ISRAEL JOURNAL OF MATHEMATICS, Vol. 47, Nos. 2–3, 1984

ON CO-κ-SOUSLIN RELATIONS†

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ABSTRACT

This is a continuation of Harrington and Shelah [3]; however, the contents of this paper are self-contained.

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§1. On the density of linear ordering

THEOREM. *Suppose P is a co-κ-Souslin relation* (*on* $\mathbb{R}$) *which is a linear order* (*so we shall denote it by* $\leqq$) *even after adding a Cohen real.*

Then either $(\mathbb{R}, \leqq)$ *has a dense subset of power* $\leqq \kappa$ or *there is a perfect set of pairwise disjoint intervals.*

† This research was partially supported by the Israel – US Binational Science Foundation.
Received March 20, 1983

REMARK. In summer 1979, Friedman and Shelah (see [1]) proved this for P a Borel relation. Shelah proved that $(\mathbb{R}, \leqq)$ cannot be Souslin: If it is, by forcing by the set of intervals, we made it to have a strictly decreasing sequence of intervals (a_i, b_i) $(i < \omega_1)$. So $\{(a_{3i}, a_{3i+2}) : i < \aleph_1\}$ is a set of $\aleph_1$ pairwise disjoint intervals. But then, by the completeness theorem for $L_{\omega_1,\omega}(Q)$ (see Keisler [4]), this holds in the original universe (we use hereby the absoluteness). So assuming $(\mathbb{R}, \leqq)$ has no countable dense sets, it has $\aleph_1$ pairwise disjoint intervals. This Friedman uses to prove the Theorem for P Borel, adapting Harrington's proof of Silver's [6] theorem, using "there are $\aleph_1$ disjoint intervals" as a "bigness" property.

That proof does not seem to apply for the present theorem.

This proof uses the method of [3] (with choice) but is represented fully.

If you have difficulties, read §2 here and/or [3] and they may be explained in more detail there.

PROOF. First we choose by induction on $i < \kappa^+$, reals a_i, b_i such that

(*) $a_i < b_i$, and for every $j < i$, $a_j < b_j \leqq a_i < b_i$, or

$a_i < b_i < a_j < b_j$ or $a_i < a_j < b_j < b_i$

and

(**) if there are κ^+ pairwise disjoint (closed intervals) then $\{(a_i, b_i) : i < \kappa^+\}$ are such intervals (i.e. for $i < j$, $b_j < a_i$ or $b_i < a_j$).

Why can we do this? If we cannot choose a_i, b_i, let $A = \{a_j, b_j : j < i\}$, then in every Dedekind cut of A, there is at most one element of $\mathbb{R} - A$ [by the order $\leqq$; more exactly, one equivalence class modulo $x \leqq y \wedge y \leqq x$], so $(\mathbb{R}, \leqq)$ has a dense subset of power $|2i| \leqq \kappa$.

Taking care of (**) is trivial.

Let $f : {}^{\omega>}\kappa \rightarrow$ "the family of open subsets of $\mathbb{R} \times \mathbb{R}$" be such that

$$\neg\, xPy \equiv (\exists \eta \in {}^{\omega}\kappa) \bigwedge_{n<\omega} [\langle x, y \rangle \in f(\eta \restriction n)].$$

Extend $(H(\kappa^{++}), \in)$ by Skolem functions and get a model $\mathfrak{C}$, and let $N < \mathfrak{C}$ be a countable elementary submodel such that $g, h, f \in N$ where $g, h : \kappa^+ \rightarrow \mathbb{R}$, $g(i) = a_i$, $h(i) = b_i$.

We define a forcing notion (t ranges over the rationals, $\bar{y}_t = \langle y_{t,i} : i < \omega \rangle$, but the formula φ involves only a finite initial segment)

$$Q = \{\varphi(x_{t_0}, \bar{y}_{t_0}, x_{t_1}, \bar{y}_{t_1}, \cdots, \bar{c}):$$

$$t_0 < \cdots < t_n \text{ in } \mathbb{Q},\ \bar{c} \in N,\ \varphi \vdash_f \text{"}x_l \text{ an ordinal } < \kappa^+\text{"},$$

$$\text{and } \mathfrak{C} \models \exists^{\kappa^+} x_{t_0} \exists \bar{y}_{t_0} \exists^{\kappa^+} x_{t_1} \exists \bar{y}_{t_1} \cdots \varphi\},$$

the order is $\varphi < \psi$ if $\psi \vdash \varphi$; we shall omit the parameters from N.

Clearly Q is equivalent to Cohen forcing, and we can naturally define a Q-name $\underset{\sim}{M}$ of an elementary extension of N, with set of elements $\{x_t, y_{t,l} : t \in \mathbb{Q}, l < \omega\}$ in which "sets of power $\leqq \kappa$" are not enlarged.

A. FACT. If $\mathfrak{C} \models (\exists^{\kappa^+} x < \kappa^+)\varphi(x, \bar{c})$, $n < \omega$, $\bar{c} \in \mathfrak{C}$ then $(\exists^{\kappa^+} x_1 < \kappa^+) \cdots (\exists^{\kappa^+} x_n < \kappa^+)$ $[\bigwedge_{l=1}^{n} \varphi(x_l, \bar{c}) \wedge$ the intervals $[g(x_l), h(x_l)]$ $(l \leqq n)$ are pairwise disjoint$]$.

If not, then easily there is (in $\mathfrak{C}$) a set $A \subseteq \kappa^+$, $|A| = \kappa^+$, $(\forall x \in A)\varphi(x, \bar{c})$ such that for no $x_1 < \cdots < x_n$ in A are the intervals $[g(x_l), h(x_l)]$ $(l \leqq n)$ pairwise disjoint. Let $\{x_i : i < m\} \subseteq A$ be such that $\{[g(x_l), h(x_l)] : l < m\}$ are pairwise disjoint, and m is maximal (hence $< n$), so for every $z \in A' \overset{\text{def}}{=} A - \{y : (\exists l) y \leqq x_l)\}$ for some $i(z) \in \{0, \cdots, m-1\}$ the interval $[g(z), h(z)]$ is not disjoint to $[g(x_{i(z)}), h(x_{i(z)})]$. By the choice of the (a_i, b_i)'s (see $(*)$), as $x_{i(z)} < z$,

$$g(x_{i(z)}) < g(z) < h(z) < h(x_{i(z)}).$$

Clearly $|A - A'| \leqq \kappa$ hence $|A'| = \kappa^+$, hence for some $l_0 < m$

$$B = \{z \in A' : i(z) = l_0\}$$

has power κ^+. For $z_1 < z_2$ in B the intervals $[g(z_1), h(z_1)]$, $[g(z_2), h(z_2)]$ cannot be disjoint [otherwise $\{x_l : l < m, l \neq l_0\} \cup \{z_1, z_2\}$ contradict the maximality of m] hence (by $(*)$ again)

$$g(z_1) < g(z_2) < h(z_2) < h(z_1),$$

so $\{g(z) : z \in B\}$ is strictly increasing; but this means there are in $(\mathbb{R}, \leqq)$ κ^+ pairwise disjoint closed intervals, and so we could have chosen the (a_i, b_i)'s to be pairwise disjoint; in this case by $(**)$ the Fact A is trivial.

