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A COMPACTNESS THEOREM FOR SINGULAR CARDINALS, FREE ALGEBRAS, WHITEHEAD PROBLEM AND TRANSVERSALS

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ABSTRACT

We prove, in an axiomatic way, a compactness theorem for singular cardinals. We apply it to prove that, for singular λ, every λ-free algebra is free; and similar compactness results for transversals and colouring numbers. For the general result on free algebras, we develop some filters on $S_\kappa(A)$. As an application we conclude that $V = L$ implies that every Whitehead group is free.

0. Introduction

Generalizing the compactness theorem is a natural question. Hanf [9] proved that the generalization "a set of sentences in $L_{\kappa,\kappa}$ which every $< \kappa$ of them have a model, has a model" usually fails; moreover, the counterexample has cardinality κ. Later the various generalizations were classified (by equivalence and implications), and the theory of large cardinals arose (with notion as κ weakly compact, measurable, compact, supercompact). Gustin asked when does $PT(\lambda,\chi)$ hold ($PT(\lambda,\chi)$ means, that if S is a family of λ sets, each of cardinality $< \chi$, and every subfamily with $< \lambda$ sets has a transversal (see Def. 2.1), then S has a transversal). The question was mentioned by Erdös and Hajnal [6, 7, problem 42 (C)]. Shelah [25] proved, in fact, that $\chi < \lambda$, $\operatorname{cf} \lambda = \aleph_0$, implies $PT(\lambda,\chi)$. Clearly not $PT(\aleph_1,\aleph_1)$, and in fact not $PT(\lambda,\lambda)$ (see [22]). Milner and Shelah [21] proved that, for regular λ, not $PT(\lambda,\chi)$ implies not $PT(\lambda^+,\chi)$, hence not $PT(\aleph_n,\aleph_1)$. It is clear (see [7], p. 279) that S^λ_χ implies not $PT(\lambda,\chi^+)$, where

> S^λ_χ: λ,χ are regular cardinals, $\chi < \lambda$, and there is a stationary set $A \subseteq \lambda$ such that $\alpha \in A \Rightarrow \operatorname{cf} \alpha = \chi$, and for any limit $\delta < \lambda$, $\delta \cap A$ is not a stationary subset of δ.

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Note that, if $\chi < \lambda$ are regular, λ is not weakly compact, and $V = L$, then S^λ_χ holds (for $\chi = \aleph_0$ by [17], for $\chi > \aleph_0$ by a slight strengthening of [17] due to A. Beler).

Note also that, when $\chi < \lambda, \lambda$ weakly compact, then $PT(\lambda, \chi)$ (see [22]).

Later the author observed that there was a great similarity between the results just mentioned and the research on the existence of λ-free non-λ^+-free (abelian) groups.

A question on this appeared in [18] and later in [8]. Specker [32] proved incompactness (i.e. existence) for abelian groups, $\lambda = \aleph_n$; Higman [14] proved, for groups, incompactness in $\aleph_1$ and compactness in strong limit λ, cf $\lambda = \aleph_0$.

Griffith [16] proved the existence of $\aleph_n$-free non-free abelian groups (whose cardinality was not necessarily $\aleph_n$). Hill [10] proved the existence of $\aleph_n$-free non-free abelian groups of cardinality $\aleph_0$, and proved the compactness result for λ when cf $\lambda = \aleph_0$. Eklof [3], Gregory [15] and the author [26] independently proved that $S^\lambda_{\aleph_0}$ implies incompactness in λ for abelian groups. Eklof [4] and Gregory [15] proved, independently, that for abelian groups, incompactness in λ, λ regular, implies incompactness in λ^+, hence proved (independently of Hill) the incompactness in $\aleph_n$. Mekler [24] generalized those results to groups, except for the last one, which he proved for $\aleph_n$ only. He also proved again the compactness result from the model theoretic theorem of Chang [2]: if M, N are $L_{\infty,\lambda}$ -eigenvalent, and of cardinality $\leqq \lambda$, and cf $\lambda = \aleph_0$, then M, N are isomorphic. Eklof and Mekler [24] proved that if κ is a compact cardinal, every κ-free (abelian) group is free. Kueker [20] proved the compactness result for freeness in any variety for strong limit λ of cofinality $\aleph_0$.

This led me to the following conjecture (only the simple cases are given, as χ can be added):

Conjecture A. The following properties of λ are equivalent:

(1) $PT(\lambda, \aleph_1)$.

(2) If G is a graph with λ vertices, and every subgraph of G spanned by $< \lambda$ vertices has colouring number $\leqq \aleph_0$, then G has colouring number $\leqq \aleph_0$.

(3) If G is a λ-free group (i.e. every subgroup of cardinality $< \lambda$ is free), then G is λ^+-free.

(4) As (3) for abelian groups.

We shall give here a positive result for singular λ for (1)–(4). Hence we have a complete answer when $V = L$ (for (2) see [25]).

By [26] (see Fuchs [8] for the group-theoretic information) we can use our compactness theorem to prove (here we deal with abelian groups):

THEOREM 0.1. (A) $(V = L)$ *Every Whitehead group is free.*

(B) *The statement "every Whitehead group is free" is independent of and consistent with ZFC.*

PROOF. (A) We prove by induction on λ that every Whitehead group A is λ^+-free.

(1) For $\lambda = \aleph_0$ the proof is well known (see e.g. [8]).

(2) λ regular: as being a Whitehead group is hereditary, and by the induction hypothesis we can assume $|A| = \lambda$ and A is λ-free, the proof in [26] works.

(3) λ singular: as being a Whitehead group is hereditary, and by the induction hypothesis A is λ-free, hence, by Theorem 2.4, A is λ^+-free.

(B) Immediate by [26] and part (A).

Hill [12] proved that λ-free abelian groups are λ^+-free when $\mathrm{cf}\,\lambda = \aleph_1$. Let us explain his proof. Let A be a λ-free group, $|A| = \chi$; then we can find an increasing and continuous sequence of pure subgroups of A,

$$A_i (i < \omega_1), \ A = \bigcup_{i < \omega_1} A_i, \ |A_i| < \lambda.$$

So A_i is free and let I_i be a basis of A_i. Choose an increasing and continuous sequence of pure subgroups of A,

$$A'_i (i < \omega_1), \ A = \bigcup_{i < \omega_1} A'_i, \ |A_i| < \lambda,$$

so that $A'_i \cap A_j$ is generated by a subset of I_j. Now choose an increasing and continuous sequence of pure subgroups of A, B_i $(i < \lambda)$, so that $|B_{i+1}/B_i| \leqq \aleph_1$ and $B_i \cap A'_j$ is generated by a subset of I'_j (I'_j a fixed basis of A'_j). Hill then proved that B_{i+1}/B_i is free. He used his theorems:

(1) ([14]) *If* $|B| = \aleph_1$, *B an abelian group,* $B = \bigcup_{i<\omega_1} B_i$, *$B_i$ increasing and continuous, B_i free, and B_{i+1}/B_i is $\aleph_1$-free, then B is free.*

(2) ([11]) *If B_n is a pure subgroup of B_{n+1}, B_n is free, then $\bigcup_{n<\omega} B_n$ is free.*

This proof is an embryonic form of Lemmas 1.8, 1.10 (see below).

SCHEME OF OUR PROOF. We first give the axioms, and prove technical Lemmas 1.1, 1.2. Lemma 1.3 is a generalization of : if A is a λ^+-free group, $B \subseteq A$, $|B| < \lambda$, then there is C, $B \subseteq C \subseteq A$, $|C| < \lambda$, C is free and A/B is λ^+-free. Then we introduce our central notion when A/B is $P_\alpha(\lambda)$-free. Lemma 1.4 is technical, and Lemmas 1.5, 1.6, 1.7 prove that the notion $P_\alpha(\lambda)$-free satisfies (variants of) the axioms VI, V, VII. In 1.8 we prove that, if λ is singular, A/B is λ-free, $|A| = \lambda$, then A/B is $P_\alpha(\mathrm{cf}\,\lambda)$-free for $\alpha \leqq \lambda$. In

1.10 we prove that, if A/B is $P_{1+\alpha}(\mu)$-free, $\mu \leqq \aleph_\alpha$ or $\alpha \geqq \omega$, then A/B is free and 1.9 proves the main point of 1.10. Lemmas 1.11, 2.1(1), (3)–15) are not really needed. In Section 2 we apply Section 1 to the cases of conjecture A, and to free algebras in general. For λ-free groups and λ-free abelian groups we get compactness for singular λ. To get this result for algebras, we deal in Section 3 with E-freeness for filters E over $S_\kappa(A)$ and get that, if rcf $\lambda \leqq \leqq \lambda$ (e.g. λ strong limit singular or $\lambda = \aleph_{\alpha+\omega}$), then without Ax I* we get that λ-free are λ^+-free. Using results from Section 3 in Section 2, we prove that if A is a λ-free algebra, cf $\lambda < \lambda$, then A is λ^+-free. We also prove that κ^+-freeness of A (or κ-freeness, for limit κ) implies $L_{\infty,\kappa}$-equivalence to a free algebra. This improves previous results of Mekler [24] (on groups and abelian groups he uses Ax I*) and Kueker [20] (for general algebras, but from $(2^\kappa)^+$-freeness he obtains L_{∞,κ^+}-equivalence, or assumes κ is strong limit). (Notice that his filter "almost all κ^+-subalgebras of A" is contained in $E_\kappa(A)$. Our filters are a combination of his filters, with the construction of models of saturation in [28] VII Section 1. Our results are stronger, as our hypotheses look like: the set of non-free $C \subseteq A$, $|C| = \kappa$ does not belong to the filter.)

Ax I* has a special role, because in some applications we use two settings: one in which it is satisfied, so we can use 1.8, and one in which it fails but $\chi_1 = \aleph_0$, so it is easier to use 1.10.

In [29] we shall give abstract form to (and strengthen) the theorem on "λ-incompactness implies λ^+-incompactness". E.g. we prove it for groups, and also generally, that $V = L$ implies the equivalence of all our properties. We prove some independence results disproving conjecture A. It is consistent that every $2^{\aleph_0}$-free group (or abelian group) is free; and it is also consistent that a graph has colouring number $\leqq \aleph_0$ iff every subgraph with $\leqq \aleph_1$ vertices has colouring number $\leqq \aleph_0$. We shall also put results from Baumgartner [1] into our scheme, and show the connection with some filters. We also prove in conjecture A that $(1) \leftrightarrow (3) \leftrightarrow (4)$[(1)].

Our results were announced in [27], [30], [31].

Conjecture B. Prove that in 3.8 we cannot omit rcf $\lambda \leqq \leqq \lambda$.

Possibly for $\alpha = \omega_1$ or $\alpha = \omega_1\omega$; when $\aleph_\alpha < 2^{\aleph_0}$ there is an $\aleph_\alpha$-free not $\aleph_{\alpha+1}$-free pair of rings.

Open Problem C. What is the situation in conjecture A for $\lambda = \aleph_{\omega+1}$? It seems likely that the result is independent.

(1) We also eliminate AxI*, so also problems B, C are solved.

OPEN PROBLEM D. Are our axioms the right ones (i.e., are there axioms, not much more complicated, applicable to cases which we cannot include in our scheme), and what is the situation for chromatic numbers and property B? (See [6], [7] problem 42.)

OPEN PROBLEM E. (1) Characterize the varieties which satisfy Ax I* (or even those for which being free is hereditary).

(2) Characterize the varieties (and the κ's) such that there is a κ (even if $V = L$) such that any κ-free algebra is free.

CONJECTURE F. Prove that for $\kappa < |A|$, Kueker filters (see e.g. [20]) (for almost all subsets of A of cardinality κ) $E_\kappa(A)$, $\underline{E}_\kappa(A)$ are distinct. Similar problems arise for $E^\mu_\kappa(A)$, $\boldsymbol{E}^\mu_\kappa(A)$ and $\underline{E}^\mu_\kappa(A)$.

REMARKS AND QUESTIONS. (1) Lemmas 1.5, 1.6, 1.7 say that $P_\alpha(\lambda)$ satisfies (variants of) the Axioms VI, V, VII, respectively, and 1.11, 1.3 are parallel to Ax III, IV and Ax VII for λ-free.

(2) We can generalize the $P_\alpha(\lambda)$-free (and add to it cardinal parameters) by replacing in Def. 1.2 (3), (iii), (iv), (v) "free", by "κ_i-free". Clearly, many lemmas generalize.

It may be interesting to investigate the generalizations suggested in (1) and (2).

(3) Clearly, 1.8 gives a much stronger result than the one needed as a hypothesis in 1.10. This may suggest that our theorem could be strengthened.

(4) Similarly to the incompactness results in [21], [4] and [16], we can show that $P_n(\aleph_n)$-freeness does not imply freeness.

(5) Clearly in 2.1 (and the conclusions) we can replace "λ-free" by "not $\underline{E}^\kappa_\kappa$-non-free", for arbitrarily large successors $\kappa < \lambda$.

(6) The notion of λ-free in our paper is weaker than the usual one (hence our results are stronger).

(7) For abelian groups, because of the theorem in [11], we can weaken the definition of $P_\alpha(\lambda)$-free, as long as we replace "subgroup" by "a pure subgroup".

(8) In Section 2, when we deal with free algebras, we can replace "Γ a set of identities" by "Γ a set of universal Horn sentences" (even for sentences in $L_{\lambda^+,\aleph_0}$; i.e., members of Γ have the form

$$(\forall \bar{y})\,[\bigwedge_i \tau_i(\bar{y}) = \sigma_i(\bar{y}) \rightarrow \tau(\bar{y}) = \sigma(\bar{y})].$$

Hence our treatment is not less general than that of Kueker [20].[2]

[2] But this free algebra is also the free algebra for some set of identities, so no generality is gained (remarked by M. Rubin).

(9) Kueker (e.g. [20]) introduces a filter on $S_\kappa(A)$; in Section 3 we add several others ($E^\mu_\kappa(A)$, $\underline{E}^\mu_\kappa(A)$, $\boldsymbol{E}^\mu_\kappa(A)$). We can easily suggest more. A better understanding of their interrelations, and relations with $<_{\mu,\lambda}$, is desirable, conjecture F is just an example.

