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Algebra Universalis

The depth of ultraproducts of Boolean algebras

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Abstract. We show that if μ is a compact cardinal then the depth of ultraproducts of less than μ many Boolean algebras is at most μ plus the ultraproduct of the depths of those Boolean algebras.

1. Introduction

Monk has looked systematically at cardinal invariants of Boolean algebras. In particular, he has looked at the relations between

$$\operatorname{inv}\Big(\prod_{i<\kappa}\mathbf{B}_i/D\Big) \text{ and } \prod_{i<\kappa}\operatorname{inv}(\mathbf{B}_i)/D,$$

i.e., the invariant of the ultraproducts of a sequence of Boolean algebras vis the ultraproducts of the sequence of the invariants of those Boolean algebras for various cardinal invariants inv of Boolean algebras. That is: is it always true that $\operatorname{inv}(\prod_{i<\kappa}\mathbf{B}_i/D) \le \prod_{i<\kappa}(\operatorname{inv}(\mathbf{B}_i/D)$? Is it consistently always true? Is it always true that $\prod_{i<\kappa}\operatorname{inv}(\mathbf{B}_i)/D \le \operatorname{inv}(\prod_{i<\kappa}\mathbf{B}_i/D)$? Is it consistenly always true? See more on this in Monk [Mo96]. Roslanowski Shelah [RoSh] deals with specific inv and with more on kinds of cardinal invariants and their relationship with ultraproducts. Monk [Mo90], [Mo96], in his list of open problems raises the question for the central cardinal invariants. Most of them have been solved by now; see Magidor and Shelah [MgSh], Peterson [Pe97], Shelah [Sh90], [Sh97], [Sh96a], [Sh00, §4], [Sh96b], [Sh01], [Sh03], Shelah and Spinas [ShSi].

This paper throws some light on problem 12 of [Mo96, p. 287] and will be continued in future work.

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Definition 1.1. For a Boolean algebra $\mathbf{B}$ let
(a) $\operatorname{Depth}(\mathbf{B}) = \sup\{\theta$: in $\mathbf{B}$ there is an increasing sequence of length $\theta\}$
(b) $\operatorname{Depth}^+(\mathbf{B}) = \sup\{\theta^+$: in $\mathbf{B}$ there is an increasing sequence of length $\theta\}$.

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Remark 1.2. So $\mathrm{Depth}^+(\mathbf{B}) = \lambda^+ \Rightarrow \mathrm{Depth}(\mathbf{B}) = \lambda$ and if $\mathrm{Depth}^+(\mathbf{B})$ is a limit cardinal then $\mathrm{Depth}^+(\mathbf{B}) = \mathrm{Depth}(\mathbf{B})$.

2. Above a compact cardinal

The following claim gives severe restrictions on any attempt to build a ZFC example for $\mathrm{Depth}(\prod_{\varepsilon<\kappa} \mathbf{B}_\varepsilon)/D > \prod_{\varepsilon<\kappa} \mathrm{Depth}(\mathbf{B}_\varepsilon)/D$. If $\mathbf{V}$ is near $\mathbf{L}$, see [Sh02] for results complimentary to §1.

Claim 2.1.

(1) *Assume*
   (a) $\kappa < \mu \leq \lambda$*;*
   (b) μ *is a compact cardinal;*
   (c) D *is an ultrafilter on* κ*;*
   (d) $\lambda = \mathrm{cf}(\lambda)$ *such that* $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$*;*
   (e) $\mathbf{B}_i\,(i < \kappa)$ *is a Boolean algebra with* $\mathrm{Depth}^+(\mathbf{B}_i) \leq \lambda$*;*
   (f) $\mathbf{B} = \prod_{i<\kappa} \mathbf{B}_i/D$*.*
   Then $\mathrm{Depth}^+(\mathbf{B}) \leq \lambda$.

(2) *Instead of* $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$*, it suffices that* $(\forall \alpha < \lambda)(|\alpha^\kappa/D| < \lambda = \mathrm{cf}(\lambda))$.

