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ON HCO SPACES. AN UNCOUNTABLE COMPACT T_2 SPACE, DIFFERENT FROM $\aleph_1 + 1$, WHICH IS HOMEOMORPHIC TO EACH OF ITS UNCOUNTABLE CLOSED SUBSPACES

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ABSTRACT

Let X be an Hausdorff space. We say that X is a **CO space**, if X is compact and every closed subspace of X is homeomorphic to a clopen subspace of X, and X is a **hereditarily CO space** (HCO space), if every closed subspace is a CO space. It is well-known that every well-ordered chain with a last element, endowed with the interval topology, is an HCO space, and every HCO space is scattered. In this paper, we show the following theorems:

THEOREM (R. Bonnet):

(a) *Every HCO space which is a continuous image of a compact totally disconnected interval space is homeomorphic to $\beta + 1$ for some ordinal β.*

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(b) *Every HCO space of countable Cantor-Bendixson rank is homeomorphic to $\alpha+1$ for some countable ordinal α.*

THEOREM (S. Shelah): *Assume $\diamondsuit_{\aleph_1}$. Then there is a HCO compact space X of Cantor-Bendixson rank ω_1 and of cardinality $\aleph_1$ such that:*

(1) *X has only countably many isolated points,*

(2) *Every closed subset of X is countable or co-countable,*

(3) *Every countable closed subspace of X is homeomorphic to a clopen subspace, and every uncountable closed subspace of X is homeomorphic to X, and*

(4) *X is retractive.*

In particular X is a thin-tall compact space of countable spread, and is not a continuous image of a compact totally disconnected interval space. The question whether it is consistent with ZFC, that every HCO space is homeomorphic to an ordinal, is open.

1. Survey of the results

DEFINITION 1.1: (a) A Boolean algebra is regarded as an algebraic structure of the form $\langle B,+,\cdot,-,0,1\rangle$ (B may be just $\{0\}$). $+$, $\cdot$, $-$ denote respectively the join, meet and complementation in B. 0^B, 1^B are used when a reference to B is needed, $\leq$ or $\leq^B$ denotes the partial ordering on B ($a\leq b$ if $a\cdot b=a$), and Δ is the symmetric difference in B, that is, $a\Delta b=(a-b)+(b-a)$.

(b) If I is an ideal in a Boolean algebra B, and $a\in B$, then $a/I \stackrel{\text{def}}{=} \{b\in B\colon b\Delta a\in I\}$. If E is a subset of B, then $E/I \stackrel{\text{def}}{=} \{a/I\colon a\in E\}$. Hence $B/I=\{a/I\colon a\in B\}$ denotes the quotient algebra, and for an ideal $J\supseteq I$ of B, J/I is an ideal of B/I.

(c) $a\in B$ is an **atom** of B if $a\neq 0$ and for every $b\in B$ such that $0\leq b\leq a$, $b=0$ or $b=a$. We denote by $\mathrm{At}(B)$ the set of atoms of B. B is **atomic** if for every non-zero $b\in B$, there is $a\in \mathrm{At}(B)$ such that $a\leq b$.

(d) A Boolean algebra B is said to be **superatomic** if every homomorphic image of B is atomic.

(e) If $a\in B$, then $B\restriction a$ denotes the Boolean algebra induced by B on $\{b\in B\colon b\leq a\}$. So $1^{B\restriction a} \stackrel{\text{def}}{=} a$ and $-x^{B\restriction a} \stackrel{\text{def}}{=} a-x$.

(f) For a subset D of B, we denote by:

- $\mathrm{cl}_B(D)$ the subalgebra of B generated by D. For instance, if J is an ideal of B, then $\mathrm{cl}_B(J) = J \cup -J$ where $-J \stackrel{\text{def}}{=} \{-b\colon b \in J\}$.
- $\mathrm{cl}_B^{\mathrm{Id}}(D)$ the ideal of B generated by D.

If B is understood from the context, we omit its mention.

(g) If E is a set, then $\wp(E)$, the power set of E, is regarded as a Boolean algebra, where $+$ and $\cdot$ are $\cup$ and $\cap$ respectively. $\mathrm{FC}(E)$ denotes the subalgebra of $\wp(E)$ of finite or cofinite subsets of E.

(h) Let $\langle C, \leq \rangle$ be a partial ordered set. We say that $\langle C, \leq \rangle$ is a **chain** if every pair of members of C are comparable, and $\langle C, \leq \rangle$ is **well-ordered** if $\langle C, \leq \rangle$ is a chain with no strictly decreasing sequence. Hence $\langle C, \leq \rangle$ is order-isomorphic to an ordinal.

(i) Let $\langle C, \leq \rangle$ be a chain with a first element denoted by 0^C (if $\langle C, \leq \rangle$ has no first element, then we must add one). Let $C^+ \stackrel{\text{def}}{=} C \cup \{\infty^C\}$ be the chain, obtained by adding a greatest element ∞^C. We denote by $B(C)$ the subalgebra of $\wp(C)$ generated by the set of $[a, b)$ for $a \in C$ and $b \in C^+$, i.e. $\mathrm{cl}_{\wp(C)}(\{[a, b)\colon a \in C \text{ and } b \in C^+\})$. $B(C)$ is called the **interval algebra** of C (see Koppelberg [12]).

B is an **ordinal algebra** if B is isomorphic to $B(C)$, where C is a well-ordered chain.

(j) A subalgebra A of B is called a **retract** of J, or of B/J, if for every $b \in B$, $|A \cap (b/J)| = 1$. An ideal J is **retractive**, if there is a retract of J. We say that B is **retractive** if every ideal of B is retractive.

(k) For a Boolean algebra B, let $\mathrm{Ult}(B)$ denote the Boolean space of B, i.e. the space of ultrafilters of B.

Let us recall classical results:

PROPOSITION 1.2: *Let J be an ideal of a Boolean algebra B, A a subalgebra of B, and π_J the canonical homomorphism from B onto B/J. The following properties are equivalent:*

(i) *A is a retract of J*

(ii) *$\pi_J \restriction A$ is an isomorphism from A onto B/J.*

(iii) *$A \cap J = \{0_B\}$ and $\mathrm{cl}_B(A \cup J) = B$.*

Day [9] (see also [12], Vol. 1, pp. 233, and [18]) has given different equivalence conditions for a Boolean algebra to be superatomic:

PROPOSITION 1.3: (Day [1977]) *Let B be a Boolean algebra. The following are equivalent:*

(i) *B is superatomic.*

(ii) *Every subalgebra of B is atomic.*

(iii) *There is no embedding from the atomless countable algebra into B.*

DEFINITION 1.4: (a) Let B be a Boolean algebra. We define, by induction on α, a sequence $\langle I_\alpha(B), D_\alpha(B)\rangle$, with the conditions $D_\alpha(B) \stackrel{\text{def}}{=} B/I_\alpha(B)$ (the algebra $D_\alpha(B)$ is called the α-th **Cantor Bendixson derivative** of B). Let $I_0(B) = \{0\}$, and thus $D_0(B) = B$. $I_1(B) \stackrel{\text{def}}{=} \mathrm{cl}_B^{\mathrm{Id}}(\mathrm{At}(B))$. Suppose that δ is a limit, and $I_\alpha(B)$ has been defined for every $\alpha < \delta$, then $I_\delta(B) = \bigcup_{\alpha<\delta} I_\alpha(B)$. Suppose that $I_\alpha(B)$ has been defined. Then $I_{\alpha+1}(B) \stackrel{\text{def}}{=} \{b \in B\colon b/I_\alpha(B) \in I_1(D_\alpha(B))\}$.

(b) A topological space X is **scattered** if every non-empty subset of X has an isolated point in its subspace topology.

Trivially, $(I_\alpha(B))_{\alpha\in\mathrm{Ord}}$ is an increasing sequence of ideals of B (Ord denotes the class of ordinals). The following additional equivalences are well-known and their proof are straightforward (see [12]).

PROPOSITION 1.5: *Let B be a Boolean algebra. The following conditions are equivalent:*

(i) *B is superatomic.*

(ii) *There is an ordinal γ such that $1_B \in I_\gamma(B)$.*

(iii) *The Boolean space $\mathrm{Ult}(B)$ of B is a scattered space.*

Clearly the first ordinal γ for which $1^B \in I_\gamma(B)$ is a successor ordinal, say $\alpha+1$, then α is denoted by $\mathrm{rk}(B)$. So $1^B \in I_{\mathrm{rk}(B)+1}(B) - I_{\mathrm{rk}(B)}(B)$ and $D_{\mathrm{rk}(B)}(B)$ is a non-trivial finite algebra isomorphic to $\wp(n)$ ($n > 0$ integer). Let $I(B)$ and $D(B)$ denote $I_{\mathrm{rk}(B)}(B)$ and $B/I(B)$ respectively. If $n = 1$, then $I(B)$ is a maximal ideal of B.

Let α be a countable ordinal and $p > 0$ be an integer. If B' and B'' are countable Boolean algebras such that $\mathrm{rk}(B') = \mathrm{rk}(B'') = \alpha$ and $D(B')$, $D(B'')$ and $\wp(p)$ are isomorphic, then B' and B'' are isomorphic algebras (see [12]: §17.2). Hence we denote by $\mathbb{B}_{\alpha,p}$ a representative Boolean algebra of this class.

DEFINITION 1.6: (a) Let X be a topological space. X is a **CO space** if every closed subspace of X is homeomorphic to a clopen subspace of X. X is an **HCO space** if every closed subspace of X is a CO space.

(b) A Boolean algebra B is a **CO algebra (HCO algebra)** if the Boolean space $\mathrm{Ult}(B)$ of B is a CO space (HCO space).

(c) B is a **thin-tall** Boolean algebra, if B is uncountable and for every $\alpha < \mathrm{rk}(B)$, $|\mathrm{At}(D_\alpha(B))| = \aleph_0$.

PROPOSITION 1.7:

(a) *Let B be a Boolean algebra.*

(1) *B is a CO Boolean algebra, if and only if every quotient is isomophic to a factor, i.e. for every ideal I of B, there is an algebra A such that B is isomorphic to $(B/I) \times A$.*

(2) *B is an HCO algebra if and only if every ideal J of B, B/J is a CO algebra.*

(b)

(1) *Every homomorphic image of a HCO algebra is a HCO algebra.*

(2) *Every finite product of CO algebras is a CO algebra.*

(3) *Let B be a finite product of Boolean algebras $(B_i)_{i<n}$. B is a HCO algebra if and only if each B_i is a HCO algebra.*

(c) *Let B be a countable superatomic algebra. Let $\beta = \mathrm{rk}(B) < \omega_1$ and $p = |D_\beta(B)|$. Then B is isomorphic to $\mathbb{B}_{\beta,p}$ and to the ordinal algebra $B(\omega^\beta \cdot p)$.*

(d) *Every ordinal algebra is a HCO algebra. In particular every finite Boolean algebra is a HCO algebra and every countable superatomic interval algebra is a HCO algebra.*

(e) *$\mathrm{FC}(X)$ is a CO algebra if and only if X is countable.*

(f) *If B is an infinite Boolean CO algebra, then B has infinitely many atoms.*

(g) *Every atomic CO algebra is superatomic.*

(h) *Every HCO algebra is superatomic.*

Proof: (a), (e) and (g) are obvious.

(b1) is a trivial consequence of the definition of a HCO algebra.

(b2) is a consequence of the fact that every ideal of $B_1 \times B_2$ has the form $I_1 \times I_2$ where I_ℓ is an ideal of B_ℓ for $\ell = 0, 1$.

(b3) First suppose that for every $i < n$, B_i is HCO. Using (b1) and the arguments of (b2), B is HCO. Conversely, if B is HCO, then B_i is HCO ($i < n$) follows from (b1) and the fact that B_i is a homomorphic image of B.

(c): see [12] (section 17.2).

(d) Let $B = B(C)$ be an interval algebra generated by a well-ordered chain. Let I be an ideal of B. Then B/I is generated by C/I ($= \{c/I\colon c \in C\}$) and C/I is a well-ordered chain too. So it sufficies to show that B is a CO algebra. Let J be an ideal of B. Then C/J is a well-ordered chain and $c \mapsto c/J$ is increasing. Therefore C/J is order-isomorphic to a (unique) initial interval L of C (we recall that $S \subseteq C$ is an initial interval of C if $c \leq s \in S$, implies $c \in S$). From the facts that:

(i) B/J is isomorphic to the interval algebra $B(C/J)$.

(ii) $B(C/J)$ is isomorphic to $B(L)$.

(iii) $B(L)$ is isomorphic to a factor of $B(C)$: this follows from the fact that the result is trivial if $L = C$. If $C - L \neq \emptyset$, then $C - L$ has a first element, and thus $B(C)$ is isomorphic to $B(L) \times B(C - L)$ (see [12], Theorem 15.11; or Lemma 2.3 below).

it follows that B/J is isomorphic to a factor of B. Now, the other parts of (d) are clear (by (c)).

(f) For every integer $n > 1$, let J be an ideal of B such that B/J is finite with n atoms; B is isomorphic to $(B/J) \times A$ for some Boolean algebra A, and thus B has at least n atoms.

(h) Let A be a quotient algebra of B, and $0 \neq a \in A$. Because $A \restriction a$ can be regarded as a quotient of B, $A \restriction a$ is a CO algebra, $A \restriction a$ contains an atom and thus A is atomic. ∎

Bekkali, Bonnet and Rubin ([4] and [5]) have shown that if B is a CO interval algebra, then B is superatomic, and if B is an interval HCO algebra, then B is isomorphic to $B(\alpha)$ for some ordinal α. Note also that S. Mazurkiewicz and W. Sierpinski [14] have shown that countable Boolean algebras are HCO if and only if they are countable ordinal algebras. We will show:

THEOREM 1.8:

(a) *Let B be an HCO algebra (and hence by Proposition 1.7(g), superatomic). If the rank of B is countable, then B is isomorphic to a countable ordinal algebra.*

(b) *Let B be a subalgebra of an interval algebra. If B is an HCO algebra, then B is isomorphic to an ordinal algebra.*

So, we prove that a HCO algebra which satisfies some additional conditions must be isomorphic to an ordinal algebra. In Theorem 1.9, we will see that it is consistent that there are HCO algebras which are not isomorphic to ordinal algebras. Indeed the counterexample that we construct is thin-tall. The counterexample of Theorem 1.9 is also retractive. So we obtain a superatomic retractive Boolean algebra with no uncountable subalgebra embeddable in an interval algebra. The fact that a thin-tall algebra does not contain an uncountable subalgebra embeddable in an interval algebra, is an easy consequence of the following result due to Rubin and Shelah which appear in Avraham and Bonnet ([1]):

If B is an infinite superatomic algebra, embeddable in an interval algebra, then $|B| = |\mathrm{At}(B)|$.

THEOREM 1.9: *Assume $\diamondsuit_{\aleph_1}$. There is a Boolean algebra B such that:*

(A1) *B a thin-tall algebra,* $\mathrm{rk}(B) = \omega_1$ *and* $D_{\omega_1}(B) = \{0,1\}$.

(A2) *B is a HCO algebra.*

(A3) *B is retractive, and for every uncountable ideal I of B, B/I is countable.*

(A4) *Every countable homomorphic image of B is isomorphic to a factor, and is isomophic to $\mathbb{B}_{\alpha,p}$ for some integer $p > 0$ and some countable ordinal α.*

(A5) *Every uncountable homomorphic image of B is isomorphic to B.*

J. Roitman ([19]) has constructed, under CH, a Boolean algebra satisfying the properties (A1), (A2), (A4) and (A5) of Theorem 1.9. (Note that (A4) and (A5) imply (A2): see the proof of Lemma 3.9.)

Comment: Let B be a Boolean algebra satisfying properties (A1)–(A5) above. Then every quotient of B has only countably many atoms. Recall that the *spread of B* is countable if every quotient of B has only countably many atoms, or equivalently, the Boolean space of B has no uncountable discrete subset. We claim that *the spread of B is countable.* For a contradiction, suppose that the spread of B is uncountable. This means that there is an ideal J of B such that B/J has uncountably many atoms. From (A5) and (A1), it follows that $\mathrm{At}(B)$ and thus $\mathrm{At}(B/J)$ is countable. Contradiction. In fact, the Boolean space $\mathrm{Ult}(B)$ of B is an Ostaszewski's space: in particular it is an uncountable scattered compact space such that every open set is countable or co-countable and every closed subset is the closure of a countable set (see Ostaszewski [16], [17], and Rudin [21]: pp. 35–36). Theorem 1.9(A5) is related to the existence of a Toronto space (see [22]: a Toronto space is an uncountable Hausdorff space which is homeomorphic to each of its uncountable subspace).

G. Gruenhage has point out that it is an easy consequence of a work of Balogh, Dow, Fremlin and Nyikos [2] that, under PFA, every HCO algebra of cardinality $\aleph_1$ is isomorphic to $B(\omega_1)$. The following problems are open.

QUESTION 1.10: (a) *In ZFC, is it consistent that every HCO algebra of cardinality* $\geq \aleph_2$ *is an ordinal algebra?*

(b) *Is a CO algebra superatomic?* ∎

To complete this introduction, let us state some recent results related to this subject:

The first is due to M. Weese ([23]):

PROPOSITION 1.11:

(a) *Let B be a Boolean algebra of cardinality $\aleph_1$ such that every uncountable homomorphic image is isomorphic to B. Then B is superatomic.*

(b) *Let B be Boolean algebra of cardinality $\aleph_1$ such that every uncountable subalgebra is isomorphic to B. Then B is superatomic. Moreover if $2^{\aleph_0} < 2^{\aleph_1}$ and B is not isomorphic to* $\mathrm{FC}(\omega_1)$, *then B is thin-tall.*

The second result, due to R. Bonnet and M. Rubin ([7]), concerns Boolean algebras B for which each uncountable subalgebra is isomorphic to B:

THEOREM 1.12: *Assume $\diamondsuit_{\aleph_1}$. Then there is a thin-tall Boolean algebra B such that every uncountable subalgebra is isomorphic to B.*

2. Proof of Theorem 1.8

Let us begin to introduce a definition and a result which are needed in the proof of Theorems 1.8 and 1.9.

DEFINITION 2.1: (a) Let B be a superatomic Boolean algebra, and $b \in B$, $b \neq 0$. Let γ be the first ordinal α such that $b \in I_\alpha(B)$. Clearly, γ is a successor ordinal, say $\beta + 1$, and we set $\mathrm{rk}^B(b) = \beta$; so $b \in I_{\mathrm{rk}^B(b)+1}(B) - I_{\mathrm{rk}^B(b)}(B)$. For instance $\mathrm{rk}^B(b) = 0$ for $b \in \mathrm{At}(B)$, and $\mathrm{rk}^B(1) = \mathrm{rk}(B)$.

(b) Let J be an ideal of a Boolean algebra B. We denote by J^c the ideal of those $b \in B$ such that $b \cdot a = 0$ for every $a \in J$.