It seems that the filters in Section 3 can be better understood as projections of filters on sets of sequences of members of $S_\kappa(A)$. E.g., let for $\delta < \kappa^+$

$$S^\delta_\kappa(A) = \{\langle A_i : i < \delta\rangle : A_i \in S_\kappa(A) : A_i \text{ increasing and continuous}\}$$

and for any function

$$g : \bigcup_{j<\delta} S^j_\kappa(A) \to S_\kappa(A)$$

$$S^\delta_\kappa(A,g) = \{\langle A_i : i < \delta\rangle \in S^\delta_\kappa(A) : g(\langle A_j : j \leqq i\rangle) \subsetneqq A_{i+1}\}.$$

The sets $S^\delta_\kappa(A,g)$ generate a filter over $S^\delta_\kappa(A)$.

(10) In Section 3 we investigate E-freeness, and "not $\underline{E}$-non-freeness", in order to get results on λ-freeness. In may be interesting to investigate these notions, and their interrelations (e.g. does E_1-freeness imply E_2-freeness), for their own sake. In this context, we can seek improvements of 3.7, 2.5, 2.6.

(11) It is easy to prove that an algebra (with countable language) which is $L_{\infty,\omega}$-free is $\aleph_1$-free, for a suitable $\underline{M}$.

Also, if $A = \bigcup_{i<\lambda} A_i$, each A_i is free, A_{i+1}/A_i is free if i is a successor or cf $i < \mu$ (see Section 2 for terminology), then A is $L_{\infty,\lambda,G(\mu)}$-free (i.e. equivalent to a free one, in the logic with the quantifiers $(\cdots(\forall \bar{X}_i)\ (\exists \bar{Y}_i)\cdots)_{i<\alpha}$, where $\alpha < \mu$ ($\bar{X}_i, \bar{Y}_i$ of length $< \lambda, \lambda$ regular) and arbitrary conjunctions. Also the converse is true, and if e.g. $V = L$, $\mu \leqq \lambda$, μ and λ are regular, we can find such A, $|A| = \lambda$, which is not $L_{\infty,\lambda,G(\mu^+)}$-free.

(12) Notice that, by 3.9, if rcf $\lambda \leqq \leqq \lambda$, then the conditions (i), (ii) are equivalent.

(i) For arbitrarily large successor $\kappa < \lambda$, A/B is not $\underline{E}^\kappa_\kappa$-non-free.

(ii) For every $\kappa < \lambda$ there are $\kappa(l)$ $(l = 1,2)$, $\kappa < \kappa(l) < \lambda$, such that A/B is not $E^{\kappa(2)}_{\kappa(1)}$-non-free.

(13) We can strengthen Section 3 a little and the appropriate part of Section 2, by replacing S_κ by $S_{<\kappa}(a) = \{b \subseteq a : |b| < \kappa\}$, mainly for regular κ.

1. The abstract setting

In the following setting U and F are essential, whereas $\underline{M}$ is for convenience. Let U be a fixed algebra, with χ_0 operations, and F be a set of pairs (A,B) where A, B are subalgebras of U or $\varnothing$ (the empty set). Let χ_2 be such that U, $F \in H(\chi_2)$ (= the family of sets which are hereditarily of cardinality $< \chi_2$) and let $\underline{M}$ be an expansion of the model $(H(\chi_2), \in, =, U, F)$ by $\leqq \chi_1$ relations and functions, and we assume $\underline{M}$ has Skolem functions. We say A is free over B or $A/B \in F$ if $(A, B) \in F$. During this section U, F, $\underline{M}$, χ_0, χ_1, χ_2 are fixed, hence "A is free over B" is not ambiguous. We assume $\chi_0 \leqq \chi_1 \leqq \chi_2$ and that the functions of U are functions of $\underline{M}$.

NOTATION. Let A, B, C, D denote subalgebras of U or empty sets; M, N will denote elementary submodels of $\underline{M}$ of cardinality χ_2. We shall not distinguish strictly between a model M and its universe $|M|$ and similarly for U, and in fact also $A, \cdots$. Let $M < N$ mean M is an elementary submodel of N, and $M < N$ mean $M < N$ and $M \in N$. For a subset V of U, cl V is the closure of V to $\emptyset$ or a subalgebra of U. We shall usually write $\bigcup_i A_i$ instead of $\mathrm{cl}(\bigcup_i A_i)$.

REMARK. To strengthen his intuition the reader may think of an example: abelian groups or transversals. For abelian groups $A/B \in F$ will mean $\mathrm{cl}(A \cup B)/B$ is a free abelian group; and the reader may read the proof that the axioms hold (in Section 2) immediately after reading the axioms.

REMARK. The theorems using Ax I* will be marked by a star *.

SET OF AXIOMS. Ax I*. *If A is free over B and $B \in N$, then $A \cap N$ is free over B.*

Ax II. *A is free over B iff $A \cup B$ is free over B; and always B is free over B.*

Ax III. *If A is free over B, and B is free over C, where $A \supseteq B \supseteq C$, then A is free over C.*

Ax IV. *If $A = \bigcup_{i<\lambda} A_i$, A_i $(i < \lambda)$ increasing and continuous, $A_0 \subseteq B$ and for $i < j < \lambda$, $A_j / A_i \cup B$ is free and λ is a regular cardinal, then A is free over B.*

Ax V. *Suppose $D \in M$, $C_i \in M$ $(i < \alpha)$, $B \subseteq D$, $A \subseteq D$, $D \subseteq C_0$, and C_i is increasing. If A is free over $(C_0 \cap M) \cup B$ and $C_i \cap M$ is free over $(C_0 \cap M) \cup D$ for $i < \alpha$, then A is free over $[(\bigcup_{i<\alpha} C_i) \cap M] \cup B$.*[3]

REMARK. Instead of $D \subseteq C_0$ we can require $M \cap (D - C_0) \subseteq B$ (just use $C_i \cup D$ instead of C_i). We shall use this version freely.

Ax VI. *If A is free over $B \cup C$, and $\{A, B, C\} \subseteq N$, then $A \cap N$ is free over $(B \cap N) \cup C$.*

Ax VII. *If A is free over B, and $\{A, B\} \subseteq N$, then A is free over $(A \cap N) \cup B$.*

[3] It seems better to replace "$C_i \cap M$ is free over $(C_0 \cap M) \cup D$" by "C_i / C_0 is free".

CLAIM 1.1. *If A_i $(i < \alpha)$ is increasing and continuous, A_0 is free over B, and A_{i+1} is free over $A_i \cup B$ (for $i < \alpha$), then $\bigcup_{i<\alpha} A_i$ is free over B.*

PROOF. By induction on α; for $\alpha = 0$, $\bigcup_{i<\alpha} A_i = \emptyset$, hence the conclusion follows by Ax II. If $\alpha = 1$, then $\bigcup_{i<\alpha} A_i = A_0$, so the conclusion is one of the hypotheses. If $\alpha = \beta + 1$, β a limit ordinal, then $\bigcup_{i<\alpha} A_i = \bigcup_{i<\beta} A_i$ (by the continuity of A_i), so the conclusion follows by the induction hypothesis. If $\alpha = \beta + 2$, then $A_\beta = \bigcup_{i<\beta+1} A_i$ is free over B by the induction hypothesis, and $A_{\beta+1} = \bigcup_{i<\alpha} A_i$ is free over $A_\beta \cup B$ by a hypothesis of the theorem, hence, by Ax III, our conclusion holds. We are left with the case α a limit ordinal, so $\lambda = \operatorname{cf} \alpha$ is a regular cardinal, and we can choose an increasing continuous sequence $\beta(i)$ $(i < \lambda)$ whose limit is α, $\beta(0) = 0$. So $A = \bigcup_{i<\lambda} A_{\beta(i)}$, $A_{\beta(i)}$ is increasing and continuous; by the induction hypothesis, $A_{\beta(j)}$ is free over $A_{\beta(i)} \cup B$ for $i < j < \lambda$. By applying Ax IV (to A'_i, where $A'_0 = \emptyset$, $A'_{1+i} = A_{\beta(i)} \cup B$) we obtain our conclusion.

DEFINITION 1.1. (1) A is λ-free over B if $\lambda > \chi_1$ and $\|N\| < \lambda$, $A \in N$, $B \in N$ implies $A \cap N$ is free over B.

(2) A is weakly λ-free over B if $\lambda > \chi_1$, and for any N, $\|N\| < \lambda$, there is an M, $\|M\| < \lambda$, $N < M$, $A \cap M$ is free over B.

(3) We also use the expression "A/B is (weakly) λ-free".

CLAIM 1.2. (1)* *A/B is λ-free iff A/B is weakly λ-free.*

(2)* *If A/B is λ-free, $B \in N$, $\|N\| < \lambda$, then $A \cap N/B$ is free.*

(3) *For $\lambda = \infty$, or even $\lambda \geqq |A|^+$, A/B is free iff it is λ-free iff it is weakly λ-free.*

(4) *If $\lambda_1 \geqq \lambda_2 > \chi_1$, and A/B is λ_1-free, then A/B is λ_2-free, hence weakly λ_2-free. Also λ_1-weakly free implies λ_2-weakly free.*

(5)* *If A_i $(i < \delta)$ is increasing, $\operatorname{cf} \delta \geqq \lambda$, A_i is λ-free over B, then $\bigcup_{i<\delta} A_i$ is λ-free over B.*

(6) *If A is free over $B \cup C$, and $\{A, B, C\} \subseteq N < M$, then $A \cap M$ is free over $(A \cap N) \cup (B \cap M) \cup C$.*

(7) *If $|A| = \lambda > \chi_1$, $A = \bigcup_{i<\lambda} A_i$, A_i is increasing and continuous, $|A_i| < \lambda$, and λ is regular, then A/B is free iff there is a closed unbounded set $S \subseteq \lambda$ such that for any i, $j \in S$, $i < j$, A_i/B and $A_j/A_i \cup B$ are free.*

PROOF. (1) For the "only if" part choose M, $N < M$, $\|M\| < \lambda$, $\{A, B\} \subseteq M$. Then by the definition of "λ-free", $A \cap M/B$ is free. On the other hand, for the "if" part, suppose $\{A, B\} \subseteq N$, $\|N\| < \lambda$; then for some M, $\|M\| < \lambda$, $N < M$, and $A \cap M/B$ is free. As $B \in N$, by Ax I* $(A \cap M) \cap N = A \cap N$ is free over B.

(2) Choose M, $N < M$, $\|M\| < \lambda$, $\{A, B\} \subseteq M$; then $A \cap M/B$ is free by the definition of "λ-free", so, by Ax I*, $(A \cap M) \cap N = A \cap N$ is free over B.

(3) Trivial.

(4) Immediate.

(5) If $B \in N$, $\|N\| < \lambda$, then for some $\alpha < \lambda$, $(\bigcup_{i<\delta} A_i) \cap N \subseteq A_\alpha$; hence $(\bigcup_{i<\delta} A_i) \cap N = A_\alpha \cap N$. But by 1.2(2), $A_\alpha \cap N/B$ is free.

(6) By Ax VII, A is free over $(A \cap N) \cup B \cup C$. As $N < M$, $A \cap N \in M$, hence, by Ax VI, $A \cap M$ is free over $(A \cap N) \cup (B \cap M) \cup C$.

(7) The "if" part follows by 1.1. For the "only if" part assume A/B is free and choose an increasing and continuous sequence N_i $(i < \lambda)$, so that $i < j \Rightarrow N_i < N_j$, $\|N_i\| < \lambda$, $\{A, B\} \subseteq N_0$, and $A \subseteq \bigcup_{i<\lambda}(A \cap N_i)$. Clearly, $S = \{i < \lambda : A \cap N_i = A_i\}$ is closed and unbounded, and if $i, j \in S$, $i < j$, then A_i/B is free by Ax VI and $A_j/A_i \cup B$ is free by 1.2(6).

Lemma 1.3*. *If A is λ-free over B, $A' \subseteq A$, and $\mu^+ < \lambda$ where $\mu = |A'| + \chi_1$, then there is N such that $\|N\| = \mu$, $A' \subseteq N$, $\{A', A, B\} \subseteq N$ and A is λ-free over $(A \cap N) \cup B$.*

Proof. Suppose there is no such N, and we shall get a contradiction. Define M_i $(i < \mu^+)$ by induction on i so that it is increasing by $<$, and $\|M_i\| = \mu$.

Choose M_0 so that $\|M_0\| = \mu$, $A' \cup \{A', A, B\} \subseteq M_0$. If M_i is defined for $i < \delta$, δ a limit ordinal, let $M_\delta = \bigcup_{i<\delta} M_i$ (it exists as M_i is increasing by $<$). Clearly the induction hypothesis is satisfied.

Suppose M_i is defined, and we shall define M_{i+1}. As M_i cannot serve as N, necessarily there is N^1 such that $\|N^1\| < \lambda$, $M_i < N^1$, and $A \cap N^1$ is not free over $(A \cap M_i) \cup B$ (using Ax I*). Choose N^2 such that $M_i < N^2$, $\|N^2\| = \mu$ and $N^1 \in N^2$.

As A is λ-free over B, and $B \in M_0 \leqq M_i < N^1$, clearly $A \cap N^1$ is free over B, and as $N^1 \in N^2$, clearly $A \cap N^1 \in N^2$; hence, by Ax VII, $A \cap N^1$ is free over $(A \cap N_1 \cap N_2) \cup B$. If $A \cap N^1 \cap N^2$ is free over $(A \cap M_i) \cup B$, then (as $A \cap M_i \subseteq A \cap N^1 \cap N^2$), by Ax III, we get that $A \cap N^1$ is free over $A \cap M_i) \cup B$, contradicting a previous hypothesis. We can conclude that $A \cap N^1 \cap N^2$ is not free over $(A \cap M_i) \cup B$. Notice that, as $\underset{\sim}{M}$ has Skolem function, $|N^1| \cap |N^2|$ is the universe of an elementary submodel of $\underset{\sim}{M}$, so call it M_{i+1}. As $|M_i| \subseteq |N^1|$, $|M_i| \subseteq |N^2|$, clearly $M_i < M_{i+1}$, and as $M_i \in N^1$, N^2, clearly $M_i \in M_{i+1}$, so $M_i < M_{i+1}$, hence the induction hypothesis holds.