B. FACT. *If $\varphi(x_{t_1}, \bar{y}_{t_1}, \cdots, x_{t_n}, \bar{y}_{t_n}) \in Q$, $t_1 < \cdots < t_n < t_{n+1} < \cdots < t_{2n}$ in $\mathbb{Q}$, and among any κ^+ ordinals $< \kappa^+$ there are $i < j$ such that $\mathfrak{C} \models \Psi(i, j)$, $m \in \{1, \cdots, n\}$,*

$$\theta = \varphi(x_{t_1}, \bar{y}_{t_1}, \cdots, x_{t_n}, \bar{y}_{t_n}) \wedge \varphi(x_{t_{n+1}}, \bar{y}_{t_{n+1}}, \cdots, x_{t_{2n}}, \bar{y}_{t_{2n}}) \wedge \psi(x_{t_m}, \bar{y}_{t_{n+m}}),$$

then $\theta \in Q$.

PROOF. We choose by induction on $\alpha < \kappa^+$, $N_\alpha < \mathfrak{C}$, $\alpha \subseteq N_\alpha$, $N \in N_0$, $\|N_\alpha\| < \kappa^+$, $\langle N_i : i \leqq \alpha \rangle \in N_{\alpha+1}$, for limit α, $N_\alpha = \bigcup_{i<\alpha} N_i$. For each α we can by induction on $l = 1, n$ choose

$$\gamma_l^\alpha \in \kappa^+ \cap (N_{n\alpha+l} - N_{n\alpha+l-1}), \qquad \bar{d}_l^\alpha \in N_{n\alpha+l}$$

such that

$(*)_l \models (\exists^{\kappa^+} x_{t_{l+1}})(\exists \bar{y}_{t_{l+1}}) \cdots (\exists^{\kappa^+} x_{t_n})(\exists \bar{y}_{t_n})\varphi(\gamma_1^\alpha, \bar{d}_1^\alpha, \cdots, \gamma_l^\alpha, \bar{d}_l^\alpha, x_{t_{l+1}}, \bar{y}_{t_{l+1}}, \cdots, x_{t_n}, \bar{y}_{t_n})$.

Note that $(*)_0$ holds as $\varphi(x_{t_1}, \bar{y}_{t_1}, \cdots) \in Q$. So to choose for l we know $(*)_{l-1}$, then we can choose first γ_l^α (as $\exists^{\kappa^+} x_{t_l}$ and the relevant parameters are in $N_{n\alpha+l}$ as $N_{n\alpha+l-1} \in N_{n\alpha+l}$ so $N_{n\alpha+l-1} \cap \kappa^+$ is "considered" by $N_{n\alpha+l}$ as a bounded subset of κ^+) and then $\bar{d}_l^\alpha$.

Lastly $\{\gamma_m^\alpha : \alpha < \kappa^+\}$ is a subset of κ^+ of power κ^+, so as $\alpha < \beta \Leftrightarrow \gamma_m^\alpha < \gamma_l^\beta$, by a hypothesis for some $\alpha < \beta$, $\models \psi(\gamma_m^\alpha, \gamma_m^\beta)$. So

$$\mathfrak{C} \models \theta[\gamma_1^\alpha, \bar{d}_1^\alpha, \gamma_2^\alpha \bar{d}_2^\alpha, \cdots, \gamma_n^\alpha \bar{d}_n^\alpha, \gamma_1^\beta, \bar{d}_1^\beta, \cdots, \gamma_n^\beta, \bar{d}_n^\beta].$$

Now we can prove by downward induction on $l = 1, 2n$ that

$$\mathfrak{C} \models \exists^{\kappa^+} x_{t_{l+1}} \exists \bar{y}_{t_{l+1}} \cdots \exists^{\kappa^+} x_{t_{2n}} \exists \bar{y}_{t_{2n}} \theta[\gamma_1^\alpha, \bar{d}_1^\alpha, \cdots, x_{t_{l+1}}, \bar{y}_{t_{l+1}}, \cdots].$$

We identify r, $\underset{\sim}{M}^{\mathrm{L}} \models$ "r a real" with a true real r' s.t. [$r'(n) = 0$ iff $\underset{\sim}{M}^{\mathrm{L}} \models$ "$r(n) = 0$"] if we identify $\mathbb{R}$ with ${}^{\omega}2$, *or* [$r' > t$ iff $\underset{\sim}{M}^{\mathrm{L}} \models r > t$] for any $t \in \mathbb{Q}$.

C. Fact. In the forcing notion $Q \times Q$ we have two names $\underset{\sim}{M}$, $\underset{\sim}{M}^{\mathrm{L}}$, $\underset{\sim}{M}^{\mathrm{R}}$, one for each Q. Now for each $t, s \in Q$

$$\Vdash_{Q \times Q} \text{"the intervals } [g(x_t), h(x_t)]^{\underset{\sim}{M}^{\mathrm{L}}}, [g(x_s), h(x_s)]^{\underset{\sim}{M}^{\mathrm{R}}} \text{ are disjoint".}$$

Proof. Otherwise there is a condition $(\varphi, \psi) \in Q \times Q$

$(\varphi, \psi) \Vdash_{Q \times Q}$ "the intervals $[g(x_t), h(x_t)]^{\underset{\sim}{M}^{\mathrm{L}}}$, $[g(x_s), h(x_s)]^{\underset{\sim}{M}^{\mathrm{R}}}$ are not disjoint".

Let

$$\varphi = \varphi(x_{t_1}, \bar{y}_{t_1}, \cdots, x_{t_n}, \bar{y}_{t_n}), \qquad \psi = \psi(x_{s_1}, \bar{y}_{s_1}, \cdots, x_{s_m}, \bar{y}_{s_m})$$

and w.l.o.g. $t \in \{t_1, \cdots, t_n\}$, $s \in \{s_1, \cdots, s_n\}$ (as we can add to φ dummy variables). So let $t = t_{n(*)}$, $s = s_{m(*)}$. Choose $t_l, s_l \in Q$ $(n < l \leqq 2n)$, such that

$$t_n < t_{n+1} < \cdots < t_{2n}, \qquad s_n < s_{n+1} < \cdots < s_{2n}.$$

By Facts A, B, $\varphi^* \in Q$ where

$$\varphi^* = \varphi^*(x_{t_1} \bar{y}_{t_1}, \cdots, x_{t_{2n}}, \bar{y}_{t_{2n}})$$
$$= \varphi(x_{t_1}, \bar{y}_{t_1}, \cdots, x_{t_n}, \bar{y}_{t_n}) \wedge \varphi(x_{t_{n+1}}, \bar{y}_{t_{n+1}}, \cdots, x_{t_{2n}}, \bar{y}_{t_{2n}})$$
$$\wedge \text{ [the intervals } [g(x_{t_{n(x)}}), h(x_{t_{n(x)}})], [g(x_{t_{n+n(x)}}), h(x_{t_{n+n(x)}})] \text{ are disjoint].}$$

