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Arch. Math. Logic 41, 207–213 (2002)
Digital Object Identifier (DOI):
10.1007/s001530000054

Mathematical Logic

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On the existence of large subsets of $[\lambda]^{<\kappa}$ which contain no unbounded non-stationary subsets*

Received: 10 December 1998 / Revised version: 2 February 1999 /
Published online: 27 March 2002 – © Springer-Verlag 2002

Abstract. Here we deal with some problems posed by Matet. The first section deals with the existence of stationary subsets of $[\lambda]^{<\kappa}$ with no unbounded subsets which are not stationary, where, of course, κ is regular uncountable $\leq \lambda$. In the second section we deal with the existence of such clubs. The proofs are easy but the result seems to be very surprising. Theorem 1.2 was proved some time ago by Baumgartner (see Theorem 2.3 of [Jo88]) and is presented here for the sake of completeness.

1. On stationary sets with no unbounded stationary subsets

The following quite answers the question of Matet indicated in the title of this section. Theorems 1.2 and 1.4 yield existence under reasonable cardinal-arithmetical assumptions. Theorem 1.5 yields consistency. Remark 1.3 recalls that under the assumption $\lambda = \lambda^{<\kappa}$ there is no such set.

1.0. Notation. $\mathscr{H}(\chi)$ is the family of sets x such that $\mathrm{TC}(x)$, the transitive closure of x, has cardinality $< \chi$, and $<^*_\chi$ is any well ordering of $\mathscr{H}(\chi)$.

1.1. Definition. *(1) For κ regular uncountable and $\kappa \leq \lambda$, let $\mathscr{E}_{\lambda,\kappa}$, the club filter of $[\lambda]^{<\kappa}$, be the filter generated by the clubs of $[\lambda]^{<\kappa}$, where a club of $[\lambda]^{<\kappa}$ (or an $\mathscr{E}_{\lambda,\kappa}$-club) is a set of the form $\mathscr{C}_{\mathscr{M}} =^{def} \{a \in [\lambda]^{<\kappa} : a = cl_{\mathscr{M}}(a) \wedge a \cap \kappa \in \kappa\}$ where $\mathscr{M}$ is a model with universe λ and countable vocabulary. Note that we could have required in the definition of $\mathscr{C}_{\mathscr{M}}$ that $\mathscr{M}$ restricted to a be an elementary submodel of $\mathscr{M}$, but it does not matter as we can expand $\mathscr{M}$ by Skolem functions. Note that, as a consequence, only the functions of $\mathscr{M}$ matter.*
(2) We call $S \subseteq [\lambda]^{<\kappa}$ stationary if it's complement $[\lambda]^{<\kappa} \setminus S$ does not belongs to $\mathscr{E}_{\lambda,\kappa}$.
(3) We call $S \subseteq [\lambda]^{<\kappa}$ unbounded if every member of $[\lambda]^{<\kappa}$ is contained in some member of S, so $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq)$ is exactly $Min\{|\mathscr{U}| : \mathscr{U} \subseteq [\lambda]^{<\kappa}$ is unbounded$\}$.
(4) In part (1) we can replace λ by any set A which includes κ.

We deal first with a special case.

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*Publication number 698 in author's list. Partially supported by the Israel Science Foundation.

1.2. Theorem. *if* $\lambda = \aleph_\omega$, $\kappa = (2^{\aleph_0})^+ < \lambda$ *and* $2^\lambda = \lambda^{\aleph_0}$ **then** *there is a stationary subset of* $[\lambda]^{<\kappa}$ *such that every unbounded subset of it is stationary.*

Proof. Let $\langle \mathscr{M}_\alpha : \alpha < 2^\lambda \rangle$ list all models with universe λ and countable vocabulary (for the sake of simplicity fix the vocabulary, having $\aleph_0$ many n-place function symbols for each n). Let $\langle a_\alpha : \alpha < 2^\lambda \rangle$ list without repetitions the countable subsets of λ.