(c) (1) Let C_1 and C_2 be two chains with first elements. We denote by $C_1 + C_2$ the chain, lexicographic sum of C_1 and C_2 (by definition, $x \leq y$ in $C_1 + C_2$ if x, y are in the same C_i and $x \leq y$ in C_i or, $x \in C_1$ and $y \in C_2$).

(2) Let $(C_\alpha)_{\alpha<\rho}$ be a family of chains indexed by an ordinal ρ. Then the lexicographic sum C of $(C_\alpha)_{\alpha<\rho}$, defined similarly to (c1), is a chain. We suppose that each C_α has a first element. Then the interval algebra $B(C_\alpha)$ is canonically isomorphic to a factor of $B(C)$.

LEMMA 2.2:

(a) *Let B be a Boolean algebra and $b \in B$, $b \neq 0$. Then for every ordinal α:*

(1) $I_\alpha(B \restriction b) = I_\alpha(B) \cap (B \restriction b)$.

(2) $D_\alpha(B) \to (D_\alpha(B \restriction b)) \times (D_\alpha(B \restriction -b))$ *defined by* $a/I_\alpha(B) \mapsto \langle a \cdot b/I_\alpha(B \restriction b), a \cdot -b/I_\alpha(B \restriction -b)\rangle$ *(for $a \in B$) is an isomorphism onto.*

(3) *If B is superatomic, then* $\mathrm{rk}^B(b) = \mathrm{rk}(B \restriction b) = \mathrm{rk}(\mathrm{cl}_B(B \restriction b))$.

(b) *Let A and B be superatomic Boolean algebras, and f be a homomorphism from B onto A. Then for every ordinal α:*

(1) $f[I_\alpha(B)] \subseteq I_\alpha(A)$.

(2) *f induces a homomorphism f_α from $D_\alpha(B)$ onto $D_\alpha(A)$ defined by*

$$f_\alpha(b/I_\alpha(B)) = f(b)/I_\alpha(A) \quad (\text{for } b \in B).$$

(3) $\mathrm{rk}(A) \leq \mathrm{rk}(B)$.

(4) *Let A be a subalgebra of B, and $a, b \in B$ such that $a \leq b$. Then* $\mathrm{rk}(A \restriction a) \leq \mathrm{rk}(A \restriction b) \leq \mathrm{rk}(A) = \mathrm{rk}^A(1)$.

(c) *Let A and B be superatomic Boolean algebras, and f be a one-to-one homomorphism from A into B. Then for every ordinal α:*

(1) $f^{-1}[I_\alpha(B)] \subseteq I_\alpha(A)$.

(2) $\mathrm{rk}(A) \leq \mathrm{rk}(B)$.

(d) *For every ideal J of an algebra B, $\mathrm{cl}^{\mathrm{Id}}_B(J \cup J^c)$ is a dense ideal of B.*

(e) *Let B be an atomic Boolean algebra, J an ideal of B and π be the canonical homomorphism from B onto B/J^c.*

(1) *If J is principal, say generated by a, then B/J is isomorphic to $B \restriction -a$, B/J^c is isomorphic to $B \restriction a$, and both are atomic.*

(2) *If J is non-principal, then:*

- $\mathrm{cl}_B(J)$ and B/J^c are atomic,
- $\mathrm{At}(\mathrm{cl}_B(J)) = \mathrm{At}(B) \cap J$,
- $\pi \restriction \mathrm{At}(\mathrm{cl}_B(J))$ *is a one-to-one function from* $\mathrm{At}(\mathrm{cl}_B(J))$ *onto the set* $\mathrm{At}(B/J^c)$,

- *$\pi \restriction \mathrm{cl}_B(J)$ is an isomorphism into B/J^c, and*
- $\mathrm{cl}_B(J) \cap J^c = \{0\}$.

Proof: (a) Simultaneously, we will prove by induction: (a1) and (a2) and

$$(\star)_\alpha\colon\quad \{a \in B \restriction b\colon a/I_\alpha(B \restriction b) \in \mathrm{At}(D_\alpha(B \restriction b))\} = \{a \in B \restriction b\colon a/I_\alpha(B) \in \mathrm{At}(D_\alpha(B))\}$$

We prove the three conditions simultaneously by induction on α; except that $(\star)_\alpha$ and $(\mathrm{a2})_\alpha$ are consequences of $(\mathrm{a1})_\alpha$. We prove the last fact first. For $(\mathrm{a2})_\alpha$, define for any $a \in B$, $h(a) = \langle a \cdot b/I_\alpha(B \restriction b), a \cdot -b/I_\alpha(B \restriction -b)\rangle$. Thus h is a homomorphism from B onto $D_\alpha(B \restriction b) \times D_\alpha(B \restriction -b)$, with kernel $\{a \in B\colon a \cdot b \in I_\alpha(B \restriction b)$ and $a \cdot -b \in I_\alpha(B \restriction -b)\}$, which is by $(\mathrm{a1})_\alpha$ $\{a \in B\colon a \cdot b \in I_\alpha(B)$ and $a \cdot -b \in I_\alpha(B)\} = I_\alpha(B)$. The existence of the indicated isomorphism is hence clear. For $(\star)_\alpha$, suppose that $c \in B \restriction b$ is such that $c/I_\alpha(B \restriction b) \in \mathrm{At}((B \restriction b)/I_\alpha(B \restriction b))$, and hence by $(\mathrm{a2})_\alpha$, $c/I_\alpha(B) \in \mathrm{At}(D_\alpha(B))$, as desired; the other inclusion in $(\star)_\alpha$ is proved similarly.

Now, we turn to the inductive step. The case $\alpha = 0$ is obvious. Suppose the conditions for α. Suppose that $c \in I_{\alpha+1}(B \restriction b)$. There is a finite subset $\{d_i\colon i < n\}$ of $B \restriction b$ such that $d_i/I_\alpha(B \restriction b) \in \mathrm{At}(D_\alpha(B \restriction b))$ $(i < n)$ and $c/I_\alpha(B \restriction b) = \sum\{d_i/I_\alpha(B \restriction b)\colon i < n\}$. It follows that $c\Delta(\sum\{d_i\colon i < n\}) \in I_\alpha(B \restriction b) \subseteq I_\alpha(B)$ by $(\mathrm{a1})_\alpha$. By $(\star)_\alpha$, $d_i/I_\alpha(B) \in \mathrm{At}(D_\alpha(B))$, for $i < n$, and thus $c \in I_{\alpha+1}(B) \cap (B \restriction b)$. Conversely, suppose that $c \in I_{\alpha+1}(B) \cap (B \restriction b)$. Then we write $c/I_\alpha(B) = \sum\{d_i/I_\alpha(B)\colon i < n\}$ with $d_i/I_\alpha(B) \in \mathrm{At}(D_\alpha(B))$. Now, $c \le b$, so $d_i/I_\alpha(B) \le c/I_\alpha(B) \le b/I_\alpha(B)$. Hence we may assume that $d_i \le b$ for $i < n$, and the argument goes as above (but backwards). The limit step of the induction is obvious.

(a3) follows directly from (a1), the definitions of $\mathrm{rk}^B(b)$ and of $\mathrm{rk}(B \restriction b)$, and from the fact that $\mathrm{cl}_B(B \restriction b) = (B \restriction b) \cup -(B \restriction b)$ is isomorphic to $(B \restriction b) \times \{0, 1\}$ if $b \ne 0, 1$.

(b1) is proved by induction. The cases $\alpha = 0$ and α limit are trivial. Now, suppose that $f[I_\alpha(B)] \subseteq I_\alpha(A)$ and f induces an homomorphism f_α from $D_\alpha(B)$ onto $D_\alpha(A)$ defined by $f_\alpha(b/I_\alpha(B)) = f(b)/I_\alpha(A)$ (for $b \in B$). Let $a \in f[I_{\alpha+1}(B) - I_\alpha(B)]$. So $a = \sum_{i<n} f(b_i)$, where the $b_i/I_\alpha(B)$'s are atoms of $D_\alpha(B)$. It suffices to show that $f(b_i) \in I_{\alpha+1}(A)$ for all $i < n$. Let $i < n$. Let $\langle a_i^0, a_i^1\rangle$ be a partition of $f(b_i)$. Hence there are $b_i^\ell \in B$ such that $f(b_i^\ell) = a_i^\ell$ for $\ell = 0, 1$. There is no loss in assuming that $\langle b_i^0, b_i^1\rangle$ is a partition of b_i. Because $b_i/I_\alpha(B)$ is an atom, $b_i^\ell/I_\alpha(B) = 0$ for some ℓ, say ℓ'. Hence, $a_i^{\ell'}/I_\alpha(A) = 0$ by the induction hypothesis. So $a \in I_{\alpha+1}(A)$.

(b2) follows from (b1). (b3) follows from (b1) and (b2). For (b4), we recall that $A \restriction a$ is a homomorphic image of A.

(c1) is proved by induction on α. The cases $\alpha = 0$ and α limit are trivial. Now, for a contradiction, suppose that $a \in f^{-1}[I_{\alpha+1}(B)] - I_{\alpha+1}(A)$. Write $a = \sum_{i<n} a_i$, where the a_i's are pairwise disjoint elements of A and $f(a_i)/I_\alpha(B) \in \mathrm{At}(D_\alpha(B))$. Because $a \notin I_{\alpha+1}(A)$, there exists disjoint elements $c_0, \dots c_n \in A$ such that $a = \sum_{i\leq n} c_i$ and $c_j/I_\alpha(A) \neq 0$ in $D_\alpha(A)$ for all $j \leq n$. Now $f(a_i) = \sum_{j\leq n} f(a_i)\cdot f(c_j)$, so there is an $\varphi(i) \leq n$ such that $f(a_i)\cdot f(c_{\varphi(i)}) \notin I_\alpha(B)$ and $f(a_i)\cdot f(c_j) \in I_\alpha(B)$ for all $j \neq \varphi(i)$. Choose $\ell \in (n+1) - \mathrm{rng}(\varphi)$. Then:

$$f(c_\ell) = \textstyle\sum_{i<n} f(c_\ell)\cdot f(a_i) \ \in I_\alpha(B).$$

Hence $c_\ell \in I_\alpha(A)$ by the induction hypothesis, contradiction.

(c2) is consequence of (c1).

(d) is trivial.

(e) The facts about principal ideals are trivial. So, assume that J is non-principal. Clearly, $\mathrm{At}(\mathrm{cl}_B(J)) \supseteq \mathrm{At}(B) \cap J$. Suppose that $a \in \mathrm{At}(\mathrm{cl}_B(J))$. If $-a \in J$, choose $x \in J$ with $-a < x$; then $0 < -x < a$, contradiction. So $a \in J$, and it is an atom of B. This proves that $\mathrm{At}(\mathrm{cl}_B(J)) = \mathrm{At}(B) \cap J$. To show that $\mathrm{cl}_B(J)$ is atomic, suppose that $0 \neq b \in \mathrm{cl}_B(J)$. If $b \in J$, it is easy to get an atom contained in b. Suppose that $-b \in J$. Choose $-b < y \in J$, and let x be an atom contained in $y \cdot b$; clearly x is as desired.

It is clear that $\pi \restriction \mathrm{At}(\mathrm{cl}_B(J))$, is one-to-one; and it is easily checked that it maps into $\mathrm{At}(B/J^c)$. Now, suppose that b/J^c is an atom of B/J^c. Thus $b \notin J^c$, so there is an $x \in J$ such that $b\cdot x \neq 0$. Let a be an atom of B such that $a \leq b\cdot x$. Then clearly $a/J^c = b/J^c$. This shows that $\pi \restriction \mathrm{At}(\mathrm{cl}_B(J))$ maps onto $\mathrm{At}(B/J^c)$. The other statements of the claim are now clear. ∎

Now, we are ready to prove Part (a) of 1.8.

2.1 Proof of Theorem 1.8(a). It suffices to prove, by induction on the rank of B, that B is countable (see for example Henkin, Monk and Tarski [10].) First, if $\mathrm{rk}(B) = 0$, i.e. B is finite, then the result is trivial. Now, let $\alpha < \omega_1$. We suppose that every HCO algebra A of rank $< \alpha$ is countable. Let B be a HCO algebra of rank α. By Proposition 1.7(b3) there is no loss in assuming that $D_\alpha(B) = \{0,1\}$. For a contradiction, assume that B is uncountable.

We claim that B has uncountably many atoms. Because B is uncountable, and

$\mathrm{rk}(B) < \omega_1$, there is $\alpha < \mathrm{rk}(B)$ such that $D_\alpha(B)$ has uncountably many atoms. Now our claim follows from the CO property.

We say that $a \in B$ is a generalized atom of B if there is β such that $a/I_\beta(B) \in \mathrm{At}(D_\beta(B))$. We will construct a sequence $(a_n)_{n<\omega}$ of pairwise disjoint generalized atoms of B such that $\sup\{\mathrm{rk}^B(a_n)+1 : n < \omega\} = \alpha$. First suppose that $\alpha = \beta+1$. From $D_\alpha(B) = \{0,1\}$, it follows that the algebra $D_\beta(B)$ is isomorphic to $\mathrm{FC}(X)$ for some infinite set X. Hence, by induction, we can choose a set $\{a_n : n < \omega\} \subseteq B$ of pairwise disjoint generalized atoms of $D_\beta(B)$. Suppose α is limit. By induction, we construct a set $\{a_n : n < \omega\}$ of pairwise disjoint elements of B such that $\sup\{\mathrm{rk}^B(a_n) + 1 : n < \omega\} = \alpha$ and each a_n is a generalized atom of $D_{\mathrm{rk}^B(a_n)}(B)$. The set $\{a_n : n < \omega\}$ is as required.

Let $J = \mathrm{cl}_B^{\mathrm{Id}}(\bigcup\{B \restriction a_n : n < \omega\})$. Obviously J is a non-principal ideal of B. Clearly $\mathrm{cl}_B(J)$ has rank α. By Lemma 2.2(e2), (c) and (b), $\mathrm{rk}(\mathrm{cl}_B(J)) \leq \mathrm{rk}(B/J^c) \leq \mathrm{rk}(B) = \alpha$, and hence $\mathrm{rk}(B/J^c) = \alpha$. Because $B \restriction a_n$ is of rank $\mathrm{rk}^B(a_n) < \alpha$, it is countable; hence so is $\mathrm{cl}_B(J)$, and so by Lemma 2.2(e2) again, B/J^c has only countably many atoms. Now B/J^c is isomorphic to a factor of B; say that B is isomorphic to $(B/J^c) \times C$. Since B/J^c has rank α and $D_\alpha(B) = \{0,1\}$ it follows that C has rank $< \alpha$ and hence C is countable. Now, because B/J^c and C have only countably many atoms, the same holds for $(B/J^c) \times C$, and thus for B, contradiction. ∎

2.2 Proof of Theorem 1.8(b). Let us begin by a simple fact on interval Boolean algebra:

Lemma 2.3:

(a) *Let C_1 and C_2 be two chains with first elements. Then $B(C_1 + C_2)$ is canonically isomorphic to $B(C_1) \times B(C_2)$.*

(b) *Let C be a chain with a first element and B be a Boolean algebra. Assume that B is isomophic to $B(C)$. Then there is a chain $C' \subseteq B$ such that C and C' are order-isomorphic, and $B = \mathrm{cl}_B(C')$.*

(c) *Let C be a subchain of a Boolean algebra B. We suppose that $0^B \in C$ (and so $0^B = 0^C$). Then $B(C)$ and $\mathrm{cl}_B(C)$ are isomorphic.*

Proof: (a) More precisely, let us recall the following fact concerning chains and Boolean algebras (see [12]: Proposition 15.11). Let C_1 and C_2 be chains with first element 0^{C_1} and 0^{C_2} respectively. Let $C = C_1 + C_2$ be the chain, lexicographic sum of C_1 and C_2 (so $c_1 < c_2$ for $c_1 \in C_1$ and $c_2 \in C_2$). Note that

C has a first element, namely 0^{C_1}. A canonical isomorphism f from $B(C)$ onto $B(C_1) \times B(C_2)$ is obtained by letting: $f(c) = \langle c \cap C_1, c \cap C_2 \rangle$. Let us remark that we identified ∞^{C_1} with 0^{C_2}. $B(C_1)$, $B(C_2)$ are factors of $B(C)$; and by identification, $B(C) \restriction C_1 = B(C_1)$ and $B(C) \restriction C_2 = B(C_2)$.

(b) and (c) are obvious. ∎

Let B be an infinite Boolean algebra of rank α. We suppose that B is embeddable in an interval algebra, and B is a HCO algebra. By Proposition 1.7(b3), we can suppose that $D_\alpha(B) = \{0,1\}$ and, from Theorem 1.8(a), $\alpha \geq \omega_1$. We will reduce the proof of Theorem 1.8(b) to:

LEMMA 2.4: *Let B be a superatomic subalgebra of an interval algebra. We suppose that B is an HCO algebra. We set $\alpha = \mathrm{rk}(B)$ and we suppose $D_\alpha(B) = \{0,1\}$.*

Then there are a chain C in B and an ideal I of B such that

(1) *C is order-isomorphic to ω^α,*
(2) *B/I is isomorphic to $B(\omega^\alpha)$,*
(3) *$\mathrm{cl}_B(C)$ is isomorphic to $B(C)$, and*
(4) *$\mathrm{cl}_B(C)$ is a retract of B/I.*

Assuming Lemma 2.4 for a moment, let us see why this finishes the proof of Theorem 1.8(b). First note that, by an easy induction, $\mathrm{rk}(B(\omega^\alpha)) = \alpha$ and $D_\alpha(B(\omega^\alpha)) = \{0,1\}$. Now, from Lemma 2.4, there is a subchain C of B and an ideal J of B such that: C is order-isomorphic to ω^α; the interval algebra $B(C)$ is isomorphic to the subalgebra $\mathrm{cl}_B(C)$ of B; and the interval algebra $B(C)$ is isomorphic to B/J. Now, B is isomorphic to $(B/J) \times A$ for some algebra A. From the facts that $\mathrm{rk}(B/J) = \mathrm{rk}(B) = \alpha$, and the fact that $D_\alpha(B) = \{0,1\}$, it follows that $\mathrm{rk}(A) < \alpha$. Because A is an homomorphic image of B, A is a HCO algebra of rank$< \alpha$. By the induction hypothesis, A is isomorphic to $B(C')$ where C' is a well-founded chain. Hence, by Lemma 2.3(a), B is generated by a well-ordered chain, namely $C' + C$. ∎

So it suffices to prove Lemma 2.4. Before continuing, let us give an example of Boolean algebra, of rank ω_1, embeddable in an interval algebra, with no uncountable chain (so, by Lemma 2.4, such an algebra is not a HCO algebra). Consider $B(\omega_1)$. Let $(a_\alpha)_{\alpha<\omega_1}$ be the strictly increasing sequence in ω_1 defined by $a_0 = 0$; $(a_\alpha, a_{\alpha+1})$ is order-isomorphic to $\alpha + 1$, for successor ordinal α; and $a_\lambda = \sup\{a_\alpha \colon \alpha < \lambda\}$ for limit ordinals $\lambda < \omega_1$. Let B be the subalgebra of

$B(\omega_1)$ consisting of $b \in B(\omega_1)$ such that b or $-b$ is disjoint from $\{a_\alpha: \alpha < \omega_1\}$, i.e. either b or $-b$ is contained in a finite union of $(a_\alpha, a_{\alpha+1})$. Then $\mathrm{rk}(B) = \omega_1$ and every subchain in B is countable.