Similarly we can show $\psi^* \in Q$ where

$$\psi^* = \psi(x_{s_1}, \bar{y}_{s_1}, \cdots, x_{s_n}, \bar{y}_{s_n}) \wedge \psi(x_{s_{n+1}}, \bar{y}_{s_{n+1}}, \cdots, x_{s_{2n}}, \bar{y}_{s_{2n}})$$

$$\wedge \text{ [the intervals } [g(x_{s_{m(*)}}), h(x_{s_{m(*)}})],\ [g(x_{s_{m+m(*)}}), h(x_{s_{m+m(*)}})] \text{ are disjoint].}$$

So $(\varphi^*, \psi^*) \in Q \times Q$ and let $G \subseteq Q \times Q$ be generic, $(\varphi^*, \psi^*) \in G_0$. As Q is equivalent to Cohen forcing also $Q \times Q$ is equivalent to Cohen forcing, hence by a hypothesis, in $V[G]$, $\leqq$, i.e. P (i.e. its definition) is still a linear order. Now for reals, i.e. in $\underset{\sim}{M}^{\mathrm{L}}[G]$, $\underset{\sim}{M}^{\mathrm{R}}[G]$ we have two definitions of $\leqq$: the one in $V[G]$, and the one as an elementary extension of N. Now if $x \in \{\mathrm{L}, \mathrm{R}\}$, $r_1, r_2 \in \underset{\sim}{M}^x[G]$, $\underset{\sim}{M}^x[G] \models r_1 < r_2$ in $M[G]$ we can find a branch of ${}^{\omega>}\kappa$ which witnesses it, and clearly it continues to witness it in $V[G]$. But in $\underset{\sim}{M}^x[G]$, $\kappa^{\underset{\sim}{M}^x[G]} \subseteq N$. So

$$I_1 = [g(x_{t_{n(*)}}), h(x_{t_{n(*)}})]^{\underset{\sim}{M}^{\mathrm{L}[G]}}, \qquad I_2 = [g(x_{t_{n+n(*)}}), h(x_{t_{n+n(*)}})]^{\underset{\sim}{M}^{\mathrm{L}[G]}}$$

are disjoint intervals (and they are intervals, i.e.

$$g(x_{t_{n(*)}}) < h(x_{t_{n(*)}}), \qquad g(x_{t_{n+n(*)}}) < h(x_{t_{n+n(*)}})).$$

Similarly

$$J_1 = [g(x_{s_{m(*)}}), h(x_{s_{m(*)}})]^{\underset{\sim}{M}^{\mathrm{R}[G]}}, \qquad J_2 = [g(x_{s_{m+m(*)}}), h(x_{s_{m+m(*)}})]^{\underset{\sim}{M}^{\mathrm{R}[G]}}$$

are disjoint intervals.

As $(\varphi, \psi) \leqq (\varphi^*, \psi^*) \in G$, and by the choice of (φ, ψ), the intervals I_1, I_2 are disjoint. Using the natural automorphisms of $Q \times Q$, I_i is disjoint to J_j for $i = 1, 2$, $j = 1, 2$. But no linear order can have four such intervals.

D. Fact. There is a perfect set of pairwise disjoint intervals.

Easy by Fact B (we do not have to really construct generic sets, just enough to compute the branches of ${}^{\omega>}\kappa$ witnessing the κ-Souslin relation).

§2. Generalizing the model theory

(A) Looking at the proofs of the theorem on number of equivalence classes, non-existence of a monotonic ω_1-sequence in a linear order, in [3] and the theorem of §1, we see that a large part is common. We try to catch this part, and phrase it here in a general way.

We do not try to see how much choice and which cardinals we need (i.e., can we replace ZFC by second-order arithmetic, etc.).

(B) We let $\mathfrak{C}$ be an expansion of some $(H(\lambda), \in)$ (by countably many relations and functions). Let κ^* be a regular cardinal. Let $A^* \in \mathfrak{C}$ be a set, $I \in \mathfrak{C}$ an ideal of subsets of A^*, which is κ^*-complete, let $\exists^* x\, \varphi(x)$ mean

$\{x \in A^* : \varphi(x)\} \notin I$. If not mentioned otherwise $I = \{B \subseteq A^* : |B| < \kappa^*\}$ and $A^* = \kappa^*$.

We choose a countable elementary submodel N of $\mathfrak{C}$. In formulas we suppress parameters from N.

We define a forcing notion Q whose members are the $\varphi = \varphi(x_{t_1}, \bar{y}_{t_1}, \cdots, x_{t_n}, \bar{y}_{t_n})$ satisfying

(a) φ is a first order formula with parameters in N,

(b) each t_l is a rational number and $t_1 < t_2 < \cdots < t_n$,

(c) $\bar{y}_{t_l} = \langle y_{t_l,0}, y_{t_l,1}, \cdots \rangle$ (we can replace it by a finite initial segment as φ is first order),

(d) $N \models \{(\exists^* x_{t_1})(\exists \bar{y}_{t_1})(\exists^* x_{t_2})(\exists \bar{y}_{t_2}) \cdots (\exists^* x_{t_n})(\exists \bar{y}_{t_n})\varphi\}$;

the order in Q is: $\varphi < \psi$ if $\psi \vdash \varphi$.

(C) If G is a generic subset of Q, we let $M[G]$ be a model with universe $|N| \cup \{x_t, y_{t,i} : t \in \mathbb{Q}, l < \omega\}$ and G giving the complete diagram of M. Clearly $N < M[G]$, and for all this it is enough that $G \subseteq Q$ is directed and not disjoint to countably many dense subsets of Q. Moreover if $a \in N$, $N \models$ "a has power $< \kappa^*$" then $(\forall b \in M[G])$ $([M \models b \in a] \Rightarrow b \in N)$ (by the κ^*-completeness of I).

NOTATION. For any index τ let

$$Q^\tau = \{\varphi(x^\tau_{t_1}, \bar{y}^\tau_{t_1}, \cdots) : \varphi(x_{t_1}, \bar{y}_{t_1}, \cdots) \in Q, \bar{y}^\tau_t = \langle y^\tau_{t,l} : l < \omega \rangle\},$$

G^τ a generic subset of Q^τ.