Now we define S as the set of subsets Y of λ of cardinality continuum such that for every $\alpha < 2^\lambda$, if a_α is a subset of Y then Y is the universe of an elementary submodel of $\mathscr{M}_\alpha$. Clearly in ω_1 steps we can "catch our tail", so proving that S is stationary (in fact if $\langle N_\zeta : \zeta < \theta \rangle$ is an increasing continuous sequence of elementary submodels of $(\mathscr{H}(\chi), \in, <^*_\chi)$, to which $\langle \mathscr{M}_\alpha : \alpha < 2^\lambda \rangle$ and $\langle a_\alpha : \alpha < 2^\lambda \rangle$ belong, each N_ζ is of the cardinality of the continuum, θ a regular uncountable $\leq 2^{\aleph_0}$ and $\langle N_\epsilon : \epsilon \leq \zeta \rangle \in N_{\zeta+1}$ for $\zeta < \theta$ (note that $N_{1+\zeta} \cap \kappa \in \kappa$ follows in this case and for limit ζ in all cases), then $\bigcup_{\zeta<\theta} N_\zeta \cap \lambda \in S$. We denote by $\mathscr{E}_{\lambda,\kappa,\theta}$ the filter generated by the family $\left\{\{\bigcup_{\zeta<\theta} N_\zeta \cap \lambda : \langle N_\zeta : \zeta < \theta \rangle \text{ is an increasing sequence of elementary submodels of } (\mathscr{H}(\chi), \in, <^*_\chi) \text{ such that } x \in N_0,\ ||N_\zeta|| < \kappa \text{ and } \langle N_\epsilon : \epsilon < \zeta \rangle \in N_{\zeta+1} : \chi > \lambda \wedge x \in \mathscr{H}(\chi)\right\}$. This filter is normal, and the set S defined above belongs to $\mathscr{E}_{\lambda,\chi,\theta}$. (On this filter see [Sh 52, section 3]).

So S is stationary. Now suppose that S_1 is an unbounded subset of S which is not stationary, so it is disjoint to some club, hence for some $\alpha < 2^\lambda$, no member of S_1 is the universe of an elementary submodel of $\mathscr{M}_\alpha$. By the unboundedness assumption, for some member Y of S_1 a_α is a subset of Y, but, by the definition of S, Y is necessarily the universe of an elementary submodel of $\mathscr{M}_\alpha$, which contradicts the choice of α. □

1.3. Remarks. (1) See more in 2.4(2).

(2) Of course, it is known that if $\lambda = \lambda^{<\kappa}$ then there is no subset of $[\lambda]^{<\kappa}$ as claimed in 1.2.

[Why? let $\{a_\alpha : \alpha < \lambda\}$ be a listing of $[\lambda]^{<\kappa}$. Now, if S is a stationary subset of $[\lambda]^{<\kappa}$, for each $\alpha < \lambda$ choose a set $b_\alpha \in S$ which includes $a_\alpha \cup \{\alpha\}$ and $b_\alpha \notin \{b_\gamma : \gamma < \alpha\}$.

Now $S_1 =^{def} \{b_\alpha : \alpha < \lambda\}$ is an unbounded subset of $S \subseteq [\lambda]^{<\kappa}$ as $a_\alpha \subseteq b_\alpha$ and is not stationary, as the mapping $b_\alpha \mapsto \alpha$ is a one to one choice function (contradicting normality).]

1.4. Theorem. *Assume that λ is a strong limit singular of cofinality σ, κ is regular and $\sigma < \kappa < \lambda$,* **then** *there is a stationary subset S of $[\lambda]^{<\kappa}$ such that every unbounded subset of it is stationary.*

Proof. Let $\langle \mathscr{M}^*_\gamma : \gamma < \lambda \rangle$ list the models with universe an ordinal $< \lambda$ and countable vocabulary, allowing partial functions. Let $\langle \mu_\epsilon : \epsilon < \mathrm{cf}(\lambda) \rangle$ be a strictly increasing sequence of cardinals with limit λ. Let $\langle \mathscr{M}_\alpha : \alpha < 2^\lambda \rangle$ list all models with universe λ and countable vocabulary. For $\alpha < 2^\lambda$ and $\epsilon < \mathrm{cf}(\lambda)$ let $\gamma_{\alpha,\epsilon}$ be the unique ordinal γ such that $\mathscr{M}_\alpha$ restricted to μ_ϵ is $\mathscr{M}^*_\gamma$ and let a_α be $\{\gamma_{\alpha,\epsilon} : \epsilon < \mathrm{cf}(\lambda)\}$.

