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# APPENDIX TO MODELS WITH SECOND ORDER PROPERTIES II TREES WITH NO UNDEFINED BRANCHES

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## Appendix: On Vaught-two-cardinal theorem

We first review some facts on two-cardinal theorems from [2].

**Definition 1.** Let $\langle \lambda_1, \mu_1\rangle \xrightarrow[\kappa]{c} \langle \lambda, \mu\rangle$ if: whenever $T$ is a (first-order) theory of cardinality $\kappa$, and every finite subset of $T$ has a $\langle \lambda_1, \mu_1\rangle$-model $M$ (i.e. $\|M\| = \lambda_1$, $|P^M| = \mu_1$) $T$ has a $\langle \lambda, \mu\rangle$-model.

**Definition 2.** Let E be an *equivalence relation* on increasing sequences of natural numbers $< n$, such that $\eta_1$ E $\eta_2$ implies $\eta_1$, $\eta_2$ have the same length.

(1) We call the pair $(n, E)$ an identity (in [2] we call it an identification)

(2) We say $\lambda \to (n, E)_\mu$, when if for every $l < \omega$ $f_l : \lambda \to \mu$ is an $l$-place function then there are $\alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \lambda$, which realize $(n, E)$ i.e.

$$\langle i(0), \dots, i(l-1)\rangle \mathrm{E} \langle j(0), \dots, j(l-1)\rangle \Rightarrow f_l(\alpha_{i(0)}, \dots) = f_l(\alpha_{j(0)}, \dots).$$

(3) Let $(m, E')$ be a subidentity of $(n, E)$ if there are $k(0) < \cdots < k(m-1) < n$ such that:
for each $i(0) < \cdots < i(l-1) < m$, $j(0) < \cdots < j(l-1) < m$.
$\langle i(0), \dots, i(l-1)\rangle \mathrm{E}' \langle j(0), \dots, j(l-1)\rangle$ implies
$\langle k(i(0)), \dots, k(i(l-1))\rangle \mathrm{E} \langle k(j(0)), \dots, k(j(l-1))\rangle$.

**Theorem 3.** *The following conditions on $\lambda_1 \geq \mu_1$, $\lambda \geq \mu$ are equivalent*

(1) $(\lambda_1, \mu_1) \xrightarrow[\aleph_0]{c} (\lambda, \mu)$,

(2) $(\lambda_1, \mu_1) \xrightarrow[\mu]{c} (\lambda, \mu)$,

(3) *there are functions $f_l : \lambda \to \mu$ ($l < \omega$, $f_l$ is $l$-place), such that if $(n, E)$ is realized, then $\lambda_1 \to (n, E)_{\mu_1}$*

**Proof.** See Shelah [2], essentially.

See there also generalizations (for some cardinals, and cardinal-like orderings). Also it is proved that $T$ (as in 1.1) has a $(\lambda, \mu)$-model in which only $\leq 2^{\aleph_0} + |T|$ (pure) types are realized, and for each order $I$, $|I| \leq \mu$, we can assume that the group of automorphism of $I$ can be embedded into the group of automorphism of the $(\lambda, \mu)$-model of $T$. If $(\lambda, \mu, |T|) \xrightarrow[\aleph_0]{c} (\lambda, \mu, |T|)$, we can assume only $\leq |T|$ types are realized.

Inspection of the proofs of the known two-cardinal theorems shows that they are usually proved by identities (see [4] for the old ones, and also [3]). However Vaught and Chang gap-one two-cardinal theorems were proved by more model-theoretic means, and no other ways of proving them is known. We find an alternative proof for Vaught two-cardinal theorem and characterize the identities $(n, \mathrm{E})$ i.e. $\aleph_1 \to (n, \mathrm{E})_{\aleph_0}$.

