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Large normal ideals concentrating on a fixed small cardinality⋆

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Abstract. The property on the filter in Definition 1, a kind of large cardinal property, suffices for the proof in Liu Shelah [LiSh484] and is proved consistent as required there (see Conclusion 6). A natural property which looks better, not only is not obtained here, but is shown to be false (in Claim 7). On earlier related theorems see Gitik Shelah [GiSh310]. On such games see e.g. [Je], [Sh-b], [Sh-f].

1 Definition. (1) Let κ be a cardinal and D a filter on κ and θ be an ordinal $\leq \kappa$ and $\mu < \chi$ but $\mu \geq 2$ and $\chi \leq \kappa$. Let $\mathrm{GM}_{\kappa,\chi,\theta,\mu}(D)$ be there following game:

a play lasts θ moves, in the ζ's move the first player chooses a function h_ζ from κ to some ordinal $\gamma_\zeta < \chi$ and the second player chooses a subset B_ζ of γ_ζ of cardinality $< \mu$.

The second player wins a play if for every $\zeta < \theta$ the set $\bigcap\{\{\beta < \kappa : h_\varepsilon(\beta) \in B_\epsilon\} : \epsilon \leq \zeta\}$ is $\neq \emptyset \operatorname{mod} D$.

(2) If $\mu = 2$ we may omit it, if $\mu = 2$ and $\chi = \kappa$ we omit χ and μ.

2 Definition. $(P, \leq, \leq_{\mathrm{pr}}) \in K_{\kappa,\chi,\theta,\mu}$ iff

1. κ is a regular cardinal.
2. $(P, \leq)$ is a forcing notion with minimal element $\emptyset$ (if in doubt we use $\leq_P$, $\emptyset_P$).
3. P satisfies the κ-c.c.
4. $\leq_{\mathrm{pr}}$ is a partial order on P such that:
   a. $p \leq_{\mathrm{pr}} q$ implies $p \leq q$
   b. any $\leq_{\mathrm{pr}}$-increasing chain of length $< \theta$ with first element $\emptyset$ in P has an $\leq_{\mathrm{pr}}$-upper bound.

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c. if $\gamma < \chi$ and $\underset{\sim}{\tau}$ is a P-name of an ordinal $< \gamma$ and $\emptyset \leq_{\rm pr} p \in P$ then for some q and $B \subseteq \gamma$ of cardinality $< \mu$ we have $p \leq_{\rm pr} q \in P$ and q forces $\underset{\sim}{\tau} \in B$.

5. for any $Y \subseteq P$ of cardinality $< \kappa$ there is $P^* \lessdot P$ of cardinality $< \kappa$ such that P/P^* satisfies condition (4), i.e. if $G^* \subseteq P^*$ is generic over V and $P/G^* \overset{\rm def}{=} \{p \in P : p$ compatible with every $q \in G^*\}$ then
   a. in P/G^*, any $\leq_{\rm pr}$-increasing sequences starting with $\emptyset$ of length $< \theta$ have an $\leq_{\rm pr}$-upper bound in P/G^*.
   b. if $p \in P/G^*$, $\emptyset \leq_{\rm pr} p$ and $\underset{\sim}{\tau}$ a P-name of an ordinal $< \gamma$ where $\gamma < \chi$ then there is a subset B of γ of cardinality $< \mu$ and p', $p \leq_{\rm pr} p' \in P/G^*$ such that p' forces $\underset{\sim}{\tau} \in B$.

2A Remark. The relation in clause 4(b) is not really stronger than having a winning strategy in the corresponding play, see [Sh250, 2.4] (or [Sh-f, XIV 2.4]).

3 Lemma. *Assume*
a. κ is a measurable cardinal with D a κ-complete ultrafilter on it
b. $(P, \leq, \leq_{\rm pr}) \in K_{\kappa,\chi,\theta,\mu}$
Then in V^P the second player wins $GM_{\kappa,\chi,\theta,\mu}(D)$

3A Remark. 1. We can replace ultrafilter by a filter in which the first player wins $\mathrm{GM}_{\theta,\kappa}$(D) [see Lemma 5].