We begin by two facts concerning interval Boolean algebras.

LEMMA 2.5: *Let D be a well-ordered subchain of a Boolean algebra B. We denote by $\{c_\theta: \theta < \rho\}$ the canonical increasing enumeration of the elements of $D \cup \{0\}$, and let $J \stackrel{\mathrm{def}}{=} \mathrm{cl}^{\mathrm{Id}}_B(\bigcup\{B \restriction (c_{\theta+1} - c_\theta): \theta < \rho\})$. We suppose that:*

(1) *$\mathrm{cl}^{\mathrm{Id}}_B(D)$ is a non-principal ideal of B.*
(2) *For $\theta < \rho$, $B \restriction (c_{\theta+1} - c_\theta)$ is isomorphic to an interval algebra, denoted by $B(C_\theta)$.*

Then $\mathrm{cl}_B(D \cup J)$ is isomorphic to the interval algebra $B(C)$ where C is the lexicographic sum of $(C_\theta)_{\theta<\rho}$.

Proof: For each $\theta < \rho$, let $B_\theta \stackrel{\mathrm{def}}{=} B \restriction (c_{\theta+1} - c_\theta)$. By Lemma 2.3(b), we can suppose that $C_\theta \subseteq B_\theta \subseteq B$. Let C be the lexicographic sum of $(C_\theta)_{\theta<\rho}$. We set $\underline{C}_\theta = \{c_\theta + b: b \in C_\theta\}$ for $\theta < \rho$ and $\underline{C} = \bigcup\{\underline{C}_\theta: \theta < \rho\}$. Then C and $\underline{C}$ are trivially order-isomorphic. Hence we consider C as a subchain of B. We claim that $\mathrm{cl}_B(D \cup J)$ and $B(C)$ are isomorphic. To see this, let us remark that, if we denote by r_θ the first element of $C_\theta \subseteq C$, then $C_\theta = [r_\theta, r_{\theta+1})$. Let $D^\star = \{r_\theta: \theta < \rho\}$ and $\psi^\star$ be the isomorphism from $B(D^\star)$ onto $\mathrm{cl}_B(D)$ defined by $\psi^\star(r_\theta) = c_\theta$. Hence $\psi^\star([r_\theta, r_{\theta+1})) = c_{\theta+1} - c_\theta$. Let ψ_θ be an isomorphism from $B(C_\theta)$ onto B_θ. Now, let $u < v$ in C. Let $\mu \leq \nu < \rho$ be such that $u \in C_\mu$ and $v \in C_\nu$. Then

$$\psi([u,v)) \stackrel{\mathrm{def}}{=} \psi_\mu([u,v) \cap C_\mu) \cup (c_\nu - c_{\mu+1}) \cup \psi_\nu([u,v) \cap C_\nu)$$

extends $\psi^\star \cup \bigcup_{\theta<\rho} \psi_\theta$, and is extendable in an isomorphism from $B(C)$ onto $\mathrm{cl}_B(D \cup J)$. ∎

Because $\mathrm{FC}(\omega)$ and $B(\omega)$ are isomorphic algebras, the proof above, can be applied to show:

LEMMA 2.6: *Let $\{b_n: n \in \omega\}$ be a set of pairwise disjoint non-zero element of a Boolean algebra B. We suppose that $B \restriction b_n$ is isomorphic to an interval algebra, denoted by $B(C_n)$, for $n \in \omega$. We set*

$$J = \mathrm{cl}^{\mathrm{Id}}_B(\bigcup\{B \restriction b_n: n \in \omega\})$$

Then $\mathrm{cl}_B(J)$ *is isomorphic to the interval algebra* $B(C)$ *where* C *is the lexicographic sum of* $(C_n)_{n<\omega}$.

Let us introduce some notions and results concerning Boolean algebras and partial ordered sets.

DEFINITION 2.7: Let B be a superatomic Boolean algebra.

Let $\alpha \leq \mathrm{rk}(B)$. We set $\widehat{\mathrm{At}}_\alpha(B) \stackrel{\mathrm{def}}{=} \{a \in B\colon a/I_\alpha(B) \in \mathrm{At}(D_\alpha(B))\}$, and $\widehat{\mathrm{At}}(B) \stackrel{\mathrm{def}}{=} \bigcup_{\alpha\leq\mathrm{rk}(B)} \widehat{\mathrm{At}}_\alpha(B)$. An element a of $\widehat{\mathrm{At}}(B)$ is called **a generalized atom** of B.

We say that H is **a complete set of generalized atoms** of B, whenever $H = \bigcup\{H_\alpha\colon \alpha \leq \mathrm{rk}(B)\}$, where for every $\alpha \leq \mathrm{rk}(B)$, $H_\alpha \subseteq \widehat{\mathrm{At}}_\alpha(B)$ and for every $g \in \mathrm{At}(B)$, there is a unique $h \in H$ such that $\mathrm{rk}^B(g\Delta h) < \mathrm{rk}^B(h) = \mathrm{rk}^B(g)$. That means that for every $\alpha \leq \mathrm{rk}(B)$, $a \mapsto a/I_\alpha(B)$ is a one-to-one function from H_α onto $\mathrm{At}(D_\alpha(B))$. Note that $0 \notin H$. Obviously, if H is a complete set of generalized atoms of B and $\alpha \leq \mathrm{rk}(B)$, then H_α denotes $\{h \in H\colon \mathrm{rk}^B(h) = \alpha\}$.

The following result seems to be well-known (see for example Bekkali [3]).

LEMMA 2.8: *Let* B *be a superatomic Boolean algebra.*

(a) *Let* $H = \bigcup_{\alpha\leq\mathrm{rk}(B)} H_\alpha$ *be a complete set of generalized atoms of* B. *Then for every ordinal* α,
$$I_\alpha(B) = \mathrm{cl}_B^{\mathrm{Id}}(\textstyle\bigcup_{\beta<\alpha} H_\beta) \subseteq \mathrm{cl}_B(\bigcup_{\beta<\alpha} H_\beta) = I_\alpha(B) \cup -I_\alpha(B).$$
(b) *In particular* $B = \mathrm{cl}_B(H)$.

Proof: (a) First let us remark that if $\bigcup_{\beta<\alpha} H_\beta \subseteq I_\alpha(B) \subseteq \mathrm{cl}_B(\bigcup_{\beta<\alpha} H_\alpha)$, then $\mathrm{cl}_B(\bigcup_{\beta<\alpha} H_\beta) = I_\alpha(B) \cup -I_\alpha(B)$. So it suffices to prove, by induction on α, that $I_\alpha(B) = \mathrm{cl}_B^{\mathrm{Id}}(\bigcup_{\beta<\alpha} H_\beta) \subseteq \mathrm{cl}_B(\bigcup_{\beta<\alpha} H_\beta)$. For $\alpha = 0$, we have $I_0(B) = \{0\} = \mathrm{cl}_B^{\mathrm{Id}}(\emptyset) \subseteq \mathrm{cl}_B(\emptyset) = \{0,1\}$. For α limit, we have
$$I_\alpha(B) = \textstyle\bigcup_{\beta<\alpha} I_\beta(B) = \bigcup_{\beta<\alpha}\left(\mathrm{cl}_B^{\mathrm{Id}}(\bigcup_{\gamma<\beta} H_\gamma)\right) = \mathrm{cl}_B^{\mathrm{Id}}(\bigcup_{\beta<\alpha} H_\beta),$$
and $\bigcup_{\beta<\alpha} I_\beta(B) \subseteq \bigcup_{\beta<\alpha} \mathrm{cl}_B(\bigcup_{\gamma<\beta} H_\gamma) = \mathrm{cl}_B(\bigcup_{\beta<\alpha} H_\beta)$. Let us consider the successor case. Now, let $a \in I_{\alpha+1}(B) - I_\alpha(B)$. Because $a/I_\alpha(B)$ is a finite sum of atoms of $D_\alpha(B)$, there is a finite subset F of H_α such that $\tilde{a} \stackrel{\mathrm{def}}{=} \sum\{h\colon h \in F\}$ verifies $a/I_\alpha(B) = \tilde{a}/I_\alpha(B)$, and thus $a\Delta\tilde{a} \in I_\alpha(B)$. Hence there are unique $u, v \in I_\alpha(B)$ such that $a = (\tilde{a} + v) - u$, $u \leq \tilde{a}$ and $v \cdot \tilde{a} = 0$. By induction hypothesis, $u, v \in \mathrm{cl}_B^{\mathrm{Id}}(\bigcup_{\beta<\alpha} H_\beta)$ (contained in $\mathrm{cl}_B(\bigcup_{\beta<\alpha} H_\beta)$), and thus $\tilde{a} + v \in \mathrm{cl}_B^{\mathrm{Id}}(\bigcup_{\beta\leq\alpha} H_\beta)$ (and $\tilde{a} + v \in \mathrm{cl}_B(\bigcup_{\beta\leq\alpha} H_\beta)$). Hence $a \in \mathrm{cl}_B^{\mathrm{Id}}(\bigcup_{\beta\leq\alpha} H_\beta)$ and $a \in \mathrm{cl}_B(\bigcup_{\beta\leq\alpha} H_\beta)$. Now, suppose that $a \in \mathrm{cl}_B^{\mathrm{Id}}(\bigcup_{\beta\leq\alpha} H_\beta)$. Then $a \leq \sum_{i<n} a_i + v$ with $v \in \mathrm{cl}_B^{\mathrm{Id}}(\bigcup_{\beta<\alpha} H_\beta)$ and $a_i \in H_\alpha$ $(i < n)$. By induction hypothesis, $v \in$

$I_\alpha(B)$, and thus $a/I_\alpha(B) \leq \sum_{i<n} a_i/I_\alpha(B)$. Therefore $a/I_\alpha(B)$ is a finite sum of atoms of $D_\alpha(B)$ and so $a \in I_{\alpha+1}(B)$. Hence $I_{\alpha+1}(B) = \mathrm{cl}_B^{\mathrm{Id}}(\bigcup_{\beta\leq\alpha} H_\beta)$.

(b) is a direct consequence of (a). ∎

Let us introduce some characterization of superatomic subalgebra of an interval algebra:

THEOREM 2.9: *Let B be a Boolean algebra. The following conditions are equivalent:*

(i) *B has a complete set G of generalized atoms such that every pair of which are either comparable or disjoint.*

(ii) *B is embeddable in an ordinal algebra.*

(iii) *B is embeddable in an interval algebra $B(C)$ generated by a well founded chain C such that $\mathrm{At}(B) = \mathrm{At}(B(C))$.*

(iv) *B is a superatomic Boolean algebra, embeddable in an interval algebra.*

Proof: For (i)⇔(ii), see Bonnet, Rubin and Si-Kaddour [8], and for (ii)⇔(iii)⇔(iv), see Avraham and Bonnet [1] (for a self-contained proof of 2.9, see also Bonnet [6]). ∎

DEFINITION 2.10: (a) Let $\langle P, \leq\rangle$ be a partial ordering, and X a subset of P. We say that:

(1) X is **cofinal** in P if for every $p \in P$, there is $q \in X$ such that $q \geq p$.

(2) X is an **initial interval** of P if $p \leq q$ and $q \in X$ implies $p \in X$, and

(3) X is an **ideal** of P if X is an initial interval of P such that for every $p, q \in X$, there is $r \in X$ such that $r \geq p, q$.

(4) Let $p, q \in X$. q is a **successor** of p in X if $p < q$ and $(p, q) \cap D = \emptyset$. We also say that p and q are **consecutive.** Moreover, if p has a unique successor, it is denoted by p^+_X or more simply p^+.

(b) Let B and G be as in Theorem 2.9(i) (so, by 2.9(iv) B is superatomic). We suppose $0 \neq 1$. Whenever $D_{\mathrm{rk}(B)}(B) = \{0, 1\}$, let:

$G^- \stackrel{\mathrm{def}}{=} \{g \in G: \mathrm{rk}^B(g) < \mathrm{rk}^B(1)\}$, and

$\mathrm{Maxid}(G^-)$ be the set of maximal ideals of the partial ordering $\langle G^-, \leq^B \restriction G^-\rangle$.

LEMMA 2.11:

(a) *Let B and G be as in Theorem 2.9*(i) *(so B is superatomic and, by Lemma 2.8, $\mathrm{cl}_B(G) = B$). We consider $\langle G, \leq\restriction G\rangle$ as a partial ordering. We recall that $B \neq \{0, 1\}$.*

(1) *G is a well-founded set, containing* $\mathrm{At}(B)$. *Moreover* $\mathrm{At}(B)$ *is the set of minimal elements of G.*

(2) *G verifies: "For every element g of G, the set $\{g' \in G\colon g' \geq g\}$ is totally ordered."*

In particular:

(3) *If D is a subchain of G, and if $d \in D$ is not a maximal element of D, then d has a unique successor d_D^+ in D.*

(b) *Let B and G be as in Theorem 2.9*(i). *We suppose that*

$$D_{\mathrm{rk}(B)}(B) = \{0,1\}.$$

Then:

(1) $\mathrm{cl}_B^{\mathrm{Id}}(G^-)$ *is a maximal ideal of B, and thus* $B = \mathrm{cl}_B(G^-)$.

(2) *Let a be an atom of B. Because* ($a2$) *and* $\mathrm{At}(B) \subseteq G^-$, *there is a unique maximal chain of G^-, containing a, denoted by C_a, and the initial interval I_a of G^- generated by C_a is an element of* $\mathrm{Maxid}(G^-)$.

(3) *Conversely, let K be an element of* $\mathrm{Maxid}(G^-)$. *Let $a \in K \cap$* $\mathrm{At}(B)$, *and C_a be the maximal chain of G^- containing a. Then $C_a \subseteq K$ and thus $I_a = K$.*

(4) *Let $K \in$* $\mathrm{Maxid}(G^-)$ *and C_a as in* (b2). *Then C_a is cofinal in K and if $g' < g''$ in C_a, then* $\mathrm{rk}^B(g') < \mathrm{rk}^B(g'')$.

(5) *Let $g_0, g_1 \in G^-$. For $i = 0, 1$, there is $K_i \in$* $\mathrm{Maxid}(G^-)$ *such that $g_i \in K_i$. If $g_0 \cdot g_1 \neq 0$, then $K_0 = K_1$. In particular for every $g \in G^-$, there is a unique $K \in$* $\mathrm{Maxid}(G^-)$ *such that $g \in K$.*

(6) *The elements of* $\mathrm{Maxid}(G^-)$ *are pairwise disjoint.*

Proof: (a) is obvious.

(b) Let B and G be as in Theorem 2.9(i). (b1) follows from Theorem 2.9(i). (b2)–(b4) are trivial. For (b5), if $a \in \mathrm{At}(B) \subseteq G^-$ is such that $a \leq g_0 \cdot g_1$, then $a \in K_0 \cap K_1$ and thus g_0 and g_1 are comparable. Assume $g_0 \leq g_1$. Then $g_0 \in K_1$, and so $K_0 = K_1$. (b6) is a direct consequence of (b5). ∎

We recall (see Definition 2.1) that for an ideal J of a Boolean algebra B, we denote by J^c the ideal of those $b \in B$ such that $b \cdot a = 0$ for every $a \in J$.

LEMMA 2.12: *Let B be a superatomic subalgebra of an interval algebra such that $D_{\mathrm{rk}(B)}(B) = \{0,1\}$, and G be a complete set of generalized atoms verifying* *the properties of Theorem 2.9(i). Let J be a non-principal ideal of B. We suppose that:*

(1) *For every $K \in \mathrm{Maxid}(G^-)$, if the set $K \cap J$ is non-empty, then either $K \subseteq J$, or $K \cap J$ has a greatest element denoted by b_K.*

(2) $J = \mathrm{cl}_B^{\mathrm{Id}}(\bigcup\{K \in \mathrm{Maxid}(G^-): K \subseteq J\} \cup \{b_K: K \not\subseteq J \text{ and } K \cap J \neq \emptyset\})$.

Then B/J^c has a retract, namely $\mathrm{cl}_B(J)$.

Proof: First $G^- \subseteq \mathrm{cl}_B(J \cup J^c)$. Let $g \in G^-$. If $g \in J \cup J^c$, there is nothing to prove. Suppose that $g \notin J \cup J^c$. Let K be the unique element of $\mathrm{Maxid}(G^-)$ such that $g \in K$. Because $g \notin J^c$, let $a \in J$ be such that $g \cdot a \neq 0$. Note that $g \cdot a \in J$. Let $\underline{a} \in \mathrm{At}(B)$ be such that $\underline{a} \leq g \cdot a$. So $\underline{a} \in J$. Because $\mathrm{At}(B) \subseteq G^-$ and $\underline{a} \leq g$; $\underline{a} \in K$. Hence $g \cdot \underline{a} = \underline{a} \in K \cap J$. So, $K \cap J$ is non-empty and thus has a greatest element b_K. Because $0 \neq g \cdot a \leq g$, $g \cdot a \leq b_K$ and $g, b_K \in G^-$; g and b_K are comparable; and because $b_K \in J$ and $g \notin J$: $b_K < g$. Let us show that $g - b_K \in J^c$. For a contradiction, suppose $g - b_K \notin J^c$. Let $d \in J$ be such that $d \cdot (g - b_K) \neq 0$. Since J is an ideal, we can suppose that $d \in J \cap \mathrm{At}(B)$, and thus $d \leq g - b_K$. Because $\mathrm{At}(B) \subseteq G^-$, $d \leq g \in K$ and $K \in \mathrm{Maxid}(G^-)$: $d \in J \cap K \cap \mathrm{At}(B)$. A contradiction follows from the facts that: $d \in J \cap K$, $d \cdot b_K = 0$ and the fact that b_K is the gratest element of $J \cap K$. Now, because $g \cdot b_K \in J$ and $g - b_K \in J^c$, we have $g = g \cdot b_K + (g - b_K) \in \mathrm{cl}_B(J \cup J^c)$.

Now, $B = \mathrm{cl}_B(\mathrm{cl}_B(J) \cup J^c)$ follows from the fact that $G^- \subseteq \mathrm{cl}_B(J \cup J^c)$ and Lemma 2.11(b1).