(3) *We can weaken clause* (e) *(for parts* (1) *and* (2)*) to*
   (g) $\{i < \kappa : \mathbf{B}_i$ *is a Boolean algebra with* $\mathrm{Depth}^+(\mathbf{B}_i) \leq \lambda\} \in D$*.*

Proof. (1): Toward a contradiction assume that $\langle a_\alpha : \alpha < \lambda\rangle$ is an increasing sequence in $\mathbf{B}$. Let $a_\alpha = \langle a_i^\alpha : i < \kappa\rangle/D$, so for $\alpha < \beta$,

$$A_{\alpha,\beta} =: \{i < \kappa : \mathbf{B}_i \models a_i^\alpha < a_i^\beta\} \in D.$$

Let E be a μ-complete uniform ultrafilter on λ.

For each $\alpha < \lambda$ let A_α be such that the set $\{\beta : \alpha < \beta < \lambda$ and $A_{\alpha,\beta} = A_\alpha\}$ is a member of E, so an unbounded subset of λ (which exists since $\lambda = \mathrm{cf}(\lambda) \geq \mu > 2^\kappa$). We choose C as follows:

$$\begin{aligned} C =: \{\, \delta < \lambda : \delta &\text{ is a limit ordinal and if } u \subseteq \delta \text{ is bounded} \\ &\text{of cardinality} \leq \kappa \text{ then } \delta = \sup(S_u \cap \delta)\,\} \end{aligned}$$

where

$$S_u =: \{\beta < \lambda : \beta > \sup(u) \text{ and } (\forall \alpha \in u)(A_{\alpha,\beta} = A_\alpha)\}.$$

As $\lambda = \mathrm{cf}(\lambda) > 2^\kappa = |D|$, for some $A_* \in D$ the set $S =: \{\alpha < \lambda : \mathrm{cf}(\alpha) > \kappa,$ and $A_\alpha = A_*\}$ is a stationary subset of λ.

As we have assumed $\lambda = \mathrm{cf}(\lambda)$ and $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$, clearly C is a club of λ. Let $\{\delta_\varepsilon : \varepsilon < \lambda\} \subseteq C$ such that δ_ε increases continuously with ε and $\delta_{\varepsilon+1} \in S$. For each $\varepsilon < \lambda$ the family $\mathfrak{A}_\varepsilon = \{S_u \cap \delta_{\varepsilon+1} \setminus \delta_\varepsilon : u \in [\delta_{\varepsilon+1}]^{\leq\kappa}\}$ is a downward κ^+-directed family of non-empty subsets of $[\delta_\varepsilon, \delta_{\varepsilon+1})$ hence there is a κ^+-complete filter E_ε on $[\delta_\varepsilon, \delta_{\varepsilon+1})$ extending $\mathfrak{A}_\varepsilon$.

For $\varepsilon < \lambda$ and $i < \kappa$ let $W_{\varepsilon,i} =: \{\beta : \delta_\varepsilon \leq \beta < \delta_{\varepsilon+1}$ and $i \in A_{\beta,\delta_{\varepsilon+1}}\}$ and let $B_\varepsilon =: \{i < \kappa : W_{\varepsilon,i} \in E_\varepsilon^+\}$. As E_ε is κ^+-complete, clearly

$$W_\varepsilon =: \bigcap\{[\delta_\varepsilon, \delta_{\varepsilon+1})\backslash W_{\varepsilon,i} : i \in \kappa\backslash B_\varepsilon\} \in E_\varepsilon,$$

hence there is $\beta \in W_\varepsilon$. If $i \in A_{\beta,\delta_{\varepsilon+1}}$ then $\{\gamma : \delta_\varepsilon \leq \gamma < \delta_{\varepsilon+1}$ and $i \in A_{\gamma,\delta_{\varepsilon+1}}\} \in E_\varepsilon^+$, so $A_{\beta,\delta_{\varepsilon+1}}$ is a subset of B_ε and belongs to D hence $B_\varepsilon \in D$. So $B_\varepsilon \cap A_* \in D$ is non-empty.