Proof. In V we define a set R, its members are sequences $\bar{p} = \langle p_\alpha : \alpha \in A^{\bar{p}} \rangle$ where $A^{\bar{p}} \in D$ and $\emptyset \leq_{\rm pr} p_\alpha \in P$ (for $\alpha \in A^{\bar{p}}$). On R we define a partial order $\leq_R$ as follows: $\bar{p} \leq_R \bar{q}$ iff $A^{\bar{q}} \subseteq A^{\bar{p}}$ and for every $\alpha \in A^{\bar{q}}$ we have $p_\alpha \leq_{\rm pr} q_\alpha$.

Clearly, in V the partial order $(R, \leq_R)$ is θ-complete.

For $G \subseteq P$ generic over V we define $R[G]$ as $\{\bar{p} : \bar{p} \in R$ and $\{\alpha \in A^{\bar{p}} : p_\alpha \in G\} \neq \emptyset \bmod D$ (in V^P, D is not a filter just a family of subsets of κ but it naturally generates a filter- just closed upward and we refer to this filter in "mod D"$\}$.

For $G \subseteq P$ generic over V and $\bar{p} \in R$ let $w[\bar{p}, G] \overset{\rm def}{=} \{\alpha \in A^{\bar{p}} : p_\alpha \in G\}$.

So $R[G] = \{\bar{p} \in R : w[\bar{p}, G] \neq \emptyset \bmod D\}$. We now prove some facts.

3B. Fact. *In $V[G]$, $(R[G], \leq_R)$ is θ-complete.*

Proof. If not then there is a P-name of a sequence of length $< \theta$, $\langle \underset{\sim}{\bar{p}}^\varepsilon : \varepsilon < \zeta \rangle$ and $r \in P$ which forces this sequence to be a counter example, so $\zeta < \theta$. So there are maximal antichains $\mathscr{T}_\varepsilon$ for $\varepsilon < \zeta$ of conditions in P forcing a value to $\underset{\sim}{\bar{p}}^\varepsilon$ (note $\underset{\sim}{\bar{p}}^\varepsilon$ is a P-name of a member of V); let Y be the set of elements appearing in some $\mathscr{T}_\varepsilon$ and r. As P satisfies the κ-c.c. clearly Y has cardinality $< \kappa$ so there is P^* as required in condition (5) of Definition 2. Let $G^* \subseteq P^*$ be generic over V and $r \in G^*$.

Now working in $V[G^*]$ we can (for each $\varepsilon < \zeta$) compute $\underset{\sim}{\bar{p}}^{\varepsilon}$ and $A^{\underset{\sim}{\bar{p}}^{\varepsilon}}$, call it then $\bar{p}^{\varepsilon}$ and A_{ε} respectively and so $\bigwedge_{\varepsilon} A_{\varepsilon} \in D$ and $A^* \overset{\text{def}}{=} \cap\{A_{\varepsilon} : \varepsilon < \zeta\}$ belongs to $D^{V[G^*]}$ (=the ultrafilter which D generates in $V[G^*]$, remember $|P^*| < \kappa$, D a κ-complete ultrafilter); also letting $w_{\varepsilon} \overset{\text{def}}{=} \{\alpha \in A^*$: there is $G \subseteq P$ generic over P extending G^* to which p^{ε}_{α} belongs$\} \in V[G^*]$ we know that in $V[G]$ we get a D-positive set $w[\underset{\sim}{\bar{p}}^{\varepsilon}, G]$ (because r forces this) hence in $V[G^*]$ the set w_{ε} is D-positive but in $V[G^*]$ we know $D^{V[G^*]}$ is an ultrafilter so necessarily w_{ε} belongs to $D^{V[G^*]}$; clearly for $\varepsilon < \zeta$, $\alpha \in w_{\varepsilon}$ we have $p^{\varepsilon}_{\alpha} \in P/G^*$. Let $B^* = A^* \cap \bigcap\{w_{\varepsilon} : \varepsilon < \zeta\}$, it is in $D^{V[G^*]}$. Now for any $\alpha \in B^*$ the sequence $\langle p^{\varepsilon}_{\alpha} : \varepsilon < \zeta\rangle$ is a $\leq_{\text{pr}}$-increasing sequence of member of P/G^* and by demand (5) (a) of Definition 2, the sequence has an $\leq_{\text{pr}}$-upper bound q_{α} (in P/G^*). Let $r_{\alpha} \in G^*$ be above r and force that this holds and moreover force some specific $q_{\alpha} \in P_{\alpha}$ is as above. So, still in $V[G^*]$, for some $C \in D$, $C \subseteq B$ and $r^* \in G^*$ we have $(\forall \alpha \in C)[r_{\alpha} = r^*]$, without loss of generality $C \in V$. As for $\alpha \in C \subseteq B$, $r^* = r_{\alpha} \Vdash$ "$q_{\alpha} \in P/\underset{\sim}{G}_{P^*}$", r^* is compatible with every q_{α} ($\alpha \in C$). By 3D below for some q^+, $r^* \leq q^+ \in P$ and $q^+ \Vdash_P$ "$\{\alpha : q^+_{\alpha} \in \underset{\sim}{G}_{P^*}\}$"$\neq \emptyset \operatorname{mod} D$. So q^+ (which is above $r \leq r^*$) force that $\bar{q} = \langle q_{\alpha} : \alpha \in C\rangle$ is an upper bound as required. (note: $\bar{q} \in V$, r^* force it is an upper bound of $\{\underset{\sim}{\bar{p}}^{\varepsilon} : \varepsilon < \zeta\}$; we need $q^+_{\alpha(*)}$ as we do not know the value of $\underset{\sim}{\bar{p}}^{\varepsilon}$. $\square$