The fact that $\mathrm{cl}_B(J) \cap J^c = \{0\}$ follows from Lemma 2.2(e2) and the fact that J is non-principal. Now the lemma is a consequence of Proposition 1.2. ∎

DEFINITION 2.13: Let $K \in \mathrm{Maxid}(G^-)$. We recall that $J = \mathrm{cl}_B^{\mathrm{Id}}(K)$ is an ideal of B, $\mathrm{cl}_B(J) = J \cup -J$. We set $\mathrm{rk}^B(K) \stackrel{\mathrm{def}}{=} \mathrm{rk}(\mathrm{cl}_B(J))$.

Now we will prove Lemma 2.4, by induction on $\mathrm{rk}(B)$.

2.2.1 Proof of Lemma 2.4: $\alpha \stackrel{\mathrm{def}}{=} \mathrm{rk}(B)$ is a successor or $\mathrm{cf}(\alpha) = \omega$.

CASE 1: *There is $K \in \mathrm{Maxid}(G^-)$ such that $\mathrm{rk}^B(K) = \mathrm{rk}(B)$.* We choose such a K. We set $J = \mathrm{cl}_B^{\mathrm{Id}}(K)$.

CLAIM 1: *J is a non-pricipal ideal and α is limit (and thus $\mathrm{cf}(\alpha) = \omega$).*

Proof: For every $g \in K$, we have $\mathrm{rk}^B(g) < \mathrm{rk}^B(1) = \mathrm{rk}(B)$. First J is non-principal. For a contradiction, suppose that J is principal, generated by a member of J, say g'. Clearly $g' \in K$. Because $\mathrm{rk}^B(g') < \mathrm{rk}^B(1)$, we have $g' \neq 0, 1$, and thus $\mathrm{cl}_B(J)$ and $(B \restriction g') \times \{0,1\}$ are isomorphic. Therefore,

$$\mathrm{rk}^B(K) \stackrel{\mathrm{def}}{=} \mathrm{rk}(\mathrm{cl}_B(J)) = \mathrm{rk}((B \restriction g') \times \{0,1\}) = \mathrm{rk}(B \restriction g') = \mathrm{rk}^B(g') < \mathrm{rk}(B) ,$$

that is contradictory with $\mathrm{rk}^B(K) = \mathrm{rk}(B)$. Hence J is non-principal. Clearly α is limit. ∎

By Lemma 2.11(b3) and (b4), let $(g_n)_{n<\omega}$ be a cofinal and strictly increasing sequence of elements of K. We set $b_0 = g_0$ and $b_{n+1} = g_{n+1} - g_n$ for $0 \leq n < \omega$. We have $J = \mathrm{cl}_B^{\mathrm{Id}}(K) = \mathrm{cl}_B^{\mathrm{Id}}(\{b_n \colon n < \omega\})$ and $\mathrm{rk}(B \restriction b_n) = \mathrm{rk}^B(b_n) < \alpha$ for $n < \omega$. By the induction hypothesis, there are well-ordered chains C_n such that $B \restriction b_n$ is isomorphic to $B(C_n)$ for $n < \omega$. Let C be the lexicographic sum of $(C_n)_{n<\omega}$. From Lemma 2.3(c), we consider C as a subchain of B. We will show that the interval algebra $B(C)$ is as required. Trivially, C is a well-ordered chain; $B(C)$ is of rank α; by Lemma 2.3(c) again, $B(C)$ and $\mathrm{cl}_B(C)$ are isomorphic; and $\mathrm{cl}_B(C) = \mathrm{cl}_B(J)$ is a retract of B/J^c, follows from Lemma 2.12. ∎

Case 2: *For every $K \in \mathrm{Maxid}(G^-)$, we have $\mathrm{rk}^B(K) < \mathrm{rk}(B)$.* We claim that:

Claim 2: *There are $\{b_n \colon n \in \omega\} \subseteq G^-$ and $\{K_n \colon n \in \omega\} \subseteq \mathrm{Maxid}(G^-)$ such that: for $m \neq n$, $b_n \in K_n$, $K_m \neq K_n$ (and thus $b_m \cdot b_n = 0$), and for every $\beta < \alpha$, there are infinitely many $n < \omega$ such that $\beta \leq \mathrm{rk}^B(b_n) < \alpha$.*

Proof: First suppose that $\alpha = \beta + 1$. Since G is a complete set of generalized atoms, $\{g/I_\beta(B) \colon g \in G^-\}$ generates $D_\beta(B)$, and because $D_{\beta+1}(B) = \{0,1\}$, it follows that $D_\beta(B)$ is isomorphic to $\mathrm{FC}(X)$ for some infinite set X. Let $\{b_n \colon n \in \omega\} \subseteq G^-$ be such that $b_n/I_\beta(B) \in \mathrm{At}(D_\beta(B))$. For $n < \omega$, let $K_n \in \mathrm{Maxid}(G^-)$ be such that $b_n \in K_n$. Then $\{b_n \colon n \in \omega\} \subseteq G^-$ and $\{K_n \colon n \in \omega\}$ are as required. Now, suppose α limit. Let $(\alpha_n)_{n<\omega}$ be a cofinal and strictly increasing sequence in α. Let $K_0 \in \mathrm{Maxid}(G^-)$ and $b_0 \in K_0$. Suppose that $\{b_i \colon i < n\}$ and $\{K_i \colon i < n\}$ satisfy the conditions above. We set $\beta_i = \mathrm{rk}^B(K_i)$. Let $\delta = \max\{\alpha_n, \beta_0, \ldots, \beta_{n-1}\}$. Let $b_{n+1} \in G^-$ be such that $b_{n+1}/I_{\delta+2}(B) \in \mathrm{At}(D_{\delta+2}(B))$ and let $K_{n+1} \in \mathrm{Maxid}(G^-)$ be such that $b_{n+1} \in K_{n+1}$. We claim that

$$(\bullet) \qquad b_{n+1} \in K_{n+1} - K_i \quad \text{for all } i < n+1$$

holds. First $\mathrm{rk}^B(b_{n+1}) = \delta + 2$. Let $i \leq n$ be given. Let $J_i = \mathrm{cl}_B^{\mathrm{Id}}(K_i)$. Note that $\beta_i = \mathrm{rk}^B(K_i) = \mathrm{rk}(\mathrm{cl}_B(J_i)) < \delta + 1$. Hence $b_{n+1} \notin K_i$. So $(\bullet)$ is proved.

Therefore $K_{n+1} \neq K_i$ for $i < n+1$. Now the other parts of the claim are clear. ∎

Now, let $J = \mathrm{cl}^{\mathrm{Id}}_B(\{b_n : n \in \omega\})$. By the induction hypothesis, there are well-ordered chains C_n such that $B \restriction b_n$ is isomorphic to $B(C_n)$ for $n < \omega$. Let C be the lexicographic sum of $(C_n)_{n<\omega}$. Then, we conclude as in Case 1. ∎

2.2.2 Proof of Lemma 2.4: $\alpha \stackrel{\mathrm{def}}{=} \mathrm{rk}(B)$ satisfies $\mathrm{cf}(\alpha) > \omega$. Let C be a chain, and D be a well-ordered subset of C. Let $\{d_\theta : \theta < \rho\}$ be the canonical increasing enumeration of elements of D. We say that D is **closed** in C if for every limit ordinal $\varphi < \rho$, we have $d_\varphi = \sup_C(\{x \in C : x < d_\varphi\})$. That means that $C = \bigcup\{[d_\theta, d_{\theta+1}) : \theta < \rho\}$.

LEMMA 2.14: *Let B be a superatomic subalgebra of an interval algebra such that $D_{\mathrm{rk}(B)}(B) = \{0,1\}$, G be a complete set of generalized atoms of B verifying the properties of Theorem 2.9(i). Let C be a chain of B. We suppose that C and $\mathrm{cl}_B(C)$ satisfy:*

1. *$C \cap G^-$ is contained in a unique $K \in \mathrm{Maxid}(G^-)$. We set $D \stackrel{\mathrm{def}}{=} C \cap K$, that is a well-ordered chain; and we denote by $\{d_\theta : \theta < \rho\}$ the canonical increasing enumeration of elements of $D \cup \{0\}$.*
2. *The set D is a maximal subchain of K, with no greatest element (and thus cofinal in K), and D is closed in C.*
3. *For $\theta < \rho$, $B \restriction (d_{\theta+1} - d_\theta) = \mathrm{cl}_B(C) \restriction (d_{\theta+1} - d_\theta)$.*

Let

$$J \stackrel{\mathrm{def}}{=} \mathrm{cl}^{\mathrm{Id}}_B\left(\bigcup\{B \restriction (d_{\theta+1} - d_\theta) : \theta < \rho\}\right).$$

Then B/J^c has a retract, namely $\mathrm{cl}_B(C)$.

Proof: Let us remark that D is closed in C means that $\mathrm{cl}_B(C) = \mathrm{cl}_B(D \cup J)$ since if $d_\theta \le c \le d_{\theta+1}$, then $c = d_\theta + (d_{\theta+1} - c)$, with $d_\theta \in D$ and $d_{\theta+1} - c \in J$. Because $D \subseteq C$, for every $c \in C$, there is a unique $\theta < \rho$ such that $d_\theta \le c < d_{\theta+1}$. Also, note that $d_0 = 0_B$, and by the hypotheses, $d_{\theta+1} - d_\theta \in \mathrm{cl}_B(C)$ for $\theta < \rho$. Let $B_\theta \stackrel{\mathrm{def}}{=} B \restriction (d_{\theta+1} - d_\theta)$.

First, we will prove that $B = \mathrm{cl}_B(\mathrm{cl}_B(C) \cup J^c)$. It suffices to show that $G^- \subseteq \mathrm{cl}_B(\mathrm{cl}_B(C) \cup J^c)$. For a contradiction, let $g \in G^-$ be such that $g \notin \mathrm{cl}_B(\mathrm{cl}_B(C) \cup J^c)$. In particular $g \in G^- - C \subseteq (G^- - K) \cup (K - C)$.

CASE 1: $g \in G^- - K$. Then, by Lemma 2.11(b6), for $h \in K$, we have $g \cdot h = 0$. In particular $g \cdot d_\theta = 0$ for $\theta < \rho$. From the definition of J, it follows that $g \in J^c$, that contradicts $g \notin J^c$.

CASE 2: $g \in K - C$. By the hypothesis (2), let γ_g be the first ordinal $\xi < \rho$ such that $g \leq d_\xi$. We claim that

$$(\bullet) \qquad \text{For } \theta < \gamma_g, \quad g \cdot d_\theta = 0.$$

For a contradiction, suppose that $g \cdot d_\theta \neq 0$ for some $\theta < \gamma_g$. Then $g \leq d_\theta$, or $g > d_\theta$. The first case contradicts the definition of γ_g. Now, suppose $d_\theta < g \leq d_{\gamma_g}$. From the fact that $\{h \in G^-: h \geq d_\theta\}$ is a chain contained in K, $d_\theta \in D$ and the fact that D is a maximal chain of K, it follows that $g \in \{h \in G^-: h \geq d_\theta\} \subseteq D = K \cap C$. That contradicts $g \in K - C$. So $(\bullet)$ holds.

Assume that γ_g is a successor, say $\gamma_g = \delta + 1$. In this case, because $g \cdot d_\delta = 0$, and $g \leq d_{\delta+1}$; $g \in B \restriction (d_{\delta+1} - d_\delta) \subseteq \mathrm{cl}_B(C) \subseteq \mathrm{cl}_B(\mathrm{cl}_B(C) \cup J^c)$. Contradiction. Suppose that γ_g is a limit ordinal. For every $\theta < \gamma_g$, we have $g \cdot d_{\theta+1} = 0$, and thus $g \cdot (d_{\theta+1} - d_\theta) = 0$; and for every $\theta \geq \gamma_g$ we have $g \leq d_\theta$ and thus $g \cdot (d_{\theta+1} - d_\theta) = 0$ too. Hence $g \in J^c$, that contradicts $g \notin \mathrm{cl}_B(\mathrm{cl}_B(C) \cup J^c)$. That proves $B = \mathrm{cl}_B(\mathrm{cl}_B(C) \cup J^c)$.

Now, we will prove that $\mathrm{cl}_B(C) \cap J^c = \{0\}$. Let $b \in \mathrm{cl}_B(C) \cap J^c$. We have $b = \sum\{u_{2k+1} - u_{2k}: k < n\}$ where $k < \omega$; with $u_i, u_j \in C \cup \{0,1\}$ and $0 \leq u_i < u_j \leq 1$ in $C \cup \{0,1\}$ for $i < j \leq 2n-1$. We will show that each $u_{2k+1} - u_{2k} = 0$. By contradiction, suppose that $u_{2k+1} - u_{2k} \neq 0$, i.e. $u_{2k} < u_{2k+1}$. By the hypothesis (2) and the fact that $D \subseteq C$, there are $\mu \leq \nu < \rho$ such that $d_\mu \leq u_{2k} < d_{\mu+1}$ and $d_\nu \leq u_{2k+1} < d_{\nu+1}$. Because $b \in J^c$, and thus $u_{2k+1} - u_{2k} \in J^c$, we have $\mu < \nu$. Hence:

$$\begin{aligned} u_{2k+1} - u_{2k} = &\big((u_{2k+1} - u_{2k}) \cdot (d_{\mu+1} - d_\mu)\big) + (d_\nu - d_{\mu+1}) \\ &+ \big((u_{2k+1} - u_{2k}) \cdot (d_{\nu+1} - d_\nu)\big) \end{aligned}$$

and thus:

$$\begin{cases} (u_{2k+1} - u_{2k}) \cdot (d_{\mu+1} - d_\mu) \in J^c, \\ d_\nu - d_{\mu+1} \in J^c, \text{ and} \\ (u_{2k+1} - u_{2k}) \cdot (d_{\nu+1} - d_\nu) \in J^c. \end{cases}$$

From the definition of J and J^c, it follows that:

$$(u_{2k+1} - u_{2k}) \cdot (d_{\mu+1} - d_\mu) = 0 = (u_{2k+1} - u_{2k}) \cdot (d_{\nu+1} - d_\nu),$$

and thus $u_{2k+1}-u_{2k}=d_\nu-d_{\mu+1}$. This imples that $d_{\mu+1}=u_{2k}$ (and $u_{2k+1}=d_\nu$), that contradicts $u_{2k}<d_{\mu+1}$. Hence $b=0$. Now, the Lemma 2.14 is a consequence of Proposition 1.2. ∎

LEMMA 2.15: *The set* Maxid(G^-) *is countable.*

Proof: By contradiction, suppose that Maxid(G^-) is uncountable. Let $(K_\alpha)_{\alpha<\omega_1}$ be a family of pairwise disjoint members of Maxid(G^-), and $a_\alpha\in K_\alpha\cap \mathrm{At}(B)$ for $\alpha<\omega_1$. We set $X=\{a_\alpha\colon \alpha<\omega_1\}$, $J=\mathrm{cl}_B^{\mathrm{Id}}(X)$ and $A=\mathrm{cl}_B(J)=\mathrm{cl}_B(X)$. Because J is a non-principal ideal of B and satisfies the hypotheses of Lemma 2.12, the subalgebra A is isomorphic to B/J^c. From the fact that FC(X) is isomorphic to A, and the fact that A is a quotient of the HCO algebra B, it follows that FC(X) is a CO algebra. By Proposition 1.7(e), X is countable, that is contradictory. Consequently Maxid(G^-) is countable. ∎

There is $K\in \mathrm{Maxid}(G^-)$ such that $\mathrm{rk}^B(K)=\alpha$, follows from $\mathrm{cf}(\alpha)\geq\omega_1$ and Lemma 2.15. Let D be a maximal chain in K, and $\{d_\theta\colon \theta<\rho\}$ be the canonical increasing enumeration of elements of $D\cup\{0\}$. Note that D has no greatest element. For each $\theta<\rho$, let $B_\theta\stackrel{\text{def}}{=}B\restriction(d_{\theta+1}-d_\theta)$, and $\alpha_\theta\stackrel{\text{def}}{=}\mathrm{rk}(B_\theta)$. Let us remark that

$$\alpha_\theta=\mathrm{rk}(B_\theta)=\mathrm{rk}^B(d_{\theta+1}-d_\theta)=\mathrm{rk}^B(d_{\theta+1})<\mathrm{rk}^B(1)=\mathrm{rk}(B)=\alpha\ .$$

By induction hypothesis, $B\restriction(d_{\theta+1}-d_\theta)$ is isomorphic to some $B(C_\theta)$, where C_θ is order-isomorphic to ω^{α_θ}. We set $\alpha=\sum\{\alpha_\theta\colon \theta<\rho\}$. Let C be the lexicographic sum of $(C_\theta)_{\theta<\rho}$ over ρ. C is order-isomorphic to ω^α, and by Lemma 2.3(c), we consider each C_θ and C as a subchain of B. We will show that the interval algebra $B(C)$ over C is as required. Trivially, $B(C)$ is of rank α. $B(C)$ and $\mathrm{cl}_B(C)$ are isomorphic algebras (that follows from Lemma 2.3(c) again). We claim that $\mathrm{cl}_B(C)$ is a retract of some quotient of B. To see this, let $J=\mathrm{cl}_B^{\mathrm{Id}}(\bigcup\{B\restriction(d_{\theta+1}-d_\theta)\colon \theta<\rho\})$. We have by the definition, $\mathrm{cl}_B(C)=\mathrm{cl}_B(D\cup J)$.

The hypotheses of Lemma 2.14 are satisfied, and thus B/J^c has a retract, namely $\mathrm{cl}_B(C)$. ∎

3. Proof of Theorem 1.9

Let us begin by a definition:

DEFINITION 3.1: A Boolean algebra B is a **good algebra** if it has the following properties:

(G1) B is thin-tall, $\text{rk}(B) = \omega_1$, and $D_{\omega_1}(B) = \{0,1\}$.

(G2) If I is an uncountable ideal of B, then B/I is countable.

(G3) There is $\beta_0(B) < \omega_1$ such that for every $\beta \geq \beta_0(B)$, there is a retract A_β of $B/I_\beta(B)$ such that $I_\beta(B)$ is generated by

$$\bigcup_{c \in \text{At}(A_\beta)} I_\beta(B) \restriction c\,.$$

Let us recall that $\diamondsuit_{\aleph_1}$ is the following axiom: " *There is a family* $(S_\alpha)_{\alpha<\omega_1}$ *of subsets of* ω_1 *such that* $S_\alpha \subseteq \alpha$ *and such that for every subset* A *of* ω_1*, the set* $\{\alpha < \omega_1 : A \cap \alpha = S_\alpha\}$ *is stationary in* ω_1".

The family $(S_\alpha)_{\alpha<\omega_1}$ is called a $\diamondsuit_{\aleph_1}$*-sequence*: $\diamondsuit_{\aleph_1}$ "captures" all subsets of ω_1 as well (see [11], [13] or [15]). The proof of Theorem 1.9 is divided into the two main parts (a) and (b) of the following result.

THEOREM 3.2:

(a) *A good Boolean algebra satisfies the requirements of Theorem 1.9.*

(b) *Assume* $\diamondsuit_{\aleph_1}$. *There is a good Boolean algebra.*

We first turn to the proof of Theorem 3.2(a).