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Let S be the set of subsets Y of λ of cardinality $< \kappa$ such that $Y \cap \kappa \in \kappa$ and if $\gamma \in Y$ then Y is closed under the partial functions of $\mathscr{M}^*_\gamma$. Clearly $S \subseteq [\lambda]^{<\kappa}$ is closed unbounded.

It is not hard to see that if $Y \in S$ includes a_α then it is the universe of a sub-model of $\mathscr{M}_\alpha$ [remember that we are using the variant Definition 1.1(1) without elementaricity]. Hence every unbounded $S' \subseteq S$ is stationary and we are done. $\square$

1.5. Theorem. *Assume that* $\sigma < \kappa = \text{cf}(\kappa) < \mu^{<\mu} < \lambda$ *(the most interesting case is where* $\sigma = \text{cf}(\lambda)$ *) and that the axiom of [Sh 80] for* μ*-complete forcing notions satisfying a strong version of the* μ^+*-cc holds for* $\leq \lambda^{<\kappa}$ *dense subsets.* **Then** *any stationary subset of* $[\lambda]^{<\kappa}$ *has an unbounded subset which is not stationary.*

Proof. Let $S \subseteq [\lambda]^{<\kappa}$. We define a forcing notion $Q = Q_S$. A member p of Q is a function from a subset of S of cardinality $< \mu$ to $\{0, 1\}$ such that there is no increasing sequence of limit length of members of the set $p^{-1}(1)$ with union a member of $p^{-1}(1)$, moreover this holds even for the union of directed systems.

Let $p \leq q$ iff q extends p and $x \subseteq y \wedge y \in p^{-1}(1) \wedge x \in q^{-1}(1) \Rightarrow x \in p^{-1}(1)$. Clearly Q is μ-complete, in fact, the union of every increasing sequence of length $< \mu$ of members of Q is in Q. Also, it is easy to see that Q satisfies μ^+-chain condition. For this proof it suffices to use Δ-systems hence also the strong version of the μ^+-chain condition (of [Sh:80]) holds.

Let $\underset{\sim}{S}_1$ be $\bigcup\{p^{-1}(1) : p \in \underset{\sim}{G}_Q\}$, it is forced to be a subset of $[\lambda]^{<\kappa}$. Clearly $\underset{\sim}{S}_1$ is not stationary as no non trivial union of directed systems of members of it belongs to it. Also, it is unbounded because the density claim is obviously satisfied. Lastly we apply the axiom of [Sh:80] to this forcing notion and we are done. (Alternatively, define a forcing notion $Q' = Q'_S$, where a member p is a one to one choice function with domain a subset of S of cardinality $< \mu$ ordered by inclusion.) $\square$

2. On clubs containing no unbounded stationary subsets

Matet has further asked whether any club of $[\lambda]^{<\kappa}$ contains an unbounded non stationary subset. Below we answer this quite completely, we shall later deal with other variants and get similar results.

2.1. Theorem. *Suppose that* κ *is regular uncountable and* $\kappa \leq \lambda$.
If $\text{cf}(\lambda) \geq \kappa$ **then** *every club of* $[\lambda]^{<\kappa}$ *contains an unbounded non-stationary subset.*

Proof. Let $\mathscr{M}$ be a model with universe λ and countable vocabulary, so $\mathscr{C} = \mathscr{C}_\mathscr{M} =^{def} \{a \in [\lambda]^{<\kappa} : a = cl_\mathscr{M}(a) \wedge a \cap \kappa \in \kappa\}$ is a club of $[\lambda]^{<\kappa}$ and every club of $[\lambda]^{<\kappa}$ has this form, so it is enough to find an unbounded non-stationary subset of $\mathscr{C}_\mathscr{M}$.

We choose an increasing continuous sequence $\langle \mathscr{M}_\alpha : \alpha < \text{cf}(\lambda)\rangle$ of elementary sub-models of $\mathscr{M}$, each of cardinality $< \lambda$ and $\mathscr{M}_\alpha \cap \kappa \in (\kappa + 1)$.

