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Lawless Order

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Abstract. R. Baer asked whether the group operation of every (totally) ordered group can be redefined, keeping the same ordered set, so that the resulting structure is an Abelian ordered group. The answer is no. We construct an ordered set $(G, \leqslant)$ which carries an ordered group $(G, \cdot, \leqslant)$ but which is *lawless* in the following sense. If $(G, *, \leqslant)$ is an ordered group on the same carrier $(G, \leqslant)$, then the group $(G, *)$ satisfies no nontrivial equational law.

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0. Introduction

An *ordered group* is a triple $(G, \cdot, \leqslant)$ such that $(G, \cdot)$ is a group, $(G, \leqslant)$ is a totally ordered set, and for all $x, y, z, w \in G$, $x \leqslant y$ implies $z \cdot x \cdot w \leqslant z \cdot y \cdot w$. In this case we will say the order $(G, \leqslant)$ *carries* the group $(G, \cdot)$, and call $(G, \leqslant)$ a *carrier* if for some group operation, $(G, \leqslant)$ carries $(G, \cdot)$. From a group theoretic standpoint, much of the interest in ordered groups centers around the question of what influence a group has on its carriers. For example, which groups have a carrier? Or, given a group, what are its possible carriers? In this paper, we study the reverse question. What influence does a carrier have on the groups it carries?

To begin with an example in which the influence of the carrier is very strong, consider the naturally ordered set of integers $(Z, \leqslant)$. This carries $(Z, +)$, and it is easily seen that any group carried by $(Z, \leqslant)$ must be cyclic and, hence, Abelian. Another example is the

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lexicographically ordered product $Z \overleftarrow{X} Z$ which carries the product of two infinite cyclic groups, and is easily seen to carry only Abelian groups. More complicated examples which, again, carry only Abelian groups are the (dense) rigid homogeneous chains of [1]. These carriers may be called *Abelian.*

A more liberal carrier is the naturally ordered set of rational numbers $(Q, \leqslant)$. This carries the obvious Abelian group $(Q, +)$. But it also carries every countable dense ordered group, including every noncyclic ordered free group. (It is well known that every free group has a carrier (see [2]). And it is easy to prove that the lack of center in a noncyclic free group forces its carrier to be dense.) Thus, although $(Q, \leqslant)$ carries Abelian groups, it is not Abelian.

At this point we note that in a certain sense the commutative law $xy = yx$ is the strongest law a nontrivial ordered group can satisfy. For if the law '$u = v$' is satisfied by some nontrivial ordered group, (where u and v are reduced words in a free group) then it is satisfied by an infinite cyclic subgroup of that group, hence also by all Abelian groups. Thus, the Abelian carriers described earlier belong to every variety of groups (defined by equational group laws) which contain a nontrivial ordered group, in the sense that all of the groups carried by these carriers belong to every such variety. The carrier $(Q, \leqslant)$, while belonging to no variety, also avoids no variety since it carries an Abelian group.

If none of the groups carried by a carrier $(G, \leqslant)$ belong to a certain variety V we may say that $(G, \leqslant)$ is *absolutely non-V*. Whether there are any such carriers is implicit in an informal question of R. Baer (1961) (see [3], p. 248), which in our terminology can be expressed as follows:

Does there exist an absolutely nonabelian carrier?

The answer is yes, as we will show in this paper. In fact, we construct a carrier whose carried groups cannot obey any equational group law at all. Thus, our carrier will be not only absolutely nonabelian; it will be absolutely lawless.

To begin, we observe that any absolutely nonabelian carrier must be uncountable. Any carrier is homogeneous – the group translations of any carried group act transitively – and so every carrier is isomorphic (as an ordered set) to a lexicographically ordered product $Z^\alpha \overleftarrow{X} D$, where D is a singleton or a dense totally ordered set and Z^α is a lexicographically ordered restricted product of copies of Z ([4]; see [5], Chapter 8). If the carrier is countable, so is D, and then D is a singleton or is isomorphic to $(Q, \leqslant)$. In either case, the carrier carries an Abelian group.

We are looking for a carrier which enforces a rather strict control on the groups it carries (to prevent them from being Abelian, for example). It is instructive to see how $(Z, \leqslant)$ controls the groups it carries. Let $+$ denote the usual addition and let $*$ denote some group operation on Z such that $(Z, *, \leqslant)$ is an ordered group. For each $n \in Z$, the mapping $t_n : x \mapsto x * n$ is an order-preserving permutation of $(Z, \leqslant)$. Moreover, the mapping $n \mapsto t_n$ embeds $(Z, *, \leqslant)$ in the group $A(Z, \leqslant)$ of all order-preserving permutations of $(Z, \leqslant)$ in such a way that $n \leqslant m$ implies $xt_n \leqslant xt_m$ for all $x \in Z$ (xt denotes the image of x under the mapping t). But every $f \in A(Z, \leqslant)$ has the form $xf = x + p$ for some

$p \in Z$. Hence, $A(Z, \leqslant)$ is isomorphic to $(Z, +)$ and so $(Z, *)$, isomorphic to a subgroup of $A(Z, \leqslant)$, must be cyclic as well. The key here is that every order-preserving permutation of $(Z, \leqslant)$ is definable in $(Z, +)$.

In general we cannot expect such tight control. For example, if $(G, \cdot, \leqslant)$ is any non-abelian ordered group, and if $g \in G$, the mapping $\phi : x \leftrightarrow g^{-1} \cdot x \cdot g$ is an order-preserving permutation of $(G, \leqslant)$ (generally nontrivial), and since $g\phi = g$, we may splice ϕ to the identity map at g, producing ψ such that

$$x\psi = \begin{cases} x\phi & \text{if } x \geqslant g \\ x & \text{if } x < g \end{cases}.$$

Then $\psi \in A(G, \leqslant)$ but ψ is only piecewise definable in $(G, \cdot)$.

The general idea behind our construction is the following. We construct a certain large ordered group $(G, \cdot, \leqslant)$ such that $(G, \cdot)$ is a free group and every order-preserving permutation of $(G, \leqslant)$ is piecewise definable in $(G, \cdot)$ (precise definition follows in next paragraph). The lawless carrier we seek is then $(G, \leqslant)$ because if $(G, *, \leqslant)$ is any ordered group on the same carrier, the mapping $g \leftrightarrow t_g$, where $xt_g = x * g$, embeds $(G, *)$ in $A(G, \leqslant)$. But knowing that the members of $A(G, \leqslant)$ are piecewise definable in the free group $(G, \cdot)$ is enough to ensure (although with some difficulty) that no nontrivial group laws are satisfied by any large subgroup.

1. A Criterion for Lawlessness

Let $(G, \cdot, \leqslant)$ be an ordered group. We will write $f \cdot g = fg$. It will also be convenient to consider the Dedekind completion $(\bar{G}, \leqslant)$ of the ordered set $(G, \leqslant)$. We note that every $\phi \in A(G, \leqslant)$ has a unique extension to $\bar{\phi} \in A(\bar{G}, \leqslant)$, and there is no need to distinguish ϕ and $\bar{\phi}$ notationally.

DEFINITION. If $(G, \cdot, \leqslant)$ is an ordered group, I an interval of $(\bar{G}, \leqslant)$, and $\phi \in A(G, \leqslant)$, ϕ is *definable on* I iff there exist $a, b \in G$ such that for all $x \in I \cap G$, $x\phi = axb$; and ϕ is *piecewise definable* iff every nontrivial open interval I of $(\bar{G}, \leqslant)$ contains a nontrivial open interval I' of $(\bar{G}, \leqslant)$ such that ϕ is definable on I'.

