\alpha < \ell g(\bar{a})\}$ is a dense open subset of $\mathbb{P}$. [Why? Given $\bar{a} = \langle a_i : i \le j\rangle$. Choose $\delta \in S_2$ such that $\delta > \sup(\kappa \cap (a_j \cup \{\beta\})$ let $c \subseteq \delta$ be countable unbounded in δ and let $a_{j+1} = a_j \cup \{\beta\} \cup c$; so trivially $\sup(a_{j+1} \cap \kappa) = \delta \in S_2$ hence $a_{j+1} \notin \mathscr{S}$. Now let $\bar{a}^+ = \langle a_i : i \le j+1\rangle$. Now check.]

So

$(*)_3$ $\Vdash_{\underset{\sim}{\mathbb{P}}}$ "$\langle \underset{\sim}{a}_i : i < \omega_1\rangle$ is an increasing continuous sequence of members of $([\lambda]^{\aleph_0})^{\mathbf{V}}\backslash\mathscr{S}$ whose union is λ" hence

$(*)_4$ $\Vdash_{\mathbb{P}}$ "$\langle \underset{\sim}{a}_i : i < \omega_1\rangle$ witness $\mathscr{S}$ is not stationary (subset) of $[\lambda]^{\aleph_0}$".

So we have finished Stage C.

Stage D. Clauses (c),(d) of 2.2 holds. Why? By Stage B and Stage C.

Stage E. Clause (b) of 2.2 holds.

So let $\chi > \lambda$, N a countable elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$ to which $\mathbb{P}$ and $p \in \mathbb{P}$ belong hence $M^*, \kappa, \lambda, S \in N$ (they are definable from $\mathbb{P}$ or demand it). In the next stage we prove

$\boxtimes$ there is a countable $M \prec (\mathscr{H}(\chi), \in <^*_\chi)$ such that $N \prec M$, $(N \cap \kappa) \trianglelefteq (M \cap \kappa)$ and $M \cap \lambda \notin \mathscr{S}$.

Let $\langle \mathcal{J}_n : n < \omega \rangle$ list the dense open subsets of $\mathbb{P}$ which belong to M. Choose by induction on n, $p_n \in N \cap \mathbb{P} : p_0 = p$, $p_n \leq_{\mathbb{P}} p_{n+1} \in \mathcal{J}_n$. So let $p_n = \langle a_i : i \leq \gamma_n \rangle$, by $(*)_1$ of Stage C the sequence $\langle \gamma_n : n < \omega \rangle$ is not eventually constant. Define q by: $q = \langle a_i : i \leq \gamma \rangle$ where $\gamma = \cup\{\gamma_n : n < \omega\}$ and $a_\gamma = M \cap \lambda$. Trivially $a_i \subseteq M \cap \lambda$ and by $(*)_2$ of Stage C clearly $a_\gamma = \cup\{a_i : i < \gamma\}$ hence $\langle a_i : i \leq \gamma \rangle$ is increasing continuous and $i \leq \gamma \Rightarrow a_i \in [\lambda]^{\leq \aleph_0}$ and $i < \gamma \Rightarrow a_i \in [\lambda]^{\aleph_0} \setminus \mathcal{S}$. So the only non trivial point is $a_\gamma \notin S$ which holds by $\boxtimes$.

Clearly $p \leq q$ and q is $(M, \mathbb{P})$-generic hence $q \Vdash$ "$N[G] \subseteq M[G]$ and $N \cap \kappa \subseteq (N[G] \cap \kappa) \subseteq M[G] \cap \kappa = M \cap \kappa$" so as $(N \cap \kappa) \lhd (M \cap \kappa)$ necessarily $(N[G] \cap \kappa) \trianglelefteq (N[G] \cap \kappa)$" as required.

Stage F. Proving $\boxtimes$.

If $N \cap \lambda \notin \underline{S}$ let $M = N$ and we are done so assume $M \cap \lambda \in \mathcal{S}$. Let $a = N \cap \lambda \in [\lambda]^{\aleph_0}$ and let $b \in [\lambda]^{\aleph_0}$ witness $a = N \cap \lambda \in \mathcal{S}$ [the rest should by now be clear but we elaborate]. Let M be the Skolem Hull in $(\mathcal{H}(\chi), \in, <^*_\chi)$ of $N \cup (b \cap \kappa)$ (exists as $<^*_\chi$ is a well ordering of $\mathcal{H}(\chi)$ so $(\mathcal{H}(\chi), \in, <^*_\chi)$ has (definable) Skolem functions).

If $\gamma \in M \cap \kappa$ then we can find a definable function f of $(\mathcal{H}(\chi), \in, <^*)$ and $x \in N$ (recall in $\mathcal{N}$ we can use m-tuple for every m) and $\alpha_1 \ldots \alpha_n \in b \cap \kappa$ such that $\gamma = f(x, \alpha_1, \ldots, \alpha_n)$. Fixing x, f the mapping $(\alpha_1, \ldots, \alpha_n) \mapsto f(x, \alpha_1, \ldots, \alpha_n)$ is an n-place function from κ to κ definable in N hence belong to N and $M^* \in N$ hence for some $\beta \in N \cap \lambda$ and $m < \omega$ we have $(\forall \alpha_1, \ldots, \alpha_n < \kappa)[f(x, \alpha_1, \ldots, a_n) = F_{n,m}(\beta, \alpha_1, \ldots, \alpha_n)]$.

But $\alpha_1, \ldots, \alpha_n \in b \cap \kappa \subseteq b$ and $\beta \in N \cap \lambda \subseteq b \cap \lambda = b$ and as b being in $\mathcal{S}$ is closed under $F_{n,m}$ clearly $\gamma = f(x, \alpha_1, \ldots, \alpha_n) = F_{n,m}(\beta, \alpha_1, \ldots, \alpha_n) \in b$ but $\gamma \in \kappa$ so $\gamma \in b \cap \gamma$. So $M \cap \kappa \subseteq b$ but of course $b \cap k \subseteq M \cap \kappa$ so $b \cap k = M \cap \kappa$. So $a \cap \kappa = (N \cap \lambda) \cap \kappa = N \cap \kappa$; but $a \cap k \lhd b \cap k$ by the choice of b so $N \cap \kappa = a \cap \kappa \lhd b \cap \kappa = M \cap \kappa$.

Lastly, $\sup(M \cap \kappa) = \sup(b \cap \kappa) \in S_2$ hence $M \cap \kappa \notin S$. So M is as required in $\boxtimes$ and we are done.

References

Content-Description:

[Sh:g] Shelah, S.: Cardinal Arithmetic, vlume 29 of Oxford Logic Guides. Oxford University Press, 1994

[Sh:f] Shelah, S.: Proper and improper forcing. Perspectives in Mathematical Logic, Springer, 1998