Let $\mathscr{U}$ be the set of $a \in \mathscr{C}$ satisfying : for some $j < \text{cf}(\lambda)$ a is included in $\mathscr{M}_{j+2}$ and has a member in $\mathscr{M}_{j+2} \setminus \mathscr{M}_{j+1}$. We shall now see that $\mathscr{U}$ is unbounded (as a subset of $[\lambda]^{<\kappa}$): if $b \in [\lambda]^{<\kappa}$ then since $\text{cf}(\lambda) \geq \kappa$ clearly for some $j < \text{cf}(\lambda)$ we have $b \subseteq \mathscr{M}_j$, let $\alpha = Min(\mathscr{M}_{j+2} \setminus \mathscr{M}_{j+1})$ and we can find a member a of $\mathscr{C}_{\mathscr{M}_{j+2}}$ which includes $b \cup \{\alpha\}$. So it is enough to prove that $\mathscr{U}$ is not stationary, that is, to find a club disjoint to it. Such a club is the set $\mathscr{C}^*$ of all $a \in \mathscr{C}$ such that for every $j < \mathrm{cf}(\lambda)$ if $a \cap \mathscr{M}_{j+1} \backslash \mathscr{M}_j \neq \emptyset$ then $a \cap \mathscr{M}_{j+2} \backslash \mathscr{M}_{j+1} \neq \emptyset$. □

2.2. Theorem. *Suppose that κ is regular uncountable and $\lambda \geq \kappa$.*

Assume $\mathrm{cf}(\lambda) < \kappa$ and either λ is strong limit or at least for every $\mu < \lambda$ there is a family of $\leq \lambda$ clubs of $[\mu]^{<\kappa}$ such that any other club of $[\mu]^{<\kappa}$ contains one of them.

Then some club of $[\lambda]^{<\kappa}$ contains no unbounded non stationary subset of $[\lambda]^{<\kappa}$.

Proof. By the assumption we can find $\langle(\alpha_\beta, \mathscr{C}_\beta) : \beta < \lambda\rangle$ such that:

(a) α_β is an ordinal $\leq \kappa + \beta < \lambda$ but $\geq \kappa$
(b) $\mathscr{C}_\beta$ is a club of $[\alpha_\beta]^{<\kappa}$
(c) if $\alpha < \lambda$ and $\mathscr{C}$ is a club of $[\alpha]^{<\kappa}$ then for some $\beta < \lambda$ we have $\alpha_\beta = \alpha$ and $\mathscr{C}_\beta \subseteq \mathscr{C}$.

Let $\mathscr{C}^* =^{def} \{a \in [\lambda]^{<\kappa} : (\forall \beta \in a)[a \cap \alpha_\beta \in \mathscr{C}_\beta]\}$.

Clearly, $\mathscr{C}^*$ is a club of $[\lambda]^{<\kappa}$. Towards contradiction, assume that $\mathscr{U}$ is an unbounded subset of $\mathscr{C}^*$ and we shall prove that it is stationary. So let $\mathscr{N}$ be a model with universe λ and countable vocabulary, and we shall find $a \in \mathscr{U} \cap \mathscr{C}_{\mathscr{N}}$ (see the proof of 2.1). Recall that by the assumption of the theorem $\sigma =^{def} \mathrm{cf}(\lambda)$ is $< \kappa$ hence we can find an increasing sequence $\langle\gamma(\zeta) : \zeta < \sigma\rangle$ of ordinals $< \lambda$ with limit λ. Let $\mathscr{C}^\zeta$ be $\{a \in [\gamma(\zeta)]^{<\kappa} : cl_{\mathscr{N}}(a) \cap \gamma(\zeta) = a\}$, clearly it is a club of $[\gamma(\zeta)]^{<\kappa}$ hence for some $\beta(\zeta) < \lambda$ we have $\alpha_{\beta(\zeta)} = \gamma(\zeta)$ and $\mathscr{C}_{\beta(\zeta)} \subseteq \mathscr{C}^\zeta$.