3C Fact. *Let $G \subseteq P$ be generic over V. In $V[G]$, if $\gamma < \chi$ and $\bar{p} \in R[G]$ and h a function from κ to γ, then for some $\bar{q}$ we have:*

a. $\bar{q} \in R[G]$

b. $\bar{p} \leq_R \bar{q}$

c. on $w[\bar{p}, G]$ the range of the function h is of cardinality $< \mu$.

Proof. Assume the conclusion fails then some $r \in G$ forces that it fails for a specific $\bar{p}$ and P-name $\underset{\sim}{h}$ (so in particular r forces that $w[\bar{p}, \underset{\sim}{G}] \neq \emptyset \operatorname{mod} D$.) Let $w^* =: \{\alpha \in A^{\bar{p}}$: the conditions r, p_{α} are compatible in P (equivalently, r does not force $\alpha \notin w[\bar{p}, \underset{\sim}{G}]$)$\}$ (so $w^* \in V$) and $w^* \in D$. Now let P^* be as in condition (5) of Definition 2 for $Y = \{r\}$ (so in particular $r \in P^*$). Now:

(*) for every $\alpha \in w^*$ there are r^*_{α} and q_{α} and B_{α} such that:

a. $r \leq r^*_{\alpha} \in P^*$.

b. $p_{\alpha} \leq_{\text{pr}} q_{\alpha}$.

c. $r^*_{\alpha} \Vdash_{P^*}$ "$q_{\alpha} \in P/\underset{\sim}{G}_{P^*}$".

d. q_{α} forces (for P) that $\underset{\sim}{h}(\alpha) \in B_{\alpha}$ and for some set $B \subseteq \gamma$ ($B \in V$), we have $|B_{\alpha}| < \mu$.

[Why? For every α in w^* we can find $G \subseteq P$ generic over V to which r and p_{α} belong (as $\alpha \in w^*$); hence $p_{\alpha} \in P/(G \cap P^*)$ hence some $r^*_{\alpha} \in G \cap P^*$ force this (for P^*) so without loss of generality $r \leq r^*_{\alpha}$ (as

$G \cap P^*$ is directed). Now apply condition (5) of Definition 2 to $G \cap P^*$, p_α and $\underset{\sim}{h}(\alpha)$ and we get some $B \subseteq \gamma$, $|B| < \mu$ and $q_\alpha \in P/(G \cap P^*)$ such that $p_\alpha \leq_{\text{pr}} q_\alpha \in P/(G \cap P^*)$ and q_α forces $\underset{\sim}{h}(\alpha) \in B$. Now increasing again r^*_α we get (*)].