So for each ε for some $i_{\delta_{\varepsilon+1}} \in A_*$ we have

$$\{\beta : \delta_\varepsilon \leq \beta < \delta_{\varepsilon+1} \text{ and } i_{\delta_{\varepsilon+1}} \in A_{\beta,\delta_{\varepsilon+1}}\} \in E_\varepsilon^+.$$

We can find $i_* \in A_*$ such that

$$Y = \{\varepsilon < \lambda : \varepsilon \text{ is an even ordinal and } i_{\delta_{\varepsilon+1}} = i_*\}$$

has cardinality λ. Let $Z = \{\delta_{\varepsilon+1} : \varepsilon \in Y\}$, so $Z \in [\lambda]^\lambda$. Now

$(*)_0$ $\varepsilon \in Y \Rightarrow A_{\delta_{\varepsilon+1}} = A_*$
(Because $\delta_{\varepsilon+1} \in S$)

$(*)_1$ $i_* \in A_* \in D$
(Trivial: note that if $\forall\alpha < \lambda, |\alpha|^{2^\kappa} < \lambda$ we can have E_ε is $(2^\kappa)^+$-complete filter so we have $B_{\delta_{\varepsilon+1}}$ instead of i_{δ_ε} so we can weaken "D ultrafilter" to: $D \subseteq \mathcal{P}(\kappa)$ is upward closed and the intersection of any two is non-empty.)

$(*)_2$ if $\alpha < \beta$ are from Z then $i_* \in A_{\alpha,\beta}$
(For let $\alpha = \delta_{\varepsilon+1}, \beta = \delta_{\zeta+1}$ so $\varepsilon < \zeta$; let

$$\mathcal{U}_1 := \{\gamma : \delta_\zeta < \gamma < \delta_{\zeta+1}, A_{\alpha,\gamma} = A_\alpha (= A_{\delta_{\varepsilon+1}})\}$$

so

$$\mathcal{U}_1 = S_{\{\delta_{\varepsilon+1}\}} \cap (\delta_\zeta, \delta_{\zeta+1}) \in \mathfrak{A}_\zeta$$
$$\mathcal{U}_1 \subseteq E_\zeta$$

and let

$$\mathcal{U}_2 := \{\gamma : \delta_\zeta \leq \gamma < \delta_{\zeta+1}, i_* \in A_{\gamma,\delta_{\zeta+1}}\} \in E_\zeta^+,$$

as this is how $i_{\delta_{\zeta+1}}$ is defined.)

So for any $\alpha < \beta$ from Z, since $\mathcal{U}_1 \in E_\zeta$ and $\mathcal{U}_2 \in E_\zeta^+$, clearly there is $\gamma \in \mathcal{U}_1 \cap \mathcal{U}_2$. Hence ($\alpha = \delta_{\varepsilon+1} < \delta_\zeta \leq \gamma < \delta_{\zeta+1} = \beta$ and) for $i = i_*$ we have $\mathbf{B}_i \models a_i^{\delta_{\varepsilon+1}} < a_i^\gamma$ (because $\gamma \in \mathcal{U}_1$) and $\mathbf{B}_i \models a_i^\gamma < a_i^{\delta_{\zeta+1}}$ (because $\gamma \in \mathcal{U}_2$) so together $\mathbf{B}_i \models a_i^{\delta_{\varepsilon+1}} < a_i^{\delta_{\zeta+1}}$. But $\alpha = \delta_{\varepsilon+1}, \beta = \delta_{\zeta+1}$, so we have gotten $\mathbf{B}_i \models a_i^\alpha < a_i^\beta$ and we are done.

(2): We change the choice of the club C. By the assumption, for each $\alpha < \lambda$ let $\langle f_\gamma^\alpha/D : \gamma < \gamma_\alpha\rangle$ be a list of the members of α^κ/D without repetitions, so $\gamma_\alpha < \lambda$. Let C be the set of all δ such that:

(i) $\delta < \lambda$ is a limit ordinal;

(ii) if $\alpha < \delta$ then $\gamma_\alpha < \delta$;

(iii) if $\alpha < \delta$ and $\gamma < \gamma_\alpha$ and $\bar{A} = \langle A_i : i < \kappa \rangle \in {}^\kappa D$ and there is $\xi \in [\delta, \lambda)$ such that $i < \kappa \Rightarrow A_{f^\alpha_\gamma(i),\xi} = A_i$, then there is $\xi \in (\alpha, \delta)$ such that $i < \kappa \Rightarrow A_{f^\alpha_\gamma(i),\xi} = A_i$.

Clearly C is a club of λ. The only additional point is

($*$) if $\delta_1 < \delta_2$ are from C and $A_{\delta_2} = A_*$, then there is $i_* \in A_*$ such that: for every $\alpha \in S \cap \delta_1$ there is $\beta \in [\delta_1, \delta_2)$ satisfying $A_{\alpha,\beta} = A_* \wedge i_* \in A_{\beta,\delta_2}$.