Now every member of $\mathscr{C}^*$ which includes $\{\beta(\zeta) : \zeta < \sigma\}$ is necessarily the universe of a submodel of $\mathscr{N}$ (just think a minute). But $\mathscr{U}$ contains a member a which includes this set as $\mathscr{U}$ is an unbounded of $[\lambda]^{<\kappa}$, and a necessarily belongs to $\mathscr{C}_{\mathscr{N}}$ as $\mathscr{U}$ is a subset of $\mathscr{C}^*$ and a include $\{\beta(\zeta) : \zeta < \sigma\}$, so we are done. □

2.3. Theorem. *Assume $\aleph_0 < \mu = \mu^{<\mu} < \lambda$ and λ is a strong limit singular and $\mu + \mathrm{cf}(\lambda) < \kappa = \mathrm{cf}(\kappa) < \lambda < \chi$ and P is any κ-c.c. forcing (e.g., adding χ Cohen subsets to μ.)* **Then** *in V^P we have:*

(a) The cardinal arithmetic is the obvious one, and
(b) There is a club subset S of $[\lambda]^{<\kappa}$ with no unbounded non-stationary subset.

Proof. The point is that the condition in 2.2 continues to hold as any new club of $[\theta]^{<\kappa}$ contains an old one when $\kappa \leq \theta < \lambda$ as the forcing satisfies the $\kappa - c.c.$ □

2.4. Theorem. *(1) Assume that $\aleph_0 < \kappa = \mathrm{cf}(\kappa) \leq \lambda$ and $\lambda = \mathrm{cf}([\lambda]^{<\kappa}, \subseteq)$.* **Then** *every stationary subset of $[\lambda]^{<\kappa}$ contains an unbounded non-stationary subset.*
(2) Assume that $\sigma < \kappa = \mathrm{cf}(\kappa) \leq \lambda$ and $\mathrm{cf}([\lambda]^\sigma, \subseteq) = gen(\mathscr{E}^\sigma_{\lambda,\kappa})$ (see below). **Then** *some $\mathscr{E}^\sigma_{\lambda,\kappa}$-club $\mathscr{C}^*$ contains no unbounded non-stationary subset.*
(3) Assume that $\aleph_0 < \kappa = \mathrm{cf}(\kappa) \leq \lambda$ and $\mathscr{E}$ a filter on $[\lambda]^{<\kappa}$ extending $\mathscr{E}_{\lambda,\kappa}$ and $\mu = gen(\mathscr{E})$ and $\langle a_\zeta : \zeta < \mu\rangle$ is a sequence of subsets of λ each of cardinality $< \kappa$ and for every $\chi > \lambda$, $x \in \mathscr{H}(\chi)$ the set $\{N \cap \lambda : N \prec (\mathscr{H}(\chi), \in), x \in N, ||N|| < \kappa, N \cap \kappa \in \kappa, (\forall \zeta < \mu)[a_\zeta \subseteq N \Rightarrow \zeta \in N]\}$ belongs to $\mathscr{E}$. **Then** *some member $\mathscr{C}^*$ of $\mathscr{E}$ contains no unbounded subset which is $= \emptyset \bmod \mathscr{E}$.*

(4) Assume that $\aleph_0 < \kappa = \mathrm{cf}(\kappa) \leq \lambda$ *and* $\mathscr{E} \subseteq \mathscr{E}'$ *are filters on* $[\lambda]^{<\kappa}$ *extending* $\mathscr{E}_{\lambda,\kappa}$ *and* $\langle \mathscr{C}_\zeta : \zeta < \mu\rangle$ *list a subset of* $\mathscr{E}'$ *which generates a filter extending* $\mathscr{E}$ *and* $\langle a_\zeta : \zeta < \mu\rangle$ *is a sequence of subsets of* λ *each of cardinality* $< \kappa$ *and for every* $\chi > \lambda, x \in \mathscr{H}(\chi)$ *the set* $\{N \cap \lambda : N \prec (\mathscr{H}(\chi), \in), x \in N, ||N|| < \kappa, N \cap \kappa \in \kappa, (\forall \zeta < \mu)[a_\zeta \subseteq N \Rightarrow N \cap \lambda \in \mathscr{C}_\zeta]\}$ *belongs to* $\mathscr{E}'$.

Then *some member* $\mathscr{C}^*$ *of* $\mathscr{E}'$ *contains no unbounded subset which is* $= \emptyset \bmod \mathscr{E}$.

We shall prove 2.4 below.