3.1 PROOF OF THEOREM 3.2(a) . We start by stating some easy facts needed in that proof.

DEFINITION 3.3: (a) Let B be an algebra, I an ideal of B, A a subalgebra of B, and $b \in B$. We set

$$I \restriction b = \{a \cdot b : a \in I\},$$
$$A \restriction b = \{a \cdot b : a \in A\}.$$

(b) For two algebras B' and B'', we denote by $B' \cong B''$ the relation: B' and B'' are isomorphic.

(c) For every superatomic Boolean algebra B and every ordinal $\beta \leq \text{rk}(B)$, we set $B_\beta \stackrel{\text{def}}{=} \text{cl}_B(I_\beta(B))$ that is $I_\beta(B) \cup -I_\beta(B)$.

(d) Let B be a superatomic Boolean algebra and A be a subalgebra of B. $A \sqsubseteq B$ if for some α, $A = \text{cl}_B(I_\alpha(B))$. Note that in this case, $D_\alpha(A) \subseteq \{0,1\}$.

The next proposition is obvious:

PROPOSITION 3.4:

(a) *Let B, A and b as in the Definition 3.3*(a). *Then:*
  (1) $I \restriction b = \{c \in I: c \leq b\}$,
  (2) $A \restriction b$ *is a subalgebra of* $B \restriction b$, *and is a homomorphic image of* A.

(b) *Let B be a superatomic Boolean algebra, A a subalgebra of B such that* $A \sqsubseteq B$ *and* $\alpha = \mathrm{rk}(A)$. *Then:*
  (1) *For every* $\gamma \leq \alpha$, $I_\gamma(A) = I_\gamma(B)$.
  (2) $I_\alpha(A) = I_\alpha(B)$ *and* $I_\alpha(A)$ *is a maximal ideal of A (and thus* $D_\alpha(A) = \{0,1\}$).

(c) $\sqsubseteq$ *is an order relation in the class of superatomic Boolean algebras.*

The proofs of all parts of the following proposition are straight-forward. For the sake of completeness, we will prove it, at the end of subsection 3.1.

PROPOSITION 3.5:

(a) *Let B be a Boolean algebra and I be an ideal of B, such that B/I is countable. Then B/I has a retract. In particular every countable algebra is retractive.*

(b) *Let B be a thin-tall Boolean algebra and J be a countable ideal of B. Then for some* $\alpha < \omega_1$, $J \subseteq I_\alpha(B)$.

(c)
  (1) *Let B be a thin-tall Boolean algebra. For every* $\beta < \omega_1$, B_β *is isomorphic to* $\mathbb{B}_{\beta,1}$.
  (2) *For* $\beta < \omega_1$ and $0 < p < \omega$, $\mathbb{B}_{\beta,p}$ *is isomorphic to a factor of B.*

(d) *Let B be a Boolean algebra and J be an ideal of B.*
  (1) *Let K be an ideal of B. If $J \subseteq K$, then*

$$\{b \in B: b/J \in I_\gamma(B/J)\} \subseteq \{b \in B: b/K \in I_\gamma(B/K)\} .$$

  (2) *Let α be an ordinal such that* $J \subseteq I_\alpha(B)$, *and* $\beta \geq \alpha \cdot \omega$. *Then*

$$\{b \in B: b \in I_\beta(B)\} = \{b \in B: b/J \in I_\beta(B/J)\} .$$

(Hence $I_\beta(B/J) = I_\beta(B)/J$.), *and there is an isomorphism g from* $D_\beta(B/J)$ *onto* $D_\beta(B)$ *such that for all* $b \in B$,

$$g((b/J)/I_\beta(B/J)) = b/I_\beta(B) .$$

(3) *For every ordinal α and β,*

$$\{b \in B\colon b/I_\alpha(B) \in I_\beta(B/I_\alpha(B))\} = I_{\alpha+\beta}(B)$$

and there is an isomorphism f from $D_\beta(D_\alpha(B))$ onto $D_{\alpha+\beta}(B)$ such that for all $b \in B$,

$$f((b/I_\alpha(B))/I_\beta(B/I_\alpha(B))) = b/I_{\alpha+\beta}(B)$$

(e) *(Isomorphism Theorem.) Let B be a Boolean algebra, I and J be ideals such that $I \supseteq J$. Then I/J is an ideal of B/J and the algebras $(B/J)/(I/J)$ and B/I are isomorphic.*

To show Theorem 3.2(a), let us begin by the retractive property:

LEMMA 3.6: *If B is a good Boolean algebra, then B is retractive.*

Proof: Let B be a good Boolean algebra. We will prove that B has the following property $(\star)$, and that property $(\star)$ implies retractiveness, where: $(\star)$ is the following property:

There are ideals $\{I_t\colon t \in T\}$ of B, and subalgebras $\{A_t\colon t \in T\}$ of B such that:

$(\star_1)$ *For every $t \in T$, B/I_t is atomic, A_t is a retract of B/I_t, I_t is generated by $\bigcup\{I_t \restriction c\colon c \in \mathrm{At}(A_t)\}$, and for every $c \in \mathrm{At}(A_t)$, $B \restriction c$ is retractive.*

$(\star_2)$ *If J is an ideal of B, then either B/J has a retract, or there is $t \in T$ such that $J \subseteq I_t$.*

Let us show that a good algebra has the property $(\star)$. Let

$$T = \{\delta + 1\colon \beta_0(B) \leq \delta < \omega_1\}\ .$$

Now, for $\delta + 1 \in T$, let $I_{\delta+1} \stackrel{\text{def}}{=} I_{\delta+1}(B)$ and $A_{\delta+1}$ be given by (G3). For $(\star_1)$, the only non-trivial part is the fact that $B \restriction a$ for $a \in \mathrm{At}(A_{\delta+1})$ is retractive. But this is a consequence of the fact that $B \restriction a$, for $a \in \mathrm{At}(A_{\delta+1})$, is countable and thus, by Proposition 3.5(a), retractive. Let us show $(\star_2)$. Let J be an ideal of B. First suppose that J is uncountable. By (G2), B/J is countable, and thus retractive, by Proposition 3.5(a). Now, suppose that J is countable. By Proposition 3.5(b), let δ be such that $J \subseteq I_\delta(B)$. So $\delta + 1 \in T$ and $J \subseteq I_{\delta+1}(B)$.

Next, we show that if B has property $(\star)$, then B is retractive. Let J be an ideal of B. By contradiction, suppose that B/J has no retract. By $(\star_2)$, let $t \in T$

be such that $J \subseteq I_t$. From $(\star_1)$, if $a \in \mathrm{At}(A_t)$, then $B \restriction a$ is retractive; let A_a be a retract of $(B \restriction a)/(J \restriction a)$ (so $A_a \subseteq B \restriction a \subseteq B$). Let

$$A \stackrel{\text{def}}{=} \mathrm{cl}_B(A_t \cup \bigcup\{A_a \colon a \in \mathrm{At}(A_t)\}) .$$

We claim that A is a retract of B/J.

Let $b \in B$. We show that $b \in \mathrm{cl}_B(A \cup J)$. Because A_t is a retract of B/I_t, let $b' \in A_t$, u^0, $u^1 \in I_t$ such that $u^0 \leq b'$, $u^1 \cdot b' = 0$ and $b = b' - u^0 + u^1$. Because $I_t = \mathrm{cl}_B^{\mathrm{Id}}(\bigcup\{I_t \restriction a \colon a \in \mathrm{At}(A_t)\})$, for $i = 0, 1$, there is a finite subset F^i of $\mathrm{At}(A_t)$ such that $u^i \in \mathrm{cl}_B^{\mathrm{Id}}(\bigcup\{I_t \restriction a^i \colon a^i \in F^i\})$. Since A_{a^i} is a retract of $(B \restriction a^i)/(J \restriction a^i)$ and $I_t \restriction a^i \subseteq B \restriction a^i$, let $b_{a^i} \in A_{a^i}$, u_{a^i}, $v_{a^i} \in J \restriction a^i \subseteq J$ such that $u_{a^i} \leq b_{a^i}$, $v_{a^i} \cdot b_{a^i} = 0$ and $u^i \cdot a^i = b_{a^i} - u_{a^i} + v_{a^i}$. Hence $u^i = b_i - u_i + v_i$ where $b_i = \sum\{b_{a^i} \colon a^i \in F^i\} \in \mathrm{cl}_B(\bigcup\{A_{a^i} \colon a^i \in F^i\})$, $u_i = \sum\{u_{a^i} \colon a^i \in F^i\} \in J$ and $v_i = \sum\{v_{a^i} \colon a^i \in F^i\} \in J$. So $b = b' - u^0 + u^1 = b' - (b_0 - u_0 + v_0) + (b_1 - u_1 + v_1) \in \mathrm{cl}_B(A_t \cup \bigcup\{A_a \colon a \in \mathrm{At}(A_t)\} \cup J)$, that finishes this part.

We show that $A \cap J = \{0\}$. It suffices to show that if $b \in A_t$ and $c \in A_a \subseteq B \restriction a$ for some $a \in \mathrm{At}(A_t)$ with $b \cdot c \in J$, then $b \cdot c = 0$. Because $a \in \mathrm{At}(A_t)$: $b \cdot a = 0$ or $a \leq b$. In the first case, $b \cdot c = 0$ since $c \leq a$. In the second case, we have $c \leq a \leq b$ and thus $b \cdot c = c \in A_a \cap (J \restriction a) = \{0\}$.

So A is a retract of B/J. ∎

Our next goal is to show that if B is good and if J is a countable ideal of B, B/J and B are isomorphic algebras.

DEFINITION 3.7: Let B be an atomic Boolean algebra, and C be a Boolean algebra. We define the Boolean algebra $C \circ B$. (Informaly speaking, $C \circ B$ is the Boolean algebra obtained by replacing every atom of B by a copy of C.) For every $a \in \mathrm{At}(B)$ and $c \in C$, let k_c^a be the member of $C^{\mathrm{At}(B)}$ such that:

$$k_c^a(b) = \begin{cases} c & \text{if } b = a, \\ 0 & \text{otherwise,} \end{cases}$$

for all $b \in \mathrm{At}(B)$. Note that the mapping $k^a \colon c \mapsto k_c^a$ is an isomorphism from C onto the factor $C_a \stackrel{\text{def}}{=} k^a[C]$ of $C^{\mathrm{At}(B)}$. Now, we define $C \circ B$ to be the subalgebra of $C^{\mathrm{At}(B)}$ generated by

$$\Big(\bigcup_{a \in \mathrm{At}(B)} C_a\Big) \cup \Big\{\sum_{\substack{a \in \mathrm{At}(B) \\ a \leq b}} 1^{C_a} \colon b \in B\Big\} .$$

Note that for every $a \in C \circ B$, there are unique $b \in B$, $u^0 = \sum_{i<m} u_i^0$ and $u^1 = \sum_{i<n} u_i^1$ such that, setting $b^\star \overset{\text{def}}{=} \sum_{\substack{a \in \mathrm{At}(B) \\ a \leq b}} 1^{C_a}$: $u_j^i \in C_{a_i}$ with $a_i \in \mathrm{At}(B)$, $u^0 \leq b^\star$, $u^0 \cdot b^\star = 0$ and $a = b^\star - u^0 + u^1$.

We need the following lemma.

LEMMA 3.8:

(a) *Let B be a good Boolean algebra, and J be a countable ideal of B. Then B/J is good.*

(b) *Let B be a good Boolean algebra. Then for every $\beta \geq \beta_0(B)$, B and $B_\beta \circ D_\beta(B)$ are isomorphic algebras.*

Proof: (a) Let J be a countable ideal of B.

B/J satisfies (G1). Let $\alpha < \omega_1$ be such that $J \subseteq I_\alpha(B)$ and $\beta = \alpha \cdot \omega$. From Proposition 3.5(d2), it follows that

$$\{b \in B: b/J \in I_\beta(B/J)\} = I_\beta(B)$$

holds. Because $B_\beta \sqsubseteq B$, it is easy to check that for $\xi < \beta$, $\mathrm{At}(D_\xi(B_\beta/J)) = \mathrm{At}(D_\xi(B/J))$. By Proposition 3.5(d2), for $\beta \leq \xi \leq \omega_1$, $D_\xi(B) \cong D_\xi(B/J)$. Hence B/J is thin-tall and $D_{\omega_1}(B/J) = \{0, 1\}$.

B/J satisfies (G2): let I be an uncountable ideal of B/J. Let K be an ideal of B such that $K/J = I$. Then $K \supseteq J$ and K is uncountable. From $(G1)_B$ and from Proposition 3.5(e), it follows that $(B/J)/I \cong B/K$ is countable.

B/J satisfies (G3): let $\alpha < \omega_1$ be such that $J \subseteq I_\alpha(B)$, and $\beta_1 \overset{\text{def}}{=} \alpha \cdot \omega$. We set $\beta_0(B/J) \overset{\text{def}}{=} \max(\beta_1 + 1, \beta_0(B))$. Let $\beta \geq \beta_0(B/J)$. Let A^β be a retract of $B/I_\beta(B)$ relative to B, be assured by $\beta_0(B)$. By Proposition 3.5(d2), $I_\beta(B/J) = I_\beta(B)/J$. Now A^β/J is a retract of $(B/J)/I_\beta(B/J)$ and $\mathrm{At}(A^\beta/J) = \{a/J: a \in \mathrm{At}(A^\beta)\}$. By (G3), $I_\beta(B/J)$ is generated by $\bigcup\{I_\beta(B/J) \restriction c: c \in \mathrm{At}(A^\beta/J)\}$.

(b) is a trivial consequence of the following claim:

CLAIM: Let J be an ideal of B. We suppose that

(1) B/J is atomic.

(2) B/J has a retract A.

(3) $J = \mathrm{cl}_B^{\mathrm{Id}}(\bigcup\{J \restriction a: a \in \mathrm{At}(A)\})$.

(4) There is an algebra D such that D is isomorphic to $\mathrm{cl}_{B\restriction a}(J \restriction a)$ for every $a \in \mathrm{At}(D)$.

Then B is isomorphic to $D \circ (B/J)$.

Proof: For $b \in B$, let $b' \in A$, u^0, $u^1 \in J$ be such that $u^0 \leq b'$, $u^1 \cdot b' = 0$ and $b = b' - u^0 + u^1$. From the hypothesis (3), for $i = 0, 1$,

$$F^i \stackrel{\text{def}}{=} \{a \in \mathrm{At}(A) \colon a \cdot u^i \neq 0\}$$

is finite. We set:

$$\varphi(b) = \Big(\sum_{\substack{a \in \mathrm{At}(B) \\ a \leq b'}} 1^{D_a} \Big) - \underline{u}^0 + \underline{u}^1$$

where $\underline{u}^i = \sum_{a \in \mathrm{At}(A)} u^i \cdot a = \sum_{a \in F^i} u^i \cdot a$ for $i = 0, 1$. Note that $\varphi(b) \in D \circ A$. It is easy to check that φ is an isomorphism from B onto $D \circ A$ that concludes the proof of our claim and of Lemma 3.8. ∎

LEMMA 3.9: *Let B be a good Boolean algebra, and J be a countable ideal of B, then B and B/J are isomorphic algebras.*

Proof: By Proposition 3.5(b), there is α such that $J \subseteq I_\alpha(B)$. Let $\beta_1 = \alpha \cdot \omega$, and $\beta_2 = \beta_0(B)$. By Lemma 3.8(a), B/J is good. So, let $\beta_3 = \beta_0(B/J)$, and let $\beta \geq \beta_1, \beta_2, \beta_3$. By Lemma 3.8(b),

(1) $B/J \cong (B/J)_\beta \circ D_\beta(B/J)$, and by Proposition 3.5(c1)

(2) $(B/J)_\beta \cong \mathbb{B}_{\beta,1}$. $D_\beta(B/J) = (B/J)/I_\beta(B/J)$. By Proposition 3.5(d2), $I_\beta(B/J) = I_\beta(B)/J$. Hence $D_\beta(B/J) = (B/J)/(I_\beta(B)/J)$. Because $J \subseteq I_\beta(B)$, by the isomorphism theorem (Proposition 3.5(e)),

(3) $D_\beta(B/J) \cong B/I_\beta(B) = D_\beta(B)$. It follows from (1), (2) and (3) that:

(4) $B/J \cong \mathbb{B}_{\beta,1} \circ D_\beta(B)$.

The same holds for $I = \{0\}$ (by the goodness of B) and thus, $B \cong \mathbb{B}_{\beta,1} \circ D_\beta(B)$. Hence by (4), B/J and B are isomorphic algebras. ∎

Let us end the proof of Theoreom 3.2(a).

(A1) follows trivially from (G1).

(A3) The first part follows from Lemma 3.6, and the second part is a consequence of (G2).

(A4) Let J be an ideal of B such that B/J is countable. Hence B/J is isomorphic to $\mathbb{B}_{\beta,p}$ for some $\beta < \omega_1$ and $1 \leq p < \omega$. (A4) follows from Proposition 3.5(c2).

(A5) Let J be an ideal of B such that B/J is uncountable. By (G2) J is countable. Now (A5) follows from Lemma 3.9.

(A2) From (A4) and (A5), it follows that B is a CO algebra. Let us show that if I is an ideal of B, then B/I is a CO algebra. If B/I is uncountable, then $B/I \cong B$ and thus is a CO algebra. If B/I is countable, then $B/I \cong \mathbb{B}_{\beta,p}$ for some $\beta < \omega_1$ and $p < \omega$, i.e. $B/I \cong B(\omega^\beta \cdot p)$. In this case the result follows from Proposition 1.7(c) and (d).

To complete the proof of Theorem 3.2(a), it remains to prove Proposition 3.5.

Proof of Proposition 3.5: (a) Let $B/I = \{b_i : i < \omega\}$. We will construct a retract $A = \{a_i : i < \omega\}$ of B/I such that $a_i/I = b_i$ for all i and $a_i \mapsto b_i$ is a one-to-one homomorphism. We can suppose that $b_0 = 0$ and $b_1 = 1$, and thus $a_0 \stackrel{\text{def}}{=} 0$ and $a_1 \stackrel{\text{def}}{=} 1$. Assume that $\{a_i : i < n\}$ are defined such that: $A_n \stackrel{\text{def}}{=} \text{cl}_B(\{a_i : i < n\})$ and $B_n \stackrel{\text{def}}{=} \text{cl}_{B/I}(\{b_i : i < n\})$ are isomorphic by $c \mapsto c/I$ for $c \in A_n$. If $b_n \in B_n$, then we set $A_{n+1} = A_n$ and $B_{n+1} = B_n$. Now suppose that $b_n \notin B_n$. Let $\{a^j : j < p\} = \text{At}(A_n)$ an $b^j = a^j/I$ for $j < p$ (hence $\text{At}(B_n) = \{b^j : j < p\}$). We define $(a^j_n)_{j<p}$ by $a^j_n = 0$ if $b^j \cdot b_n = 0$, $a^j_n = a^j$ if $b^j \leq b_n$; next if $b^j \cdot b_n \neq 0$ and $b^j - b_n \neq 0$, then we choose $a^j_n \in B$ such that $a^j_n \leq a^j$, $a^j_n/I = b^j \cdot b_n$. Hence $(a^j - a^j_n)/I = b^j - b_n$. We set $a_n = \sum_{j<p} a^j_n$, $A_{n+1} \stackrel{\text{def}}{=} \text{cl}_B(\{a_i : i \leq n\})$ and $B_{n+1} \stackrel{\text{def}}{=} \text{cl}_{B/I}(\{b_i : i \leq n\})$. Clearly $b_n = a_n/I$ and, by the construction, A_{n+1} is a retract of $B_{n+1}/(I \cap B_{n+1})$ containing A_n. Hence $A = \bigcup_{n<\omega} A_n$ is a retract of B/I.