(Why does ($*$) hold? If not, then for every $i \in A_*$ there is $\alpha_i \in S \cap \delta_1$ satisfying $\beta \in [\delta_1, \delta_2) \wedge A_{\alpha_i,\beta} = A_* \Rightarrow i \notin A_{\beta,\delta_2}$. Let $f \in {}^\kappa\alpha$ be defined by $f(i) = \alpha_i$ if $i \in A_*$, $f(i) = 0$ otherwise. So for some $\gamma < \gamma_{\delta_1}$ we have $f = f^{\delta_1}_\gamma \bmod D$, hence $A =: \{i \in A_* : f(i) = f^{\delta_1}_\gamma(i)\} \in D$. As $\kappa < \mu$ and D is μ-complete there is $\xi_1 \in (\delta_2, \lambda)$ such that $i < \kappa \Rightarrow A_{f^{\delta_1}_\gamma(i),\xi_1} = A_{f^{\delta_1}_\gamma(i)}$. Hence by the choice of C there is $\xi_2 \in (\delta_1, \delta_2)$ such that $i < \kappa \Rightarrow A_{f^{\delta_1}_\gamma(i),\xi_2} = A_{f^{\delta_1}_\gamma(i),\xi_1} = A_{f^{\delta_1}_\gamma(i)}$. But $i \in A \Rightarrow f^{\delta_1}_\gamma(i) = f(i) = \alpha_i \in S \Rightarrow A_{\alpha_i,\xi_2} = A_{f^{\delta_1}_\gamma(i),\xi_2} = A_{f^{\delta_1}_\gamma(i)} = A_*$ so $i \in A \Rightarrow A_{\alpha_i,\xi_2} = A_*$. Now $A_{\xi_2,\delta_2} \in D$, hence there is $i_* \in A_* \cap A_{\xi_1,\delta_2}$, and for it we get contradiction.)

Of course, the set of such i_*'s belongs to D.

(3): Obvious. □

Conclusion 2.2. *Let μ be a compact cardinal. If $\kappa < \mu$, D is an ultrafilter on κ, and $\mathbf{B}_i$ is a Boolean algebra for $i < \kappa$, then:*

(a) *if D is a regular ultrafilter then* $\mathrm{Depth}(\prod_{i<\kappa} \mathbf{B}_i/D) \le \mu + \prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D$;

(b) *this holds if $\kappa = \aleph_0$.*

Proof. If this fails, let $\lambda = (\mu + \prod_{i<\kappa} \mathrm{Depth}(\mathbf{B}_i)/D)^+$, so λ is a regular cardinal $> \mu$ and $(\forall \alpha < \lambda)[|\alpha^\kappa/D| < \lambda]$ (see below) and $\lambda \le \mathrm{Depth}(\prod_{i<\kappa} \mathbf{B}_i/D)$, so by 2.1 we get a contradiction. □

Remark 2.3. (1) Actually we prove that if μ is a compact cardinal, $\kappa < \mu \le \lambda = \mathrm{cf}(\lambda)$ and $\mathbf{c} : [\lambda]^2 \to \kappa$, then we can find an increasing sequence $\langle \alpha_\varepsilon : \varepsilon < \lambda \rangle$ of ordinals $< \lambda$ and $i, j < \kappa$ such that for every $\varepsilon < \zeta < \lambda$ for some γ satisfying $\alpha_\varepsilon < \gamma < \alpha_\zeta$ we have $\mathbf{c}\{\alpha_\varepsilon, \gamma\} = i, \mathbf{c}\{\gamma, \alpha_\zeta\} = j$ (the result follows using $\mathbf{c} : [\lambda]^2 \to D$).

(2) We use i_* rather than some $B \in D$ in order to help clarify what we need.

(3) Note that if D is a normal ultrafilter on $\kappa > \aleph_0$ and $\langle \lambda_i : i < \kappa \rangle$ is increasing continuous with limit $\lambda, i < \kappa \Rightarrow \prod_{j \le i} \lambda_j < \lambda_{i+1}$ then $\lambda = \prod_{i<\kappa} \lambda_i/D$ but $\lambda^\kappa/D > \lambda$. This is essentially the only reason for the undesirable extra assumption "D is regular" in 2.2.