$M[G^\tau]$ is defined similarly. For $\varphi = \varphi(x_{t_1}, \bar{y}_{t_1}, \cdots) \in Q$ let

$$\varphi^\tau = \varphi(x^\tau_{t_1}, \bar{y}^\tau_{t_1}, \cdots).$$

Let $\bar{z}_t = \langle x_t \rangle ^\wedge \bar{y}_t$; $\exists^* \bar{z}_t$ means $\exists^* x_t \exists \bar{y}_t$. Let $\bar{s}$, $\bar{t}$ denote increasing sequences from $\mathbb{Q}$; if $\bar{t} = \langle t_1, \cdots, t_n \rangle$ then

$$\bar{z}_{\bar{t}} = \bar{z}_{t_1} {}^\wedge \cdots {}^\wedge \bar{z}_{t_n}; \qquad \exists^* \bar{z}_{\bar{t}} \text{ means } \exists^* \bar{z}_{t_1} \cdots \exists^* \bar{z}_{t_n}.$$

Let

$$\bar{z}_{t,i} = \langle x_{t,i} \rangle ^\wedge \bar{y}_{t,i}, \quad \bar{y}_{t,i} = \langle y_{t,0,i}, y_{t,1,i}, \cdots \rangle, \quad \bar{z}_{\bar{t},i} = \bar{z}_{t_1,i} {}^\wedge \bar{z}_{t_2,i} {}^\wedge \cdots {}^\wedge \bar{z}_{t_n,i}.$$

(D) We now describe the construction:

We shall define by induction on $k < \omega$, for every $\eta \in {}^k 2$, $\varphi_\eta = \varphi_\eta(\cdots x_t, \bar{y}_t, \cdots)_{t \in U(n)}$, such that:

(a) $U(n)$ is a finite subset of $\mathbb{Q}$,

(b) $U(n) \subseteq U(n+1)$,

(c) for $l < k$, $\varphi_\eta \vdash \varphi_{\eta \upharpoonright l}$.

We call the set of such $\bar{\varphi} = \langle \varphi_\eta : \eta \in {}^{\omega>}2 \rangle$, Φ. A natural topology is defined on Φ.

We shall prove that various facts holds for "almost all $\bar{\varphi}$", i.e., for all but a first category set; later on we usually ignore the "exceptional" $\bar{\varphi}$'s.

(E) For $\bar{\varphi} \in \Phi$, $\eta \in {}^{\omega}2$ let $G^{\eta}_{\bar{\varphi}} = \{\varphi_{\eta \restriction k} : k < \omega\}$.

It is easy to prove that for every η, $\bar{\varphi}$, $G^{\eta}_{\bar{\varphi}}$ is directed, and for every dense $D \subseteq Q$, for almost all $\bar{\varphi}$ for every η, $G^{\eta}_{\bar{\varphi}} \cap D \neq \emptyset$ (note the order of quantification).

(F) Hence for almost all $\bar{\varphi}$, $M[G^{\eta}_{\bar{\varphi}}]$ is as in (C) (for all $\eta \in {}^{\omega}2$). We denote the elements of $M[G^{\eta}_{\bar{\varphi}}]$ by $x^{\eta}_{l}, \bar{y}^{\eta}_{l} = \langle y^{\eta}_{l,l} : l < \omega \rangle$ to avoid confusion. Let $M^{\eta}_{\bar{\varphi}} = M[G^{\eta}_{\bar{\varphi}}]$. Note: if $M^{\eta}_{\bar{\varphi}}$ "says" x is a natural number or a real, then it really is (or at least we can consider it as such). Clearly if ψ is a κ-Souslin relation on reals, whose definition belongs to N, $\kappa < \kappa^*$, and $M^{\eta}_{\bar{\varphi}} \models \psi[r_1, \cdots, r_n]$ then really (in V)$\models \psi[r_1, \cdots, r_n]$ (note that $M^{\eta}_{\bar{\varphi}} \models$ "x is in ${}^{\omega>}\kappa$" then $x \in N$ hence really $x \in {}^{\omega>}\kappa$).

(G) Let ψ be κ-Souslin relations on reals, in N.

For almost all $\bar{\varphi}$ the following holds:

for arbitrarily large $k < \omega$, for every distinct $\nu_1, \cdots, \nu_m \in {}^{k}2$, $\bar{t}_1, \cdots, \bar{t}_m$ increasing and $\bar{t}_i \subseteq \{\pm l/n : l, n < k\} \cap U(k)$, for some $\psi' \in \{\psi, \neg\psi\}$, $f_1, \cdots, f_m \in N$ function into $\mathbb{R}$ (or ${}^{n}\mathbb{R}$)

$$\langle \varphi^{1}_{\nu_1}, \cdots, \varphi^{m}_{\nu_m} \rangle \Vdash_{Q^1 \times Q^2 \times \cdots \times Q^m} \text{“}\psi'(f_1(\bar{z}^{\nu_1}_{\bar{t}_1}), f_2(\bar{z}^{\nu_2}_{\bar{t}_2}), \cdots, f_m(\bar{z}^{\nu_m}_{\bar{t}_m}))\text{”}$$

(for this the "κ-Souslinity" is not necessary).

(H) In (G)'s notation when $\psi' = \psi$ (for almost all $\bar{\varphi}$'s) if $\nu_l \lessdot \eta_l \in {}^{\omega}2$ then $\models \psi[f_1(\bar{z}^{\eta_1}_{\bar{t}_1}), f_2(\bar{z}^{\eta_2}_{\bar{t}_2}), \cdots]$.

Proof. There is a $Q^1 \times Q^2 \times \cdots \times Q^m$ name $\underset{\sim}{\eta}$ of $\eta \in {}^{\omega}\kappa$ which witnesses the satisfaction of ψ (for some specific $k, v_1, \cdots, \bar{t}_1, \cdots$).

So it is sufficient that for arbitrarily large $l < \omega$

(*) if $\rho_1, \cdots, \rho_m \in {}^{l}2$, $\nu_l < \eta_l$ then for some $\eta \in {}^{l}\kappa$

$$(\varphi^{1}_{\rho_1}, \cdots, \varphi^{m}_{\rho_m}) \Vdash_{Q^1 \times \cdots \times Q^m} \text{“}\underset{\sim}{\eta} \restriction l = \eta\text{”}$$

and this holds for almost every $\bar{\varphi}$.

(I) Now we come to the point which in each application is the heart of the matter. So we assume (remember we can add to φ^l and ψ dummy variables)

$$(\varphi^1, \cdots, \varphi^m) \Vdash_{Q^1 \times Q^2 \times \cdots \times Q^m} \text{“}\psi(f_1(\bar{z}^{1}_{\bar{t}}), \cdots, f_m(\bar{z}^{m}_{\bar{t}}))\text{”},$$

$\varphi_l = \varphi_l(\bar z_{\bar l})$, ψ a conjunction of κ-Souslin and negation of κ-Souslin formulas for $\kappa < \kappa^*$. *Then*

(α) for every generic $G^1 \times \cdots \times G^m \subseteq Q^1 \times \cdots \times Q^m$, there are $\bar s_i$ $(i < \omega)$, $\bar s_0 < \bar s_1 < \cdots$ such that

(a) $M[G^l] \models \varphi^i[\bar z^{\,l}_{\bar s_{mj+i}}]$ when $j < \omega$, $i = 1, m$;

(b) if $j_l \equiv l \bmod m$ for $l = 1, \cdots, m$ then $\models \psi[f_1(\bar z^{\,1}_{\bar s_{j_1}}), \cdots, f_m(\bar z^{\,1}_{\bar s_{j_m}})]$;

(c) moreover *if* $j_l < \omega$, $j_l = l \bmod m$, and $i_1, \cdots, i_m$ is a permutation of $\{1, \cdots, m\}$ *then*

$$\models \psi[f_1(\bar z^{\,i_1}_{\bar s_{j_1}}), \cdots, f_m(\bar z^{\,i_m}_{\bar s_{j_m}})].$$

REMARK. Note that $f_l(\bar z^{\,i}_{\bar s_j})$ is computed in $M[G^i]$.