2.5. Definition. *(1)* $\mathscr{E}^\sigma_{\lambda,\kappa}$ *is the filter on* $[\lambda]^{<\kappa}$ *generated by the* $\mathscr{E}^\sigma_{\lambda,\kappa}$*-clubs, where*
(2) An $\mathscr{E}^\sigma_{\lambda,\kappa}$*-club is, for some* $\chi > \lambda$ *and* $x \in \mathscr{H}(\chi)$*, the set*
$\mathscr{C}^\sigma_{\lambda,\kappa}[\chi, x] =^{def} \{N \cap \lambda : N \prec (\mathscr{H}(\chi), \in, <^*_\chi), ||N|| < \kappa, N \cap \kappa \in \kappa, (\forall X \in [N]^\sigma)[X \in N], x \in N\}$
(Actually, it is clear that we can assume, without loss of generality, that $\chi = (\lambda^\kappa)^+$*. For this definition to be meaningful, i.e., for the filter to be non trivial, we have to assume that* $(\forall \alpha < \kappa)(|\alpha|^\sigma < \kappa)$*).*
(3) For a filter D let $gen(D)$ *be the minimal cardinality of s subset which generates D, this is the same as the cofinality of D under inverse inclusion.*

We note the following easy monotonicity properties:

2.6. Observation. (1) If for some filter $\mathscr{E}$ extending $\mathscr{E}_{\lambda,\kappa}$ we have "there is a member of $\mathscr{E}$ (or just a member of $\mathscr{E}^+$) such that every unbounded subset belongs to $\mathscr{E}^+$", **then** there is a stationary subset of $[\lambda]^{<\kappa}$ such that every unbounded subset of it is stationary .
(2) Similarly we can replace $\mathscr{E}_{\lambda,\kappa}$ by any filter $\mathscr{E}' \supseteq \mathscr{E}$.

2.7. Remark. (1) So for $\sigma = \aleph_0$ we get that $\mathscr{E}^\sigma_{\lambda,\kappa} = \mathscr{E}_{\lambda,\kappa}$.
(2) Note that, of course, for every λ we have $2^\lambda \geq gen(\mathscr{E}_{\lambda,\kappa})$ and for λ strong limit of cofinality $< \kappa$ we have $gen(\mathscr{E}_{\lambda,\kappa}) = 2^\lambda$.

Proof of 2.4.

(1) As in 1.3(2).
(2) Let $\mu =^{def} \mathrm{cf}([\lambda]^\sigma, \subseteq)$ and $\langle \mathscr{C}_\zeta = \mathscr{C}^\sigma_{\lambda,\kappa}[\chi_\zeta, x_\zeta] : \zeta < \mu\rangle$ list a family which generates $\mathscr{E}^\sigma_{\lambda,\kappa}$, and let $\chi^* =^{def} \sup\{2^{\chi^+_\zeta} : \zeta < \mu\}$ and $\chi = 2^{2^{\chi^*}}$.

Let $\langle a_\zeta : \zeta < \mu\rangle$ list with no repetition a cofinal subset of $[\lambda]^\sigma$ of cardinality μ, lastly let $y \in \mathscr{H}(\chi)$ code $\sigma, \kappa, \lambda, \mu$ and the three sequences $\langle \chi_\zeta : \zeta < \mu\rangle$, $\langle x_\zeta : \zeta < \mu\rangle$, $\langle a_\zeta : \zeta < \mu\rangle$ and, pedantically $\langle <_{\chi_\zeta} : \zeta < \mu\rangle$ and let $\mathscr{C}^* =^{def} \mathscr{C}^\sigma_{\lambda,\kappa}[\chi, y]$.