Now $M = \bigcup_{i<\mu^+} M_i < \underset{\sim}{M}$, $\|M\| = \mu^+ < \lambda$, and of course $\{A, B\} \subseteq M$, hence (by Definition 1.1) $A \cap M$ is free over B. If $i < j < \mu^+$ and $A \cap M_j/(A \cap M_i) \cup B$ is free, then, as $(A \cap M_i) \cup B \in M_{i+1}$, by Ax I*,

$(A \cap M_j) \cap M_{i+1} = A \cap M_{i+1}$ is free over $(A \cap M_i) \cup B$, contradicting our construction. So for $i < j < \lambda$, $A \cap M_j / (A \cap M_i) \cup B$ is not free. Noticing $|A \cap M_j| \leqq \mu < \mu^+$, we get a contradiction by 1.2(7).

DEFINITION 1.2. We define when A is $P_\alpha(\lambda)$-free over B (or A/B is $P_\alpha(\lambda)$-free); where α is an ordinal, λ usually a regular cardinal, but sometimes a limit ordinal. We define by induction on α:

(1) $\alpha = 0$. Any pair A/B is $P_0(\lambda)$-free.

(2) $\alpha = \delta$ a limit ordinal. The pair A/B is $P_\delta(\lambda)$-free iff for every $\beta < \alpha$, A/B is $P_\beta(\lambda)$-free.

(3) $\alpha = \beta + 1$. The pair A/B is $P_\alpha(\lambda)$-free if it has a $P_\beta(\lambda)$-decomposition. A $P_\beta(\lambda)$-decomposition of A/B is a sequence A_i $(i < \delta)$ such that:

(i) A_i is increasing and continuous, $\bigcup_{i<\delta} A_i \subseteq A$ and $\mathrm{cf}\ \delta = \mathrm{cf}\ \lambda$, $A_0 \subseteq B$.

(ii) For $i \leqq j < \delta$, A_{j+1} is $P_\beta(\lambda)$-free over $A_i \cup B$.

(iii) A is free over $\bigcup_{i<\delta} A_i \cup B$.

(iv) A_{i+1} is free over B (for $i < \delta$).

(v) For $i < j < \delta$, A_{j+1} is free over A_{i+1}.

REMARK. Notice that the definition depends on U and F only (and not on $\underline{M}$).

DEFINITION 1.3. The $P_\beta(\lambda)$-decomposition is called standard if $\delta = \mathrm{cf}\ \lambda$.

CLAIM 1.4. (1) *If A/B is $P_\alpha(\delta)$-free*, $\gamma \leqq \alpha$, cf $\delta' =$ cf δ, *then A/B is $P_\gamma(\delta')$-free. The same holds for decompositions.*

(2) *If A/B has a $P_\alpha(\lambda)$-decomposition, then A/B has a standard $P_\alpha(\lambda)$-decomposition.*

PROOF. (1) We prove it by induction on α. For $\alpha = 0$ it is self-evident, and for α limit it follows by the induction hypothesis and the definition.

For $\alpha = \beta + 1$ let A_i $(i < \delta_1)$ be a $P_\beta(\delta)$-decomposition of A/B, cf $\delta_1 =$ cf δ. Then by the induction hypothesis we see (checking Def. 1.2) that for every $\zeta \leqq \beta$ it is a $P_\zeta(\delta')$-decomposition of A/B, hence A/B is $P_{\zeta+1}(\delta')$-free. If γ is a successor, choose ζ such that $\zeta + 1 = \gamma$, if $\gamma = 0$ there is nothing to prove, and for γ limit it should be clear from the definition.

(2) Let A_i $(i < \delta)$ be a $P_\alpha(\lambda)$-decomposition of A/B. Choose an increasing continuous sequence of ordinals $j(i)$ $(i < \mathrm{cf}\ \lambda)$ which converge to δ so that, for i successor, $j(i)$ is a successor and $j(0) = 0$. Clearly, $A_{j(i)}$ $(i < \mathrm{cf}\ \lambda)$ is a $P_\alpha(\lambda)$-decomposition (check Def. 1.2), hence a standard one.

LEMMA 1.5. *If* $A/B \cup C$ *is* $P_\alpha(\lambda)$*-free* (λ *a regular cardinal*) *and* $\{A, B, C, \lambda, \alpha\} \cup \alpha \subseteq N$ *and* $\lambda \cap N$ *is an initial segment of* λ, *then* $A \cap N/(B \cap N) \cup C$ *is* $P_\alpha(\delta^*)$*-free where* $\delta^* = \lambda \cap N =$ *the order type of* $\lambda \cap N =$ *the first ordinal not in* N.

PROOF. By induction on α;

$\alpha = 0$: There is nothing to prove.

α limit: Immediate, by the induction hypothesis.

$\alpha = \beta + 1$: Let A_i $(i < \lambda)$ be a standard $P_\beta(\lambda)$-decomposition of A/B such that $\langle A_i : i < \lambda \rangle \in N$ (clearly it exists as $N < \underline{M}$). Clearly, $\langle A_i \cap N : i < \delta^* \rangle$ is increasing, continuous and of length of cofinality cf δ^* and

$$\bigcup_{i<\delta^*} (A_i \cap N) \subseteq A \cap N, \quad A_0 \cap N \subseteq (B \cap N) \cup C,$$

hence (i) from Definition 1.2(3) holds. For (ii) use the induction hypothesis. As for (iv), $\{A_{i+1}, B, C\} \subseteq N$, so, as $A_{i+1}/B \cup C$ is free, by Ax VI also $A_{i+1} \cap N/(B \cap N) \cup C$ is free. We can prove (v) similarly. As for (iii), notice that $(A_{i+1} - A_i) \cap N \neq \emptyset$ implies $i \in N$, hence

$$\bigcup_{i<\delta^*} (A_i \cap N) = \Big(\bigcup_{i<\lambda} A_i \Big) \cap N,$$

so (iii) also is easy by Ax VI.

LEMMA 1.6. *Suppose* $D \in M$, $C_i \in M$ $(i < \gamma)$ *and* $B \subseteq D$, $A \subseteq D$, $D \subseteq C_0$, *and* C_i *is increasing,* λ *a regular cardinal.*

If A *is* $P_\alpha(\lambda)$*-free over* $(C_0 \cap M) \cup B$ *and* $C_i \cap M$ *is free over* $(C_0 \cap M) \cup D$, *then* A *is* $P_\alpha(\lambda)$*-free over* $(\bigcup_{i<\gamma} C_i \cap M) \cup B$.

REMARK. We can replace the demand $D \subseteq C_0$ by $M \cap (D - C_0) \subseteq B$ (just replace C_i by $C_i \cup D$). We shall use this remark freely.

PROOF. We shall prove by induction on α;

$\alpha = 0$: The conclusion says nothing.

α limit: The conclusion follows by the induction hypothesis.

$\alpha = \beta + 1$: Let A_i $(i < \lambda)$ be a standard $P_\beta(\lambda)$-decomposition of $A/(C_0 \cap M) \cup B$, and we shall prove that it is also a $P_\beta(\lambda)$-decomposition of $A/(\bigcup_{i<\gamma} C_i \cap M) \cup B$. We check the conditions of Definition 1.2(3).

Condition (i) is obvious. Let $i < j < \lambda$, j a successor, then A_j is $P_\beta(\lambda)$-free over $A_i \cup (C_0 \cap M) \cup B$. By the induction hypothesis with A_j, $A_i \cup B$ replacing A, B, respectively, we get that A_j is $P_\beta(\lambda)$-free over $A_i \cup [\bigcup_{i<\gamma} C_i \cap M] \cup B$, hence condition (ii) holds.

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By the choice of A_i $(i<\lambda)$, A is free over $\bigcup_{i<\lambda} A_i \cup (C_0 \cap M) \cup B$, so by Ax V (with $\bigcup_{i<\lambda} A_i \cup B$ replacing B), A is free over $\bigcup_{i<\lambda} A_i \cup (\bigcup_{i<\lambda} C_i \cap M) \cup B$, hence condition (iii) holds.

Conditions (iv) and (v) can be proved similarly.

LEMMA 1.7. *Suppose λ is regular, $\lambda > \chi_1$, and $\alpha \cup \lambda \cup \{A, B, \alpha, \lambda\} \subseteq N$. If A is $P_\alpha(\lambda)$-free over B, then A is $P_\alpha(\lambda)$-free over $(A \cap N) \cup B$.*

PROOF. By induction on α;

$\alpha = 0$ or α limit: Immediate.

$\alpha = \beta + 1$: Let $\langle A_i : i < \lambda \rangle \in N$ be a $P_\beta(\lambda)$-decomposition of A/B, and we shall show that it is a $P_\beta(\lambda)$-decomposition of $A/(A \cap N) \cup B$. As $\lambda \subseteq N$, for each $i < \lambda$, $A_i \in N$.

Let us check the conditions of Def. 1.2(3).

Condition (i) is immediate. As $\{A, B, \bigcup_{i<\lambda} A_i\} \subseteq N$ and $A/\bigcup_{i<\lambda} A_i \cup B$ is free, by Ax VII A is free over $(A \cap N) \cup \bigcup_{i<\lambda} A_i \cup B$, so condition (iii) holds.

For $i < \lambda$, A_{i+1}/B is free, $A_{i+1} \in N$, hence, by Ax VII, $A_{i+1}/(A_{i+1} \cap N) \cup B$ is free. Now for $j > i$, $A_{j+1}/A_{i+1} \cup B$ is free, hence, by Ax VI, $A_{j+1} \cap N/A_{i+1} \cup B$ is free. So we can apply Ax V with N, A_{i+1}, B, $A_{i+1} \cup B$,λ, $A_{i+\zeta+1}(\zeta < \lambda)$ for M, A, B, D, γ, $C_\zeta(\zeta < \gamma)$, respectively (notice that A_{i+1}, B, $A_{i+\zeta+1} \in N$). We get that A_{i+1} is free over

$$\Big(\bigcup_{\lambda > j \geq i} A_{j+1} \cap N\Big) \cup B = \Big(\bigcup_{j<\lambda} A_j \cap N\Big) \cup B.$$

Now clearly $A/\bigcup_{i<\lambda} A_i \cup B$ is free, hence, by Ax VI, $A \cap N/[\bigcup_{i<\lambda} A_i \cap N] \cup B$ is free. Now apply again Ax V with A_{i+1}, B, $A_{i+1} \cup B$, 2, $\langle \bigcup_{i<\lambda} A_i, A \rangle$ for A, B, D, γ, $\langle C_i : i < \gamma \rangle$, respectively. Hence $A_{i+1}/(A \cap N) \cup B$ is free, so condition (iv) holds.

The proof of condition (v) is similar, and also the proof of (ii)—using 1.6 instead of Ax V.

THEOREM 1.8. (1)* *If μ is a singular cardinal, $\mu > \chi_1$,* cf *$\mu = \lambda$, and A is μ-free over B, $|A| = \mu$, then A/B is $P_\alpha(\lambda)$-free for every $\alpha \leqq \mu$.*

(2) *If μ is as above, $|A| = \mu$; and for every N, $\|N\| < \mu$, there is M, $N < M$, $\|M\| < \mu$ such that A is μ-free over $(A \cap M) \cup B$; and $A \cap M/B$ is free, then A/B is $P_\alpha(\lambda)$-free for every $\alpha \leqq \mu$.*

(3) *Suppose that μ is as above, $|A| = \mu$, A/B is μ-free; and there are N_i $(i<\lambda)$ such that $\lambda \cup \{A, B\} \subseteq N_0$, $i < j \Rightarrow N_i < N_j$, and for limit $\delta < \lambda$, $N_\delta = \bigcup_{i<\delta} N_i$, and $\|N_i\| < \mu$ and $A \subseteq \bigcup_{i<\lambda} N_i$.*

Suppose also that $A/(A \cap N_i) \cup B$ is μ-free or *for every $i < j < \lambda$, $A \cap N_{i+1}/B$ and $A \cap N_{j+1}/(A \cap N_{i+1}) \cup B$ are free. Then A/B is $P_\alpha(\lambda)$-free for every $\alpha \leqq \mu$.*

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PROOF. (1), (2) We prove by induction on α;

$\alpha = 0$ or α limit: For $\alpha = 0$ we have nothing to prove, and for α limit the proof follows by the induction hypothesis.

$\alpha = \beta + 1$: Let $A = \{a_j : j < \mu\}$, $\mu = \Sigma_{i<\lambda} \mu_i$, μ_i increasing, $\mu_i < \mu$.

Define by induction on $i < \lambda$, a model N_i so that $\|N_i\| < \mu$, $j < i \Rightarrow N_j < N_i$. For $i = 0$, choose any N_0, $\|N_0\| < \mu$, $\lambda \cup \{A, B\} \subseteq N_0$. For limit i, $N_i = \bigcup_{j<i} N_j$ (exists by the induction hypothesis, which clearly continues to hold, and $\|N_i\| < \mu$ as $i < \lambda = \mathrm{cf}\ \mu$).

If N_i is defined, choose N_{i+1} so that $N_i < N_{i+1}, \{a_j : j < \mu_i\} \subseteq N_{i+1}, \|N_{i+1}\| < \mu$ and A is μ-free over $(A \cap N_{i+1}) \cup B$ (possible in (1) by 1.3, in (2) by an assumption).

We shall prove that $\langle A \cap N_i : i < \lambda \rangle$ is a $P_\beta(\lambda)$-decomposition of A/B. Denote $A_i = A \cap N_i$, and let us check the conditions of Def. 1.2(3).

Condition (i) is trivially true, and condition (iii) holds by Ax II as

$$A = \{a_i : i < \mu\} = \bigcup_{i<\lambda} \{a_i : i < \mu_i\} \subseteq \bigcup_{i<\lambda} (A \cap N_{i+1}) = \bigcup_{i<\lambda} A_i \subseteq \bigcup_{i<\lambda} A_i \cup B.$$

Also condition (iv) holds (by Def. 1.1). By the choice of N_{i+1}, $A/(A \cap N_{i+1}) \cup B$ is μ-free, and, for $j > i + 1$, $\{A, B, N_{i+1}\} \subseteq N_j$, hence, by Def. 1.1, $A \cap N_j/(A \cap N_{i+1}) \cup B$ is free, so condition (v) holds.