**Definition 4.** The identity we get from $(n, \mathrm{E})$ by doubling at $k$, is $(m, \mathrm{E}')$ where

(1) $m = n + n - k$

(2) $\mathrm{E}'$ is minimal such that:

(A) $\eta_1 \mathrm{E} \eta_2$ implies $\eta_1 \mathrm{E}' \eta_2$

(B) if $i(0) < \cdots < i(r-1) < k \leq i(r) < \cdots < i(l-1) < n$

then:

$$\langle i(0), \ldots, i(r-1), i(r), \ldots, i(l-1) \rangle$$

and

$$\langle i(0), \ldots, i(r-1), i(r+n-k), \ldots, i(l-1+n-k) \rangle$$

are $\mathrm{E}'$ equivalent.

**Definition 5.** $\Gamma = \Gamma(\aleph_1, \aleph_0)$ is the *smallest set of identities*, containing the trivial set of identities, containing the trivial identity $\langle (1, \mathrm{E}) \rangle$ and closed under doubling and subidentities.

**Lemma 6.** *If* $(n, \mathrm{E}) \in \Gamma$, $\lambda > \mu$, *then* $\lambda \to (n, \mathrm{E})_\mu$.

**Proof.** Clearly we can assume $\lambda = \mu^+$; and by the definition of $\Gamma$, it suffices to prove:

(*) if we get $(m, \mathrm{E}')$ by doubling $(n, \mathrm{E})$ at $k$, and $\mu^+ \to (n, \mathrm{E})_\mu$ then $\mu^+ \to (m, \mathrm{E}')_\mu$.

We use the notation of 1.2, 1.4. So $f_l : \lambda^+ \to \lambda$ is an $l$-place function, and let $\mu = (\lambda^+, f_0, f_1, f_2, \ldots \alpha)_{\alpha < \lambda, l < \omega}$. It is well known that

$$S = \{\alpha < \lambda^+ : \alpha \geq \lambda;\ M \restriction \alpha \text{ is an elementary submodel of } M\}$$

is a closed unbounded subset of $\lambda^+$; hence its order-type is $\lambda^+$, hence by the hypothesis on $(n, \mathrm{E})$ there are in $S$ $\alpha_0 < \cdots < \alpha_{n-1} < \lambda^+$ which realizes the identity $(n, \mathrm{E})$. Let

$$\varphi(x_0, \ldots, x_{n-1}) = \bigwedge_{i<n-1} x_i < x_{i+1} \wedge \bigwedge\{f_l(x_{i(0)}, \ldots, x_{i(l-1)}) = \alpha :$$
$$M \models f_l(\alpha_{i(0)}, \ldots, \alpha_{i(l-1)}) = \alpha\}.$$

Clearly $M \models \varphi[\alpha_0, \ldots, \alpha_{n-1}]$ hence

$$M \models (\exists y_k \cdots y_{n-1})\varphi(\alpha_0, \ldots, \alpha_{k-1}, y_k, \ldots).$$

Necessarily

$$A = \{\beta < \lambda^+ : M \models (\exists y_{k+1}, \ldots, y_{n-1})\varphi(\alpha_0, \ldots, \alpha_{k-1}, \beta, y_{k+1}, \ldots)\}$$

is an unbounded subset of $\lambda^+$ (otherwise there is a least upper bound, which necessarily belongs to $M \upharpoonright \alpha_k$, so $\alpha_k$ is bigger than it and $\alpha_k \in A$, contradiction). So there is $\alpha'_k > \alpha_{n-1}$ in $A$, hence there are $\alpha'_{k+1}, \ldots, \alpha'_{n-1}$ such that

$$M \models \varphi[\alpha_0, \ldots, \alpha_{k-1}, \alpha'_k, \ldots, \alpha'_{n-1}].$$

So necessarily

$$\alpha_0 < \cdots < \alpha_{k-1} < \alpha_k < \cdots < \alpha_{n-1} < \alpha'_k < \cdots < \alpha'_{n-1};$$

and it is easy to check that this sequence realizes $(m, \mathrm{E}')$

**Definition 7.** A *primitive formula* is a formula of the form $\wedge \varphi_i$ where $\varphi_i$ is

(i) $R(x_1, \ldots)x_i$ variable

or

(ii) $f(x_1, \ldots) = c$, $x_i$ a variable $c$ an individual constant

or

(iii) a negation of a formula as in (i).