Clearly $\mathscr{C}^*$ is a $\mathscr{E}^\sigma_{\lambda,\kappa}$-club, so to finish assume that $\mathscr{U}$ is an unbounded subset of $\mathscr{C}^*$ (that is, cofinal in $[\lambda]^{<\kappa}$) which is not stationary, hence there is $\zeta < \mu$ such that $\mathscr{C}_\zeta$ is disjoint to $\mathscr{U}$. As $\mathscr{U}$ is unbounded, there is $a \in \mathscr{U}$ such that $a_\zeta \subseteq a$. But $\mathscr{U} \subseteq \mathscr{C}^*$ hence $a \in \mathscr{C}^*$ so there is a model N as in the definition of $\mathscr{C}^*$ (see [2.5(2)) which witnesses $a \in \mathscr{C}^*$ so in particular $a =^{def} N \cap \lambda \in \mathscr{C}^*$. As $a_\zeta \subseteq a \subseteq N$ and $|a_\zeta| < \sigma$ (by the choice of $\langle a_\xi : \xi < \mu\rangle$), necessarily we have $a_\zeta \in N$ but $\langle a_\zeta : \zeta < \mu\rangle \in N$ is without repetitions, hence $\zeta \in N$, and since the sequences

$\langle \chi_\zeta : \zeta < \mu \rangle$, $\langle x_\zeta : \zeta < \mu \rangle$ belong to N clearly $\chi_\zeta, x_\zeta, <_{\chi_\zeta} \in N$ hence N witnesses that $N \cap \lambda \in \mathscr{C}_\zeta$ so $a \in \mathscr{C}_\zeta$ contradicting $a \in \mathscr{U}$ and the choice of ζ.
(3),(4) Proof similar to Part (2). □

The following gives somewhat more than "Theorem 2.2 applied for the club filter still holds under weaker version where $\{a \in [\lambda]^{<\kappa} : a \cap \kappa \in \kappa\}$ is not required to belong to the filter".

2.8. Definition. *(1) We say that $\mathfrak{C}$ is a club filter (λ, κ)-definor if it is a function such that:*

(a) Its domain is the family of models $\mathscr{M}$ with universe a subset of λ which includes κ and with countable vocabulary, $\mathfrak{C}_{\mathscr{M}}$ is a family of submodels of $\mathscr{M}$ of cardinality $< \kappa$.
(b) If $\mathscr{M}$ is a submodel of $\mathscr{N}$ then $\mathfrak{C}_{\mathscr{M}} \subseteq \mathfrak{C}_{\mathscr{N}}$.
(c) If $\mathscr{M}, \mathscr{N}$ are models as above and $\mathscr{M}$ is interpretable in $\mathscr{N}$ with the parameters from A then $a \in \mathfrak{C}_{\mathscr{N}} \wedge A \subseteq a$ implies $a \cap \mathscr{M} \in \mathfrak{C}_{\mathscr{M}}$.
(d) If $\mathscr{M}$ is as above, and $\mathscr{M}_1$ is a submodel of $\mathscr{M}$ which includes κ then $\mathfrak{C}_{\mathscr{M}}$ has a member a which is contained in $\mathscr{M}_1$.

(2) For $\mathfrak{C}$ as above and a subset Y of λ which includes κ, the filter $\mathscr{E}[Y] = \mathscr{E}[Y, \mathfrak{C}]$ is the family of subsets of $[Y]^{<\kappa}$ which include some $\mathfrak{C}_{\mathscr{M}}$ with $\mathscr{M}$ as above with universe Y; any such set is called an $\mathscr{E}[Y]$-club of $[Y]^{<\kappa}$. If $Y = \lambda$ we may write just $\mathscr{E}$. If $\mathscr{M}$ is a model with countable vocabulary and universe Y, $\kappa \subseteq Y \subseteq \lambda$, we may write $\mathscr{E}[\mathscr{M}]$ for $\mathscr{E}[Y]$.
(3) If for every such Y we have $[Y]^{<\sigma} \in \mathscr{E}[Y]$ then we call $\mathscr{E}$ a $(\lambda, \kappa, \sigma)$-definor and may consider only $[Y]^{<\sigma}$.
(4) We say $\mathscr{U} \subseteq [Y]^{<\kappa}$ is $\mathscr{E}$-unbounded if $\{a \in [Y]^{<\kappa} : (\exists b \in \mathscr{U})[a \subseteq b]\} \in \mathscr{E}[Y]$.
(5) We say $\mathscr{U} \subseteq [Y]^{<\kappa}$ is $\mathscr{E}$- stationary if $\mathscr{U} \neq \emptyset \bmod \mathscr{E}[Y]$.