THEOREM 1.1. *If $(G, \cdot, \leqslant)$ is an uncountable ordered group, and $(G, \cdot)$ is a free group on generators $Y \subseteq G$, and Y contains a countable subset which is dense in $(G, \leqslant)$, and every $\phi \in A(G, \leqslant)$ is piecewise definable then $(G, \leqslant)$ is lawless; that is, if $(G, *, \leqslant)$ is an ordered group with carrier $(G, \leqslant)$, then $(G, *)$ belongs to no proper variety of groups.*

In the second part of this paper, we construct an ordered group satisfying the hypotheses of the theorem. Before starting the proof of the theorem, it may help the reader understand the argument if we sketch the proof under the (false) assumption that every permutation of $(G, \leqslant)$ induced by a translation in $(G, *)$ is actually definable (via $(G, \cdot)$) on the entire interval $(G, \leqslant)$ – not just piecewise definable. For each $g \in G$, choose $a_g, b_g \in G$ such that for all $x \in G$, $x * g = a_g x b_g$. Let $Y(g)$ denote the unique finite set of free generators from Y which occur in the reduced expression for g as a member of the free group $(G, \cdot)$. Let us now show, for example, that $(G, *)$ is not Abelian. Since G is uncountable, there are $f, g \in G$ such that either (i) $Y(a_f) \not\subseteq Y(a_g)$ and $Y(a_g) \not\subseteq Y(a_f)$, or (ii) $Y(b_f) \not\subseteq Y(b_g)$ and $Y(b_g) \not\subseteq Y(b_f)$. Then either $a_g^{-1}a_f^{-1}a_g a_f \neq e$ (in case (i)) or $b_f b_g b_f^{-1} b_g^{-1} \neq e$ (in case (ii)). Choose

$$c \in Y \backslash (Y(a_f) \cup Y(b_f) \cup Y(a_g) \cup Y(b_g)).$$

Then

$$c * f * g * f^{-1} * g^{-1} = a_g^{-1}a_f^{-1}a_g a_f c b_f b_g b_f^{-1} b_g^{-1} \neq c$$

in either case. Therefore, $f * g * f^{-1} * g^{-1} \neq e$, so $(G, *)$ is not Abelian.

To prove the theorem we must struggle with the problems posed by the less generous hypothesis of piecewise definability. If we are given some $w(x_1, x_2, \ldots, x_p) = w(\bar{x})$, a nontrivial reduced group word in variables $x_1, x_2, \ldots, x_p$, we wish to find $g_1, g_2, \ldots, g_p \in G$ such that the substitution $g_i \leftrightarrow x_i$ results in $w(\bar{g}) \neq e$ in $(G, *)$. Since $(G, *)$ is embedded in $A(G, \leqslant)$, we will compute $w(\bar{g})$ in $A(G, \leqslant)$ as successive applications of the piecewise defineable maps which are the right translations in $(G, *)$ determined by the various g_i. Because we want to refer the computation back to $(G, \cdot)$, we will need to find a sequence of intervals $K_1, K_2, \ldots, K_k$ of $(G, \leqslant)$ such that among the substitution we are interested in, each g_i (or g_i^{-1}, as appropriate) is defineable on each K_j and maps K_j into K_{j+1}. For example, consider again the word $w(\bar{x}) = x_1 x_2 x_1^{-1} x_2^{-1}$. We need intervals K_1, K_2, K_3, K_4 such that for some large subset $F \subseteq G$, and for each $g \in F$, right translation by g (in $(G, *)$) is definable on K_1 and $K_1 * g \subseteq K_2$; right translation by g is definable on K_2 and $K_2 * g \subseteq K_3$; right translation by g^{-1} is definable on K_3, and $K_3 * g^{-1} \subseteq K_4$; and right translation by g^{-1} is definable on K_4. Then for any substitutions $g \leftrightarrow x_1, f \leftrightarrow x_2$, with $f, g \in F$, we can compute $K_1 * g * f * g^{-1} * f^{-1}$ in $(G, \cdot)$. Indeed, if $x * g = a_1 x b_1$ for $x \in K_1$, $x * f = a_2 x b_2$ for $x \in K_2$, $x * g^{-1} = a_3 x b_3$ for $x \in K_3$, and $x * f^{-1} = a_4 x b_4$ for $x \in K_4$, then for $x \in K_1, x * g * f * g^{-1} * f^{-1} = a_4 a_3 a_2 a_1 x b_1 b_2 b_3 b_4$, these products computed in $(G, \cdot)$. Another difficulty should now be apparent. We can choose, f, g so that either a_1 and a_2 are very different, or b_1 and b_2 are very different. But we have no independent control over a_3 and a_4, nor b_3 and b_4, since for example, a_1 and a_3 are both associated with the same g.

Our first goal is to produce a rather long (relative to $w(\bar{x})$) sequence of intervals I_i such that for a large subset $F \subseteq G$, if $f \in F$ then f is definable on I_i and $I_i * f \subseteq I_{i+1}$. We will worry later about f^{-1}.

LEMMA 1.2. *Suppose the hypotheses of Theorem* 1 *are satisfied. Let n be a positive integer, and let F be an uncountable subset of* $A(G, \leqslant)$. *Then there is an uncountable subset H of F, and nontrivial intervals* $I_0, I_1, \ldots, I_n$ *of* $(G, \leqslant)$ *such that for each* $0 \leqslant i \leqslant n$, *and each* $h \in H$, *h is definable on* I_i, *and for* $i < n$, $I_i h \subseteq I_{i+1}$.

Proof. Let $f \in F$, and let X be a countable dense subset of $(G, \leqslant)$. It will suffice to show that there are intervals I_i with end points in X such that f is definable on I_i and $I_i f \subseteq I_{i+1}$. For then, as there are only countably many arrays $I_0, \ldots, I_n$ of such intervals, the same array works for some uncountable subset H of F. First, the definability. Let

J_0 be any nontrivial interval such that f is definable on J_0. Proceeding by induction, let J_{i+1} be any nontrivial subinterval of $J_i f$ such that f is definable on J_{i+1}. Next, the containment. Let I_n be any nontrivial subinterval of J_n with end points in X. Proceeding by 'reverse induction', let I_{i-1} be any nontrivial subinterval of $I_i f^{-1}$ with end points in X. □

Now we want to compute $w(\bar{g})$ by traveling among subintervals of I_i for some suitably long sequence $I_0, \ldots, I_n$ satisfying the conclusion of Lemma 1.2. We will need to be able to start at a more or less arbitrary I_m (m not too close to 0 or n), and we will step up or down in a manner determined by the form of $w(\bar{x})$. Specifically, we can write $w(x_1, \ldots, x_p) = z_0^{\epsilon(0)} z_1^{\epsilon(1)} \ldots z_q^{\epsilon(q-1)}$ where each $z_j \in \{x_1, \ldots, x_p\}$ and $\epsilon(j) \in \{-1, 1\}$. The sequence $(\epsilon(0), \ldots, \epsilon(q-1))$ determines the direction of travel. If we begin in $K_0 \subseteq I_m$ then we want (the substitution for) z_0 to map: $K_0 * z_0^{\epsilon(0)} \subseteq I_{m+1}$ if $\epsilon(0) = 1$ and $K_0 * z_0^{\epsilon(0)} \subseteq I_{m-1}$ if $\epsilon(0) = -1$. Generally, if we have arrived in $K_j \subseteq I_m$, then we want $K_j * z_j^{\epsilon(j)} \subseteq I_{m'+\epsilon(j)}$. Thus, if $\sigma(j) = \Sigma_{i<j}\epsilon(i)$ then after j steps we will be in $I_{m+\sigma(j)}$.

LEMMA 1.3. *Suppose the conclusion of Lemma* 1.2 *holds for* I_i, $0 \leqslant i \leqslant n$ *and* H. *Let* $(\epsilon(0), \ldots, \epsilon(q-1))$ *be a sequence such that* $\epsilon(j) \in \{-1, 1\}$ *for each* j, *let* $\sigma(0) = 0$ *and* $\sigma(j) = \Sigma_{i<j}\epsilon(i)$. *Further assume* $\sigma(q) = 0$. *Let* m *be an integer such that* $0 \leqslant m + \sigma(j) \leqslant n$ *for all* j, $0 \leqslant j \leqslant q$. *Then there is an uncountable subset* S *of* H *and there are nontrivial intervals* K_j, $0 \leqslant j \leqslant q$ *such that* $K_j \subseteq I_0 f^{m+\sigma(j)}$, *and for every* $f \in S$, $f^{\epsilon(j)}$ *is defineable on* K_j, *and* $K_j f^{\epsilon(j)} \subseteq K_{j+1}$ *if* $j < q$.