So we can find for $\alpha \in w^*$, r_α, q_α and B_α as in $(*)$, (all in V); let $A^* \subseteq w^*$ be such that $A^* \in D$ and $\langle B_\alpha : \alpha \in w^* \rangle$ is constant on A^* and also r_α is constantly r^* (note: D is κ-complete, $w^* \in D$, and κ is strongly inaccessible hence $|\gamma|^{<\mu} < \kappa$ and $|P^*| < \kappa$. Now some q^+, satisfying $r^* \leq q^+ \in P$, forces that $\langle q_\alpha : \alpha \in A^* \rangle$ is in $R[G]$ by fact 3D below and so clearly is as required in the Fact 3C. $\square$

3D Observation. *Assume $\bar{p} = \langle p_\alpha : \alpha \in A \rangle \in R$ and $r \in P$ is compatible (in P) with every p_α (for $\alpha \in A$). Then some r^*, $r \leq r^* \in P$, force that $\bar{p} \in R[\underset{\sim}{G}_P]$.*

Proof. Let $\mathscr{T}$ be a maximal antichain of P above r such that for every $q \in \mathscr{T}$ we have either $q \Vdash_P$ "$w[\bar{p}, \underset{\sim}{G}_P]$ is a subset of A_q" where $A_q \subseteq \kappa$ and $\kappa \setminus A_q \in D$ *or* $q \Vdash_P$ " $w[\bar{p}, \underset{\sim}{G}_P] \neq \emptyset \operatorname{mod} D$.

So $\mathscr{T}$ has cardinality $< \kappa$ and if the conclusion fails then always the first possibility holds; now we let $B \overset{\text{def}}{=} \bigcap \{\kappa \setminus A_q : q \in \mathscr{T}\}$, clearly it belongs to D. Now there is $\alpha \in B \cap A$ (as $B \cap A \in D$) and there is $r^* \in P$ above r and above p_α (exist by assumption); now r^* force that $\alpha \in w[\bar{p}, \underset{\sim}{G}_p] \subseteq A_q \subseteq \kappa \setminus B$, contradiction. $\square$

3E *Continuation of the Proof of Lemma 3:* immediate for the Facts 3B, 3C. $\square$

Now we shall use it:

4 Lemma. (From Gitik [Gi] Sect. 3, relaying on Sect. 1 there, in different terminology). *Assume $\chi < \kappa$, $\theta < \kappa$ a regular cardinal, κ is a measurable cardinal of order $\theta + 1$ (i.e. there is a coherent sequence of ultrafilters on κ of length $\theta + 1$, see [Gi, Sect. 3 p. 293], with D an ultrafilter on κ appearing in the θ'th place in the appropriate sequence.*

Then for some forcing notion P we have

(a) *P of cardinality κ, $\Vdash_P$ "κ is strongly inaccessible".*
(b) $\{\delta : \Vdash_P \text{"cf}(\delta) = \theta\text{"}\} \in D$
(c) *$P \in K_{\kappa,\chi,\theta,2}$ (in particular P satisfies the κ-c.c., $\leq_{\text{pr}}$ for P is called $\leq_E$ in [Gi] (called Easton)*
(d) *For some $\leq_{\text{pr}}$ Condition (4) of Definition 2 is satisfied by P (for $\mu = 2$). Moreover, given any $\chi^* < \kappa$ and $Y \subseteq P$ of cardinality $< \kappa$ we can find $P^* \lessdot P$ as in clause (5) of Definition 2 replacing θ and χ by χ^* (for any strongly inaccessible non measurable $\kappa_1 < \kappa$ we can "divide" the forcing at κ_1).*

5 Claim. *Under the assumptions of Lemma 4, if* $\theta + \chi \le \mu = \mu^{<\mu} < \kappa$ *let* $Q = P * (\mathrm{Levy}(\mu, <\kappa))^{V^P}$ *defining* $(p_1, q_1) \le_{\mathrm{pr}} (p_2, q_2)$ *iff* $p_1 \le_P p_2$ *and* $p_2 \Vdash_P$ "$q_1 \le q_2 \in \mathrm{Levy}(\mu, <\kappa)^{V^P}$"
Then $Q \in K_{\kappa,\chi,\theta,2}$ *and in* V^Q, $\kappa = \mu^+ = 2^\mu$.

Proof. Easy. □

5A Remark. Actually in the conclusion of Claim 5 we can weaken $\theta + \chi \le \mu$ to $\theta^+ + \chi \le \mu^+$ hence in the conclusion $\chi = \mu^+ (= \kappa)$ is o.k. This applies also to Conclusion 6.