Thus only condition (ii) is left; let $i < j < \lambda$, j successor, so $N_i < N_j$ and $\{A, B\} \subseteq N_0 \subseteq N_i$, hence, by 1.7 and the induction hypothesis ($= A/B$ is $P_\beta(\lambda)$-free),

$$A/(A \cap N_i) \cup B \quad \text{is} \quad P_\beta(\lambda)\text{-free},$$

hence, by 1.5 (noting $N_i \in N_j$, hence $A \cap N_i \in N_j$),

$$(A \cap N_j) \quad \text{is } P_\beta(\lambda)\text{-free over} \quad [(A \cap N_i) \cap N_j] \cup B = (A \cap N_i) \cup B,$$

so we prove condition (ii).

(3) We can prove it similarly.

LEMMA 1.9. *Suppose* $\alpha \cup \{\lambda, \alpha, A, B, C\} \subseteq N_0$, $N_0 < N_1$, $\lambda > \chi_1$, λ *regular,* $N_\epsilon \cap \lambda = \delta_\epsilon < \lambda$, $\|N_\epsilon\| = |\delta_\epsilon| > \lambda$, $|A| = \lambda$ *and* $\mathrm{cf}\ \delta_1 = \omega$.

If A is $P_{1+\alpha}(\lambda)$-free over $B \cup C$, then $A \cap N_1$ is $P_\alpha(\mathrm{cf}\ \delta_0)$-free over $(A \cap N_0) \cup (B \cap N_1) \cup C$.

PROOF. We prove by induction on α;

$\alpha = 0$ or α limit: Immediate.

$\alpha = \beta + 1$: Let $\langle A_i : i < \lambda \rangle \in N_0$ be a $P_{1+\beta}(\lambda)$-decomposition of $A/B \cup C$, and we shall prove that $A_i^1 = A_i \cap N_1$ $(i < \delta_0)$ is a $P_\beta(\mathrm{cf}\ \delta_0)$-decomposition of $A \cap N_1/(A \cap N_0) \cup (B \cap N_1) \cup C$.

Condition (i) follows immediately, and for condition (ii) use the induction hypothesis: for $i < j < \delta_0$, j a successor, $\{A_j, A_i\} \subseteq N_0$, so use our theorem with $\beta, A_j, A_i \cup B, C, N_0, N_1$ instead of $\alpha, A, B, C, N_0, N_1$, respectively. We get that

$$A_j \cap N_1/(A_j \cap N_0) \cup [(A_i \cap N_1) \cup (B \cap N_1)] \cup C$$

is $P_\beta(\mathrm{cf}\ \delta_0)$-free.

Now we use Lemma 1.6, with $\beta, \mathrm{cf}\ \delta_0, N_0, A_j \cap N_1, (A_i \cap N_1) \cup (B \cap N_1) \cup C$, $A_j \cup B \cup C, \delta_0, \langle A_{j+\zeta} : \zeta < \delta_0 \rangle$ for $\alpha, \lambda, M, A, B, D, \gamma, \langle C_\zeta : \zeta < \gamma \rangle$, respectively (notice that $j < \delta_0, \zeta < \delta_0 \Rightarrow j + \zeta < \delta_0$; the assumption, $A/(C_0 \cap M) \cup B$ is $P_\alpha(\lambda)$-free, corresponds to the statement above; and to "$C_i \cap M/(C_0 \cap M) \cup D$ is free", corresponds "$A_{j+\zeta+1} \cap N_0/(A_j \cap N_0) \cup B \cup C$ is free", which holds by Ax VI, because $\{A_{j+\zeta+1}, A_j, B, C\} \subseteq N_0$ and $A_{j+\zeta+1}/A_j \cup B \cup C$ is free by Df. 1.2(3) as j is a successor). So we get that $A_j \cap N_1$ is $P_\beta(\mathrm{cf}\ \delta_0)$-free over

$$\left(\bigcup_{\zeta < \delta_0} A_{j+\zeta+1} \cap N_0\right) \cup [(A_i \cap N_1) \cup (B \cap N_1) \cup C].$$

Notice that

$$\bigcup_{\zeta < \delta_0} A_{j+\zeta+1} = \bigcup_{\zeta < \delta_0} A_\zeta, \quad \text{and} \quad \bigcup_{\zeta < \delta_0} A_\zeta \cap N_0 = \bigcup_{\zeta < \lambda} A_\zeta \cap N_0.$$

Using 1.6 with β, $\mathrm{cf}\ \delta_0$, $A_j \cap N_1$, $(A_i \cap N_1) \cup (B \cap N_1) \cup C$, $A_j \cup B \cup C$, 2, $\langle \bigcup_{\zeta<\lambda} A_\zeta, A \rangle$, N_0 for $\alpha, \lambda, A, B, D, \gamma, \langle C_\zeta : \zeta < \gamma \rangle, M$, respectively, we get that $A_j \cap N_1$ is $P_\beta(\mathrm{cf}\ \delta_0)$-free over

$$(A \cap N_0) \cup [(A_i \cap N_1) \cup (B \cap N_1) \cup C)]$$

$$= (A_i \cap N_1) \cup [(A \cap N_0) \cup (B \cap N_1) \cup C],$$

so condition (ii) holds.

The proof of conditions (iv) and (v) is similar, using Ax V instead of Lemma 1.6.

So we are left with condition (iii) and we have to prove that $A \cap N_1$ is free over

$$\left(\bigcup_{i<\delta_0} A_i \cap N_1\right) \cup (A \cap N_0) \cup (B \cap N_1) \cup C.$$

We will rely on the hypothesis $\mathrm{cf}\ \delta_1 = \aleph_0$.

First we prove that, if $A^*, B^*, C^* \in N_1$, $A^*/B^* \cup C^*$ is $P_1(\lambda)$-free, then $A^* \cap N_1$ is free over $(A^* \cap N_1) \cup (B^* \cap N_1) \cup C^*$. Let $\langle A_i^* : i < \lambda \rangle \in N_1$ be a $P_0(\lambda)$-decomposition of $A^*/B^* \cup C^*$, and choose $j(n) < \delta_1$, $j(n) < j(n+1)$, $\delta_1 = \bigcup_{n<\omega} j(n)$, $j(n+1)$ successor, $j(0) = 0$. By Def. 1.2(3), (iii), (iv), (v),

$$A^*/\bigcup_{i<\lambda} A_i^* \cup B^* \cup C^* \quad \text{and} \quad A^*_{j(n+1)}/A^*_{j(n)} \cup B^* \cup C^*$$

are free. Hence, by Ax VI, $A^* \cap N_1$ is free over

$$\left(\bigcup_{i<\lambda} A_i^* \cap N_1\right) \cup (B^* \cap N_1) \cup C^*,$$

and $A^*_{j(n+1)} \cap N_1$ is free over

$$(A^*_{j(n)} \cap N_1) \cup (B^* \cap N_1) \cup C^*.$$

By 1.1 it follows that $A^* \cap N_1$ is free over $(B^* \cap N_1) \cup C^*$, as

$$\bigcup_{i<\lambda} A_i^* \cap N_1 = \bigcup_{i<\delta_1} A_i^* \cap N_1 = \bigcup_{n<\omega} A_{j(n)} \cap N_1.$$

Now we return to (iii). Choose j_n, $\delta_0 \leqq j_n < j_{n+1} < \delta_1$, $j_0 = \delta_0$, j_{n+1} is a successor, $\bigcup_{n<\omega} j_n = \delta_1$; denote $A' = \bigcup_{i<\lambda} A_i$ and notice $A' \cap N_e = A_{\delta_e} \cap N_e$ for $e = 1, 2$. As $\langle A_i : i < \lambda \rangle$ is a $P_{1\text{-}\beta}(\lambda)$-decomposition of A/B, $A_{j(1)}/A_{j(0)} \cup B \cup C$ is $P_{1\text{-}\beta}(\lambda)$-free, hence, by 1.4 (1), it is $P_1(\lambda)$-free, hence, by the previous observation,

$$A_{j(1)} \cap N_1/(A_{j(0)} \cap N_1) \cup (B \cap N_1) \cup C$$

is free. Clearly,

$$A_{j(n+2)}/A_{j(n+1)} \cup B \cup C$$

is free, hence

$$A_{j(n+2)} \cap N_1/(A_{j(n+1)} \cap N_1) \cup (B \cap N_1) \cup C$$

is free. Hence, by Claim 1.1,

$$\bigcup_n A_{j(n)} \cap N_1/(A_{j(0)} \cap N_1) \cup (B \cap N_1) \cup C$$

is free. Notice that

$$\bigcup_n A_{j(n)} \cap N_1 = A_{\delta_1} \cap N_1 = A' \cap N_1,$$

hence $A' \cap N_1$ is free over $(A_{\delta_0} \cap N_1) \cup (B \cap N_1) \cup C$.

Now we apply Ax V with $N_0, A' \cup B \cup C, A_{\delta_1} \cap N_1, (A_{\delta_0} \cap N_1) \cup (B \cap N_1) \cup C$, 2, $\langle A', A \rangle$ for M, D, A, B, γ, $\langle C_i : i < \gamma \rangle$, respectively.

Let us check each hypothesis of Ax V. Firstly, $A_{\delta_1} \cap N_1$ is free over

$$(A' \cap N_0) \cup [(A_{\delta_0} \cap N_1) \cup (B \cap N_1) \cup C]$$

by the previous result, as

$$A_{\delta_1} \cap N_1 = A' \cap N_1, \quad \text{and} \quad A' \cap N_0 = A_{\delta_0} \cap N_0 \subseteq A_{\delta_0} \cap N_1.$$

Secondly,

$$A \cap N_0/(A' \cap N_0) \cup (A' \cup B \cup C)$$

is free by Ax VI, as $\{A, A', B, C\} \subseteq N_0$ and A is free over

$$A' \cup B \cup C = A' \cup (A' \cup B \cup C).$$

Hence, we can conclude that $A' \cap N_1 = A_{\delta_1} \cap N_1$ is free over

$$(A \cap N_0) \cup [(A_{\delta_0} \cap N_1) \cup (B \cap N_1) \cup C].$$

Now $A/A' \cup B \cup C$ is free, and $\{A, A', B, C\} \subseteq N_0 < N_1$, hence, by 1.2(6), $A \cap N_1$ is free over

$$(A \cap N_0) \cup (A' \cap N_1) \cup (B \cap N_1) \cup C.$$

Combining this with the previous result, we get by Ax III that $A \cap N_1$ is free over

$$(A_{\delta_0} \cap N_1) \cup (A \cap N_0) \cup (B \cap N_1) \cup C,$$

so we prove condition (iii), hence 1.9.

THEOREM 1.10. *Suppose* $\chi_1 = \aleph_0$, *or there is* $\zeta < \chi_1^+$ *such that* (*) *for every regular* $\mu \leqq \chi_0$, *if* A/B *is* $P_\zeta(\mu)$ *free and* $|A| \leqq \chi_1$, *then* A/B *is free. If* $\chi_1 = \aleph_0$ *let* $\zeta = 1$.

(A) *Then for every ordinal* α, *every* $P_{\zeta+\alpha}(\aleph_\alpha)$*-free pair is free.*

(B) *If* $\zeta \geqq \omega \wedge \alpha = 0$ *or* $\zeta < \omega \wedge \alpha = \omega$, *then every* $P_{\zeta+\alpha}(\lambda)$*-free pair is free.*

PROOF. If $\chi_1 = \aleph_0$, the condition (*) above is satisfied; because $\mu \leqq \aleph_0$ (so $\mu = \aleph_0$) and $\langle A_n : n < \omega \rangle$ is a standard $P_1(\lambda)$-decomposition of B, $A_0 \subseteq B$, $A_{n+1}/A_n \cup B$ are free, hence, by 1.1, A/B is free.

Now we prove 1.10 by induction on α, and for a fixed α, by induction $\mu = |A|$, and let in (A) $\lambda = \aleph_\alpha$, and for fixed α and μ, we prove by induction on λ.

Case I. $|A| = \mu > \lambda + \chi_1$. Choose an increasing (by $<$) and continuous sequence $N_i (i < \mu)$ so that $\|N_i\| < \mu$, $A \subseteq \bigcup_{i<\mu} N_i$, $N_i \in N_{i+1}$ and $\lambda \cup \{\lambda, A, B\} \subseteq N$. By 1.5, $A \cap N_0/B$ is $P_{\zeta+\alpha}(\lambda)$-free, and by 1.7, A is $P_{\zeta+\alpha}(\lambda)$-free over $(A \cap N_i) \cup B$, hence, by 1.5, $(A \cap N_{i+1})$ is $P_{\zeta+\alpha}(\lambda)$-free over $(A \cap N_i) \cup B$. By the induction hypothesis, $A \cap N_0/B$ and $A \cap N_{i+1}/(A \cap N_i) \cup B$ are free (for $i < \lambda$), hence, by 1.1, A/B is free.

Case II. $|A| = \mu < \lambda$. Let $\langle A_i : i < \lambda \rangle$ be a $P_1(\lambda)$-decomposition of A/B, hence, for some $i(0) < \lambda$, $A_{i(0)} = \bigcup_{i<\lambda} A_i$; so $A/A_{i(0)+1} \cup B$ and $A_{i(0)+1}/B$ are free, $A_{i(0)+1} \subseteq A$, hence, by Ax III, A/B is free.

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Case III. (a) $|A| = \mu \geqq \lambda$, $\mu \leqq \chi_0$. Our conclusion follows by (*) (if $\chi_0 = \aleph_0$, we have proved (*), otherwise we have assumed it).

(b) $|A| = \mu = \lambda$, λ singular. In this case A/B is also $P_{\zeta+\alpha}(\mathrm{cf}\ \lambda)$-free (by 1.4(1)); hence our conclusion follows by the induction hypothesis.