Proof. As before, it suffices to show that for each $f \in H$ there is a sequence of K_j's with end points in X satisfying the desired conclusion. Let K_q be any nontrivial subinterval of $I_0 f^m$ with end points in X. If we have found $K_q, K_{q-1}, \ldots, K_{q-t}$, satisfying the conclusion then let K_{q-t-1} be any nontrivial subinterval of $K_{q-t} f^{-\epsilon(q-t-1)}$ with end points in X. Note that whenever f is definable on I as $xf = axb$, then f^{-1} is definable on I_f as $yf^{-1} = a^{-1}yb^{-1}$. We now have $K_{q-t-1} \subseteq I_{m+\sigma(q-t-1)}$ and $K_{q-t-1} f^{\epsilon(q-t-1)} \subseteq K_{q-t}$. Also, f is definable on K_{q-t-1}. Moreover, if $\epsilon(q-t-1) = -1$, then $K_{q-t-1} \subseteq K_{q-t} f \subseteq I_{m+\sigma(q-t)} f$, and so f^{-1} is definable on K_{q-t-1}. □

We now prove Theorem 1.1. We wish to show that $(G, *)$ satisfies no law of the form $w(\bar{x}) = e$. We consider the form $w = z_0^{\epsilon(0)} \ldots z_{q-1}^{\epsilon(q-1)}$, $z_j \in \{x_1, \ldots, x_p\}$, $\epsilon(j) \in \{-1, 1\}$, and let $\sigma(0) = 0$ and $\sigma(j) = \Sigma_{i<j}\epsilon(i)$. We may assume $\sigma(q) = 0$, for otherwise any substitution of a nontrivial $g \in G$ for each of the x_i shows that $w(g, g, \ldots, g) = g^{\sigma(q)} \neq e$ since $(G, *)$ is torsion free. We may also assume $z_j \neq z_{j+1}$, $0 \leqslant j < q-1$, for we can replace each x_i by a product $x_{i1}x_{i2}$ of two independent variables (so that all x_{ik} are different) producing w', and if a group satisfies $w = e$ then it satisfies $w' = e$ as well.

Since $\sigma(q) = 0$, $q = 2s$ for some integer s. Let $n = (s+1)^2$. The right translations $x \to x * g$ form an uncountable subset F of $A(G, \leqslant)$, and so there are intervals I_i, $0 \leqslant i \leqslant n$, and an uncountable subset H of F satisfying the conclusion of Lemma 1.2. Fix a dense set Y of free generators of $(G, \cdot)$. For each $h \in H$ and each $0 \leqslant i \leqslant n$, h is definable on I_i, and so for all $x \in I_i$, $x * h = a_i^h x b_i^h$ for certain a_i^h, $b_i^h \in G$. For each

$h \in H$, let $Y(h)$ be a finite subset of Y such that for all i, $a_i^h, b_i^h \in \langle Y(h)\rangle$ = the subgroup of $(G, \cdot)$ generated by $Y(h)$. Applying the Δ-lemma ([6]; p. 49) and passing to an uncountable subset of H we may assume all the $Y(h)$ have the same cardinality and there is a finite set Δ such that for $f \neq g$, $Y(f) \cap Y(g) = \Delta$. Again passing to an uncountable subset, we may assume if for some f and i, $a_i^f \in \langle\Delta\rangle$ then for all g, $a_i^g = a_i^f$, and likewise if for some f and i, $b_i^f \in \langle\Delta\rangle$ then for all g, $b_i^g = b_i^f$.

We note that translations $x * g$ and $x * f$ by different g, f must differ at every point. Hence, no two members of H agree at any point. In particular, if $f \neq g$ then for each i either $a_i^f \neq a_i^g$ or $b_i^f \neq b_i^g$ (otherwise f and g agree on I_i). A consequence of our choices in the previous paragraph is, therefore, that if $b_i^f \in \langle\Delta\rangle$ then $a_i^f \notin \langle\Delta\rangle$, and vice-versa. This leads to the consideration of two cases.

Case 1. Suppose that for some integer r, $0 \leqslant r < r + s \leqslant n$, and if $r \leqslant i \leqslant r + s$ then $b_i^f \notin \langle\Delta\rangle$ for all $f \in H$. That is, there is some stretch of $s + 1$ integers for which all the b's are outside $\langle\Delta\rangle$. We are going to compute $w = z_0^{\epsilon(0)} \dots z_{q-1}^{\epsilon(q-1)} = z_0^{\epsilon(0)} \dots z_{2s-1}^{\epsilon(2s-1)}$ beginning at the point $m = r + s - \max \sigma(j)$. Note that since $\sigma(0) = 0 = \sigma(q)$, and $\sigma(j) - \sigma(j+1) = \pm 1$, $\max \sigma(j) - \min \sigma(j) \leqslant s$. Hence, $m + \min \sigma(j) \geqslant r$ and, therefore, $r \leqslant m + \sigma(j) \leqslant r + s$ for all j. The hypotheses of Lemma 1.3 are satisfied. Therefore, S and K_j of the conclusion of Lemma 1.3 exist. We now substitute $g_i \leftrightarrow x_i$ where $\{g_1, \dots, g_p\}$ are different members of S. Then $w(\bar{g}) = f_0^{\epsilon(0)} \dots f_{q-1}^{\epsilon(q-1)}$ where $f_j \in \{g_1, \dots, g_p\}$ and $f_j \neq f_{j+1}$, $0 \leqslant j < q - 1$. Observe that $f_j^{\epsilon(j)}$ is definable on $K_j \subseteq I_{m+\sigma(j)}$ and since $r \leqslant m + \sigma(j) \leqslant r + s$, if $x * f_j^{\epsilon(j)} = a_{(j)} x b_{(j)}$ for all $x \in K_j$, then $b_{(j)} \notin \langle\Delta\rangle$. Choose $c \in (K_0 \cap Y) \backslash \cup Y(g_i)$. Then $c * w(\bar{g}) = a_{(q-1)} \dots a_{(0)} c b_{(0)} \dots b_{(q-1)}$. But $f_{j-1} \neq f_j \neq f_{j+1}$, and so $Y(f_{j-1}) \cap Y(f_j) = \Delta = Y(f_j) \cap Y(f_{j+1})$. Moreover, $b_{(j)} \in \langle Y(f_j)\rangle \backslash \langle\Delta\rangle$. Therefore, some member of $Y(f_j)$ appears in the reduced form of $b_{(j)}$ but not in that of $b_{(j-1)}$ nor that of $b_{(j+1)}$. Since this is true for each j (with appropriate modification at $j = 0$ and $j = q - 1$), $b_{(0)} \dots b_{(q-1)} \neq e$. It follows that $c * w(\bar{g}) \neq c$. The substitutions were taken from an isomorphic copy of $(G, *)$ in $A(G, \leqslant)$. Hence, $(G, *)$ does not satisfy $w(\bar{x}) = e$ in this case.

Case 2. For every integer r with $0 \leqslant r < r + s \leqslant n$ there exists $r \leqslant i \leqslant r + s$ and $f \in H$ such that $b_i^f \in \langle\Delta\rangle$. It follows that for all $f \in H$, $a_i^f \notin \langle\Delta\rangle$. Consider an uncountable set H' consisting of elements g such that $g = f_{g,0} f_{g,1} \dots f_{g,s}$ with $f_{g,j} \in H$ and all $f_{g,j}$ different.

Let $I_i' = I_{is+i}$, $0 \leqslant i \leqslant s$. Then for $g \in H'$, g is definable on I_i' and $I_i' g \subseteq I_{i+1}'$. For $x \in I_i'$, $x * g = c_i^g x d_i^g$ where $d_i^g = b^{(is+i)} b^{(is+i+1)} \dots b^{(is+i+s)}$ is a product of b_k^f corresponding to various $f = f_{g,j}$ on intervals I_{is+i+j}, and likewise, c_i^g is a product of a_k^f's. By assumption, for some k, $is + i \leqslant k \leqslant is + i + s$, $a_k^f \notin \langle\Delta\rangle$. It follows that $c_i^g \notin \langle\Delta\rangle$. We now have the hypotheses of Case 1 for I_i', H', and $r = 0$, but with $c_i^g \notin \langle\Delta\rangle$ instead of $b_i^g \notin \langle\Delta\rangle$. The $Y(g)$'s, $g \in H'$ form a Δ-system with the same Δ as before. The proof can now be completed just as in Case 1, except that one will have to deal with the left multipliers c_i^g instead of the right multipliers b_i^f.