(b) Let J be a countable ideal of B. For each $a \in J$, $a \in I_{\text{rk}^B(a)+1}(B)$. Let $\alpha = \sup\{\text{rk}^B(a) + 1 : a \in J\}$. Then $J \subseteq I_\alpha(B)$.

(c1) Because B_β is a countable algebra and $D_\beta(B) = \{0,1\}$, then B_β and $\mathbb{B}_{\beta,1}$ are isomorphic.

(c2) Let $\mathbb{B}_{\beta,p}$ be given. For $i < p$, let $c_i \in B$ be such that $c_i/I_\beta(B) \in \text{At}(D_\beta(B))$. We can suppose that for $i < j < p$, $c_i \cdot c_j = 0$. Clearly $\mathbb{B}_{\beta,1}$ is isomorphic to $B \upharpoonright c_i$ and thus $\mathbb{B}_{\beta,p}$ is isomorphic to $B \upharpoonright \sum_{i<p} c_i$.

(d1) By induction on γ. The case $\gamma = 0$ or γ limit, are trivial. For the successor case, suppose that $(b/J)/I_\gamma(B/J)$ is an atom of $(B/J)/I_\gamma(B/J)$; we would like to show that $(b/K)/I_\gamma(B/K)$ is zero, or is an atom of $(b/K)/I_\gamma(B/K)$. For $b \in B$, we recall that $\langle b^0, b^1 \rangle$ is a *partition* of b if $b = b^0 + b^1$ with $b^0 \cdot b^1 = 0$. For a partition $\langle b^0, b^1 \rangle$ of b, we set $i_\gamma(J, b^0, b^1) = \{\ell \in \{0,1\} : b^\ell/J \in I_\gamma(B/J)\}$. $i_\gamma(K, b^0, b^1)$ is defined similarly. The fact that $(b/J)/I_\gamma(B/J)$ is an atom of $(B/J)/I_\gamma(B/J)$ means that for every partition (b^0, b^1) of b, $i_\gamma(J, b^0, b^1) \neq \emptyset$. Our induction hypothesis implies that $i_\gamma(J, b^0, b^1) \subseteq i_\gamma(K, b^0, b^1)$. Hence $i_\gamma(K, b^0, b^1) \neq \emptyset$ for every partition $\langle b^0, b^1\rangle$ of b. So, if $|i_\gamma(K, b^0, b^1)| = 2$, then $(b/K)/I_\gamma(B/K)$ is zero, and if $|i_\gamma(K, b^0, b^1)| = 1$, then $(b/K)/I_\gamma(B/K)$ is an atom of $(B/K)/I_\gamma(B/K)$.

(d2) Suppose that $J \subseteq I_\alpha(B)$. We set $\beta = \alpha \cdot \omega$. For every $\ell < \omega$, we have, using (d1) twice:

$$\begin{aligned}\{b \in B\colon b/J \in I_{\alpha\cdot\ell}(B/J)\} &\subseteq \{b \in B\colon b/I_\alpha(B) \in I_{\alpha\cdot\ell}(B/I_\alpha(B))\}\\ &= I_{\alpha\cdot(\ell+1)}(B)\\ &\subseteq \{b \in B\colon b/J \in I_{\alpha\cdot(\ell+1)}(B/J)\}\ ;\end{aligned}$$

from this the desired conclusion follows.

(d3) is proved similarly by induction.

(e) is a classical result on commutative rings and left to the reader. ∎

3.2 Proof of Theorem 3.2(b).

3.2.1 Preliminaries. We will give a proof of Theorem 3.2(b), with $\beta_0(B) = 0$ in the the part $(G3)$ of our definition of a good Boolean algebra.

The following result can be found in Bonnet, Rubin and Si-Kaddour [8]:

THEOREM 3.10: *Let $\langle\lambda_\alpha\rangle_{\alpha\leq\rho}$ be a sequence of cardinals, with $\lambda_\rho = 1$ and $\lambda \overset{\text{def}}{=} \lambda_0$. Let $\{a_{\alpha,v}\colon v < \lambda_\alpha\ ;\ \alpha < \rho\} \subseteq \wp(\lambda)$ satisfy the following properties. For $\beta \leq \rho$, let B_β be the subalgebra of $\wp(\lambda)$ generated by $\{a_{\alpha,v}\colon v < \lambda_\alpha\ ;\ \alpha < \beta\}$, and J_β be the ideal of B_β generated by $\{a_{\alpha,v}\colon v < \lambda_\alpha\ ;\ \alpha < \beta\}$. We suppose:*

(H1) *$a_{0,i} = \{i\}$ for every $i < \lambda$.*

(H2) *$a_{\alpha,v} \notin B_\alpha$ for every $v < \lambda_\alpha$ and $\alpha < \rho$.*

(H3) *$a_{\alpha,v} \cap a_{\alpha,\mu} \in J_\alpha$ for every $\alpha < \rho$ and $v < \mu < \lambda_\alpha$.*

(H4) *For every $\alpha < \beta < \rho$ and $v < \lambda_\alpha$, $\mu < \lambda_\beta$, we have $a_{\alpha,v} \cap a_{\beta,\mu} \in B_{\alpha+1}$.*

(H5) *For every $\alpha < \rho$, $v < \lambda_\alpha$ and $\beta < \alpha$ there are infinitely many μ's in λ_α such that $a_{\beta,\mu} - a_{\alpha,v} \in J_\beta$.*

Let $B \overset{\text{def}}{=} B_\rho$. Then:

(1) *J_α is a maximal ideal of B_α.*

(2) *$J_\alpha = I_\alpha(B_\beta)$ for $\beta \geq \alpha$, (in particular J_α is an ideal of B) and*

$$\mathrm{At}(D_\alpha(B_\beta) = \{a_{\alpha,v}/I_\alpha(B_\beta)\colon v < \lambda_\alpha\} \quad \text{for } \beta \geq \alpha.$$

In particular:

(3) *$J_\alpha = I_\alpha(B)$ and for $\alpha \leq \beta$, $B_\alpha \sqsubseteq B_\beta \sqsubseteq B$.*

(4) *$\mathrm{At}(D_\alpha(B)) = \{a_{\alpha,v}/I_\alpha(B)\colon v < \lambda_\alpha\}$ for $\alpha < \rho$.*

Let us remark that (H3) and (H4) imply:

(H6) *For every* $\langle \alpha, v\rangle \in \Omega \overset{\text{def}}{=} \bigcup_{\beta<\kappa}\{\beta\} \times \lambda_\beta$, $B \restriction a_{\alpha,v} \subseteq B_{\alpha+1}$.

Indeed, the set $\{b \in B\colon a_{\alpha,v} \cap b \in B_{\alpha+1}\}$ contains all the generators of B by (H3) and (H4), and is clearly closed under the Boolean operations, so it consists of all the B, as desired.

In what follows, we set

$$J_\alpha \overset{\text{def}}{=} I_\alpha(B_\alpha) = I_\alpha(B_{\omega_1})$$

for $\alpha \leq \omega_1$. We will construct $\{c_{\alpha,i}\colon i < \omega, \alpha < \omega_1\}$ satisfying the conditions of Theorem 3.10.

Let $(\lambda_\alpha)_{\alpha<\omega_1}$ be the canonical enumeration of limit countable ordinals. Hence $\lambda_\alpha = \omega \cdot (1 + \alpha)$, (for instance $\lambda_0 = \omega$). Note that if $\omega^\alpha = \alpha$, then $\lambda_\alpha = \alpha$ (for informations, see J. G. Rosenstein [20]: exercice 3.34). We set $\mathrm{Lim}(\omega_1) = \{\lambda_\alpha\colon \alpha < \omega_1\}$, and

$$D_0 \overset{\text{def}}{=} \{\lambda_\alpha \in \mathrm{Lim}(\omega_1)\colon \lambda_\alpha = \alpha\}$$

that is a club (closed unbounded subset) of ω_1.

In order to use $\diamondsuit_{\aleph_1}$, we shall also have a one-to-one correspondance φ_α from λ_α onto the subalgebra B_α of $\wp(\omega)$ generated by $\{c_{\beta,i}\colon i < \omega, \beta < \alpha\}$; in such a way $\varphi_\beta \subseteq \varphi_\alpha$ for $\beta < \alpha$. Note that $0^{\wp(\omega)} = \emptyset$, $1^{\wp(\omega)} = \omega$; $B_0 = \mathrm{FC}(\omega)$, and $\varphi[\lambda_\alpha] = B_\alpha$ for all $\alpha < \omega_1$. We set $\varphi \overset{\text{def}}{=} \bigcup_{\alpha<\omega_1} \varphi_\alpha$, that is a one-to-one function from ω_1 onto $B \overset{\text{def}}{=} B_{\omega_1}$.

Let $(S_\alpha)_{\alpha<\omega_1}$ be a $\diamondsuit_{\aleph_1}$-sequence. Let us remark that:

(1) $\alpha \in D_0$ if and only if $\varphi[\alpha] = B_\alpha$.

(2) If $\alpha \in D_0$, then $\varphi[S_\alpha] \subseteq B_\alpha$.

LEMMA 3.11:

(a) *Let* $a, c \in B$ *be such that* $a/J_\alpha \in \mathrm{At}(D_\alpha(B))$. $\mathrm{rk}^B(a - c) < \mathrm{rk}^B(a)$ $(= \alpha)$ *if and only if* $a/J_\alpha = (a \cap c)/J_\alpha \leq c/J_\alpha$.

(b) *Let* $\{c_{\alpha,i}\colon i < \omega, \alpha < \omega_1\} \subseteq \wp(\omega)$ *satisfy the hypotheses of Theorem* 3.10, *and the following property:*

(•) *For every* $\alpha < \omega_1$, *if*

(i) $\alpha \in D_0$ *(so* $\alpha = \lambda_\alpha$ *is limit and* $\varphi[\alpha] = B_\alpha$*)* ,

(ii) $\varphi[S_\alpha] \subseteq J_\alpha$,

(iii) *If* $a \in \varphi[S_\alpha]$, *then* $a/J_{\mathrm{rk}^{B_\alpha}(a)} \in \mathrm{At}(D_{\mathrm{rk}^{B_\alpha}(a)}(B_\alpha))$, *and*

(iv) $\sup\{\mathrm{rk}^{B_\alpha}(a)\colon a \in \varphi[S_\alpha]\} = \alpha$;

then for every $i < \omega$, and every $c \in c_{\alpha,i}/J_\alpha$, there is $a \in \varphi[S_\alpha]$ such that $\mathrm{rk}^{B_\alpha}(a-c) < \mathrm{rk}^{B_\alpha}(a)$.

Then B satisfies:

(1) *B is thin-tall and $D_{\omega_1}(B) = \{0,1\}$.*

(2) *If J is an uncountable ideal of B, then B/J is countable.*

Proof: (a) is trivial.

(b1) follows from the conclusions of Theorem 3.10.

(b2) Let J be an ideal of B. We suppose that B/J is uncountable. We will show that there is $\alpha < \omega_1$ such that $J \subseteq J_\alpha$ and thus J is countable (since trivially J_α is countable).

CLAIM 1: *For every $a \in B$, if $\mathrm{rk}^B(a) = \alpha < \omega_1$, there are unique n, $i_0, \ldots, i_{n-1} < \omega$ and u, $v \in J_\alpha$ such that $a = \tilde{a} - u + v$, with $\tilde{a} \overset{\text{def}}{=} \sum_{k<n} c_{\alpha,i_k}$, $u \le \tilde{a}$ and $v \cdot \tilde{a} = 0$.*

Proof: Clear. ∎

CLAIM 2: *Let $\xi < \omega_1$ and $d \in \mathrm{At}(D_\xi(B/J))$. There are $\alpha(d) < \omega_1$ and $i(d) < \omega$ such that:*

(1) *$a(d) \overset{\text{def}}{=} c_{\alpha(d),i(d)}$ satisfies $(a(d)/J)/I_\xi(B/J) = d$ and*

(2) *If $v \in J_{\alpha(d)} \restriction a(d)$ (so $v \le a(d)$), then $(v/J)/I_\xi(B/J) = 0$.*

Proof: Similar to that of Lemma 2.8. Let

$$\alpha(d) = \min(\{\alpha < \omega_1 \colon \exists\, a \in B\ ,\ \mathrm{rk}^B(a) = \alpha \text{ and } (a/J)/I_\xi(B/J) = d\})\ ,$$

and a be such that $\mathrm{rk}^B(a) = \alpha(d)$ and $(a/J)/I_\xi(B/J) = d$. By Claim 1, $a = \tilde{a} - u^0 + v^1$ with $\tilde{a} = \sum_{k<n} c_{\alpha(d),i_k}$, $u^0 \le \tilde{a}$, $u^1 \cdot \tilde{a} = 0$ and u^0, $u^1 \in J_{\alpha(d)}$. From the definition of $\alpha(d)$ and from $\mathrm{rk}^B(u^i) < \alpha(d)$ $(i = 0,1)$, we have $(u^i/J)/I_\xi(B/J) = 0$. Hence we can suppose that $a = \tilde{a} = \sum_{k<n} c_{\alpha,i_k}$. Because $d \in \mathrm{At}(D_\xi(B/J))$, it follows that for some $\ell < n$, say $i(d)$, $(c_{\alpha,i_\ell}/J)/I_\xi(B/J) = d$. We set $i(d) = i_\ell$ and $a(d) \overset{\text{def}}{=} c_{\alpha(d),i(d)}$. Then, $\alpha(d)$ and $i(d)$ are as required: (2) follows from the fact that $d \in \mathrm{At}(D_\xi(B/J))$ and from the definition of $\alpha(d)$. ∎

End of the proof of Lemma 3.11(b2). With the notation of Claim 2, for each d, we choose $a(d)$, and let $A \overset{\text{def}}{=} \bigcup_{\xi<\omega_1} \{a(d) \colon d \in \mathrm{At}(D_\xi(B/J))\}$, that is, by the definition, a set of generalized atoms of B. We recall that φ is a one-to-one function from ω_1 onto B.

Let $S \subseteq \omega_1$ be such that $\varphi[S] = A$. The set

$$\begin{aligned} D_1 &= \{\delta \in D_0\colon\ \delta = \sup\{\mathrm{rk}^B(\varphi(\nu))\colon \nu \in S \text{ and } \mathrm{rk}^B(\varphi(\nu)) < \delta\}\} \\ &= \{\delta \in D_0\colon\ \delta = \sup\{\mathrm{rk}^B(b)\colon b \in A \text{ and } \mathrm{rk}^B(b) < \delta\}\} \end{aligned}$$

is a club of ω_1. By $\diamondsuit_{\aleph_1}$, there is $\alpha \in D_1$ such that $S \cap \alpha = S_\alpha$. Hence α satisfies the hypotheses (i)–(iv) of (•).

We will show that $J \subseteq J_{\alpha+1}$. By contradiction, let $b \in J - J_{\alpha+1}$. Then $b/J_\alpha \neq 0$. Then, there are $i < \omega$ and $u \in J_\alpha$ such that $c_{\alpha,i} - u \leq b$. Hence $c_{\alpha,i} - u \in J$. From (•), the choice of α, and from the definition of A, S and S_α, it follows that there is $\nu \in S_\alpha$ such that $a(d) \overset{\text{def}}{=} \varphi(\nu)$ verifies

$$\mathrm{rk}^B(a(d) - (c_{\alpha,i} - u)) < \alpha(d) \overset{\text{def}}{=} \mathrm{rk}^B(a(d)).$$

Hence $a(d) - (c_{\alpha,i} - u) \in J_{\alpha(d)}$. Let ξ be such that $d \in D_\xi(B/J)$. We have:

(1) $((a(d) - (c_{\alpha,i} - u)/J)/I_\xi(B/J) = 0$ by Claim 2(2),

(2) $((a(d)/J)/I_\xi(B/J) \neq 0$ since $d \in \mathrm{At}(D_\xi(B/J)$.

Therefore $(a(d) \cdot (c_{\alpha,i} - u)/J)/I_\xi(B/J) \neq 0$, and thus $a(d) \cdot (c_{\alpha,i} - u) \notin J$. That implies $c_{\alpha,i} - u \notin J$. Contradiction. ∎

3.2.2 The inductive step. Our induction hypothesis $\star(\alpha)$ (for $\alpha < \omega_1$) is as follows:

$\star_1(\alpha)$ *For every $\gamma \leq \beta \leq \alpha$, B_β and J_γ satisfy the hypotheses* (H1)–(H5) *of Theorem* 3.10.

$\star_2(\alpha)$ *For every $\gamma < \beta \leq \alpha$, there is a retract $B_{\beta,\gamma}$ of B_β/J_γ.*
For every $\lambda < \alpha$ limit, if $\beta < \lambda$, then $B_{\lambda,\gamma} = \bigcup_{\gamma<\beta<\lambda} B_{\beta,\gamma}$.

$\star_3(\alpha)$ *For every $\gamma < \beta' < \beta'' \leq \alpha$, $B_{\beta',\gamma} \subseteq B_{\beta'',\gamma}$.*

$\star_4(\alpha)$ *For every $\gamma \leq \alpha$, every finite subset σ of γ and every $a \in J_\gamma - \bigcup\{J_\delta\colon \delta < \gamma\}$ there is $\underline{a} \supseteq a$ such that $\underline{a} \in J_\gamma \cap \bigcap\{B_{\alpha,\beta}\colon \beta \in \sigma\}$.*

First, let us show that $\star_3(\alpha)$ implies that for $\gamma < \beta_1 < \beta_2 \leq \alpha$, $B_{\beta_1,\gamma} = B_{\beta_2,\gamma} \cap B_{\beta_1}$. This is a direct consequence of Part (a) of the following lemma:

LEMMA 3.12:

(a) *Let B_1, B_2 be Boolean algebras such that $B_1 \subseteq B_2$, I an ideal of B_2, $A_2 \subseteq B_2$ a retract of B_2/I, and $A_1 \subseteq B_1$ be a retract of $B_1/(I \cap B_1)$. If $A_1 \subseteq A_2$, then $A_1 = A_2 \cap B_1$.*

(b) *Moreover, we suppose that A_1 is atomic and that there is an atomic subalgebra C of B_2 such that:*

(1) $A_2 = \mathrm{cl}_{B_2}(A_1 \cup C)$.