**Claim 8.** There are $l$-place functions $f_l : \omega \to \omega_0$, such that

(1) $M = (\omega, <, f_0, f_1, \ldots, \alpha)_{\alpha<\omega}$ satisfies:

if $\varphi(x_0, \ldots)$ is primitive, and

$$M \models \varphi(a_0, a_1, \ldots, a_{n-1}) \wedge \bigwedge_{i<n-1} a_i < a_{i+1}$$

and $k < n$, then for every $b$ there are $a'_k, \ldots, a'_{n-1}$, such that $b < a'_k < a'_{k+1} < \cdots < a'_{n-1}$ and

$$M \models \varphi[a_0, \ldots, a_{k-1}, a'_k, \ldots, a'_{n-1}].$$

(2) all identities realized are in $\Gamma$.

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**Proof.** Trivial; we define by induction on $n$, finite sets $\Phi_n$ of conditions of the form $f_l(a_0, \ldots, a_{l-1}) = f(c_0, \ldots, c_{l-1})$ ($a_i, c_i \in \omega$) such that $\Phi_n \subseteq \Phi_{n+1}$, and $\Phi_n$ does not imply an identity not in $\Gamma$, and such that some instances of 1.8(1) are satisfied. For example we can define $\Phi_0 = \phi$, and if $\Phi_n$ is defined, all natural numbers appearing in it are $< r-1 < \omega$ ($r$ minimal), then

$$\Phi_{n+1} = \Phi_n \cup \{f_l(a_0, \ldots, a_{k-1}, a_k, \ldots, a_{l-1}) = f_l(a_0, \ldots, a_{k-1}, a_k + r(b+1), \ldots, a_{n-1} + r(b+1)): k < l, l < r, a_0 < \cdots < a_{k-1} < b \leq a_k < \cdots < a_{l-1}\}.$$

now we define $f_l$, so that $f_l(a_0, \ldots) = f_l(b_0, \ldots)$ iff this equality belongs to some $\Phi_n$.

**Lemma 9.** *Suppose $M$ is a model, its universe is $\omega$, and it has countably many functions and relations, including $<$ (the natural ordering) and $\alpha < \omega$ as individual constants, such that condition (1) of 1.8 is satisfied.*

*Then there is a model $N$ such that:*

(1) *$N$ extends $M$, and its universe is $\omega_1$,*

(2) *$<^N$ is the usual ordering,*

(3) *if $\varphi(\bar{x})$ is primitive and $N \models (\exists \bar{x})\varphi(\bar{x})$, then $M \models (\exists \bar{x})\varphi(\bar{x})$.*

**Proof.** This is a case of Theorem 13 in [1]. (Let $A_n$ be $\{0, \ldots, n-1\}$).

**Proof of Vaught's two-cardinal theorem.** The theorem states: for all $\lambda > \mu \geq \aleph_0$ $\langle \lambda, \mu \rangle \to \langle \aleph_1, \aleph_0 \rangle$. (That means: it $T$ is a theory in a countable language and has a $\langle \lambda, \mu \rangle$ model then it has an $\langle \aleph_1, \aleph_0 \rangle$ model).
We get a little more. For a countable theory $T$: if every finite subset of $T$ has a $\langle \lambda, \mu \rangle$ model, then $T$ has an $\langle \aleph_1, \aleph_0 \rangle$ one. ($\langle \lambda, \mu \rangle \xrightarrow[\aleph_0]{c} \langle \aleph_1, \aleph_0 \rangle$).

By Theorem 3 it is enough to prove that there are functions $f_l : \aleph_1 \to \aleph_0$ $l < \omega$), $f_l$ is $l$-place, such that if $(n, E)$ is realized then $\lambda \to (n, E)_\mu$. By 8, 9, there are such functions realizing only the identities of $\Gamma$, and by 6 it is enough.

## References


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