2. The Construction

In this section we construct an ordered group $(G, \cdot, \leqslant)$ satisfying the hypotheses of Theorem 1.1. Because the construction is rather complicated, we begin with a general outline. First, we construct a certain ordered group $(H, \cdot, \leqslant)$ with $(H, \cdot)$ a product of free groups and $(H, \leqslant)$ a dense ordered subset of the real line $(\mathbf{R}, \leqslant)$. We then construct $(G, \cdot, \leqslant)$ as a certain dense ordered subgroup of $(H, \cdot, \leqslant)$. The automorphisms of $(G, \leqslant)$ will arise, therefore, as restrictions of automorphisms of $(\mathbf{R}, \leqslant)$. The automorphisms of $(\mathbf{R}, \leqslant)$ are listed $\{\phi_\alpha : 0 \leqslant \alpha < 2^{\aleph_0}, \alpha \text{ even}\}$. We then construct G as a union of subgroups G_α of $(H, \cdot)$ with $|G_\alpha| < 2^{\aleph_0}$, where, in constructing $G_{\alpha+1}$, α even, we try to eliminate ϕ_α from being an automorphism of the resulting $(G, \leqslant)$, if possible. This is not always possible, of course, but the remaining automorphisms ϕ_α will necessarily be of *type III*, which means that for all y in a certain large dense subset Y_α of $(H, \leqslant)$, $y\phi_\alpha \in \langle G_\alpha, y\rangle$ and $y \in \langle G_\alpha, y\phi_\alpha\rangle$, the angle brackets denoting generated subgroup of $(H, \cdot)$. In Lemma 2.2, we prove that if ϕ_α is of type III, then ϕ_α is definable at each point of Y_α, that is, for each $y \in Y_\alpha$ there are $a, b \in G_\alpha$ such that $y\phi_\alpha = a \cdot y^{\pm 1} \cdot b$. In the discussion following Lemma 2.4, we show this implies ϕ_α is piecewise definable on $(G, \cdot, \leqslant)$.

Now for the details. First, we construct H. For each natural number n, let F_n be a free group on K_n, $|K_n| = n + 1$, and let $H = \Pi F_n$. As mentioned earlier, any free group admits an order under which it becomes an ordered group. For each F_n fix an arbitrary such order, and then order H lexicographically. The resulting structure $(H, \cdot, \leqslant)$ is then an ordered group. It is clear that $(H, \leqslant)$ has a countable dense subset (the restricted product, for example), and so we may consider $(H, \leqslant)$ to be an ordered subset of the real line $(R, \leqslant)$. We list all automorphisms of $(\mathbf{R}, \leqslant)$ as $\{\phi_\alpha : 0 \leqslant \alpha < 2^{\aleph_0}, \alpha \text{ even}\}$. We can also write the elements of H as $y = (y(0), y(1), \ldots), y(n) \in F_n$.

Let U be a nonprincipal ultrafilter on ω. Let $Y = \{y \in H : \{n : y(n) \in K_n\} \in U\}$, that is, y is 'almost everywhere' a free generator of F_n. We define an equivalence relation on H with classes $[y]$ such that $[y] = [y']$ if $\{n : y(n) = y'(n)\} \in U$. It follows that every nontrivial open interval of $(H, \leqslant)$ contains $2^{\aleph_0}$ pairwise nonequivalent members of Y (see [7], p. 128, Theorem 3.12). Let $s = (s(0), s(1), \ldots, s(n))$ be a finite sequence with each $s(i) \in F_i$, and let $I_s = \{y \in H : y(i) = s(i), 0 \leqslant i \leqslant n\}$. Then I_s is an open interval of $(H, \leqslant)$ and the collection of all such I_s forms a base for the order topology of $(H, \leqslant)$. It is clear that each equivalence class $[y]$ is dense in $(H, \leqslant)$. The equivalence relation $\equiv$ determines a quotient group $H/\!\equiv$ (an ultraproduct in which the classes $\{[y] : y \in Y\}$ form a free generating set for the subgroup they generate. Hence, for each $a \in \langle Y\rangle$, the set $S(a) = \{[y] : y \in Y$ and $[y]$ occurs in the reduced expression for $[a]\}$ is unique. For $a \notin \langle Y\rangle$, $S(a)$ is empty. Moreover, any pairwise inequivalent subset of Y is free in H.

We will inductively construct G_α and R_α for each $0 \leqslant \alpha < 2^{\aleph_0}$ such that G_α is subgroup of $(H, \cdot)$, R_α is a set of $\equiv$-classes, $|G_\alpha| < 2^{\aleph_0}$, $|R_\alpha| < 2^{\aleph_0}$, and the following conditions hold:

(1) G_0 is generated by a countable dense set of pairwise inequivalent members of Y and $R_0 = \{[y] : y \in G_0 \cap Y\}$.

(2) For all $\alpha \leqslant \beta$, $G_\alpha \subseteq G_\beta$ and $R_\alpha \subseteq R_\beta$.

(3) For limit ordinals λ, $G_\lambda = \cup_{a<\lambda} G_\alpha$ and $R_\lambda = \cup_{\alpha<\lambda} R_\alpha$.

(4) For all α, there exists $b_\alpha \in Y$ such that $G_{\alpha+1} \subseteq \langle G_\alpha, b_\alpha\rangle$ and $[b_\alpha] \in R_{\alpha+1}\backslash R_\alpha$.

(5) If α is odd, $G_{\alpha+1} = \langle G_\alpha, b_\alpha\rangle$.

(6) $G = \cup G_\alpha$.

Any G constructed so that (1)–(6) are true will be uncountable (G contains the uncountable set $\{b_\alpha : \alpha \text{ odd}\}$), and will be contained in the group obtained by adjoining to G_0 the independent elements $\{b_\alpha\}$. Thus, $G_0 \subseteq G \subseteq G_0 * \langle b_\alpha : 0 \leqslant \alpha < 2^{\aleph_0}\rangle$. The product $*$ is a free product. By the Kurosh subgroup theorem (see [8]), G is a free group on a set including the free generators of G_0. In particular, the free generating set for G contains a countable dense subset.

It remains to ensure that every member of $A(G, \leqslant)$ is piecewise definable. We arrange this by making the construction satisfy (1)–(6) in the following special way. Suppose that (1)–(5) hold for all ordinals less than α, and (1)–(3) hold for all ordinals $\leqslant \alpha$. We are ready to find b_α and construct $G_{\alpha+1}$ and $R_{\alpha+1}$.

If α is odd, let b_α be any member of Y such that $[b_\alpha] \notin R_\alpha$, let $G_{\alpha+1} = \langle G_\alpha, b_\alpha\rangle$ and $R_{\alpha+1} = R_\alpha \cup \{[b_\alpha]\}$.

If α is even there are three cases.

Case I. Suppose there exists $b \in Y$ such that $[b] \notin R_\alpha$ and $b\phi_\alpha \notin \langle G_\alpha, b\rangle$. Then let $b_\alpha = b$, $G_{\alpha+1} = \langle G_\alpha, b\rangle$, and $R_{\alpha+1} = R_\alpha \cup \{[b]\} \cup S(b\phi_\alpha)$.

Case II. Suppose there is no b of the type in Case I, but there exists $b \in Y$ such that $[b] \notin R_\alpha$ and $b \notin \langle G_\alpha, b\phi_\alpha\rangle$. Then let $b_\alpha = b$, $G_{\alpha+1} = \langle G_\alpha, b\phi_\alpha\rangle$, and $R_{\alpha+1} = R_\alpha \cup \{[b]\}$.