(c) $|A| = \mu = \lambda$, λ regular $> \chi_1$. Choose $N_i (i < \lambda)$ so that $N_i < N_{i+1} < \underline{M}$, $N = \bigcup_{i<\delta} N_i$, $\|N_i\| \leqq |i|^+ \chi_1 < \lambda$, $\{\lambda, \zeta, A, B\} \subseteq N_0$, $A \subseteq \bigcup_{i<\lambda} N_i$, and $\lambda \cap N_i$ is an initial segment of λ of cardinality $\|N_i\|$, such that, for successor i, it has cofinality $\aleph_0$.

Notice that, if $\beta \in N_i$, $1 + \beta \leqq \zeta + \alpha$, then, by 1.5, $A \cap N_1/B$ is P_β (cf $(\lambda \cap N_i)$)-free [as A/B is $P_{1+\beta}(\lambda)$-free, $\beta \leqq 1 + \beta$] and by 1.9 $A \cap N_{i+1}$ is P_β(cf $(\lambda \cap N_i)$)-free over $(A \cap N_i) \cup B$ [as A/B is $P_{1+\beta}(\lambda)$-free]. Hence, by the induction hypothesis and 1.2, $A \cap N_0/B$, $A \cap N_{i+1}/(A \cap N_i) \cup B$ are free, so, by 1.1, A/B is free.

LEMMA 1.11*. *If A_i $(i < \alpha)$ is increasing and continuous, A_0/B is λ-free and $A_{i+1}/A_i \cup B$ is λ-free, then $\bigcup_{i<\alpha} A_i/B$ is λ-free.*

PROOF. As in 1.1, it suffices to prove the parallels of Ax III and Ax IV.

(1) If $C \subseteq B \subseteq A$ and A/B and B/C are λ-free, then A/C is λ-free.

For suppose $C \in N$, $\|N\| < \lambda$, choose M, $\|M\| < \lambda$, $N < M$, A, B, $C \in M$. So by the definition, $A \cap M$ is free over B. Choose M_1, $\|M_1\| < \lambda$, $M < M_1$; then by Ax VI, $(A \cap M) \cap M_1 = A \cap M$ is free over $(B \cap M_1) \cup C$; and as B/C is λ-free, $B \cap M_1$ is free over C. So, by Ax III, $(A \cap M) \cup (B \cap M_1)$ is free over C. As $C \in M$, also

$$[(A \cap M) \cup (B \cap M_1)] \cap M = A \cap M$$

is free over C; so we finish by 1.2 (1).

(2) If A_i $(i < \mu)$ is increasing and continuous, μ regular, $A_0 \subseteq B$. and for $i < j < \mu$, $A_j/A_i \cup B$ is λ-free, then $\bigcup_{i<\mu} A_i/B$ is λ-free.

Clearly, it suffices to prove this for regular $\lambda > \chi_0$. If $\mu \geqq \lambda$, the conclusion holds by 1.2 (5), so we can assume $\mu < \lambda_0$. If $\|N\| < \lambda$, $B \in N$, choose $\|M\| < \lambda$ so that $A_i \in M$ (for $i < \mu$). Then, by Ax VI, for $i < j < \mu$, $(A_j \cap M)/(A_i \cap M) \cup B$ is free, and $A_0 \subseteq B$.[(4)] Hence, by Claim 1.1, $(\bigcup_{i<\mu} A_i) \cap M = \bigcup_{i<\mu}(A_i \cap M)$ is free over β, so clearly (2) holds by 1.2 (1).

2. Applications

MAIN THEOREM 2.1. *Suppose U, F, A, B are given (as in Section 1) and*

(1) $|A| = \lambda$, λ *is singular*;

[(4)] Seemingly incorrect, but the lemma is true under stronger Axioms of [29].

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(2) *There are* $\underline{M}$, χ_1, $\chi_2 < \lambda$ *such that*

(α) Ax I–Ax VII *are satisfied and* A/B *is* λ*-free* or

β) Ax II–Ax VII *are satisfied and the assumption of* 1.8 (2) *or* (3) *holds*;

(3) *There are* $\underline{M}'$, χ'_1, χ'_2 *such that* $\chi'_1 = \aleph_0$ or *for some* $\zeta < (\chi'_1)^*$ *condition* (*) *of* 1.10 *holds, and* Ax II–Ax VII *are satisfied*;

Then A/B *is free.*

Proof. Using assumptions (1) and (2), we get by 1.8 that A/B is P_α (cf λ)-free for any $\alpha \leqq \lambda$. Remembering that this notion depends on U, F only, we can use 1.10 (in the context of $\underline{M}'$) and we get that A/B is free.

Transversals

Definition 2.1. A transversal of a family S of sets, in a one-to-one choice function, i.e. $a \in S \to f(a) \in S$, $a \neq b \in S \Rightarrow f(a) \neq f(b)$.

Let S be a family of subsets of V, each of cardinality $\leqq \chi_1$. Assume without loss of generality that $S \cap V = \emptyset$, and let $U = S \cup V$, and $A/B \in F$ if there is a one-to-one choice function of $(A - B) \cap S$, whose range is $\subseteq (A - B) \cap V$. (Notice that V is an algebra in a trivial way: it has no operations.)

Let $V, S \in H(\chi_2)$, and $\underline{M}' = (H(\chi_2), \in, V, S)$ and $\underline{M}^* = (M', i)_{i<\chi_1}$. It is easy to check that all the assumptions of 2.1 hold. E.g., Ax I* holds for $\underline{M}^*$ because if $a \in S$, $a \in A \cap N$, then $a \subseteq N$, hence, if f is the choice function showing A/B is free, $f \restriction A \cap N$ shows $A \cap N/B$ is free. Hence, clearly,

Conclusion 2.2. *If* $|S| = \lambda > \chi_1$, λ *singular, S a family of sets of cardinality* $= \chi_1$ *and every* $S' \subseteq S$, $|S'| < \lambda$ *has a transversal, then S has a transversal.*

Colouring numbers

Definition 2.2. A graph G has colouring number $\leqq \lambda$ if there is a well ordering $<$ of its set of vertices $V(G)$ so that

$$|\{b < a : b \in V(G),\ (a, b) \in E(G)\}| < \lambda$$

for each $a \in V(G)$, where $E(G)$ is the set of edges of G.

Let χ_0 be a cardinal, G a graph, $U = V(G)$, and $A/B \in F$ if for every $a \in A - B$,

$$\{b \in B : (a, b) \in E(G)\}| < \chi_0,$$

and the restriction of G to $A - B$ has colouring number $\leqq \chi_0$. Let $G \in H(\chi_2)$, $\underline{M} = (H(\chi_2), \in G, i)_{i<\chi_0}$. Notice that, if $|A| \leqq \chi_0$, and for every $a \in A$,

$$|\{b \in B : (a, b) \in E(G)\}| < \chi_0,$$

then A/B is free (choose any ordering of $A - B$ of order-type $\leqq \chi_0$). Hence, if $|A| \leqq \chi_0$ and A/B is $P_1(\mu)$-free for some μ, then A/B is free (by the way, this implies that if A_i $(i < \alpha, \mathrm{cf}\ \alpha < \chi_0^+)$ is increasing, A_i/B is free, then $\bigcup_{i<\alpha} A_i/B$ is free).

So, by 2.1,

CONCLUSION 2.3. *If G is a graph of cardinality λ, λ singular, $\lambda > \chi_0$, and every subgroup of cardinality $< \lambda$ has colouring number $\leqq \chi_0$, then G has colouring number $\leqq \chi_0$.*

Free algebras

Let U be an algebra with χ_0 operators, which satisfies a set Γ of identities (see 0(8)). We say that $I \subseteq U$ is free over a subalgebra A of U, if for any algebra U' (of the same similarity type) which satisfies Γ, $A \subseteq U'$, and function $f: I \to U'$, f can be extended to a homomorphism $g: \mathrm{cl}\,(A \cup I) \to U'$, g is the identity over A. For simplicity let $\chi_0 = \aleph_0$.

We call an algebra A λ-free (for a fixed Γ) if any subalgebra of it of cardinality $< \lambda$ is free (this does not coincide with our definition in Section 1, but is stronger, so our result surely holds). We say I is a free base of A/B (A, B are $\emptyset$ or subalgebras of U) if I is free over B and $\mathrm{cl}\,(A \cup B) = \mathrm{cl}\,(B \cup I)$. Let

$$F = \{A/B: \text{there is a free base of } A/B\}.$$

We shall prove now that, for any appropriate $\underline{M}$, χ_1, χ_2, axioms II–VIII are satisfied.

Ax II: Trivial (the free basis of B/B is the empty set).

Ax III: Let I, J be free bases of A/B, B/C respectively. Then $I \cup J$ is a free basic of A/B.

Ax IV: Let I_α be a free basis of $A_{\alpha+1}/A_\alpha \cup B$, then $\bigcup_{\alpha<\lambda} I_\alpha$ is a free basis of A/B.

Ax V: Let I be a free basis of $A/(C_0 \cap M) \cup B$ and suppose I is not a free basis of $A/(\bigcup_{i<\alpha} C_i \cap M) \cup B$.

Clearly,

$$\mathrm{cl}\left[I \cup \left(\bigcup_{i<\alpha} C_i \cap M\right) \cup B\right] = \mathrm{cl}\left[A \cup \left(\bigcup_{i<\alpha} C_i \cap M\right) \cup B\right],$$

hence I is not free over $(\bigcup_{i<\alpha} C_i \cap M) \cup B$, hence, for some i, I is not free over $(C_i \cap M) \cup B$. Let J be a free basis of $C_i \cap M/C_0 \cap M$. As $J \subseteq C_i \cap M$ is free over $C_0 \cap M$, it is free over C_0, hence over $(C_0 \cap M) \cup D$. So J is a free basis of $C_i \cap M/(C_0 \cap M) \cup D$ and, as

$$I \subseteq \mathrm{cl}\,(A \cup (C_0 \cap M) \cup B) \subseteq \mathrm{cl}\,((C_0 \cap M) \cup D),$$

clearly $I \cup J$ is free over $(C_0 \cap M) \cup B$, hence I is free over

$$\mathrm{cl}\,[J \cup (C_0 \cap M) \cup B] \supseteq \mathrm{cl}\,[(C_i \cap M) \cup B],$$

a contradiction.

Ax VI: There should be $I \in N$ which is a free basis of $A/B \cup C$. So for every $a \in A \cup B \cup C$, $a \in N$, there are $a_i \in I \cap N$, $b_i \in B \cap N$, $c_i \in C \cap N$ and term τ so that $a = \tau(a_1, \cdots, b_1, \cdots, c_1, \cdots)$. Hence

$$\mathrm{cl}\,[(A \cap N) \cup (B \cap N) \cup (C \cap N)] = \mathrm{cl}\,[(I \cap N) \cup (B \cap N) \cup (C \cap N)].$$

It is also clear that $I \cap N$ is free over $B \cup C$, hence also over $(B \cap N) \cup C$. So $I \cap N$ is a free basis of $A \cap N/(B \cap N) \cup C$.

Ax VII: There is $I \in N$ such that I is a free basis of A/B. Let $J = I - I \cap N$; then clearly

$$\mathrm{cl}\,(A \cup B) = \mathrm{cl}\,(I \cup B) = \mathrm{cl}\,(J \cup (I \cap N) \cup B) = \mathrm{cl}\,(J \cup (A \cap N) \cup B)$$

and J is free over

$$\mathrm{cl}\,((I \cap N) \cup B) = \mathrm{cl}\,((A \cap N) \cup B).$$

So J is a free basis of A over

$$\mathrm{cl}\,((I \cap N) \cup B) = \mathrm{cl}\,((A \cap N) \cup B).$$

So J is a free basis of $A/(A \cap N) \cup B$.

Clearly for abelian groups Ax I* holds, hence

CONCLUSION 2.4. *If A is a λ-free abelian group of cardinality λ, λ singular (so Γ is the set of identities of abelian groups), then A is free.*

Do groups satisfy Ax I*?

For abelian groups, A is free over B iff the quotient group A/B is free, so there are no problems. However, checking the proof of a somewhat more general theorem, appearing in [19], Vol. II p. 17, we find easily that groups satisfy Ax I*.

Unfortunately, e.g. not every subring of a free ring is free. It is not known to me whether 2.1 (2) (β) (α) may hold. This motivates Section 3, so now we assume knowledge of it. By 3.8, if A is λ-free, λ regular, $|A| = \lambda$, and rcf $\lambda \leqq \ \leqq \lambda$, then A is free, so e.g. for λ strong limit this holds. But using the particular properties of λ-free algebras we can get a better result which, however, seemingly does not generalize to pairs. Let from now on $B = \emptyset$.

DEFINITION 2.3. $P_\kappa(A, C)$ means that $|C| \leqq \kappa$, $\kappa \geqq \chi_0$, C is free and there is a free algebra with κ^+ generators C', $C \subseteq C'$, C'/C is free and $(A, a)_{a \in C}$, $(C', a)_{a \in C}$ are L_{∞, κ^+}-equivalent (see e.g. [2] on $L_{\infty, \lambda}$).

DEFINITION 2.4. A is $L_{\infty,\kappa}$-free if it is $L_{\infty,\kappa}$-equivalent to the free algebra with κ generators (see e.g. [2]).

LEMMA 2.5. (A) *Let $\mu \leqq \kappa$, $\chi_0 < \kappa < |A|$, and $R(A, E^{\mu}_{\kappa}) = \infty$, then A is L_{∞,κ^+}-free.*

(B) *If $\chi_0 \leqq \kappa_1 < \kappa < |A|$, A is $L_{\infty,\kappa}$-free, then A is L_{∞,κ_1}-free.*

(C) *If A is $L_{\infty,\kappa}$-free, $\chi_0 < \kappa(1) < \kappa < |A|$, $\kappa(1)$ is regular, then*

$$\{C \in S_{\kappa(1)}(A) : C <_{\infty,\kappa(1)} A\} \in E^{\kappa(1)}_{\kappa(1)}(A),$$

where $<_{\infty,\kappa(1)}$ means being an $L_{\infty,\kappa(1)}$-elementary submodel.