(2) *We assume that there is ideal* $J_1 \supseteq \mathrm{At}(A_1)$ *of* A_1 *such that for every* $a_1 \in J_1$ *and every* $c \in \mathrm{At}(C)$, $a_1 \cdot c \in A_1$.

Then A_2 *is atomic and* $\mathrm{At}(A_2) = \mathrm{At}(A_1)$.

(c) *Let* α *be a limit ordinal. Suppose that:*

(1) *For every* $\beta < \alpha$, B_β *and* J_β *satisfies the hypotheses* (H1) – (H5) *of Theorem 3.10.*

(2) *For every* $\gamma \leq \beta < \alpha$, B_β / J_γ *has a retract* $B_{\beta,\gamma}$.

(3) *For every* $\gamma \leq \beta < \alpha$, $B_{\beta,\gamma} \subseteq B_{\beta+1,\gamma}$.

(4) *For every limit* $\lambda < \alpha$ *and every* $\gamma < \lambda$, $B_{\lambda,\gamma} = \bigcup_{\beta<\lambda} B_{\beta,\gamma}$.

For $\gamma < \alpha$, *let* $B_{\alpha,\gamma} \stackrel{\text{def}}{=} \bigcup_{\beta<\alpha} B_{\beta,\gamma}$. *Then* $B_{\alpha,\gamma}$ *is a retract of* B_α / J_γ.

Proof: (a) First, suppose that $B_1 = B_2$. It suffices to show that $A_2 \subseteq A_1$. Let $a_2 \in A_2$. Let $a_1 \in A_1$ be such that $a_1 \Delta a_2 \in I$. Since $a_1 \in A_1 \subseteq A_2$, we have $a_1 \Delta a_2 \in A_2$ and thus $a_1 \Delta a_2 = 0$. So $A_1 \ni a_1 = a_2$. In the general case, it suffices to show that $A_2 \cap B_1$ is a retract of $B_1/(I \cap B_1)$. Trivially $A_2 \cap B_1 \cap I = \{0\}$, and $B_1 = \mathrm{cl}_{B_1}((A_1 \cup (I \cap B_1)) \subseteq \mathrm{cl}_{B_1}((A_2 \cap B_1) \cup (I \cap B_1))$, and thus $\mathrm{cl}_{B_1}((A_2 \cap B_1) \cup (I \cap B_1)) = B_1$. Now (a) follows from Proposition 1.2.

(b) Clearly

$$\mathrm{At}(\mathrm{cl}_{B_2}(A_1 \cup C)) = \{a_1 \cdot c \colon a_1 \in \mathrm{At}(A_1),\ c \in \mathrm{At}(C) \text{ and } a_1 \cdot c \neq 0\}\ .$$

Let $a_2 \in \mathrm{At}(A_2)$. Let us show that $a_2 \in \mathrm{At}(A_1)$. a_2 has the form $a_1 \cdot c$, with $a_1 \in \mathrm{At}(A_1)$ and $c \in \mathrm{At}(C)$. By (2), $a_1 \cdot c \in A_1$, and thus $a_1 = a_1 \cdot c = a_2$. Hence $\mathrm{At}(A_2) \subseteq \mathrm{At}(A_1)$. Let $a_1 \in \mathrm{At}(A_1) - \mathrm{At}(A_2)$. This means that there are $a_2 \in \mathrm{At}(A_2)$ and $c \in \mathrm{At}(C)$ such that $0 \neq a_2 \cdot c < a_1$. As $a_2 \cdot c \in A_1$, we obtain a contradiction. So $\mathrm{At}(A_1) = \mathrm{At}(A_2)$.

(c) is obvious. ∎

Because we will construct a subalgebra $B = B_{\omega_1}$ of $\wp(\omega)$, the operation $\cdot$ and $+$ are denoted by $\cap$ and $\cup$.

It is easy to construct $\langle B_\alpha, J_\alpha, (B_{\alpha,m})_{m<\alpha}\rangle_{\alpha\leq\omega}$ satisfying $\star(\omega)$. Let $\alpha < \omega_1$ be limit and suppose that for every $\delta < \alpha$, $(B_\beta)_{\beta\leq\delta}$, $(J_\gamma)_{\gamma\leq\delta}$, and $(B_{\beta,\gamma})_{\gamma<\beta\leq\delta}$ are defined and verify $\star(\delta)$. We set $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$, $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$ and $B_{\alpha,\gamma} = \bigcup_{\beta<\alpha} B_{\gamma,\beta}$. From Lemma 3.12(c), it follows that $(B_\beta)_{\beta\leq\alpha}$, $(J_\gamma)_{\gamma\leq\alpha}$, and $(B_{\beta,\gamma})_{\gamma<\beta\leq\alpha}$ satisfy $\star(\alpha)$.

Let $\omega \leq \alpha < \omega_1$. We suppose that B_α, J_α and for $\beta < \alpha$, J_β and $B_{\alpha,\beta}$ are constructed, and satisfy $\star(\alpha)$. We must construct $B_{\alpha+1}$, $J_{\alpha+1}$ and $B_{\alpha+1,\beta}$ for $\beta < \alpha + 1$.

Let $\{\beta_k: k < \omega\}$ be a fixed enumeration of α.

For $i < \omega$, we will define $c_{\alpha,i}$ as a union of $c^n_{\alpha,i}$, for $i < n < \omega$. To do this, we will define an increasing sequence $\mathbb{P} = \langle p^n \rangle_{n<\omega}$ of conditions. A condition p^n has the form $\langle a^n, \{c^n_{\alpha,i}: i < n\}\rangle$, where n is an integer, and $\langle a^n, \{c^n_{\alpha,i}: i < n\}\rangle$ satisfies:

(P1) $a^n \in J_\alpha$.

(P2) $a^n \in \bigcap\{B_{\alpha,\beta_k}: k \leq n\}$.

(P3) $\{c^n_{\alpha,i}: c^n_{\alpha,i} \neq \emptyset \text{ and } i < n\}$ is a partition of a^n.

For two conditions $p^m = \langle a^m, \{c^m_{\alpha,i}: i < m\}\rangle$ and $p^n = \langle a^n, \{c^n_{\alpha,i}: i < n\}\rangle$ (in $\mathbb{P}$), we set $p^m \leq p^n$ if $m \leq n$ and if:

(P4) $a^m \subseteq a^n$.

(P5) $c^n_{\alpha,i} \cap a^m = c^m_{\alpha,i}$ for every $i < k \leq m$.

(P6) $c^n_{\alpha,i} - a^m \in \bigcap\{B_{\alpha,\beta_k}: k \leq m\}$ for every $i < n$,

hold.

Before continuing, let us note an easy fact:

LEMMA 3.13: *Let $p^q = \langle a^q, \{c^q_{\alpha,i}: i < q\}\rangle$ for $q \leq n$ satisfy* (P1)–(P6) *(so the conditions are pairwise comparable). Then:*

(P7) *For every $i < p$, and $m \leq p$, $c^p_{\alpha,i} \cap (a^p - a^m) \in \bigcap\{B_{\alpha,\beta_k}: k \leq m\}$.*

(P8) *If $i < p \leq m$, then $c^p_{\alpha,i} - a^m = \emptyset$. Hence* (P6) *is trivial in this case.*

(P9) *If $i < p$ and $m < p$, then $c^p_{\alpha,i} - a^m = c^p_{\alpha,i} - c^m_{\alpha,i}$.*

(P10) *If $m \leq p$, then $a^p - a^m \in \bigcap\{B_{\alpha,\beta_k}: k \leq m\}$.*

(P11) *$\leq$ is an order relation on $\mathbb{P}$.*

Proof: (P7) From (P3) and (P6), it follows that:

$$c^p_{\alpha,i} \cap (a^p - a^m) = c^p_{\alpha,i} - a^m \in \bigcap\{B_{\alpha,\beta_k}: k \leq m\}.$$

(P8) $c^p_{\alpha,i} \subseteq a^p \subseteq a^m$ by (P3) and (P4).

(P9) follows from the fact that $c^p_{\alpha,i} - a^m = c^p_{\alpha,i} - (c^p_{\alpha,i} \cap a^m) = c^p_{\alpha,i} - c^m_{\alpha,i}$.

(P10) is a consequence of (P3) and (P5).

(P11) The only non-trivial part is the fact that the property (P6) is transitive, but this follows from (P9) and (P5). ∎

The case of a legal guess (defined in 3.14) ensures that our construction will satisfies the hypotheses (•) of Lemma 3.11(b).

DEFINITION 3.14: S_α is a *legal guess* whenever:

(1) $\alpha = \lambda_\alpha \in D_0$ (and thus $\varphi[\alpha] = B_\alpha$).

(2) $\varphi[S_\alpha] \subseteq J_\alpha \subsetneq B_\alpha = \varphi[\alpha]$,

(3) For every $a \in \varphi[S_\alpha]$, $a/J_{\mathrm{rk}^{B_\alpha}(a)}$ is an atom of $B/J_{\mathrm{rk}^{B_\alpha}(a)}$.

(4) For every $\beta < \alpha$, there is $e \in \varphi[S_\alpha]$ such that $\mathrm{rk}^{B_\alpha}(e) > \beta$.

LEMMA 3.15:

(a) *Suppose that α is a limit ordinal, B_α has been defined, and S_α is a legal guess. There is an increasing sequence of conditions $\mathbb{P} = \{p^n : n < \omega\}$ with $p^n = \langle a^n, \{c^n_{\alpha,i} : i < n\}\rangle$ such that $\mathbb{P}$ satisfies the following density requirements:*

(R1) *For every $a \in J_\alpha$, there is $n < \omega$ such that $a \subseteq a^n$.*

(R2) *For every $i < \omega$ and $\beta < \alpha$, there is $n < \omega$ such that $c^n_{\alpha,i} \notin J_\beta$.*

(R3) *For every $i < \omega$, $n < \omega$ and $\beta < \alpha$, there are $k > n$ and $e \in \varphi[S_\alpha]$ such that $e \notin J_\beta$ and $\mathrm{rk}^{B_\alpha}(e - (c^k_{\alpha,i} - a^n)) < \mathrm{rk}^{B_\alpha}(e)$.*

(b) *Suppose that α is a limit, B_α has been defined, and S_α is not a legal guess. There is an increasing sequence of conditions $\mathbb{P} = \{p^n : n < \omega\}$ with $p^n = \langle a^n, \{c^n_{\alpha,i} : i < n\}\rangle$ such that $\mathbb{P}$ satisfies the following density requirements:*

(R1) *For every $a \in J_\alpha$, there is $n < \omega$ such that $a \subseteq a^n$.*

(R2) *For every $i < \omega$ and $\beta < \alpha$, there is $n < \omega$ such that $c^n_{\alpha,i} \notin J_\beta$.*

(c) *Suppose that α is a successor ordinal and B_α has been defined. We set $\alpha = \beta + 1$. There is an increasing sequence of conditions $\mathbb{P} = \{p^n : n < \omega\}$ with $p^n = \langle a^n, \{c^n_{\alpha,i} : i < n\}\rangle$ such that $\mathbb{P}$ satisfies the following density requirements:*

(R1) *For every $a \in J_\alpha$, there is $n < \omega$ such that $a \subseteq a^n$.*

(R2) *For every $i < \omega$ and $r < \omega$, there is $n < \omega$ such that $c^n_{\alpha,i}/J_\beta$ is the union of at least r atoms of B_α/J_β.*

Proof of Lemma 3.15(a): The claims 1, 2 and 3 are stated to satisfy the density requirements (R1), (R2) and (R3) respectively.

CLAIM 1: *Let $\{\langle a^j, \{c^j_{\alpha,i} : i < j\}\rangle : j < n\}$ satisfy* (P1)–(P6), *and let $a \in J_\alpha$. Then, there are a^n and $\{c^n_{\alpha,i} : i < n\}$ such that $a \subseteq a^n$ and $\{\langle a^j, \{c^j_{\alpha,i} : i < j\}\rangle : j \leq n\}$ satisfy* (P1)–(P6).

Proof: Let a^n be such that $a^n \in J_\alpha$, $a^n \supseteq a \cup a^{n-1}$ and $a^n \in \bigcap\{B_{\alpha,\beta_j} : j \leq n\}$. The existence of a^n follows from $\star_4(\alpha)$. Let $c^n_{\alpha,n-1} = a^n - a^{n-1}$, and $c^n_{\alpha,i} = c^{n-1}_{\alpha,i}$

for every $i < n-1$. Trivially (P1)–(P5) are satisfied. Let us prove (P6). Because (P2) holds for a^{n-1} and a^n, we have $c^n_{\alpha,n-1} = a^n - a^{n-1} \in \bigcap\{B_{\alpha,\beta_k}: k \leq n-1\}$. Hence, for $\ell \leq n$, $c^n_{\alpha,n-1} - a^\ell = a^n - a^{n-1} \in \bigcap\{B_{\alpha,\beta_k}: k \leq n-1\} \subseteq \bigcap\{B_{\alpha,\beta_k}: k \leq \ell\}$. Next, suppose $i < n-1$ and $\ell \leq n$. If $c^n_{\alpha,i} - a^\ell = \emptyset$, then $c^n_{\alpha,i} - a^\ell \in \bigcap\{B_{\alpha,\beta_k}: k \leq \ell\}$. If $c^n_{\alpha,i} - a^\ell \neq \emptyset$, then $\ell < n$ and $c^n_{\alpha,i} - a^\ell = c^{n-1}_{\alpha,i} - a^\ell \in \bigcap\{B_{\alpha,\beta_k}: k \leq \ell\}$ by the induction hypothesis. ∎

CLAIM 2: *Let* $\{\langle a^j, \{c^j_{\alpha,i}: i < j\}\rangle: j < n\}$ *satisfy* (P1)–(P6), *let* $p < n-1$ *and* $\beta < \alpha$. *Then, there are* a^n *and* $\{c^n_{\alpha,i}: i < n\}$ *such that* $a \supseteq a^n$ *and* $\{\langle a^j, \{c^j_{\alpha,i}: i < j\}\rangle: j \leq n\}$ *satisfy* (P1)–(P6), *and* $c^n_{\alpha,p} \notin J_\beta$.

Proof: Let $a \in J_\alpha - J_\beta$ be such that $a \cap a^{n-1} = \emptyset$. By $\star_4(\alpha)$, let a^n be such that $a^n \in J_\alpha$, $a^n \supseteq a \cup a^{n-1}$ and $a^n \in \bigcap\{B_{\alpha,\beta_j}: j \leq n\}$. We set $c^n_{\alpha,p} = c^{n-1}_{\alpha,p} \cup (a^n - a^{n-1})$, $c^n_{\alpha,n-1} = \emptyset$, and for every $i < n-1$ and $i \neq p$, we set $c^n_{\alpha,i} = c^{n-1}_{\alpha,i}$. Trivially (P1)–(P5) holds. Now, (P6) holds. Because (P2) holds for a^{n-1} and a^n, $(a^n - a^{n-1}) \in \bigcap\{B_{\alpha,\beta_j}: j \leq n-1\}$, and thus $c^n_{\alpha,p} - a^{n-1} = a^n - a^{n-1} \in \bigcap\{B_{\alpha,\beta_j}: j \leq n-1\}$. If $\ell < n-1$, then $c^n_{\alpha,p} - a^\ell = (a^n - a^{n-1}) \cup (c^{n-1}_{\alpha,p} - a^\ell) \in \bigcap\{B_{\alpha,\beta_j}: j \leq \ell\}$. ∎

CLAIM 3: *Let* $\{\langle a^j, \{c^j_{\alpha,i}: i < j\}\rangle: j < n\}$ *satisfy* (P1)–(P6), *and let* $p < n-1$ *and* $\beta < \alpha$. *Then, there are* a^n *and* $\{c^n_{\alpha,i}: i < n\}$, *and there is* $e \in \varphi[S_\alpha]$ *such that* $\{\langle a^j, \{c^j_{\alpha,i}: i < j\}\rangle: j \leq n\}$ *satisfy* (P1)–(P6), $e \notin J_\beta$ *and* $\mathrm{rk}^{B_\alpha}(e - (c^n_{\alpha,p} - a^{n-1})) < \mathrm{rk}^{B_\alpha}(e)$.

Proof: Let $\gamma < \alpha$ be such that $a^{n-1} \in J_\gamma$. Since S_α is a legal guess, there is $e \in \varphi[S_\alpha]$ such that $e \in J_\alpha - (J_\beta \cup J_\gamma)$ (so $e \notin J_\beta$). By $\star_4(\alpha)$, let a^n be such that $a^n \in J_\alpha$, $a^n \supseteq a^{n-1} \cup e$ and $a^n \in \bigcap\{B_{\alpha,\beta_j}: j \leq n\}$. We set $c^n_{\alpha,p} = c^{n-1}_{\alpha,p} \cup (a^n - a^{n-1})$, $c^n_{\alpha,n-1} = \emptyset$, and for every $i < n-1$ such that $i \neq p$, we set $c^n_{\alpha,i} = c^{n-1}_{\alpha,i}$. As in the proof of Claim 2, $\{a^j: j \leq n\}$, $\{c^k_{\alpha,i}: i < k \leq n\}$ satisfy (P1)–(P6). Next, $\mathrm{rk}^{B_\alpha}(e \cap a^n) = \mathrm{rk}^{B_\alpha}(e)$ follows from the fact that $\mathrm{At}(D_{\mathrm{rk}^{B_\alpha}(e)}(B) \ni e/J_{\mathrm{rk}^{B_\alpha}(e)} \leq a^n/J_{\mathrm{rk}^{B_\alpha}(e)}$. By the choice of e, the fact that $a^{n-1} \in J_\gamma$, and the definition of $c^n_{\alpha,p}$; we have $\mathrm{rk}^{B_\alpha}(e - (c^n_{\alpha,p} - a^{n-1})) < \mathrm{rk}^{B_\alpha}(e)$. ∎

End of the proof of Lemma 3.15(a): Let $a_0 = \emptyset$ $(= 0)$. Then a_0, $\{c^0_{0,i}: i < 0\}$ satisfy (P1)–(P6). In order that (i), (ii) and (iii) will hold, we have to fulfil $\aleph_0$ tasks. We make an ω-sequence of taskes such that every task appears infinitely many times. By claims 1–3 the taskes can be fulfilled, that ends Part (a).