Case III. Suppose there is no b of the types in Cases I or II. Choose any $b \in Y$ such that $[b] \notin R_\alpha$ and let $b_\alpha = b$, $G_{\alpha+1} = \langle G_\alpha, b\rangle$, and $R_{\alpha+1} = R_\alpha \cup \{[b]\}$.

Suppose now that G has been constructed in this way. We claim that every automorphism of $(G, \leqslant)$ is piecewise definable. Because G is dense in R, it suffices to show that if ϕ_α restricts to an automorphism of $(G, \leqslant)$ then ϕ_α is piecewise definable.

Suppose ϕ_α is of type I, that is, the construction of b_α, $G_{\alpha+1}$, and $R_{\alpha+1}$ was as in Case I. Then $b_\alpha \in G_{\alpha+1} \subseteq G$, so also $b_\alpha\phi_\alpha \in G$. Since G is contained in a group generated by $G_{\alpha+1}$ and certain b_λ such that $[b_\lambda] \notin R_{\alpha+1}$, $b_\alpha\phi_\alpha$ is a product of elements of $G_{\alpha+1}$ with such b_λ's. If any such b_λ actually occurs in a reduced product for $b_\alpha\phi_\alpha$, then $[b_\lambda] \in S(b_\alpha\phi_\alpha)$. But this is impossible because $S(b_\alpha\phi_\alpha) \subseteq R_{\alpha+1}$. Therefore, $b_\alpha\phi_\alpha \in G_{\alpha+1}$, a contradiction. Hence, ϕ_α is not of type I.

Suppose ϕ_α is of type II. Then $b_\alpha\phi_\alpha \in G_\alpha \subseteq G$, so also $b_\alpha = (b_\alpha\phi_\alpha)\,\phi_\alpha^{-1} \in G$. As before, b_α is a product of elements of $G_{\alpha+1}$ with b_λ's such that $[b_\lambda] \notin R_{\alpha+1}$. But since $S(b_\alpha) = \{[b_\alpha]\} \subseteq R_{\alpha+1}$, no such b_λ's occur, and $b_\alpha \in G_{\alpha+1}$, a contradiction. Hence ϕ_α is not of type II either.

Thus, if ϕ_α restricts to an automorphism of G, ϕ_α is of type III, which implies that for every $y \in Y$ such that $[y] \notin R_\alpha$, both $y\phi_\alpha \in \langle G_\alpha, y\rangle$ and $y \in \langle G_\alpha, y\phi_\alpha\rangle$. Lemmas 2.1–2.4 and the ensuing discussion imply that for every nontrivial open interval I of $(G, \leqslant)$ there is a nontrivial subinterval I', $\epsilon \in \{-1, 1\}$, and $a, b \in G$ such that for all $x \in I'$, $x\phi_\alpha = ax^\epsilon b$. Because ϕ_α preserves order, $\epsilon = +1$. Hence, ϕ_α is piecewise definable.

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In order to show that any automorphism of type III is piecewise definable, we first need the following lemma which applies to *any* groups G and H.

LEMMA 2.1 *Let G be a subgroup of a torsion free group H; $x \in H$; x free over G. If $w \in \langle G, x\rangle$ and $x \in \langle G, w\rangle$ then $w = g_1 x^{\epsilon} g_2$ where $g_i \in G$, $\epsilon = \pm 1$.*

Proof. $w = g_1 x^{\epsilon_1} g_2 x^{\epsilon_2} \ldots g_n x^{\epsilon_n} g_{n+1}$, $\epsilon_i = \pm 1$, $g_i \in G$ and if $\epsilon_{i-1}\epsilon_i = -1$ then $g_i \neq e$. Then w is reduced as a word in the free product of G with $\langle x\rangle$. (*) $x = h_1 w^{\delta_1} h_2 w^{\delta_2} \ldots h_m w^{\delta_m} h_{m+1}$, $\delta_j = \pm 1$, $h_j \in G$ and if $\delta_{j-1}\delta_j = -1$ then $h_j \neq e$.

Let $N = \Sigma_{i=1}^{n} \epsilon_i$ and $M = \Sigma_{j=1}^{m} \delta_j$. Since x is free over G, a count of the x's on both side of (*) shows that $1 = MN$. Hence, $N = \pm 1$. This implies that n is odd, say $n = 2p - 1$.

Rewriting (*) using the original expression for w gives an expression of the form

$$\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n} k_{n(j-1)+i}\, x_{i,j}\right)\right) k_{nm+1}\, x_{1,m+1} = e \ (\dagger)$$

where $x_{i,j} \in \{x, x^{-1}\}$; each k is either one of the g's or a product of one of the g's with one or two h's.

Because we are in the free product of G and $\langle x\rangle$, the formal expression ($\dagger$) must reduce by successive cancellation of adjacent letters, and each $x_{i,j}$ must cancel with another $x_{i',j'}$. Let ϕ denote some specific cancellation procedure which reduces the left-hand side of ($\dagger$) to e and let $x_{i,j}\,\phi\, x_{i',j'}$, mean that $x_{i,j}$ cancels with $x_{i',j'}$ in the procedure ϕ:

Because of the way the cancellation must proceed, it has the following basic property: if $a\,\phi\, b$ and $c\,\phi\, d$ and c is between a and b (in the natural sequence of letters in the expression ($\dagger$)), then also d is between a and b.

Since w is reduced, for no i, i', j does $x_{i,j}\phi x_{i',j}$. Now consider the 'middle' term $x_{p,j}$ for each $j < m$. For no i does $x_{p,j}\,\phi\, x_{i,j+1}$. For suppose it does. Each member of $\{x_{r,j} : r > p\}$ must cancel with a unique member of $\{x_{s,j+1} : s < i\}$. The first set has $p - 1$ members, the second $i - 1$. Therefore $i = p$. It follows from $x_{p,j}\phi x_{p,j+1}$ that $\delta_j \delta_{j+1} = -1$, so that $h_{j+1} \neq e$ and for each $i > p$, $x_{i,j}\,\phi\, x_{p-i,j+1}$. In particular, $x_{n,j}\,\phi\, x_{1,j+1}$, which can only happen if $k_{nj+1} = e$. Either $\delta_j = 1$ and $\delta_{j+1} = -1$ or $\delta_j = -1$ and $\delta_{j+1} = 1$. We assume the former, both cases being similar. Then $e = k_{nj+1} = g_{n+1} h_{j+1} (g_{n+1})^{-1}$, a contradiction. Thus the claim is established.

At most, one of the middle $x_{p,j}$, $j \leqslant m$ cancels with $x_{1,m+1}$. Hence, if $m > 1$, some $x_{p,j}$, $j \leqslant m$ must cancel some $x_{i,t}$, $t \leqslant m$. Between $x_{p,j}$ and $x_{i,t}$ will be a 'nearest' pair $x_{p,j'}\ \phi x_{i',t'}$. But then $t' = j' + 1$ or $j' - 1$, which we have shown impossible. Hence $m = 1$ and $x = h_1 w^{\delta_1} h_2 = h_1 (g_1 x^{\epsilon_1} \ldots x^{\epsilon_n} g_{n+1})^{\delta_1} h_\alpha$. Since we are in the free product and w is reduced, it must be that $w = g_1 x^{\epsilon_1} g_2$. □

LEMMA 2.2. *If ϕ_α is of type III in the construction of G, then for each $y \in Y$ such that $[y] \notin R_\alpha$, there exist $a, b \in G_\alpha$ and $\epsilon \in \{-1, 1\}$ such that $y\phi_\alpha = a y^{\epsilon} b$.*

Proof. Immediate from Lemma 2.1 since y is free over G_α. □

This lemma shows that ϕ_α is 'pointwise' definable on a large subset of H. Our next goal is to spread this to intervals.