Claim 2.4. (1) *In* 2.1 *instead of "$\mu \in (\kappa, \lambda]$ is a compact cardinal" it suffices to demand $\circledast_{\kappa^+, 2^\kappa, \lambda}$ where:*

$\circledast_{\sigma,\theta,\lambda}$ *if* $\mathbf{c} : [\lambda]^2 \to \theta$ *then we can find a stationary* $S \subseteq \lambda$ *and* $\gamma < \theta$ *such that for every* $u \in [S]^{<\sigma}$ *the set* $S_u = \{\beta < \lambda : (\forall \alpha \in u)[\mathbf{c}\{\alpha, \beta\} = \sigma]\}$ *is unbounded in* λ.

(2) *If* μ *is supercompact* $\sigma < \theta = \mathrm{cf}(\theta) < \mu < \lambda = \mathrm{cf}(\lambda)$ *and* $\mathbb{Q}$ = *adding* μ *Cohen subsets of* θ, *then in* $\mathbf{V}, \circledast_{\sigma,\mu,\lambda}$ *holds (even* $\circledast_{\sigma,\mu_1,\lambda}$ *if* $\mu_1^{<\sigma} < \lambda$ *in* $\mathbf{V}$*).*

In 2.4 we cannot get such results for $\kappa > \mu$, because for μ supercompact Laver indestructible and regular $\lambda > \kappa \geq \mu$ we can force $\{\delta < \lambda : \mathrm{cf}(\delta) > \mu\}$ to have a square preserving the supercompactness.

Claim 2.5. *Assume* $\lambda = \mathrm{cf}(\lambda) > \kappa^+$ *and* $\kappa = \mathrm{cf}(\kappa)$, *and there is a square on* $S = \{\delta < \lambda : \mathrm{cf}(\delta) \geq \kappa\}$ *(see* 2.6 *below). Then:*

(a) *there is a sequence* $\langle \mathbf{B}_i : i < \kappa \rangle$ *of Boolean algebras such that:*
  (α) $\mathrm{Depth}^+(\mathbf{B}_i) \leq \lambda$;
  (β) *for any uniform ultrafilter* D *on* κ, $\mathrm{Depth}^+(\prod_{i<\kappa} \mathbf{B}_i/D) > \lambda$;

(b) *the proof of* [Sh02, 5.1] *can be carried over.*

Where we have:

Definition 2.6. For $\lambda = \mathrm{cf}(\lambda) > \aleph_0, S \subseteq \lambda = \sup(S)$ we say that S has a square when we can find S^+ and $\langle C_\alpha : \alpha \in S^+ \rangle$ such that:

(a) $S \backslash S^+$ is not a stationary subset of λ;
(b) C_α is a closed subset of α;
(c) $\beta \in C_\alpha \Rightarrow \beta \in S \cap C_\beta = C_\alpha \cap \beta$;
(d) we stipulate $C_\alpha = \{\emptyset\}$ for $\alpha \notin S^+$.

Proof of 2.5. As in [Sh02, 5.1], using $\bar{C} = \langle C_\alpha : \alpha \in S^+ \rangle$ from 2.6 instead, $\langle \mathrm{acc}(C_\alpha) : \alpha < \lambda^+ \rangle$. The only change being that in the proof of [Sh02, Fact 5.3] in case 3, we have just $\mathrm{cf}(\alpha) \leq \kappa$ and we let $\langle \beta_\zeta : \zeta < \mathrm{cf}(\alpha)$ be increasing continuous with limit α. If $\mathrm{cf}(\alpha) < \kappa$, we can find $\varepsilon(*) < \kappa$ such that $\zeta_1 < \zeta_2 < \kappa \Rightarrow \beta_{\zeta_1} \in A_{\beta_{\zeta_2}, \varepsilon(*)}$ and let $A_{\alpha,\varepsilon} = \emptyset$ if $\varepsilon < \varepsilon(*)$ and $A_{\alpha,\varepsilon} = \cup\{A_{\beta_\zeta,\varepsilon} : \zeta < \mathrm{cf}(\kappa)\}$ if $\varepsilon \in [\varepsilon(*), \kappa)$. □

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