(β) Suppose in addition that $I^* = \{A \subseteq A^* : |A| < \kappa^*\}$, $r_l(\xi) \in \{1, \cdots, m\}$ for $\xi \in \{1, \cdots, j\}$, $\theta_l = \theta_l(\cdots x_{t_i}, \bar y_{t_i} \cdots)_{i=1,nmj}$ $(l = 1, m)$ are such that:

(*) if $a^{\alpha,l}_1, \bar b^{\alpha,l}_1, \cdots, a^{\alpha,l}_n, \bar b^{\alpha,l}_n \in \mathfrak{C}$, $\mathfrak{C} \models \varphi_l[a^{\alpha,l}_1, \bar b^{\alpha,l}_1, \cdots]$ (for $\alpha < \kappa$), the $a^{\alpha,l}_\beta$ $(\alpha < \kappa, \beta = 1, n)$ are distinct, *then* for each l for some $\alpha(1) < \cdots < \alpha(j)$

$$\models \theta_l[\cdots a^{\alpha(m\xi+\zeta),\zeta}_{nm\xi+n\zeta+r}, b^{\alpha(m\xi+\zeta),\zeta}_{nm\xi+n\zeta+r} \cdots]_{\xi=1,j;\zeta=1,m;r=1,n}.$$

Then for some generic $G^1 \times \cdots \times G^m \subseteq Q^1 \times \cdots \times Q^m$ s.t. (in addition to (α) (a) and (α) (b)):

(d) for $l = 1, \cdots, m$

$$\models \theta_l[\cdots x^{m\xi+\zeta,\zeta}_{nm\zeta+n\zeta+r}, \bar y^{m\xi+\zeta,\zeta}_{nm\zeta+r\zeta+r} \cdots]_{\substack{\xi=1,j\\ \zeta=1,m\\ r=1,n}}$$

The proof is like that of (2) (B)

§3. Variants and consequences

(A) Here we remark on various variations of §2.

(A1) We can use the quantifier $\exists^*$ directly (and not the ideal): so instead of $\mathcal{P}(A^*) - I$ we can have any family of subsets of A^* such that if $(\exists^* x)[\varphi_1(x) \vee \varphi_2(x)]$ then $\exists^* x\, \varphi_1(x)$ or $\exists^* x\, \varphi_2(x)$.

(A2) In [3] we use an M generated by one element x; here it is generated by $\{x_t \bar y_t \in \mathbb{Q}\}$ (but use more the Skolem functions, which are not needed here). I do not see any difference (the phrasing of §2, (I) will be a little different).

(A3) In (I)(β) instead $I^* = \{A \subseteq A^* : |A| < \kappa^*\}$; if $A^* = \kappa^*$ we can also use $\{A \subseteq \kappa^* : A$ not stationary$\}$. Also for any I^* we can just demand that the suitable conditions exist.

(A4) We can phrase §2 as a partition theorem.

(A5) In §2, (I), instead of discussing $V^{Q^1\times\cdots\times Q^m}$, we can look at a game in which the players build $G^1\times\cdots\times G^m \subseteq Q^1\times\cdots\times Q^m$, each giving a finite approximation. The outcome of a play is determined by the satisfaction of some $\psi(\bar{z}^1_{\bar{t}_1},\cdots,\bar{z}^m_{\bar{t}_m})$. We shall require that such games are determined.

(B)(a) How do we use §2? We want to find a perfect set of reals satisfying something.

We define A^*, I^* (usually the standard one). And take a $\bar{\varphi}$ which is like almost all $\bar{\varphi}$'s. Then we want to show that $\{x^\eta_0 : \eta \in {}^\omega 2\}$ (or something similar) is as required. We assume not, and use (I) ((α) or (β)) to show that in $V^{Q^1\times\cdots\times Q^m}$ there are some reals satisfying some κ-Souslin and co-κ-Soulsin formulas $\kappa<\kappa^*$. $Q^1\times\cdots\times Q^m$ is equivalent to Cohen forcing, and so in $V^{Q^1\times\cdots\times Q^m}$ supposedly we got something forbidden by a hypothesis in the specific theorem we are trying to prove.

(B)(b) Instead of assuming that even after adding Cohen reals there are no reals $r_1,\cdots,r_n$ satisfying $\bigwedge_{i=1}^m \psi_i[r_1,\cdots,r_n]\wedge\bigwedge_{i=m+1}^n \psi_i[r_1,\cdots,r_n]$ (ψ_i κ-Souslin), we can demand

(*) if $f_1,\cdots,f_n$ define $\psi_1,\cdots,\psi_n$, r a real, then in V there is a real generic over $L[r,f_1,\cdots,f_n]$. This holds if $\mathbb{R}$ is not the union of κ nowhere-dense sets.

If each φ_i is Σ^1_2 hence $\aleph_1$-Souslin, we can omit $f_1,\cdots,f_n$ in (*), and then, of course, $\aleph_1^{L[r]}<\aleph_1$ is sufficient.

(C) THEOREM. *If $\leqq$ is a co-κ-Souslin relation on $\mathbb{R}$, $(\mathbb{R},\leqq)$ is a quasi-linear order (even after adding a Cohen real), then there is no (strictly) increasing sequence of length κ^+.*

REMARK. This essentially appears in [3] (for $\kappa=\aleph_0$); $x\leqq y\wedge y\leqq x$ may be here a non-trivial equivalence relation.

PROOF. We use (2) with $\kappa^*=\kappa^+$, $A^*=\{r_i : i<\kappa^*\}$ strictly increasing, i and the definition of $\leqq$ is in N, I^* standard. Let $\bar{\varphi}\in\Phi$ be as usual.