LEMMA 2.3. *If F is a free group on $X = \{x_0, x_1, \ldots\}$ with elements $a_0, b_0, a_1, b_1 \in \langle X \setminus \{x_0, x_1\}\rangle$, and there are $a, b \in F$ and $\epsilon, \epsilon_0, \epsilon_1 \in \{-1, 1\}$ such that $ax_0^\epsilon b = a_0 x_0^{\epsilon_0} b_0$ and $ax_1^\epsilon b = a_1 x_1^{\epsilon_1} b_1$, then $\epsilon = \epsilon_0 = \epsilon_1$, $a_0 = a_1$, and $b_0 = b_1$.*

Proof. We can assume a, b are reduced words in X and a_0, a_1, b_0, b_1 reduced words in $X \setminus \{x_0, x_1\}$. Since the right side of $ax_0^\epsilon b = a_0 x_0^{\epsilon_0} b_0$ has no x_1, the total degree of x_1 in ab is 0. Similarly, the total degree of x_0 in ab is 0. It follows that $\epsilon = \epsilon_0 = \epsilon_1$. By taking inverses if necessary, we can assume $\epsilon = 1$. If the displayed x_0 on the left-hand side of $ax_0 b = a_0 x_0 b_0$ does not cancel, then $ax_0 b$ is reduced. Hence, a, b contain no x_1. Therefore, $ax_1 b$ is also reduced, and so $a = a_0 = a_1$ and $b = b_0 = b_1$. We arrive at the same conclusion if the displayed x_1 on the left side of $ax_1 b = a_1 x_1 b_1$ does not cancel.

In the remaining case, we may assume that the length of a is minimal for these equations with given a_0, b_0, a_1, b_1, and that both the displayed x_0 and x_1 on the left-hand side cancel. Then either $a = a' x_0^{-1}$ and $b = x_1^{-1} b'$ where a' and b' are reduced and a' does not end in x_0 and b' does not begin in x_1, or similarly with x_0 and x_1 exchanged. We assume the former. Consider $ax_0 b = a' x_0^{-1} x_0 x_1^{-1} b' = a' x_1^{-1} b'$. Since $a' x_1^{-1} b'$ has no x_1 in its reduced form $(a_0 x_0 b_0)$ the displayed x_1^{-1} must cancel, and since $x_1^{-1} b' = b$ is reduced, $a' = a'' x_1$. Similarly, $b' = x_0 b''$. Then $a_0 x_0 b_0 = ax_0 b = a'' x_0 b''$ and $a_1 x_1 b_1 = ax_1 b = a'' x_1 b''$. But the length of a'' is less than the length of a, a contradiction.

LEMMA 2.4. *Assume ϕ_α is of type III. Let I be a nontrivial open interval of $\bar{G}$ such that ϕ_α is not definable on any nontrivial open subinterval of I. If I_0 and I_1 are nontrivial open subintervals of I, there are nontrivial open intervals $I_0' \subseteq I_0$ and $I_1' \subseteq I_1$ such that if $x_0 \in I_0' \cap H$ and $x_1 \in I_1' \cap H$, and $x_0 \phi_\alpha = ax_0^\epsilon b$ ($\epsilon \in \{-1, 1\}$) then $x_1 \phi_\alpha \neq ax_1^\epsilon b$.*

Proof. Choose any $y_0 \in I_0 \cap Y$ such that $[y_0] \notin R_\alpha$. By Lemma 2.2 $y_0 \phi_\alpha = a_0 y_0^{\epsilon_0} b_0$ for some a_0, $b_0 \in G_\alpha$ and $\epsilon_0 \in \{-1, 1\}$. Choose $y_1 \in I_1 \cap Y$ such that $[y_1] \notin R_\alpha$ and $[y_0] \neq [y_1]$. Since $[y_1] \cap I_1$ is dense in I_1, ϕ_α cannot be defined the same way at each point of $[y_1]$, otherwise by continuity ϕ_α is definable on I_1. Hence, we may assume $y_1 \phi_\alpha \neq a_0 y_1^\epsilon b_0$. Nevertheless, $y_1 \phi_\alpha = a_1 y_1^{\epsilon_1} b_1$ for some a_1, $b_1 \in G_\alpha$ and $\epsilon_1 \in \{-1, 1\}$. By the construction of G_α, G_α is contained in a group G' generated by a set X_α of pairwise nonequivalent elements of Y such that for all $x \in X_\alpha$, $[x] \in R_\alpha$. Let Z be a finite subset of X_α such that a_0, a_1, b_0, $b_1 \in \langle Z \rangle$. Choose a natural number n such that $y_0(n)$, $y_1(n)$, and all $z(n)$ ($z \in Z$) are distinct members of K_n. For $i = 0,1$ consider the finite sequences $s_i = (s_i(j) : 0 \leqslant j \leqslant n)$ with

$$s_i(j) = a_i(j)\,(y_i(j))^{\epsilon_i} b_i(j) \quad \text{and} \quad y_i \mid n = (y_i(j) : 0 \leqslant j \leqslant n).$$

Then

$$y_i \in J_i = I_i \cap I_{y_i \mid n} \cap I_{s_i} \phi_\alpha^{-1}.$$

Let I_i' be any nontrivial open subinterval of J_i. Suppose, by way of contradiction, that for some $x_i \in I_i'$ ($i = 0,1$), and some a, b, ϵ, $x_i \phi_\alpha = ax_i^\epsilon b$. Then $x_i(n) = y_i(n)$ and $x_i \phi_\alpha \in I_{s_i}$. Therefore,

$$a(n)\,(y_i(n))^\epsilon b(n) = a_i(n)\,(y_i(n))^{\epsilon_i} b_i(n), \quad i = 0,1.$$

But this contradicts Lemma 2.3.

Lemma 2.4 has an obvious generalization for any finite set of nontrivial open subintervals $I_0, I_1, \ldots, I_n$ of I.

Under the assumption that ϕ_α of type III is not piecewise definable, we will get a contradiction by showing that Player I has a winning strategy in the following game.

THE ULTRAFILTER GAME

We are given the ultrafilter U. Players I and II alternately choose the terms of an increasing sequence of positive integers. Let $m_0 = 0$. Player I chooses any m_1. Player II then chooses any $m_2 > m_1$, etc. Player I wins the game if $\cup_{n<\omega} [m_{2n}, m_{2n+1}) \in U$.

This leads to a contradiction because there is no winning strategy in the ultrafilter game. (A proof of this fact is included as Lemma 2.6.)

Here is a winning strategy for Player I under the assumption that ϕ_α is of type III but not piecewise definable. While playing the game, Player I will simultaneously build a finitely branching tree T, consisting of finite sequences s such that $s(n) \in F_n$ for each n (where F_n is the free group on K_n used in the construction of H.) Let $|s|$ denote the length of the finite sequence s. The tree T will be constructed in stages so that when m_n is a play of the game (by either player) Player I constructs $T_{m_n} = \{s \in T : |s| = m_n\}$.

We observe again that the intervals I_s, for finite sequences s form a base for the topology of H. Let I be a nontrivial open interval of H such that ϕ_α is not definable on any nontrivial open subinterval of I. Player I begins by choosing a finite sequence s such that $I_s \subseteq I$ and then makes the first play, $m_1 = |s|$, and sets $T_{m_1} = \{s\}$. In general, suppose Player I has played m_{2n-1} and has constructed $T_{m_{2n-1}}$, and that Player II responds by playing m_{2n}. Then Player I first constructs $T_{m_{2n}}$ as the set of all sequences of length m_{2n} which extend sequences from T_{2n-1} and such that for each $m_{2n-1} \leqslant i < m_{2n}$, $s(i) \in K_i$. Player I then considers the subintervals $I_t \subseteq I$, $t \in T_{m_{2n}}$. Applying Lemma 2.4 (rather, its generalization) to this set of intervals, for each $t \in T_{m_{2n}}$, he can find an extension t' so that $I_t \subseteq I_t$ and such that if $x_0 \in I_{t'_0}$ and $x_1 \in I_{t'_1}$ for $t_0 \neq t_1$, then if $x_0\phi_\alpha = ax_0^\epsilon b$, then $x_1\phi_\alpha \neq ax_1^\epsilon b$. It may be assumed that all t' have the same length m_{2n+1}, which Player I now plays. He then constructs $T_{m_{2n+1}}$ consisting of all t', and awaits the next move of Player II.