2.9. Theorem. *Suppose that $\mathfrak{C}$ is a club filter (λ, κ)-definor. We shall abbreviate $\mathscr{E}[\lambda, \mathfrak{C}]$ by $\mathscr{E}$.*
(1) Assume that: every $\mathscr{M}$ as above, can be represented as a union of an increasing chain of submodels $\langle \mathscr{M}_\zeta : \zeta < \delta \rangle$ with δ a limit ordinal, such that $\{a \in [\lambda]^{<\kappa} : (\exists \zeta < \delta)[a \subseteq \mathscr{M}_\zeta]\} \in \mathscr{E}$ and $(\forall \zeta < \delta)$ $(\forall a \in \mathfrak{C}_{\mathscr{M}_\zeta})$ $(\exists b \in \mathfrak{C}_{\mathscr{M}_{\zeta+1}})$ $(a \subseteq b \wedge b \nsubseteq \mathscr{M}_\zeta)$.
Then *every $\mathscr{E}$-club of $[\lambda]^{<\kappa}$ contains an $\mathscr{E}$-unbounded non $\mathscr{E}$-stationary subset.*
(2) Assume $\mathrm{cf}(\lambda) < \kappa$ and λ is strong limit (or at least for every $\mu < \lambda$ we have there is a family of $\leq \lambda$ $\mathscr{E}[\mu, \mathfrak{C}]$-clubs of $[\mu]^{<\kappa}$ such that any other $\mathscr{E}[\mu, \mathfrak{C}]$-club of $[\mu]^{<\kappa}$ contains one of them, in other words $\lambda \geq \mathrm{cf}(\mathscr{E}[\mu, \mathfrak{C}], \supseteq)$), and for any subset Y of λ of cardinality $\leq \mathrm{cf}(\lambda)$ the family $\{a \in [\lambda]^{<\kappa} : Y \subseteq a\}$ belongs to the filter $\mathscr{E}$, and if $\langle \mu_\zeta : \zeta < \mathrm{cf}(\lambda) \rangle$ is an increasing sequence of ordinals $> \kappa$ with limit λ then for every $\mathscr{E}$-club $\mathfrak{C}$ of $[\lambda]^{<\kappa}$ there is a sequence $\langle \mathscr{C}_\zeta : \zeta < \mathrm{cf}(\lambda) \rangle$ with $\mathscr{C}_\zeta$ being a $\mathscr{E}[\mu, \mathfrak{C}]$-club such that $(\forall a \in [\lambda]^{<\kappa})(\forall \zeta < \mathrm{cf}(\lambda))[a \cap \mu_\zeta \in \mathscr{C}_\zeta]$ implies $a \in \mathscr{C}$.
Then *some $\mathscr{E}$-club of $[\lambda]^{<\kappa}$ contains no $\mathscr{E}$-unbounded non $\mathscr{E}$-stationary subset of $[\lambda]^{<\kappa}$.*

Proof. Similar to the previous ones, e.g.
(1) Let $\mathscr{C}$ be a $\mathscr{E}$-club, say $\mathscr{C} = \mathscr{E}_{\mathscr{M}}$, and let $\langle \mathscr{M}_\zeta : \zeta < \delta \rangle$ be as on (1)'s assumption. Now we choose
$\mathscr{U} =^{def} \{N : \text{ for some } \zeta < \delta, N \subseteq \mathscr{M}_{\zeta+2} \text{ and } N \setminus \mathscr{M}_{\zeta+1} \neq \emptyset\}$.
Now □
$\mathscr{U} \subseteq \mathfrak{C}_{\mathscr{M}}$ as $\mathfrak{C}_{\mathscr{M}} \subseteq \cup\{\mathfrak{C}_{\mathscr{M}_\zeta} : \zeta < \delta\}$.
$\mathscr{U}$ is $\mathscr{E}$-unbounded by the last clause of the assumption. $\mathscr{U}$ is not $\mathscr{E}$-stationary as we can use $(\mathscr{M}, F)$, Dom(F)=$\mathscr{M}$, $[x \in \mathscr{M}_\zeta \setminus \cup \{\mathscr{M}_\varepsilon : \varepsilon < \zeta\} \Rightarrow F(x) =$ Min$(\mathscr{M}_{\zeta+1} \setminus \mathscr{M}_\zeta)$. (so (c), (d) of 2.8 (1) were not used, in (2) clause (d) is not used).

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