So for $\eta\neq\nu$, x^η_0, x^ν_0 are comparable, so w.l.o.g. $x^\eta_0\leqq x^\nu_0$; so $\neg(x^\nu_0<x^\eta_0)$ hence for some $(\varphi^1_1,\varphi^2_2)\in Q^1\times Q^2$, $(\varphi^1_1,\varphi^2_2)\Vdash$"$\neg(x^2_0<x^1_0)$" (as it cannot force the negation) hence $(\varphi^1_1,\varphi^2_2)\Vdash_{Q^1\times Q^2}$"$x^1_0\leqq x^2_0$". By §2 (I)($\alpha$)(b) for some $(\psi^1,\psi^2)\in Q^1\times Q^2$ and $t_0<t_1$, $(\psi^1,\psi^2)\Vdash_{Q^1\times Q^2}$"$x^1_{t_1}\leqq x^2_{t_0}\wedge x^2_{t_1}\leqq x^1_{t_0}$". But we know that

$$(\psi^1,\psi^2)\Vdash_{Q^1\times Q^2}\text{“}x^1_{t_0}<x^1_{t_1}\wedge x^2_{t_0}<x^2_{t_1}\text{”}.$$

So (ψ^1,ψ^2) forces that $x^1_{t_0}<x^1_{t_1}\leqq x^2_{t_0}<x^2_{t_1}\leqq x^1_{t_0}$, contradiction.

(D) THEOREM. *Suppose $M = (\mathbb{R}, \cdots, f_i, \cdots)_{i<\omega}$ is an algebra, $d(-,-)$ a semi-metric on $\mathbb{R}$. Suppose each relation (τ a term of $L(M)$) $d(x, \tau(y_1, \cdots, y_n)) < \varepsilon$ (ε rational) is κ-Soulsin. If $\mathbb{R}$ has no dense subset of power κ then for some $\varepsilon > 0$ ($\varepsilon \in \mathbb{Q}$) there are $x_\eta \in \mathbb{R}$ ($\eta \in {}^\omega 2$), such that the distance of each x_η to the subalgebra generated by $\{x_\nu : \nu \in {}^\omega 2, \nu \neq \eta\}$ is $> \varepsilon$.*

We assume, of course, that adding a Cohen real does not change the hypothesis.

REMARKS. (1) We can apply it to an algebra by using a trivial $d : d(x, y)$ is 1 if $x \neq y$, 0 if $x = y$. This case, for Borel relations, appears in Friedman [1].

(2) This also holds for a metric space (no functions), also for Banach spaces (operations $x + y$, $x - y$, qy ($q \in \mathbb{Q}$)), and we get $d(x_\eta, \mathrm{Sp}\{x_\nu : \nu \in {}^\omega 2 - \{\eta\}) \geqq \varepsilon$ (clearly adding rx ($r \in \mathbb{R}$) does not change), and we can replace ε by 1.

PROOF. Let $\kappa^* = \kappa^+$; as $\mathbb{R}$ has no dense subset of power κ, there is $A^* = \{x_i : i < \kappa\}$, $d(x_i, x_j) > \varepsilon_i$ for every $j < i$, w.l.o.g. $\varepsilon_i = \varepsilon$.

Let $\bar{\varphi}$ be as usual, $\eta_1, \cdots, \eta_n \in {}^\omega 2$ (distinct), τ a term. We shall show $d(x_0^{\eta_n}, \tau(x_0^{\eta_1}, \cdots, x_{n-1}^{\eta_{n-d}})) \geqq \varepsilon/2$. Otherwise (by §2(I)(α)) for some generic $G \times \cdots \times G^n \subseteq Q \times \cdots \times Q^n$ and $t_0 < t_1$,

$$d(x_{t_l}^n, \tau(x_0^1, \cdots, x_0^{n-1})) < \varepsilon/2 \qquad (l = 0, 1).$$

But also (see §2)

$$d(x_{t_0}^n, x_{t_1}^n) \geqq \varepsilon.$$

This contradicts d being semi-metric.

§4. Subsets of uncountable cardinals

(A) Can we replace $\aleph_0$ by some other cardinal χ (and $\mathbb{R}$ by $\bar{\mathcal{P}}(\chi)$)? Clearly we have a chance for positive results only for χ strong limit of cofinality $\aleph_0$. The results are quite weak.

(B) It is not hard to prove, and it is known, that

THEOREM. *If $A^* \subseteq \mathcal{P}(\chi)$ is κ-Souslin of power $> \kappa$, $\chi < \kappa < \chi^{\aleph_0}$ then $|A| = \chi^{\aleph_0}$.*

PROOF. Let λ be large enough, $N \prec (H(\lambda), \in)$, $\lambda \subseteq N$, $H(\chi) \subseteq N$, $N = \bigcup_{l<\omega} N_l$, $\|N_l\| < \chi$, $\mathcal{P}(N_l) \subseteq N_{l+1}$. And $f : {}^{\omega>}\kappa \to$ "family of open subsets of $\mathcal{P}(\chi)$" exemplify A being κ-Souslin, $f \in N$. We can find elementary extensions N_η of N for $\eta \in {}^\omega\chi$ and $X_\eta \in N_\eta$, $N_\eta \models$ "$X_\eta \subseteq \chi$" and so identifying X_η and $\{i < \chi : N_\eta \models i \in X_\eta\}$ the X_η are distinct. Just let $I^* = \{B \subseteq A^* : |B| \leqq \kappa\}$. (See Magidor, Shelah and Stavi [5] on related problems.)

(C) Another approach is to consider relations on $\mathcal{P}(\chi)$ which are Borel (i.e., the closure of the family of open subsets by complementation and countable intersection). As a result we shall get a perfect set (i.e., like ${}^{\omega}2$ not ${}^{\omega}\chi$). Remembering §3(A5) it is not hard to repeat §2 (and the consequences).

So it is sufficient to show we cannot have a too long Borel well-ordering.

(D) Can we, in (C), replace Borel by analytic? (The quantification is $(\exists B \subseteq \chi)$.) The problem is to make the games determined which holds if there are enough $B^{\#}$ (B a set of ordinals). But we know they exist if G.C.H. fails for strong limit cardinal $\geqq \operatorname{Sup} B$.

(E) Another approach is to replace Borel by $\Sigma_0^{\chi} = \Pi_0^{\chi}$ which we define to be $\{\{A \in \mathcal{P}(\chi) : (\chi, A, \alpha)_{\alpha \in \chi} \models \varphi\} : \varphi \in L_{\chi^+,\omega}\}$ and Σ_n^{χ}, Π_n^{χ} are defined naturally.

The parallel of §2 holds if we replace Cohen forcing by $\mathrm{Col}(\aleph_0, \chi) = \{f : f$ a finite function from ω to χ).

§5. A consistent counterexample

We would have liked to remove the hypothesis "even after adding a Cohen generic real, R still defines, e.g., an equivalence relation". Unfortunately

5.1. THEOREM. ZFC + "$2^{\aleph_0} = \aleph_2$" + "*there is a co-$\aleph_1$-Souslin equivalence relation E with $2^{\aleph_0}$ equivalence classes but with no perfect set of pairwise non-E-equivalent elements*" *is consistent.*

PROOF. We start with a universe $V \models 2^{\aleph_0} = \aleph_2$ and we shall define a finite support iteration $\langle P_i, Q_i : i < \omega_1 \rangle$ of forcing satisfying the c.c.c., letting $V_i = V^{P_i}$. In V^{P_i} a sequence $\bar{B}_i = \langle B_j : j < \omega i \rangle$ will be defined, such that B_i is a Borel function from ${}^{\omega}2$ to ${}^{\omega}2$. We define relations R_i, R_i^- on ${}^{\omega}2$: and xR_i^-y *iff* $\bigvee_m [y = B_{\omega i+m}(x)]$. Then we define $xR_iy \stackrel{\text{def}}{=} (\forall j < i)[\neg xR_j^- y \wedge \neg yR_j^- x]$. Now R_{ω_1} will be the relation we need. Clearly R_i^-, R_i are Borel relations and R_{ω_1} is a co-$\aleph_1$-Souslin relation. However it is not apriori clear that R_{ω_1} is an equivalence relation. Note that R_i^- is (defined) in $V^{P_{i+1}}$, R_i in V^{P_i}.