We now show that Player II cannot win this game. Suppose, by way of contradiction, that $\cup [m_{2n}, m_{2n+1}) \notin U$, so that $\cup [m_{2n+1}, m_{2n+2}) \in U$. Consider the set B of *branches* of T, that is, those members $y \in H$ such that each initial segment of y of length m_i belongs to T. We see that $B \subseteq Y$ and that each equivalence class of Y contains a member of B. There are $2^{\aleph_0}$ equivalence classes of Y which are not in R_α, and if we choose a member of B in each, we will have $2^{\aleph_0}$ elements $y \in B$ such that $[y] \notin R_\alpha$. By Lemma 2.2, for each such y there exist $a, b \in G_\alpha$, $\epsilon \in \{-1, 1\}$ such that $y\phi_\alpha = ay^\epsilon b$. If $y_0 \neq y_1$ then for some finite sequences $t'_0 \neq t'_1 \in T$, $y_0 \in I_{t'_0}$ and $y_1 \in I_{t'_1}$. These were constructed so that if $y_0\phi_\alpha = a_0 y_0^{\epsilon_0} b_0$ and $y_1\phi_\alpha = a_1 y_1^{\epsilon_1} b_1$, then $(a_0, \epsilon_0, b_0) \neq (a_1, \epsilon_1, b_1)$. Thus, we have a set of such (a, ϵ, b) of cardinality $2^{\aleph_0}$. But this is impossible, since $|G_\alpha| < 2^{\aleph_0}$.

The contradiction shows that any ϕ_α of type III must be piecewise definable on G. We have proved:

THEOREM 2.5. *There exists an ordered group* $(G, \cdot, \leqslant)$ *such that any ordered group* $(G, *, \leqslant)$ *on the same carrier is lawless.*

LEMMA 2.6. *Neither player has a winning strategy in the ultrafilter game (when U is any nonprincipal ultrafilter).*

Proof. Since both cases are similar, we suppose Player I has a winning strategy. Player II can now win the game by playing two simultaneous games, the principal game, and an auxiliary game. Suppose Player I, following the supposed strategy, plays m_1 in the principal game. Then we insist that Player I begins the auxiliary game with the same play, $k_1 = m_1$. Player II plays $k_2 = k_1 + 1$ in the auxiliary game, and Player I following the strategy responds with some $k_3 > k_2 > m_1$. Player II plays $m_2 = k_3$ in the principal game. Thereafter, when Player I plays m_{2i+1} in the principal game, Player II plays $k_{2i+2} = m_{2i+1}$ in the auxiliary game, and when Player I plays k_{2i+1} in the auxiliary game, Player II plays $m_{2i} = k_{2i+1}$ in the principal game. Since Player I has followed the winning strategy in the auxiliary game, $\cup_{n=0}^{\infty} [k_{2n}, k_{2n+1}) \in U$. Since U is not principal, also $\cup_{n=2}^{\infty} [k_{2n}, k_{2n+1}) \in U$. But

$$\bigcup_{n=0} [m_{2n+1}, m_{2n+2}) \supseteq \bigcup_{n=1}^{\infty} [m_{2n+1}, m_{2n+2}) = \bigcup_{n=2}^{\infty} [k_{2n}, k_{2n+1}) \in U.$$

Hence, $\cup_{n=0}^{\infty} [m_{2n+1}, m_{2n+2}) \in U$, and Player II has won the principal game.

3. Consistency Results

In this section we discuss the number of separable lawless orders and whether there can exist separable lawless orders of cardinality $< 2^{\aleph_0}$.

THEOREM 3.1. *There are* $2^{2^{\aleph_0}}$ *pairwise nonisomorphic separable lawless orders.*

Proof. We must refine the construction in Section 2. The notation is the same. Choose $\mathscr{E} = \{E_\alpha : \alpha < 2^{\aleph_0}\}$ such that for all α, E_α is a set of $2^{\aleph_0}$ $\equiv$-equivalence classes; and if $\alpha \neq \beta$, then $E_\alpha \cap E_\beta = 0$. Suppose $S \subseteq 2^{\aleph_0}$. Let $\{\nu_\alpha : \alpha < 2^{\aleph_0},\ \alpha \text{ odd}\}$ enumerate S (possibly with repetition). We can now choose G^S as in Section 2 where '(4)' is replaced by '$(4)_S$ for all α there is $b_\alpha \in \cup_{\alpha \in S} \cup E_\alpha \backslash \cup R_\alpha$ so that $G_{\alpha+1} \subseteq \langle G_\alpha, b_\alpha \rangle$, and $[b_\alpha] \notin R_{\alpha+1}$' and '(5)' is replaced by '$(5)_S$ if α is odd and b_α is as above then $[b_\alpha] \in E_{\nu_\alpha}$ and $G_{\alpha+1}^S = \langle G_\alpha^S, b_\alpha \rangle$'. We can assume for all S, S', $G_0^S = G_0^{S'}$. Exactly the same proof shows the construction of G^S yields a lawless order. Further, if $S \neq S'$ then $G^S \neq G^{S'}$.

Since each G^S is dense any order isomorphism from G^S to $G^{S'}$ induces an automorphism of R. So for any $S \subseteq 2^{\aleph_0}$ there are at most $2^{\aleph_0}$ S' such that $G^S \simeq G^{S'}$.

For any unexplained set-theoretic notions and results in what follows, the reader may consult [6].

THEOREM 3.2. *It is consistent that* $2^{\aleph_0} > \aleph_1$ *and for all* $\aleph_1 \leqslant \kappa \leqslant 2^{\aleph_0}$ *there are* 2^κ *pairwise nonisomorphic separable lawless orders.*

Proof. Let F_n and K_n be as in Section 2. Let P be the following partially ordered set: $\{\{\alpha_0\} \times s_0, \ldots, \{\alpha_m\} \times s_m\}$ is an element of P if for all i, $\alpha_i < 2^{\aleph_0}$ and there is some n so that each s_i is a sequence of length n and for all $j < n$, $s(j) \in F_j$. P is ordered in the obvious way. (P is really the poset for adding $2^{\aleph_0}$ Cohen reals.) Suppose $F = \{f_\alpha : \alpha < 2^{\aleph_0}\}$ is a P-generic set. We work in $V[F]$. Let $H = \Pi F_n$. So for all α, $f_\alpha \in H$. Fix X an uncountable subset of F. Let $G_X = \langle f : f \in X \rangle$. View G_X as an ordered subgroup of H. We will show G_X, as an order, is lawless. More exactly we will show G_X satisfies the hypotheses of Theorem 1.1. The genericity of X guarantees G_X is a separable dense linear order. Also if we let $Y_f = \{n : f(n) \in K_n\}$, then $\{Y_f : f \in X\}$ has the finite intersection property (and can be extended to a nonprincipal ultrafilter). So G_X is a free group.

It remains to show every order automorphism of G_X is piecewise definable. Suppose $\tilde{\varphi}$ is a name so that $0 \Vdash$ '$\tilde{\varphi}$ is in order automorphism of $\tilde{G}_X$'. (We use $\tilde{G}_X$ and $\tilde{f}$ to denote the obvious names for G_X and f.) To simplify the notation we will show there is a condition p and an interval I so that $p \Vdash$ '$\tilde{\varphi}$ is definable on $\check{I}$'. It is easy to modify this proof to show the desired result. For each f choose p_f and a reduced word $w_f(f_{0(f)} \ldots f_{n(f)})$ so that $p_f \Vdash \tilde{\varphi}(\tilde{f}) = w_f(\tilde{f}_{0(f)} \ldots \tilde{f}_{n(f)})$.