Proof of Lemma 3.15(b) *and* (c): Similar. ∎

3.2.3 End of the proof. We set:

- For $i < \omega$, let $c_{\alpha,i} \bigcup_{\substack{n\in\omega \\ n>i}} c^n_{\alpha,i}$.
- $B_{\alpha+1} = \mathrm{cl}_{\wp(\omega)}(B_\alpha \cup \{c_{\alpha,i}\colon i < \omega\})$.
- $B_{\alpha+1,\alpha} = \mathrm{cl}_{\wp(\omega)}(\{c_{\alpha,i}\colon i < \omega\})$.
- For $\beta < \alpha$, let k be the unique index such that $\beta_k = \beta$. (We recall that $\{\beta_k\colon k < \omega\}$ is an enumeration of α.) Let

$$B_{\alpha+1,\beta} = \mathrm{cl}_{\wp(\omega)}(B_{\alpha,\beta} \cup \{c_{\alpha,i} - a^k\colon i < \omega\})\ .$$

Note that for $k > i$ $c_{\alpha,i} - a^k = c_{\alpha,i} - c^k_{\alpha,i}$ and for $k \le i$ $c_{\alpha,i} - a^k = c_{\alpha,i}$. We will prove that:

LEMMA 3.16: *With the above notations, we have:*

(a)

(1) $\bigcup\{c_{\alpha,i}\colon i < \omega\} = \omega$

(2) *If* $i \ne j$ *then* $c_{\alpha,i} \cap c_{\alpha,j} = \emptyset$.

(3) $\{c_{\beta,i}\colon i < \omega$ and $\beta \le \alpha\}$ *satisifies the hypotheses* (H1)–(H5) *of Theorem* 3.10.

(b) $B_{\alpha+1,\alpha}$ *is a retract of* $B_{\alpha+1}/J_\alpha$.

(c) *For* $\beta < \alpha$, $B_{\alpha+1,\beta}$ *is a retract of* $B_{\alpha+1}/J_\beta$.

(d) *For every* $a \in J_{\alpha+1} - J_\alpha$ *and every finite subset* σ *of* $\alpha+1$, *there is* $\underline{a} \in J_{\alpha+1}$ *such that* $\underline{a} \supseteq a$ *and* $\underline{a} \in \bigcap\{B_{\alpha+1,\beta}\colon \beta \in \sigma\}$.

Proof: (a1) and (a2) follows from (P3) and (R1).

(a3). It suffices to verify (H1)–(H5) for $\beta = \alpha$. (H2) is a consequence of (R2), and (H3) follows from (a2). (H4) holds: let $\gamma < \alpha$ and $j < \omega$. Then $c_{\gamma,j} \in J_\alpha$. By (R1), $c_{\gamma,j} \subseteq a^n$ for some n, and thus $c_{\gamma,j} \cap c_{\alpha,i} \subseteq a^n \in J_\alpha$. Therefore $c_{\gamma,j} \cap c_{\alpha,i} \in J_\alpha \subseteq B_\alpha$. (H5) holds: let $i < \omega$ and $\beta < \alpha$. For every $\ell \ne i$, there is j_ℓ such that $c_{\beta,j_\ell} \cap c_{\alpha,\ell} \notin J_\beta$, and thus $c_{\beta,j_\ell} - c_{\alpha,i} \in J_\beta$. Moreover, if $\ell' \ne \ell''$, then $j_{\ell'} \ne j_{\ell''}$. So (H5) holds.

(b) Trivially $c_{\alpha,i} \in B_{\alpha+1}$. $c_{\alpha,i}/J_\alpha \ne 0/J_\alpha$ in $B_{\alpha+1}/J_\alpha$ follows from (R2) and from the definition of $c_{\alpha,i}$. Now, it is easy to check that $B_{\alpha+1,\alpha}$ is a retract of $B_{\alpha+1}/J_\alpha$.

(c) *In what follows,* β *and* k *are such that* $\beta = \beta_k$.

First, we show that $\mathrm{cl}_{p(\omega)}(J_\beta \cup B_{\alpha+1,\beta}) = B_{\alpha+1}$. We have:

$$B_\alpha \subseteq \mathrm{cl}_{p(\omega)}(J_\beta \cup B_{\alpha,\beta}) \subseteq \mathrm{cl}_{p(\omega)}(J_\beta \cup B_{\alpha+1,\beta})$$

So, it suffices to show that $c_{\alpha,i} \in \mathrm{cl}_{p(\omega)}(J_\beta \cup B_{\alpha+1,\beta})$. We have:
(1) $c_{\alpha,i} - c^k_{\alpha,i} = c_{\alpha,i} - a^k \in B_{\alpha+1,\beta}$. By (P3), $c^k_{\alpha,i} \subseteq a^k \in J_\alpha$, and thus $c^k_{\alpha,i} \in J_\alpha \subseteq \mathrm{cl}_{p(\omega)}(J_\alpha) = B_\alpha = \mathrm{cl}_{p(\omega)}(J_\beta \cup B_{\alpha,\beta})$. Therefore:
(2) $c^k_{\alpha,i} \in \mathrm{cl}_{p(\omega)}(J_\beta \cup B_{\alpha,\beta})$. Hence

$$c_{\alpha,i} = (c_{\alpha,i} - c^k_{\alpha,i}) \cup c^k_{\alpha,i} \in \mathrm{cl}_{p(\omega)}(J_\beta \cup B_{\alpha+1,\beta}) \ .$$

That shows that $\mathrm{cl}_{p(\omega)}(J_\beta \cup B_{\alpha+1,\beta}) = B_{\alpha+1}$.

In order to prove that $J_\beta \cap B_{\alpha+1,\beta} = \{0\}$, we need the following result.

SUBLEMMA 3.17:

(a) *Let $b \in J_\alpha \cap B_{\alpha,\beta}$, and $i < \omega$. Then $b \cap (c_{\alpha,i} - a^k) \in B_{\alpha,\beta}$.*

(b) *Let $\beta < \alpha$. Then $\mathrm{At}(B_{\alpha,\beta}) = \mathrm{At}(B_{\alpha+1,\beta})$.*

(c) *Let $d \in J_\beta$. Suppose that d has the form $b \cap (c_{\alpha,i} - a^k)$ with $b \in B_{\alpha,\beta}$. Then $b \in J_\alpha$.*

(d) *Let $d \in J_\beta$. Suppose that d has the form $b \cap \bigcap_{i\in F} -(c_{\alpha,i} - a^k)$ with $b \in B_{\alpha,\beta}$ and $F \subseteq \omega$ finite. Then:*
   (1) *$b \in J_\alpha$ and*
   (2) *d has the form $b \cap \bigcap_{i\in G}(c_{\alpha,i} - a^k)$ for some finite subset G of ω.*

Proof: (a) From (R1) and (P4), there is $n > k$ such that $b \subseteq a^n$.

$$b \cap (c_{\alpha,i} - a^k) = c_{\alpha,i} \cap (a^n - a^k) \cap b = (c^n_{\alpha,i} - a^k) \cap b$$

since $c_{\alpha,i} \cap (a^n - a^k) = (c_{\alpha,i} \cdot a^n) - (c_{\alpha,i} \cdot a^k) = c^n_{\alpha,i} - a^k$. By (P6) (and the fact that $\beta = \beta_k$), we have $c^n_{\alpha,i} - a^k \in B_{\alpha,\beta}$, and because $b \in B_{\alpha,\beta}$, $b \cap (c_{\alpha,i} - a^k) \in B_{\alpha,\beta}$.

(b) Let $C = \mathrm{cl}_{B_{\alpha+1}}(\{c_{\alpha,i} - a^k : i < \omega\})$. Clearly C is an atomic subalgebra of $B_{\alpha+1}$ and every atom of C has the form $c_{\alpha,i} - a^k$. Now (b) follows from the fact that $B_{\alpha+1,\beta} = \mathrm{cl}_{B_{\alpha+1}}(B_{\alpha,\beta} \cup C)$, from Part (a) and Lemma 3.12(b).

(c) By contradiction $b \in B_\alpha - J_\alpha$ and thus $-b \in J_\alpha$. By Part (a), we have $(-b) \cap (c_{\alpha,i} - a^k) \in B_{\alpha,\beta}$. From the facts that $d \in J_\beta \subseteq B_\alpha$ and the fact that

$$\begin{aligned} c_{\alpha,i} - a^k &= (b \cap (c_{\alpha,i} - a^k)) \cup ((-b) \cap (c_{\alpha,i} - a^k)) \\ &= d \cup ((-b) \cap (c_{\alpha,i} - a^k)) \ , \end{aligned}$$

it follows that $c_{\alpha,i} - a^k \in B_\alpha$. Because $c^k_{\alpha,i} \subseteq a^k \in J_\alpha \subseteq B_\alpha$, we have $c^k_{\alpha,i} \in J_\alpha \subseteq B_\alpha$. Consequently $c_{\alpha,i} = (c_{\alpha,i} - c^k_{\alpha,i}) \cup c^k_{\alpha,i} \in B_\alpha$. Contradiction.

(d1): $b \in J_\alpha$. We have:

(1) $d/J_\alpha = 0^{B_{\alpha+1}/J_\alpha}$ since $d \in J_\beta \subseteq J_\alpha$.

(2) $(c_{\alpha,i} - a^k)/J_\alpha = c_{\alpha,i}/J_\alpha \in \mathrm{At}(B_{\alpha+1}/J_\alpha)$, since $a^k \in J_\alpha$. By (1),

(3) $0^{B_{\alpha+1}/J_\alpha} = d/J_\alpha = (b/J_\alpha) \cdot (\prod_{i \in F}(-c_{\alpha,i}/J_\alpha)$.

From (1)–(3), it follows that

(4) $b/J_\alpha \neq 1^{B_{\alpha+1}/J_\alpha}$. But

(5) $b/J_\alpha = 0^{B_{\alpha+1}/J_\alpha}$ or $1_{B^{\alpha+1}/J_\alpha}$ since $b \in B_\alpha = J_\alpha \cup -J_\alpha \subseteq B_{\alpha+1}$.

From (4) and (5), it follows that $b/J_\alpha = 0^{B_{\alpha+1}/J_\alpha}$, i.e. $b \in J_\alpha$. So (d1) is proved.

(d2): d has the form $b \cap \bigcap_{i \in G}(c_{\alpha,i} - a^k)$ for some finite subset G of ω. We have

$$\begin{aligned} d &= b \cap (\bigcap_{i \in F} -(c_{\alpha,i} - a^k)) = \bigcap_{i \in F} b - (c_{\alpha,i} - a^k) \\ &= (\bigcap_{i \in F}(b - c_{\alpha,i} - a^k)) \cup (b \cap a^k) . \end{aligned}$$

Because: $a^k \in B_{\alpha,\beta}$ (this follows from (P2) and $\beta = \beta_k$) and $b \in B_{\alpha,\beta}$; we have $b \cap a^k \in B_{\alpha,\beta}$. Because $b \cap a^k \subseteq d \in J_\beta$, we have $b \cap a^k \in J_\beta \cap B_{\alpha,\beta}$. Therefore $b \cap a^k = 0$. By (R1), let $n < \omega$ be such that $b \subseteq a^n$. We have $b \subseteq a^n = \bigcup_{i<n} c^n_{\alpha,i} \subseteq c_{\alpha,i}$. Hence

$$d = \bigcap_{i \in F}(b - c_{\alpha,i} - a^k) = \bigcup_{\substack{i<n \\ i \notin F}} b \cap (c_{\alpha,i} - a^k) .$$

That finishes the proof of Sublemma 3.17. ∎

We end the proof of 3.16(c). To show that $J_\beta \cap B_{\alpha,\beta} = \{0^{B_{\alpha+1}}\}$, it suffices to show that if $d \in J_\beta$ has the form $b \cap \bigcap_{i \in F} b_i$ with $b \in B_{\alpha,\beta}$, $F \subseteq \omega$ finite and $b_i = c_{\alpha,i} - a^k$ or $-(c_{\alpha,i} - a^k)$, then $d = 0^{B_{\alpha+1}}$. Let $b \cap \bigcap_{i \in F} b_i$ be such a d. Note that:

(1) If $b_i = c_{\alpha,i} - a^k$ and $b_j = c_{\alpha,j} - a^k$ with $i \neq j$, then $b_i \cap b_j = \emptyset$, and thus $d = 0 = \emptyset$.

(2) If $b_i = c_{\alpha,i} - a^k$ and $b_j = -(c_{\alpha,j} - a^k)$ with $i \neq j$, then $b_i \subseteq -b_j$, and thus $d \subseteq b \cap \bigcap_{i \in F - \{j\}} b_i$.

Therefore, there is no loss in assuming that d has the form

(i) : $b \cap (c_{\alpha,i} - a^k)$ or

(ii) : $b \cap \bigcap_{i \in F} -(c_{\alpha,i} - a^k)$.

We will distinguish the two cases:

CASE 1: $d = b \cap (c_{\alpha,i} - a^k)$. We have $b \in B_{\alpha,\beta} \subseteq B_\alpha = I_\alpha \cup -I_\alpha$. By Sublemma 3.17(c), $b \in J_\alpha$. Now, $d \in B_{\alpha,\beta}$ by Sublemma 3.17(a). Therefore $d \in J_\beta \cap B_{\alpha,\beta}$, and thus $d = 0$.

CASE 2: $d = b \cap \bigcap_{i \in F} -(c_{\alpha,i} - a^k)$. By Sublemma 3.17(d), d has the form $b \cap \bigcap_{i \in G}(c_{\alpha,i} - a^k)$ for some finite subset G of ω. But in this case, we can suppose that d has the form $b \cap (c_{\alpha,i} - a^k)$, and the arguments go as in Case 1.

Now (c) is a consequence of Proposition 1.2.

(d) Let $a \in J_{\alpha+1} - J_\alpha$, and $\sigma \subseteq \alpha + 1$ finite. There is no loss in assuming $\alpha \in \sigma$ (note that this implies that we must find $\underline{a}$ as a finite sum of $c_{\alpha,i}$).

First, we claim that

(d′) *There is* $n^a < \omega$ *such that* $a \subseteq \bigcup\{c_{\alpha,i}: i < n^a\}$.

By definition, $a \subseteq b \cup \bigcup\{c_{\alpha,i}: i < r\}$ where $b \in J_\alpha$. From (R1), let s be such that $b \subseteq a^s$. From $(P3)$, $a^s \subseteq \bigcup\{c^s_{\alpha,i}: i < s\}$. Consequently $a \subseteq \bigcup\{c_{\alpha,i}: i < n^a\}$ where $n^a = \max(r, s)$.

Next, we claim that

(d″) *For every* $\beta < \alpha$ *there is* $n_\beta < \omega$ *such that for every* $q \geq n_\beta$, $\bigcup\{c_{\alpha,i}: i < q\} \in B_{\alpha+1,\beta}$

Let $k < \omega$ be such that $\beta_k = \beta$. We set $n_\beta = k + 1$. Let $q \geq n_\beta$ be fixed. By (P3), we have $a^{k+1} \subseteq \bigcup\{c^{k+1}_{\alpha,i}: i < k+1\}$. Hence $\bigcup\{c_{\alpha,i}: i < q\} = \bigcup\{(c_{\alpha,i} - a^{k+1}) \cup a^{k+1}: i < q\}$. By (P2) $a^{k+1} \in B_{\alpha,\beta_k} = B_{\alpha,\beta}$. Now $c_{\alpha,i} - a^{k+1} = (c_{\alpha,i} - a^k) - (a^{k+1} - a^k)$. Because $\beta_k = \beta$ and the definition of $B_{\alpha+1,\beta_k}$, we have $(c_{\alpha,i} - a^k) \in B_{\alpha+1,\beta_k}$. By Lemma 3.13(P10), $(a^{k+1} - a^k) \in B_{\alpha,\beta_k}$. Consequently $\bigcup\{c_{\alpha,i}: i < q\} \in B_{\alpha+1,\beta_k} = B_{\alpha+1,\beta}$.

We will finish the proof of (d). Let $n = \max(n^a, \max(\{n_\beta: \beta \in \sigma\}))$ and $\underline{a} = \bigcup\{c_{\alpha,i}: i < n\}$. So $\underline{a} \in J_{\alpha+1}$. By (d′), $a \subseteq \underline{a}$; and by (d″), $\underline{a} \in \bigcap\{B_{\alpha+1,\beta}: \beta \in \sigma\}$. This finishes the proof of (d), and of Lemma 3.16. ∎

To finish the proof of Theorem 3.2(b), it remains to show:

LEMMA 3.18:

(a) *The induction hypothesis* $\star(\alpha + 1)$ *holds.*

(b) $\{c_{\beta,i}: i < \omega ; \beta < \alpha+1\}$ *verifies the hypotheses* (H1)–(H5) *of Theorem* 3.10 *(therefore* $B \stackrel{\text{def}}{=} \bigcup\{B_\alpha: \alpha < \omega_1\}$ *verifies the conclusions of Theorem* 3.10*).*

(c) *B is good (i.e. verifies* (G1), (G2) *and* (G3)*), that shows that B is as required in Theorem* 3.2(b).

Proof: (a) $\star_\ell(\alpha+1)$ holds trivially for $\ell = 1, 2$; $\star_3(\alpha+1)$ follows from Lemma 3.16 (b) and (c); and $\star_4(\alpha+1)$ is a consequence of $\star_4(\alpha)$ and of Lemma 3.16 (d).

(b) is a consequence of Lemma 3.16(a3).

(c). Let us show that B is good.

(G1) follows from Part (a) and Theorem 3.10.

(G2) is a consequence of Lemma 3.11(b). Let us verify that the hypothesis (•) of 3.11(b) holds. Let $\alpha < \omega_1$ be such that the hypotheses (i)–(iv) of (•) holds. Then S_α is a legal guess. Let $i < \omega$ and $c \in c_{\alpha,i}/J_\alpha$. We must find $a \in \varphi[S_\alpha]$ such that $\mathrm{rk}^{B_\alpha}(a - c) < \mathrm{rk}^{B_\alpha}(a)$. There is no loss in assuming that $c = c_{\alpha,i} - u$ with $u \in J_\alpha$. By Lemma 3.15(a) (R1) there is $n < \omega$ such that $u \subseteq a^n$, and thus we can suppose that $c = c_{\alpha,i} - a^n$. By (R3), for $\beta < \alpha$, there are $k > n$ and $a \in \varphi[S_\alpha]$ such that $\mathrm{rk}^{B_\alpha}(a - (c^k_{\alpha,i} - a^n)) < \mathrm{rk}^{B_\alpha}(a)$. Since $c = c_{\alpha,i} - a^n \supseteq c^k_{\alpha,i} - a^n$, the conclusion of (•) holds.

(G3). Let $a \in J_\alpha$. By (R1), let $n < \omega$ be such that $a \subseteq a^n$. By (P1) and (P3), $\bigcup_{i<n} c^n_{\alpha,i} = a^n \in J_\alpha$. Therefore $a \subseteq \bigcup_{i<n} c_{\alpha,i}$ and $a = \bigcup_{i<n}(a \cap c^n_{\alpha,i}) \in \mathrm{cl}^{\mathrm{Id}}_B(\bigcup_{i<n} J_\alpha \restriction c_{\alpha,i})$. ∎

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