We shall define by induction on $i < \omega_1$ (in V^{P_i}) a c.c.c. forcing notion Q_i, an equivalence relation E_i on $({}^{\omega}2)^{V^{P_i}}$, and a sequence $\langle F_\alpha^i : \alpha < \omega_1 \rangle$ of functions such that:

(a) for $j < i$, $E_i \restriction ({}^{\omega}2)^{V^{P_j}} = E_j$, moreover if $x \in ({}^{\omega}2)^{V^{P_j}}$ then $x/E_i \subseteq ({}^{\omega}2)^{V^{P_j}}$;

(b) Q_i has power $\aleph_2$, and satisfies the c.c.c.;

(c) each F_α^i is a one-place function from $({}^{\omega}2)^{V^{P_i}}$ to itself, $\neg(xE_iF_\alpha^i(x))$ and

$$\neg(xE_iy) \Rightarrow \bigvee_{\alpha < \omega_1} (x = F_\alpha^i(y) \vee y = F_\alpha^i(x));$$

(d) for odd i if $x = F^j_\alpha(y)$, $j < i$, $\alpha < i$ then $xR_i^- y$ (in $V^{P_{i+1}}$);

(e) for even i for every perfect non-empty set $A \subseteq {}^\omega 2$ in V^{P_i}, whose definition is in V^{P_j} for fome $j < i$, there are $x, y \in A$, $xE_i y$;

(f) if $j < i$, $xR_j^- y$ then $\neg\, xE_i y$ (in V^{P_i});

(e) E_0 has at least two equivalence classes.

Case I: *i is odd*

First we choose any E_i satisfying (a). Second we define F^i_α $(\alpha < \omega_2)$ as explained below. Third let $\{A^i_\alpha : \alpha < \omega_2\}$ list all perfect non-empty sets $A \subseteq {}^\omega 2$ in V^{P_i}, $A^i_{2\alpha} = A^i_{2\alpha+1}$ and let Q_i force, for each $\alpha < \aleph_2$, a generic real inside A^i_α, i.e.,

$$Q_i = \{g : g \text{ is a finite function with domain } \subseteq \omega_2, \text{ such that for each } \alpha \in \operatorname{Dom} g,\ f(\alpha) \in {}^\omega 2 \text{ and } (\exists \eta \in A^i_\alpha)[g(\alpha) \triangleleft \eta]\}.$$

Let $\underset{\sim}{r}^i_\alpha = \bigcup\{g(\alpha) : g \in \underset{\sim}{G}^{Q_i}\}$.

Case II: *i is even*

We define E_i: for $x, y \in V^{P_i}$, $xE_i y$ iff $xE_{i-1}y$ (so $x, y \in V^{P_{i-1}}$), or $x = y$ or for some $\alpha < \omega_2$, $\{x, y\} = \{r^{i-1}_{2\alpha+1}, r^{i-1}_{2\alpha}\}$.

(This takes care of (e).) Second we define F^i_α $(\alpha < \omega_2)$ as explained below. Now we take care of (d).

Let $\mathcal{P}$ be the set of f, f a finite one-to-one function from ${}^{\omega \geqq}2$ to ${}^{\omega\geqq}2$, preserving the order $\triangleleft$, i.e., for $\eta, \nu \in \operatorname{Dom} F$, $\eta \triangleleft \nu$ iff $f(\eta) \triangleleft f(\nu)$ ($\triangleleft$ is an initial segment), $l(\eta) = \omega$ iff $l(f(\eta)) = \omega$, and if $\eta \neq \nu \in \operatorname{Dom} f$ then for some $\rho \in \operatorname{Dom} f$, $\rho \triangleleft \eta$, $\rho \ntriangleleft \nu$ and

$$Q_i = \{\langle f_0, \cdots, f_n \rangle : \text{for each } l < n,\ f_l \in \mathcal{P} \text{ and if } x \in (\operatorname{Dom} f_l) \cap ({}^\omega 2)^{V^{P_{i-1}}} \text{ then } f_l(x) \in \{F^j_\alpha(x) : j < i, \alpha < i\}\}.$$

The order is $\langle f_0, \cdots, f_n \rangle \leqq \langle f'_0, \cdots, f'_m \rangle$ *iff* $n \leqq m$, and $f_l \subseteq f'_l$ for $l \leqq n$.

As the number of possible values of $f_l(x)$ is countable Q_i satisfies the c.c.c., and in $V^{P_{i+1}}$ we will define $B_{\omega i+k}$ as the unique continuous function extending

$$\bigcup\{f_k : \text{for some } n \geqq k \text{ and } f_0, \cdots, f_{k-1}, f_{k+1}, \cdots, f_n,\ \langle f_0, \cdots, f_n \rangle \in \underset{\sim}{G}^{Q_i}\}.$$

The only point left is defining F^i_α $(\alpha < \omega_2)$. So let $({}^\omega 2)^{V^{P_i}} = \{r^i_\xi : \xi < \omega_2\}$, and for each $\xi < \omega_2$ we define $\langle F^i_\alpha(r^i_\xi) : \alpha < \omega_1 \rangle$ such that

$$\{r^i_\zeta : \zeta < \xi, \neg\, r^i_\xi E_i r^i_\zeta\} \subseteq \{F^i_\alpha(r^i_\xi) : \alpha < \omega_2\} \subseteq \{r^i_\zeta : \zeta < \omega_2, \neg\, r^i_\zeta E_i r^i_\xi\}$$

as $|\{\zeta : \zeta < \xi\}| \leqq \aleph_1$ and by (g) this can be done trivially.

§6. More on partial order

In summer 1983, in answer to a question of D. Marker, we proved the following, which was added to the paper after it had already been sent to the press.

THEOREM. *Suppose $\mathcal{P}$ is a co-κ-Souslin relation (on $\mathbb{R}$) which is a partial quasi-order (i.e., satisfies transitivity) even after adding a Cohen real. We denote the relation by $\leqq : x \leqq y$.*

Suppose further that there is no perfect set of reals which are pairwise incomparable. Then for every sequence $\langle a_i : i < \kappa^+ \rangle$ of reals there are sets X, $Y \subseteq \kappa^+$ each of power κ^+, such that for every $i \in X$ and $j \in Y$, $a_i \leqq a_j$.