(D) *If $\chi_0 \leqq \kappa(1) < \kappa < |A|$, $C <_{\infty,\kappa} A$, $P_{\kappa(1)}(A, C_0)$, $C_0 \subseteq C$, C is free, then C/C_0 is free.*

PROOF. Let D_κ be the free algebra with κ-generators.

(A) Let W be the set of functions f such that: f is an isomorphism from a subalgebra A_1 of A onto a subalgebra D_1 of D_{κ^+}, D_1 and D_{κ^+}/D_1 free, $|A_1| = \kappa$ and $R^{\mu}_{\kappa}(A_1) = \infty$. (We first assume $R^{\mu}_{\kappa} = \infty$.)

By well known results, it suffices to prove that: for any $f \in W, A_1 \subseteq A$, $D_1 \subseteq D_{\kappa^+}$, $|A| + |D_1| \leqq \kappa$, there is $f' \in W$ extending f such that $A_1 \subseteq \mathrm{Dom}\, f'$, $D_1 \subseteq \mathrm{Range}\, f'$. For this it suffices to prove that, if $|A_0| = \kappa$, $R^{\mu}_{\kappa}(A_0) = \infty$, $A_0 \subseteq A_1 \subseteq A$, $|A_1| \leqq \kappa$, then for some $A_2 \subseteq A$, $A_1 \subseteq A_2$, $R^{\mu}_{\kappa}(A_2) = \infty$ and A_2/A_0 is free, and generated by κ-generators. Except for "generated by κ-generators", this holds by 3.4. Using 3.10 and $\boldsymbol{R}^{\mu}_{\kappa}$ instead, we get our requirements easily.

(B) Well known.

(C) Clearly, for every $C \in S_{\kappa(1)}(A)$ there is $C' \in S_{\kappa(1)}(A)$, $C \subseteq C'$ and $C' <_{\infty,\kappa(1)} A$ (there is a sentence in $L_{\infty,\kappa}$ saying that, and D_κ satisfies it). It is also easy to see that if $C_i <_{\infty,\kappa(1)} A$, $(i < \kappa(1))$, C_i increasing, then $\bigcup_i C_i <_{\infty,\kappa(1)} A$; hence our conclusion is easy.

(D) We can assume $A = C$. We define by induction on $n < \omega$ sets C_n such that $|C_n| = \kappa(1)$, $C_n \subseteq C_{n+1}$, $P_{\kappa(1)}(A, C_{2n})$, C_{2n+2}/C_{2n} is free and $C_{2n+1} = A \cap N_n$, $N_n < \underline{M}$, $A \in N_n < N_{n+1}$. This is easy, and $A / \bigcup_{n<\omega} C_n$ is free by Ax VII, $\bigcup_{n<\omega} C_n / C_0$ is free by 1.1, hence, by Ax III, A/C_0 is free.

THEOREM 2.6. (A) *If $\lambda > \chi_0$ is singular, A is λ-free, then, for arbitrarily large $\kappa < \lambda$, A is not $\underline{E}^{\kappa^+}_{\kappa^+}$-non-free.*

(B) *If $\chi_0 \leqq \kappa < \lambda = |A|$, A is not $\underline{E}^{\kappa^+}_{\kappa^+}$-non-free, then A is L_{∞,κ^+}-equivalent to the free algebra with κ^+-generators.*

(C) *If A is L_{∞,κ^+}-equivalent to the free algebra with κ^+-generators for every $\kappa < \kappa_0$, then A is L_{∞,κ_0}-equivalent to the free algebra with κ_0-generators.*

(D) *If $|A| = \lambda$, singular and A is $L_{\infty,\lambda}$-free, then A is free.*

REMARK. Note that each part gives in fact, as a conclusion, the hypothesis of the next part.

PROOF. (A) Trivial.

(B) By 3.4 and 2.5 (A).

(C) Let W be the set of functions f such that: f is an isomorphism from $A^1 \subseteq A$ onto $D^1 \subseteq D_{\kappa_0}$, $|A^1| < \kappa_0$, D^1 and D_{κ_0}/D^1 are free, and $P_\kappa(A, A^1)$ where $\kappa = |A^1|$.

Now, for every $f \in W$, and $A^2 \subseteq A$, $|A^2| < \kappa_0$, choose $\kappa < \kappa_0$, $|\mathrm{Dom}\, f \cup A^2| < \kappa$, and $A^3 <_{\infty,\kappa} A$, $|A^3| = \kappa$, $P_\kappa(A, A^3)$ and $\mathrm{Dom}\, f \cup A^2 \subseteq A^3$ (possible as A is $L_{\infty,\kappa^{++}}$-free). So A^3 is free and by 2.5(D) $A^3/\mathrm{Dom}\, f$ is free, so we can extend f to $f' \in W$, $\mathrm{Dom}\, f' = A^3$. As clearly free basis of $A^3/\mathrm{Dom}\, f$ has cardinality κ, we are finished, as in the proof of 2.5(A).

(D) Similar to 3.7, assuming this time $P_{|C_i|}(A, C_i)$ for successor i.

CONCLUSION 2.7. *For singular λ, any λ-free A is λ^+-free.*

Subalgebras of free algebra

The context here is just as in the preceding sub-section ("free algebras"), i.e., we have the same U, χ_0, χ_1, χ_2, $\underset{\sim}{M}$ but:

$$F' = \{A/B: \text{there is an algebra } C \text{ (satisfying } \Gamma),$$

$$A \cup B \subset C,\ C/B \text{ is free in the previous sense}\}$$

(we have to assume U is quite "big").

THEOREM 2.8. *All axioms, including* Ax I*, *hold, provided the class of algebras satisfying Γ has the amalgamation property. Hence for λ singular, λ-freeness implies λ^+-freeness (for pairs).*

Generalizing transversals

Like Mirski (23), we can replace transversals by "independent transversals" relative to some independence structure satisfying some natural requirement. We shall need a special case in [29].

THEOREM 2.9. *Suppose $S = \{\langle Q_i, P_i\rangle: i < \lambda\}$, where $|Q_i| \leqq \chi_0$, P_i a family of subsets of Q_i, $|P_i| < \chi_2$.*

We say S has a transversal if we can choose $t_i \in P_i$ $(i < \lambda)$ which are pairwise disjoint. If $\chi_0 + \chi_1 + \mathrm{cf}\ \lambda < \lambda$, and every $S' \subseteq S$, $|S'| < \lambda$ has a transversal, then S has a transversal.

We leave the proof to the reader.

3. On "almost all subsets of cardinality κ"

Let μ denote a regular cardinal, $\chi_1 \leqq \kappa < \lambda$. We return to the setting of Section 1, and let A, B be fixed, $|A| = \lambda$.

DEFINITION 3.1. For any set a and $\kappa \leqq |a|$, let $S_\kappa(a) = \{b : b \subseteq a, |b| = \kappa\}$.

DEFINITION 3.2. (A) An expansion $\underset{\sim}{M}^*$ of $\underset{\sim}{M}$ is called a κ-expansion if it is an expansion by $\leqq \kappa$ relations and functions, and A, B, i ($i \leqq \kappa$) are individual constants of $\underset{\sim}{M}^*$.

(B) N_i ($i < \alpha$) is an $\underset{\sim}{M}^*$-sequence if it is increasing (by $<$) and continuous and for every $i < \alpha$, $\langle N_j : j \leqq i \rangle \in N_{i+1}$ and $N_i < \underset{\sim}{M}^*$.

(C) For any $\kappa < \lambda$ (and $\mu \leqq \kappa$) let $\underset{\sim}{E}_\kappa(A)$ $[\underset{\sim}{E}^\mu_\kappa(A)]$ be the filter generated by the sets $A \subseteq S_\kappa(A)$ called its generators such that, for some κ-expansion $\underset{\sim}{M}^*$ of $\underset{\sim}{M}$,

$$S = \underset{\sim}{S}_\kappa(\underset{\sim}{M}^*)\,[S = \underset{\sim}{S}^\mu_\kappa(\underset{\sim}{M}^*)]$$

where

(1) $\underset{\sim}{S}_\kappa(\underset{\sim}{M}^*) = \Big\{ A \cap \bigcup_{i<\alpha} N_i :\ N_i\ (i < \alpha)$ is an $\underset{\sim}{M}^*$-sequence, $\|N_i\| = \kappa,\ \alpha < \kappa^+,\ \alpha = \kappa\mu,$ where $\mu = \operatorname{cf}\alpha \Big\}$ ($\kappa\mu$-ordinal multiplication),

(2) $\underset{\sim}{S}^\mu_\kappa(\underset{\sim}{M}^*) = \Big\{ A \cap \bigcup_{i<\alpha} N_i : N_i\ (i < \alpha)$ is an $\underset{\sim}{M}^*$-sequence, $\alpha = \kappa\mu \Big\}$.

DEFINITION 3.3. $E_\kappa(A)$, $E^\mu_\kappa(A)$ are defined as in Def. 3.2 (C), only replacing $\alpha = \kappa\mu$ by $\operatorname{cf}\alpha = \mu$ in (2) and omitting $\alpha = \kappa\mu$ in (1). Similarly we define $S_\kappa(A)$, $S^\mu_\kappa(A)$.

LEMMA 3.1. *Let $\mu \leqq \kappa < \lambda$ (μ regular).*

(A) *The filters $E_\kappa(A)$, $E^\mu_\kappa(A)$, $\underset{\sim}{E}_\kappa(A)$, $\underset{\sim}{E}^\mu_\kappa(A)$ are non-trivial (i.e., the empty set is not in the filter), not principal, and κ^+-complete. Moreover, the intersection of any $\leqq \kappa$ generators includes a generator.*

(B) *$S \in E_\kappa(A)$ iff for every $\mu \leqq \kappa$, $S \in E^\mu_\kappa(A)$, and $S \in \underset{\sim}{E}_\kappa(A)$ iff for every $\mu \leqq \kappa$, $S \in \underset{\sim}{E}^\mu_\kappa(A)$.*

(C) $E_\kappa(A) \subseteq E^\mu_\kappa(A)$, $\underset{\sim}{E}_\kappa(A) \subseteq \underset{\sim}{E}^\mu_\kappa(A)$, $E_\kappa(A) \subseteq \underset{\sim}{E}_\kappa(A)$, $E^\mu_\kappa(A) \subseteq \underset{\sim}{E}^\mu_\kappa(A)$.

(D) *If in* Def. 3.2(C) (2), *we replace $\alpha = \kappa\mu$ by $\alpha = \mu$, we get the same filter. A similar result holds for $\underset{\sim}{E}_\kappa(A)$.*

(E) *The filters depend on A and the cardinal parameters, but not on $\underset{\sim}{M}^*$, B.*

(F) *If the language L^* of $\underset{\sim}{M}^*$ contains only finitely many symbols, except for the individual constants, $\underset{\sim}{M}^+$ is an expansion of $\underset{\sim}{M}^*$, then for any $\underset{\sim}{M}^+$*

sequence $\langle N^+_\alpha : \alpha < \alpha_0 \rangle$, *if* N^*_α *is the* L^**-reduct of* N^+_α, *then* $\langle N^+_\alpha : \alpha < \alpha_0 \rangle$ *is an* $\underset{\sim}{M}^*$*-sequence.*

(G) *In* Def. 3.2(C), *it suffices to take* $\underset{\sim}{M}^*$ *as mentioned above; so we need not distinguish strictly between* $N < \underset{\sim}{M}^*$ *and* $|N|$.

PROOF: left to the reader. We shall use this lemma freely.

DEFINITION 3.4. (A) For limit cardinal λ, let rcf λ (revised confinality of λ) be the first (regular) cardinal μ such that for some $\lambda_0 < \lambda$, for every singular cardinal λ_1, $\lambda_0 < \lambda_1 < \lambda \Rightarrow \mathrm{cf}\ \lambda_1 < \mu$.

(B) $\lambda^{<\mu} = \Sigma_{0 \leq \kappa < \mu} \lambda^\kappa$.

(C) $\mu \leqq \leqq \lambda$ if $\lambda_1 < \lambda$ implies $\lambda_1^{<\mu} < \lambda$.

(D) Let $\kappa(1) < \kappa$, then κ is near $\kappa(1)$ if $\kappa(1) < \mu \leqq \kappa \wedge \mathrm{cf}\ \mu \leqq \kappa(1) \Rightarrow \kappa^{\mathrm{cf}\,\mu} = \kappa$.

LEMMA 3.2. (A) $\mathrm{cf}\ \lambda \leqq \mathrm{rcf}\ \lambda \leqq \lambda$, *and if* $\mathrm{rcf}\ \lambda \leqq \leqq \lambda$, *then there is* $\lambda_0 < \lambda$ *such that for* $\kappa(1)$, κ:

$$\lambda_0 < \kappa(1) < \kappa = \kappa^{<\mathrm{rcf}\lambda} < \lambda \text{ implies } \kappa \text{ is near } \kappa(1).$$

(B) *If* $\|N\|^{|\alpha|} = \|N\|$ *and* $N < M$, *then for every* $a_j \in N$ $(j < \alpha)$ *the sequence* $\langle a_j : j < \alpha \rangle$ *belongs to* M. *If* $a \in N$, $|a| = \kappa$, *then* $a \subseteq N$ *iff* $\kappa \subseteq N$.

(C) *If* $\kappa = \kappa^{|\alpha|}$, $\alpha < \mathrm{cf}\ \delta$, $\underset{\sim}{M}^*$ *is a* κ*-expansion of* $\underset{\sim}{M}$, $\langle N_i : i < \delta \rangle$ *is an* $\underset{\sim}{M}^*$*-sequence and* $a_j \in \bigcup_{i<\delta} N_i$ $(j < \alpha)$, *then* $\langle a_j : j < \alpha \rangle \in \bigcup_{i<\alpha} N_i$.

(D) *Let* $\kappa(1) < \kappa$, κ *is near* $\kappa(1)$, $\underset{\sim}{M}^*$ *a* κ*-expansion of* $\underset{\sim}{M}$, $\langle N_i : i < \delta \rangle$ *an* $\underset{\sim}{M}^*$*-sequence*:

(1) *If* $a \subseteq N_i$, $|a| \leqq \kappa(1)$, γ *is the order type of* $\{\kappa' : \kappa(1) < \kappa' \leqq \kappa,\ \mathrm{cf}\ \kappa' \leqq \kappa(1)\}$, *then there is* $a_1 \in N_{i+\gamma+1}$, $a \subseteq a_1$, $|a_1| = \kappa(1)$.