By passing to an uncountable subset Y and applying the Δ-lemma, we can assume there are $w, f_{\beta_0}, \ldots, f_{\beta_m}$, and $f_{0(f)}, \ldots, f_{n(f)}$ so that: for all

$$f \in Y,\ p_f \Vdash \tilde{\varphi}(\tilde{f}) = w(\tilde{f}_{\beta_0}, \ldots, \tilde{f}_{\beta_m}, \tilde{f}_{0(f)} \ldots \tilde{f}_{n(f)});$$

and if $f \neq f'$, then $\{0(f), \ldots, n(f)\} \cap \{0(f'), \ldots, n(f')\} = 0$. Further we can assume that each p_f is of the form $\{\{\beta_0\} \times s_{\beta_0}, \ldots, \{\beta_m\} \times s_{\beta_m}, \{i(f)\} \times s, \{0(f)\} \times s_0, \ldots, \{n(f)\} \times s_n\}$ for fixed $s_{\beta_0}, \ldots, s_{\beta_m}, s, s_1, \ldots, s_n$. (Here $i(f) = \alpha$ iff $f = f_\alpha$.) Note our definition of P implies all the sequences have the same length.

We claim $n = 0$ and $i(f) = 0(f)$. Suppose not. First since $\tilde{\varphi}$ is the name of an automorphism $n \geqslant 0$. So we can assume $i(f) \neq 0(f)$ for all $f \in Y$. We can assume s has length $k > m + 1 + n$. Take $f \neq g \in Y$. Let $q = \{\{\beta_0\} \times s'_{\beta_0}, \ldots, \{\beta_m\} \times s'_{\beta_m}, \{i(f)\} \times s', \{i(g)\} \times s', \{0(f)\} \times s'_{0(f)}, \ldots, \{n(f)\} \times s'_{n(f)}, \{0(g)\} \times s'_{0(g)}, \ldots, \{n(g)\} \times s'_{n(g)}\}$ be an extension of $p_f \cup p_g$ so that length $(s') = k + 1$ and $w(s'_{0(f)}(k), \ldots, s'_{n(f)}(k)) < w(s'_{0(g)}(k), \ldots, s'_{n(g)}(k))$. So $q \Vdash \tilde{\varphi}(\tilde{f}) < \tilde{\varphi}(\tilde{g})$. But we can now choose q' extending q so that $q' \Vdash \tilde{g} < \tilde{f}$. This is a contradiction. So if we take any $f \in Y$, $p_f \Vdash$ 'for all α if for all $i < k$, $\tilde{f}_\alpha(i) = s(i)$, then $\tilde{\varphi}(\tilde{f}_\alpha) = w(\tilde{f}_{\beta_0}, \ldots, \tilde{f}_{\beta_m}, \tilde{f}_\alpha)$'. Since the set of such f_α must be dense in I_s, $p_f \Vdash$ 'for all $x \in I_s$, $\tilde{\varphi}(x) = w(\tilde{f}_{\beta_0}, \ldots, \tilde{f}_{\beta_m}, x)$'. Refining this argument slightly, we can show; for any order automorphism φ of G and nonempty open interval I there is a nonempty open subinterval I' and a word w (with coefficients from G) so that for all $x \in I'$, $\varphi(x) = w(x)$. Consider any order automorphism φ and open interval I. Choose intervals $I_0 \subset I$ and I_1 and words $w_0(x)$, $w_1(x)$ with coefficients in G so that: $\varphi(I_0) \subseteq I_1$; for all $x \in I_0$, $\varphi(x) = w_0(x)$; and for all $x \in I_1$, $\varphi^{-1}(x) = w_1(x)$. Choose Z a finite subset of X such that every coefficient in w_0 and w_1 is in $\langle Z \rangle$. Let $f \in (X \backslash Z) \cap I_0$. Applying Lemma 2.1 to f and $\langle Z \rangle$, there are $g, h \in \langle Z \rangle$ so that $w(x) = gxh$.

We have shown for any uncountable $X \subseteq F$, G_X is a lawless order. It remains to verify that for all $\aleph_1 \leqslant \kappa \leqslant 2^{\aleph_0}$ there are 2^κ nonisomorphic lawless orders. If G_X is isomorphic to G_Y then there is a countable subset $Z \subseteq F$ so that $Y \subseteq V[X][Z]$. Hence, if the symmetric difference of X and Y is uncountable, then $G_X \not\cong G_Y$. □

THEOREM 3.3. *It is consistent with ZFC, that every separable carrier of cardinality* $< 2^{\aleph_0}$ *carries an Abelian group and, hence, that there exist no separable lawless orders of cardinality* $< 2^{\aleph_0}$.

Proof. By [9] we can assume $2^{\aleph_0} = \aleph_2$ and any two separable $\aleph_1$-dense linear orders without endpoints are isomorphic. (An order is $\aleph_1$-dense if for all $a < b$, $|\{x : a < x < b\}| = \aleph_1$.) Suppose G is a separable lawless order of cardinality $\aleph_1$. Define an equivalence relation $\equiv$ on G by $a \equiv b$ if $|\{x : a < x < b \text{ or } b < x < a\}| \leqslant \aleph_0$. Since G is separable, each equivalence class is denumerable. Because G carries a group, $\mathrm{Aut}(G)$ is transitive. If some (hence every) equivalence class were not a singleton, then G would not be separable. Clearly there are nonlawless separable orders of cardinality $\aleph_1$; e.g., any subgroup of $\mathbf{R}$ of cardinality $\aleph_1$.

4. Concluding Remarks

In this section we collect some remarks on generalizations and open questions.

We first note that the proof of Theorem 1.1 actually shows not only that no $(G, *)$ belongs to a proper variety, but also no uncountable subgroup of any $(G, *)$ belongs to a proper variety.

It is possible to associate laws of other structures (than ordered groups) with ordered sets. One immediate generalization concerns *right ordered groups*, structures $(G, \cdot, \leqslant)$ such that $(G, \leqslant)$ is a totally ordered set, $(G, \cdot)$ is a group, and $x \leqslant y$ implies $x \cdot z \leqslant y \cdot z$. In such a structure, right translation is an automorphism of $(G, \leqslant)$, and this is enough to make all our theorems work. Thus, the ordered group $(G, \cdot, \leqslant)$ constructed in Section 2 has the property that if $(G, *, \leqslant)$ is any *right* ordered group on the same carrier $(G, \leqslant)$ then $(G, *)$ satisfies no nontrivial group law. In [10] we show how the construction of this paper can easily be modified to produce a right ordered group $(G, \cdot, <)$ such that $(G, <)$ carries no ordered group.

Pursuing a somewhat different line, we may consider ordered semigroups. In the construction of Section 2 we begin with ordered free semigroups S_n instead of free groups F_n and proceed as before. The result is an ordered semigroup $(S, \cdot, \leqslant)$ such that if $(S, *, \leqslant)$ is any ordered semigroup on the same carries them $(S, *)$ satisfies no nontrivial semigroup law. We omit the details, but note that one must deal with endomorphisms of $(S, \leqslant)$ rather than automorphisms, and with the more complicated form of definability which, for groups, was simplified by Lemma 2.1.

Ordered groups form a subclass of lattice ordered groups, a class whose language involves the group operation together with the lattice operations $\vee$ and $\wedge$. We might ask whether the order we construct is lawless even with respect to the laws of lattice ordered groups (except for those satisfied by all totally ordered groups). The answer is it will certainly depend on the orders chosen on the free groups F_n which begin the construction. For our original purpose, these orders were irrelevant.

Question: Is there an ordered group $(G, \cdot, \leqslant)$ such that every ordered group $(G, *, \leqslant)$ on the same carrier is lawless in the language of lattice ordered groups (except for those laws satisfied by every ordered group)?

Returning to the original question of Baer, it is easy to prove that the variety of nilpotent class 2 ordered groups is minimal over the Abelian variety. Is it possible to construct an order which is 'just barely absolutely nonabelian'?

Question: Does there exist a nilpotent class 2 ordered group $(G, \cdot, \leqslant)$ such that every ordered group $(G, *, \leqslant)$ on the same carrier is nonabelian?

We noted in the introduction that certain orders, such as $(Z, \leqslant)$ uniquely determine the group structures which they carry (to within isomorphism). The order we constructed in Section 2 does not uniquely determine the group it carries – if it did, we would have been done in a hurry since the group is free. In fact, it has many convex normal subgroups and each of these gives rise to a different group structure obtained as the direct product of the normal subgroup with its quotient.

Question: Is there an order which carries a unique group, and such that that group is nonabelian?

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