REMARKS. (1) We can replace κ^+ by any regular $\kappa_1 > \kappa$.

(2) We could use the general method of §2, but we present it as in §1.

PROOF. Suppose $\langle a_i : i < \kappa^+ \rangle$ is a counterexample to the conclusion.

Let λ be a regular, large enough cardinality. Let N be an elementary submodel of $\mathfrak{C} = (H(\lambda), \in, \leqq, <^*)$ which is countable and to which $\langle a_i : i < \kappa^+ \rangle$ belongs, where $<^*$ is a well-ordering of $H(\lambda)$.

Let $\mathfrak{C}$ be an expansion of $(H(\lambda), \in, \leqq)$ by Skolem functions and N a countable elementary submodel of $\mathfrak{C}$.

As $\leqq$ is a co-κ-Souslin relation, there is a function f from ${}^{\omega >}\kappa$ to the family of open subsets of $\mathbb{R} \times \mathbb{R}$ such that

$$x \leqq y \textit{ iff } \text{for no } \eta \in {}^{\omega}\kappa, \quad (\forall n)[\langle x, y \rangle \in f(\eta \restriction n)].$$

Let

$$Q = \{\varphi(\bar{x}) : \varphi \text{ a (first order) formula with parameters from } N \text{ (in its language)} \\ \{x \in N : \models \varphi[x]\} \text{ is an unbounded subset of } \kappa^+\}$$

with the order: $\varphi_1 \leqq \varphi_2$ iff $N \models (\forall x)[\varphi_2(x) \rightarrow \varphi_1(x)]$.

We consider Q as a forcing notion; clearly it is equivalent to Cohen forcing. For $G \subseteq Q$ generic and variable y we can define an elementary extension $N[G, y]$ of N which is the Skolem hull of $N \cup \{y\}$, and for $\varphi(x) \in Q$

$$N[G, y] \models \varphi[y] \qquad \text{iff } \varphi(x) \in Q.$$

We can also define a real $\tau = \tau_G$: so $\tau \in {}^{\omega}2$, $\tau(n) = 0$ iff $[a_x(n) = 0] \in G$.

If we force by $Q \times Q$ we get a generic subset $G_1 \times G_2$ and two elementary extensions of N, say $N[G_1, y_1]$, $N[G_2, y_2]$. Now we may ask (in $V[G_1 \times G_2]$) whether $\tau_{G_1} \leqq \tau_{G_2}$ holds.

By standard methods it will be enough to prove

MAIN FACT. If $\varphi(x) \in Q$ then there is $\langle \varphi^1(x), \varphi^2(x) \rangle \in Q \times Q$ such that
(a) $\varphi^1 \geqq \varphi$, $\varphi^2 \geqq \varphi$ (in Q),
(b) $\langle \varphi^1, \varphi^2 \rangle \Vdash_{Q \times Q}$ "$\tau_{G_1} \not\leqq \tau_{G_2}$ and $\tau_{G_2} \not\leqq \tau_{G_1}$".

PROOF. We can use the forcing notion $Q \times Q \times Q$ (calling the generic set $G_1 \times G_2 \times G_3$). So $\langle \varphi, \varphi, \varphi \rangle \in Q \times Q \times Q$. Hence there is a condition $\langle \varphi_1, \varphi_2, \varphi_3 \rangle$,

$$\langle \varphi, \varphi, \varphi \rangle \leqq \langle \varphi_1, \varphi_2, \varphi_3 \rangle \in Q \times Q \times Q,$$

which, for each of the following statements, force it or its negation:

$$\tau_{G_l} \leqq \tau_{G_m} \qquad (l = 1,2,3 \text{ and } m = 1,2,3).$$

Clearly already $\langle \varphi_l, \varphi_m \rangle$ forces this (in the right naming).

By absoluteness, if τ_{G_l}, τ_{G_m} are forced to be $\leqq$-incomparable, then $\langle \varphi_l, \varphi_m \rangle$ is as required. So we assume this does not hold. By symmetry, w.l.o.g.

$$\langle \varphi_1, \varphi_2, \varphi_3 \rangle \Vdash_{Q \times Q \times Q} \text{“}\tau_{G_1} \leqq \tau_{G_2} \leqq \tau_{G_3}\text{”}.$$

Let $X_l = \{i < \kappa^+ : \varphi_l(a_i)\}$, so clearly $|X_l| = \kappa^+$. Also, as $\langle a_i : i < \kappa^+ \rangle$ is a counterexample to the conclusion, X_1, X_3 cannot serve as X, Y there; and moreover, for every $\alpha < \kappa^+$, $X_1 - \alpha$, $X_3 - \alpha$ cannot serve as X, Y there. So we can find $X \subseteq X_1$ of power κ^+ and order (of the ordinals) preserving $h : X \to X_3$ such that $a_i \not\leqq a_{h(i)}$ for $i \in X$. We can assume $h \in N$.

Now $\langle \varphi_1, \varphi_2, \varphi_3 \rangle \leqq \langle x \in X, y_2, h^{-1}(x) \in X \rangle \in Q_1 \times Q_2 \times Q_3$ and suppose $G_1 \times G_2 \times G_3 \subseteq Q \times Q \times Q$ is generic (over V),

$$\langle x \in X, \varphi_2(x), h^{-1}(x) \in X \rangle \in G_1 \times G_2 \times G_3.$$

By our assumption on $\langle \varphi_1, \varphi_2, \varphi_3 \rangle$, $\tau_{G_1} \leqq \tau_{G_2} \leqq \tau_{G_3}$. Now let $G_1^* = \{\varphi(x) : \varphi(h^{-1}(x)) \in G_1\}$, $G_3^* = \{\varphi(x) : \varphi(h(x)) \in G_3\}$. Clearly $G_3^* \times G_2 \times G_1^* \subseteq Q \times Q \times Q$ is generic (over V).

As $\varphi_3 \in G_1^*$, $\varphi_1 \in G_3^*$, clearly $\tau_{G_3^*} \leqq \tau_{G_2} \leqq \tau_{G_1^*}$. So together $\tau_{G_1} \leqq \tau_{G_1^*}$. But in $N[G_1, y]$, τ_{G_1}, $\tau_{G_1^*}$ are represented by a_y, $a_{h(y)}$ respectively (i.e., $\tau_{G_1^*}(n) = 0$ iff $N[G_1, y] \models$ "$a_{h(y)}(n) = 0$" and similarly for y). Also, by h's definition $N[G, y] \models$ "$a_y \not\leqq a_{h(y)}$". As $f \in N$ (and its choice), this implies that really (in $V[G_1 \times G_2 \times G_3]$) $\tau_{G_1} \not\leqq \tau_{G_1^*}$, contradicting the previous paragraph.

ACKNOWLEDGEMENT

The author thanks R. M. Solovay for pointing out an inaccuracy in the proof of 5.1 (in the definition of $\mathscr{P}$ we demanded that f preserve the level, hence the c.c.c. fails).

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