(2) *If* $a \subseteq \bigcup_{i<\delta} N_i$, $|a| \leqq \kappa(1) < \mathrm{cf}\ \delta$ [*so* γ (*From* (1)) *divides* δ], *then there is* $a_1 \in \bigcup_{i<\delta} N_i$, $a \subseteq a_1$, $|a_1| = \kappa(1)$.

PROOF. (A) Immediate.

(B) As $N \in M$, $|N| \in M$, hence S, the set of sequences of length α of members of $|N|$, belongs to $|M|$. As $\langle a_j : j < \alpha \rangle \in S \in M$, $|S| = \|N\|$, it suffices to prove the second phrase (then use it twice: $|N| \subseteq |M|$ implies $\|N\| \subseteq |M|$, which implies $S \subseteq |M|$).

So let $N < \underset{\sim}{M}$, $a \in N$, $|a| = \kappa$. As the function car, car $(a) = |a|$, is definable in $\underset{\sim}{M}$, $\kappa = |a| \in N$, and as $\underset{\sim}{M} \models |a| = \kappa$, also $N \models |a| = \kappa$, and so a one-to-one function from κ onto a, $\in N$, hence $\kappa \subseteq |N|$ iff $a \subseteq N$.

(C) Immediate by (B).

(D) (1) For $\kappa_2 \geqq \kappa(1)$ let $\rho(\kappa_2)$ be the order type of

$$\{\alpha : \kappa(1) < \aleph_\alpha \leqq \kappa_2,\ \mathrm{cf}\ \aleph_\alpha \leqq \kappa(1)\}.$$

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Now we prove by induction on κ_2, $\kappa(1) \leqq \kappa_2 \leqq \kappa$, that, if $a \subseteq a^*$, $a^* \in N_i$, $|a^*| \leqq \kappa_2$, then, for some $a' \in N_{i+\rho}$ $(\rho = \rho(\kappa_2))$, $a \subseteq a'$, $|a'| \leqq \kappa(1)$. If $\kappa_2 = \kappa(1)$ this is trivial. If cf $\kappa_2 > \kappa(1)$, there is in N_i a one-to-one function f from κ_2 onto a^*, so for some $i < \kappa_2$, $a \subseteq \{f(j): j < i\}$, and we can use the induction hypothesis. If $\mu =$ cf $\kappa_2 \leqq \kappa(1)$, then $a^* = \bigcup_{j<\mu} a^*_j$, $a^*_j \in N_i$, $|a^*_j| < |a^*|$. So by the induction hypothesis there are $a'_j \in N_{i+\rho-1}$, $a \cap a^*_j \subseteq a'_j \subseteq a^*_j$, $|a'_j| \leqq \kappa$ (1). By (B) $\langle a'_j: j < \mu \rangle \in N_{i+\rho}$ $(\rho = \rho(\kappa_2))$, hence $\bigcup_{j<i} a'_j \in N_\rho$ is the desired set.

(D) (2) Immediate by (D) (1).

LEMMA 3.3. *For regular $\kappa < \lambda$, $\underset{\sim}{E}^{\kappa}_{\kappa}(A)$ is generated also by the sets $\underset{\sim}{S}^*_\kappa(\underset{\sim}{M}^*)$ (for χ_1-expansion $\underset{\sim}{M}^*$ of M, $\chi_1 < \kappa$) where*

$$\underset{\sim}{S}^*_\kappa(\underset{\sim}{M}^*) = \left\{ A \cap \bigcup_{i<\kappa} N_i : N_i \ (i < \kappa) \text{ an } \underset{\sim}{M}^*\text{-sequence, and } \|N_i\| < \kappa \right\}.$$

PROOF. We should prove that every set of the form $\underset{\sim}{S}^\kappa_\kappa(\underset{\sim}{M}^*)$ includes one of the form $\underset{\sim}{S}^*_\kappa(\underset{\sim}{M}^*)$ and vice versa ($\underset{\sim}{M}^*$—a κ-expansion of $\underset{\sim}{M}$).

PART I. $\underset{\sim}{S}^\kappa_\kappa(\underset{\sim}{M}^+) \subseteq \underset{\sim}{S}^*_\kappa(\underset{\sim}{M}^*)$, where $\underset{\sim}{M}^+$ is the expansion of $\underset{\sim}{M}^*$ by $P = \{N: N < \underset{\sim}{M}^*\}$ and $i\,(i \leqq \kappa)$. For every $A^* \in \underset{\sim}{S}^\kappa_\kappa(\underset{\sim}{M}^+)$ there is an $\underset{\sim}{M}^+$-sequence $\langle N'_i: i < \delta \rangle$, such that $A^* = A \cap \bigcup_{i<\delta} N'_i$, $\delta = \kappa\kappa$. Let $N_i = N'_{\kappa i}$, then $\langle N_i: i < \kappa \rangle$ is also an $\underset{\sim}{M}^+$-sequence and $A^* = A \cap \bigcup_{i<\kappa} N_i$.

There is a function g_1, definable in $\underset{\sim}{M}$, so that $g_1(a)$ $(a \in \underset{\sim}{M})$ is a one-to-one function from $|a|$ onto a. There is a function g_2 definable in $\underset{\sim}{M}^+$, so that for $\alpha < i < \kappa$, $\langle N_j: j < i \rangle$ an $\underset{\sim}{M}^*$-sequence $\|N_j\| = \kappa$, $g_2(\langle N_j: j < i \rangle, \alpha)$ depends on α $\langle N_j: j < \alpha \rangle$ only, has the form $\langle M^\alpha_\beta: \beta < \kappa \rangle$, M^α_β $(\beta < \kappa)$ is increasing and continuous, $\|M^\alpha_\beta\| < \kappa$, $\bigcup_\beta M^\alpha_\beta = N_\alpha$, $M^\alpha_\beta < \underset{\sim}{M}^*$, for $\gamma < \alpha$ $|M^\gamma_\beta| \subseteq |M^\alpha_\beta|$ and for limit α, $M^\alpha_\beta = \bigcup_{\gamma<\alpha} M^\gamma_\beta$ and $\langle N_j: j \leqq \gamma \rangle \in M^{\gamma+1}_0$ when $\gamma < \alpha$.

Then $N^*_i = \bigcup_{\alpha,\beta<i} M^\alpha_\beta$ $(i < \kappa)$ is an $\underset{\sim}{M}^*$-sequence, $\|N^*_i\| < \kappa$, $A^* = A \cap \bigcup_{i<\kappa} N^*_i$, so $A^* \in \underset{\sim}{S}^*_\kappa(\underset{\sim}{M}^*)$.

Part II. $\underset{\sim}{S}^*_\kappa(\underset{\sim}{M}^+) \subseteq \underset{\sim}{S}^\kappa_\kappa(\underset{\sim}{M}^*)$ for appropriate M^+. By 3.1 (D) there is a κ-expansion $\underset{\sim}{M}^1$ such that

$$S^1 = \left\{ A \cap \bigcup_{i<\kappa} N_i : N_i \ (i < \kappa) \quad \text{an } \underset{\sim}{M}^1\text{-sequence,} \quad \|N_i\| = \kappa \right\}$$

is a subset of $\underset{\sim}{S}^\kappa_\kappa(\underset{\sim}{M}^*)$. Using parametrization, there is a χ_1-expansion $\underset{\sim}{M}^2$ of $\underset{\sim}{M}$, so that all relations, functions and constants of $\underset{\sim}{M}^1$ are first-order definable in $(\underset{\sim}{M}^2, i)_{i<\kappa}$. Hence $S^2 \subseteq S^1$, where

$$S^2 = \left\{ A \cap \bigcup_{i<\kappa} N_i : N_i \ (i < \kappa) \text{ is an } M^2\text{-sequence, } \|N_i\| = \kappa,\ \kappa \subseteq N_0 \right\}.$$

We can assume that $\underline{M}^2$ has Skolem-functions. Let $\underline{M}^+$ be the expansion of $\underline{M}^2$ by a function g, so that for $N < \underline{M}^2$, $\|N\| < \kappa$, $g(N)$ is the Skolem-hull of $|N| \cup \kappa$ in $\underline{M}^2$. It suffices to prove that $\underline{S}^*_\kappa(\underline{M}^+) \subseteq S^2$.

So let

$$A^* = A \cap \bigcup_{i<\kappa} N, N_i \ (i < \kappa) \quad \text{an } \underline{M}^+\text{-sequence}, \quad \|N_i\| < \kappa.$$

As $N_i < N_{i+1}$, $\kappa \cap N_i$, is strictly increasing, and $\kappa \subseteq \bigcup_{i<\kappa} N_i$, hence $N_j \subseteq g(N_j) \subseteq \bigcup_{i<\kappa} N_i$, hence $\bigcup_{i<\kappa} N_i = \bigcup_{i<\kappa} g(N_i)$, hence $A \cap \bigcup_i N_i = A \cap \bigcup g(N_i)$. As $\langle N_j : j \leqq i \rangle \in N_{i+1}$, also $\langle g(N_j) : j \leqq i \rangle \in N_{i+1}$, hence $\langle g(N_j) : j \leqq i \rangle \in g(N_i)$. Clearly $g(N_j)$ is increasing and continuous, so we are finished.

Definition 3.3. (A) The pair A/B is E-free (E, or $E(A)$, is a filter over a family of subsets of A) if

$$\{C : C \in \cup E, C/B \text{ is free}\} \in E.$$

(B) We can replace "free" by any other property.

Remark. Obvious monotonicity results hold.

Definition 3.4. (A) For every $\mu \leqq \kappa < \lambda$, $C \in S_\kappa(A)$, A, $|A| = \lambda$, and B, and filter E over $S_\kappa(A)$, we define the rank $R(C, E)$ as an ordinal or ∞, so that

(1) $R(C, E) \geqq \alpha + 1$ iff C/B is free and

$$\{D \in S_\kappa(A) : C \subseteq D, D/C \cup B \text{ is free and } R(D, E) \geqq \alpha\} \neq \emptyset \bmod E.$$

(2) $R(C, E) \geqq \delta$ ($\delta = 0$ or δ limit) iff C/B is free and $\alpha < \delta$ implies $R(C, E) \geqq \alpha$ (more exactly, we should write $R(C, E; A/B)$).

(B) $R(A/B, E) = \sup\{R(C, E) : C \in S_\kappa(A)\}$.

(C) $R^\mu_\kappa(C) = R(C, E^\mu_\kappa)$ and $R^\mu_\kappa = R^\mu_\kappa(A/B) = R(A/B, E^\mu_\kappa)$; $\underline{R}^\mu_\kappa$, $\mathbf{R}^\mu_\kappa$ are defined similarly.

Lemma 3.4. *Suppose $\kappa^+ < \lambda$, $\mu \leqq \kappa$, A/B is not $\underline{E}^{\kappa^+}_\kappa$-non-free and $S_1 \in \underline{E}^{\kappa^+}_\kappa(A)$, $S_2 \in E^\mu_\kappa(A)$, $\underline{M}^*$ a κ-expansion of $\underline{M}$.*

Then $R^\mu_\kappa = \infty$, moreover for every κ-expansion $\underline{M}^*$ of $\underline{M}$ there are $C \in S_2$ and $D \in S_1$ and $N < \underline{M}^*$, $\|N\| = \kappa$ such that $D \in N$, $C = D \cap N$ and $R^\mu_\kappa(C) = \infty$.

Proof. If $C \in S_\kappa(A)$, $0 \leqq R^\mu_\kappa(C) < \infty$, then there is a generator $S(C) \in E^\mu_\kappa(A)$, $S(C) = S^\mu_\kappa(M^*_C)$, such that for $D \in S(C)$, $D/C \cup B$ is not free *or* $R^\mu_\kappa(D) < R^\mu_\kappa(C)$. If C/B is not free or $R^\mu_\kappa(C) = \infty$, let M^*_C be any

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κ-expansion of $\underset{\sim}{M}$, and let $S_2 = S^{\mu}_{\kappa}(\underset{\sim}{M}^2)$. Let $\underset{\sim}{M}^+$ be a κ-expansion of $\underset{\sim}{M}$, expanding $\underset{\sim}{M}^*$, $\underset{\sim}{M}^2$ and having the relations P, P^2 where

$$P = \{(C, N): C \in S_\kappa(A),\ N < \underset{\sim}{M}^*_C,\ \|N\| < \chi_2\},$$

$$P_2 = \{N: N < \underset{\sim}{M}^2,\ \|N\| < \chi_2\}.$$

As

$$\{D \in S_{\kappa^+}(A): D/B \text{ is free}\} \neq \emptyset \bmod \underset{\sim}{E}^{\kappa^+}_{\kappa^+}(A)$$

and $S_1 \in \underset{\sim}{E}^{\kappa^+}_{\kappa^+}(A)$ and (by 3.3) $\underset{\sim}{S}^*_{\kappa^+}(M^+) \in \underset{\sim}{E}^{\kappa^+}_{\kappa^+}(A)$; there is D such that:

(1) D/B is free.

(2) $D \in S_1$.

(3) $D = A \cap \bigcup_{i<\kappa^+} N_i$, N_i $(i < \kappa^+)$ is an $\underset{\sim}{M}^+$-sequence and $\|N_i\| \leqq \kappa$, so without loss of generality $\|N_i\| = \kappa$, $\kappa \subseteq N_i$.

Let $A^*_i = D \cap N_i$, so $A^*_i \in N_{i+1}$, and let $N = \bigcup_{i<\kappa^+} N_i$. Clearly $\langle N_i : i < \kappa^+\rangle$ is also an $\underset{\sim}{M}^2$-sequence, hence for each $\delta < \kappa^+$, $\langle N_i : i < \delta\rangle$ is an $\underset{\sim}{M}^2$-sequence, hence, if κ divides δ, cf $\delta = \mu$, then $A^*_\delta \in S_2$. If $C \in N_i$, $C \in S_\kappa(A)$, then for every $j > i$, $j < \kappa^+$ there is a model $N^i_j < \underset{\sim}{M}^*_C$, $\|N^i_j\| = \kappa$, $|N_j| \subseteq |N^i_j|$ and $N^i_j \in N_{j+1}$, hence $N^i_j \subseteq N_{j+1}$.

Hence, for any limit ordinal δ, $i < \delta < \kappa^+$ implies $N_\delta < \underset{\sim}{M}^*_C$. Clearly $\langle N_j : i < j < \kappa^+, j \text{ limit}\rangle$ is an $\underset{\sim}{M}^+$-sequence, hence it is an $\underset{\sim}{M}^*_C$-sequence, hence, if $i < \delta < \kappa^+$, δ is limit, κ^2 divides δ, cf $\delta = \mu$, then $A^*_\delta \in S(C)$. As D/B is free, by 1.2(7) there is a closed unbounded subset of κ^+, W, such that for $i, j \in W$, $i < j$, $A^*_j/A^*_i \cup B$ is free and A^*_i/B is free. We can assume that each $i \in W$ is divisible by κ^2. Hence, if $i, j \in W$, $i < j$, cf $j = \mu$, $R^{\mu}_{\kappa}(A^*_i) < \infty$, then $R^{\mu}_{\kappa}(A^*_j) < R^{\mu}_{\kappa}(A^*_i) < \infty$ (by the definition of $S(C)$). So, if for some $i \in W$, $R^{\mu}_{\kappa}(A^*_i) < \infty$, cf $i_n = \mu$, $i_n \in W$, $i < i_n < i_{n+1}$, then $R^{\mu}_{\kappa}(A^*_{i_n})$ is an infinite decreasing sequence of ordinals, a contradiction. Hence $i \in W$ implies $R^{\mu}_{\kappa}(A^*_i) = \infty$. Let $D = \bigcup_{i<\kappa^+} A^*_i$, and choose $N < \underset{\sim}{M}^*$, $D \in N$, $N \cap \bigcup_{i<\kappa^+} A^*_i = A^*_\delta$, $\delta \in W$, cf $\delta = \mu$, and $C = A^*_\delta$. So we are finished.

Lemma 3.5. (A) *If* $\mu \leqq \kappa < \lambda$, $C \in S_\kappa(A)$, $R^{\mu}_{\kappa}(C) = \infty$, $S \in E^{\mu}_{\kappa}(A)$, *then for some* $D \in S$, $C \subseteq D$, $R^{\mu}_{\kappa}(D) = \infty$ *and* $D/C \cup B$ *is free.*

(B) *The same holds for any filter over* $S_\kappa(A)$.

Proof. (A) As $S_\kappa(A)$ is a set, for some ordinal $\alpha_0 < |S_\kappa(A)|^+$, for no $C \in S_\kappa(A)$ is $R^{\mu}_{\kappa}(C) = \alpha_0$. We can easily prove that $R^{\mu}_{\kappa}(C) \geqq \alpha_0$ iff $R^{\mu}_{\kappa}(C) = \infty$. Using the definition we get our assertion.

(B) The same proof.

LEMMA 3.6. *Suppose $\mu(1) \leqq \kappa(1) < \mu \leqq \kappa < \lambda$, $R^{\mu(1)}_{\kappa(1)}(C) = \infty$, κ is near $\kappa(1)$ and A/B is not $\underline{E}^{\mu}_{\kappa}$-non-free. Then $A/C \cup B$ is not $\underline{E}^{\mu}_{\kappa}$-non-free, moreover the set of $D \in S_{\kappa}(A)$ such that "D/B is free implies $D/C \cup B$ is free" belongs to $E^{\mu}_{\kappa}(A)$.*

REMARK. (1) The lemma holds also when $\mu(1) = \kappa(1) = \mu = \aleph_0 \leqq \kappa < \lambda$.
(2) We can replace $\underline{E}^{\mu}_{\kappa}$, E^{μ}_{κ}.

PROOF. Let $\underline{M}^*$ be a κ-expansion of $\underline{M}$, including C as an individual constant and (by 3.5 it exists) the function g, so that if $R^{\mu}_{\kappa}(C) = \infty$, $C_0 \subseteq C_1 \in S_{\kappa}(A)$, then $C_1 \subseteq g(C, C_1) \in S_{\kappa}(A)$, $\infty = R^{\mu}_{\kappa}(g(C, C_1))$ and we shall prove that every $D \in S^{\mu}_{\kappa}(\underline{M}^*)$ satisfies our conclusion.

So let $\langle N_i : i < \delta \rangle$ be an $\underline{M}^*$-sequence, $\|N_i\| = \kappa$, κ divides δ, and $D = A \cap \bigcup_{i<\delta} N_i$, and let $N = \bigcup_{i<\delta} N_i$, cf $\delta = \mu$.

We define M_n, C_n, by induction on n, so that

(1) $\qquad C_0 = C, \qquad |C_n| = \kappa(1), \ \|M_n\| = \kappa(1),$

(2) $\qquad R^{\mu(1)}_{\kappa(1)}(C_n) = \infty$ and $C_{n+1}/C_n \cup B$ is free, and $C_n \subseteq D$,

(3) $\qquad M_n < M_{n+1} < \underline{M}, \qquad D \in M_0,$

(4) $\qquad C_n \subseteq D \cap M_n \subseteq C_{n+1}, \qquad C_n \in M_n, \ C_n \in N.$

For $n = 0$ there is no problem. For $n + 1$, by 3.2 (D) there is $a \in \bigcup_{i<\delta} N_i$, $D \cap M_n \subseteq a$, $|a| = \kappa(1)$. Now let $C_{n+1} = g(C_n, a \cap A)$. Now it is easy to define $\underline{M}_n$.

By 1.1 $\bigcup_{n<\omega} C_n/C \cup B$ is free, and as $\bigcup_{n<\omega} M_n < \underline{M}$, by Ax VII D is free over $(D \cap \bigcup_n M_n) \cup B = \bigcup_{n<\omega} C_n \cup B$. Hence, by Ax III, $D/C \cup B$ is free.

THEOREM 3.7. *Suppose $\lambda = |A|$ is singular,* rcf $\lambda \leqq \leqq \lambda$, *and moreover, for arbitrarily large successors $\kappa < \lambda$, $\kappa^{<\mathrm{rcf}\lambda} = \kappa$ and A/B is not $\underline{E}^{*}_{\kappa}$-non-free, and $(*)$ of 1.10. Then A/B is free.*

REMARK. The condition rcf $\lambda \leqq \leqq \lambda$ holds, e.g. if λ is strong limit or if $\lambda = \aleph_{\alpha+\omega}$.

PROOF. We can find $\lambda_i < \lambda$ $(i < \mathrm{cf}\,\lambda)$ such that λ_i is increasing, and, denoting $\kappa(i) = \lambda_i^+$, $\kappa(i)^{<\mathrm{rcf}\lambda} = \kappa(i)$ and A/B is not $\underline{E}^{\kappa(i)}_{\kappa(i)}$-non-free and for $i < j$, $\kappa(j)$ is near $\kappa(i)$, λ_i (see 3.2(A)).
By 3.4, for every $\mu \leqq \lambda_i$, $R^{\mu}_{\lambda_i} = \infty$. Let $A = \{a_i : i < \lambda\}$.

Now we define, by induction on $i < \mathrm{cf}\,\lambda$, sets $C_i \subseteq A$ and models N_i such that $A \cap N_i = C_i$, $N_i < N_{i+1}$ for $i \neq 0$, and

$$|C_{i+1}| = \lambda_{i+1}, R^{\aleph_0}_{\lambda_{i+1}}(C_{i+1}) = \infty, \{a_j : j < \lambda_i\} \subseteq C_{i+1};$$

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and N_i is increasing and continuous, and for $j < i$, $C_{j+1}/C_{j-1} \cup B$ is free.

For $i = 0$, let $C_0 = 0$ and for limit $\delta < \mathrm{cf}\,\lambda$, $N_\delta = \bigcup_{i<\delta} N_i$, $C_\delta = \bigcup_{i<\delta} C_i$. Suppose N_i, C_i is defined and let us define C_{i+1}, N_{i+1}. Let $C^1 = C_i \cup \{a_i : i < \lambda_i\}$; by 3.6,

$S_0 = \{D \in S_{\kappa(i+1)}(A)$: for any $j < i$, D/B free$\rightarrow D/C_{j+1} \cup B$ is free$\} \in E^{\kappa(i+1)}_{\kappa(i+1)}(A)$ (because $E^{\kappa(i+1)}_{\kappa(i+1)}(A)$ is $\kappa(i+1)^+$-complete). Let

$$\underset{\sim}{M}^* = (\underset{\sim}{M}, \langle N_j : j < i\rangle, j)_{j \leq \lambda_i},$$

and

$$S_1 = S_0 \cap \{D : D = A \cap N, N < \underset{\sim}{M}^*, \| N \| = \kappa(i+1)\},$$

so clearly $S_1 \in E^{\kappa(i+1)}_{\kappa(i+1)}(A)$.

Now apply 3.4 (λ_{i+1} stands for κ), so there exists $D \in S_1$, C, N such that

$$C = D \cap N, D \in N, R^{\mu}_{\kappa}(C) = \infty, N < \underset{\sim}{M}^*.$$

As $D \in S_1$, $D = A \cap N'$, $N' < \underset{\sim}{M}^*$. Clearly, for $j < i$, $C_{j+1} \in N$, and as $D \in S_0$, $D/C_{j+1} \cup B$ is free. hence $C/C_{j+1} \cup B$ is free.

Let $N_{i+1} = N \cap N'$, then $N_{i+1} < \underset{\sim}{M}^*$, hence $N_i < N_{i+1}$, $\langle N_j : j \leq i\rangle \in N_{i+1}$, and $C = A \cap N_{i+1}$, so all our demands are satisfied.

So $A = \bigcup_i C_i$, C_i is increasing and continuous, $C_0 = \varnothing$, C_{i+1}/B is free (as $R^{\mu}_{\kappa}(C_{i+1}) = \infty$) and for $j < i$, $C_{i+1}/C_{j+1} \cup B$ is free. So A/B is $P_1(\mathrm{cf}\,\lambda)$-free, and, as in 1.8, we can prove by induction on $\alpha \leq \lambda$ that it is $P_\alpha(\mathrm{cf}\,\lambda)$-free. By 1.10, A/B is free.

Conclusion 3.8. *If $|A| = \lambda$, λ is singular,* $\mathrm{rcf}\,\lambda \leq \leq \lambda$, *and A/B is λ-free, then A/B is free provided that for arbitrarily large $\kappa < \lambda$, $\kappa^{<\mathrm{rcf}\,\lambda} = \kappa$.*

We can notice also

Lemma 3.9. *If κ is near $\kappa(1)$, $\mu(1) \leq \kappa(1) < \mu \leq \kappa < \lambda$, $\kappa^{<\mu(1)} = \kappa$ and A/B is $E^{\mu(1)}_{\kappa(1)}$-non-free, then A/B is E^{μ}_{κ}-non-free.*

Proof. Let $\underset{\sim}{M}^*$ be a κ-expansion of $\underset{\sim}{M}$ so that whenever N_i ($i < \delta = \mu' = \mu(1)$) are as in Def. 3.2(C), $(A \cap \bigcup_{i<\delta} N_i)/B$ is not free (see 3.1(D)). Let $P = \{N : N < \underset{\sim}{M}^*, \|N\| = \kappa\}$, and it suffices to prove that for any $A^* = A \cap \bigcup_{i<\delta} N^*_i$, N^*_i, δ, $(\underset{\sim}{M}^*, P)$ as in Def. 3.2(C), A^*/B is not free; let $N^* = \bigcup_{i<\delta} N^*_i$. We define M^e_i, $e = 1, 2$, by induction on $i < \mu$, so that M^e_i ($i < \mu$) is increasing and continuous, $\|M^e_i\| = \kappa(1)$, $M^1_i < N^*$, $N^* \in M^2_i < \underset{\sim}{M}^+$, $\langle M^1_j : j \leq i\rangle \in M^1_{i+1}$, and $M^1_i \cap A \subseteq M^2_i \cap A \subseteq M^1_{i+1} \cap A$.

For defining M^1_{i+1} use 3.2(C) and 3.2(B).

By the choice of $\underset{\sim}{M}^*$, $A \cap \bigcup_{i<\mu'} M^1_i/B$ is not free. But if A^*/B is free, as $N^* \in \bigcup_{i<\mu'} M^2_i < \underset{\sim}{M}^+$, $A^* \in \bigcup_{i<\mu'} M^2_i$, hence, by Ax VI, $A^* \cap \bigcup_{i<\mu'} M^2_i/B$ is free. But $A^* \cap \bigcup_{i<\mu'} M^2_i = A \cap \bigcup_{i<\mu'} M^1_i$, a contradiction.

DEFINITION 3.5. $E^{\mu}_{\kappa}(A)$ is the filter over $S_{\kappa}(A)$ generated by the sets $S^{\mu}_{\kappa}(\underline{M}^*)$, $\underline{M}^*$ a κ-extension of $\underline{M}$, where

$$S^{\mu}_{\kappa}(\underline{A}^*) = \{D : \langle N_i : i < \kappa^+\rangle \text{ an } \underline{M}^*\text{-sequence},$$

$$\|N_i\| = \kappa, \text{ and there is an } \underline{M}^*\text{-sequence}$$

$$\langle M_i : i < \delta\rangle, \langle N_i : i < \kappa^+\rangle \in M_0, \operatorname{cf}\delta = \mu.$$

$$M_i \cap \bigcup_j N_j = N_{\alpha(i)}, \text{ and } D = A \cap \bigcup_{i<\delta} N_{\alpha(i)},$$

$$\delta \text{ divisible by } \kappa\}.$$

LEMMA 3.10. *If $\kappa^+ < \lambda$, $\mu \leqq \kappa$, A/B is not $\underline{E}^{\kappa^+}_{\kappa}$-non-free and $S \in E^{\mu}_{\kappa}(A)$, then for some $C \in S_{\kappa}(A)$, $R(C, E^{\mu}_{\kappa}) = \infty$.*

PROOF. The same as for 3.4.

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