5B Remark. Of course Claim 5 and Definition 2 are formulated so that we get consistency results justifying the name of the paper. We formulate below (Conclusion 6) the one used in Liu Shelah [LiSh484].

6 Conclusion. *Assume* $0 = n_0 < n_1 < n_2 < \ldots < n_\ell, n_\ell + 1 < n_{\ell+1}$, $\kappa_{\ell+1}$ *is a measurable of order* $\theta_\ell + 1$ *and for simplicity GCH holds and stipulate* $\kappa_0 = \aleph_0$ *and* $\theta_{\ell+1} < \kappa_{\ell+1}$ *is regular for* $\ell < \omega$, *moreover* $\theta_{\ell+1} \le \kappa_\ell^{+(n_{\ell+1} - n_\ell)}$ *for simplicity.*

Then there is a forcing notion P *of cardinality* $\le 2^{\Sigma_{\ell<\omega} \kappa_{\ell+1}}$ *which preserves* $\mathrm{cf}(\theta_{\ell+1}) = \theta_{\ell+1}$, *makes* $\kappa_{\ell+1}$ *to* $\aleph_{n_{\ell+1}}$ *and preserves* $(\kappa_\ell)^{+i}$ *if* $i < n_{\ell+1}$, *preserves G.C.H. and for* $\ell < \omega$ *in* V^P *the second player wins* $GM_{\aleph_{n_{\ell+1}}, \aleph_{n_{\ell+1}-1}, \theta_{\ell+1}, 2}(D_{\ell+1})$ *for some* $D_{\ell+1} \in V$, *a normal ultrafilter on* $\kappa_{\ell+1}$ *of order* $\theta_\ell + 1$.

Proof. We use iteration $\langle P_i, Q_i : i < \omega\rangle$ described as follows: Q_ℓ=the forcing notion from Lemma 5 (for $\kappa = \kappa_{\ell+1}$, $\theta = \theta_{\ell+1}$, $\mu = \kappa_\ell^{+(n_{\ell+1} - n_\ell)}$ and $\chi_{\ell+1} = \kappa_{\ell+1}^{+(n_{\ell+1} - n_\ell - 1)}$), the limit is a full support for pure extensions of the $\emptyset$ and finite support otherwise (for the Levy collapse all conditions are pure extensions of $\emptyset$). The checking is standard. □

Discussion. We shall now prove that for a natural strengthening of Definition 2, we cannot get consistency results. Specifically we cannot, in the game in Definition 2, let player I just decrease the present D-positive set.

7 Definition. (1) Let κ be a cardinal and D a filter on κ and θ be an ordinal $\le \kappa$. Let $\mathrm{GM}^*_\theta(D)$ be the following game:

a play lasts θ moves; in the ζ's move

first player chooses a subset A_ζ of κ, $A_\zeta \ne \emptyset \bmod D$ such that: if $\zeta = 0$, $A_\zeta \subseteq \kappa$ and if $\zeta = \varepsilon + 1$ then $A_\zeta \subseteq B_\varepsilon$ and if ζ is a limit ordinal then $A_\zeta = \cap\{A_\varepsilon : \varepsilon < \zeta\}$

and then *the second player* chooses a subset B_ζ of A_ζ satisfying $B_\zeta \ne \emptyset \bmod D$.

A player wins the play if he has no legal move (can occur only to the first player in a limit stage), if the play lasts θ moves then the second player wins.

8 Definition. Let λ be regular countable, $S \subseteq \lambda$; we say that there is a ($\le \theta$)-square for S if: there is a set S^+, and sequence $\langle C_\alpha : \alpha \in S^+\rangle$ such that:

a. $S \subseteq S^+ \subseteq \lambda$

b. for $\beta \in C_\alpha$ (so $\alpha \in S^+$) we have: $\beta \in S^+$ and $C_\beta = \beta \cap C_\alpha$.

c. $\mathrm{otp}(C_\alpha) \le \theta$ for $\alpha \in S^+$.

d. if $\delta \in S$ is a limit ordinal then $\delta = \sup(C_\delta)$

e. C_α is a closed subset of α.

9 Claim. *1) Assume λ is regular $> \theta$, D is a normal filter on λ^+ to which $\{\delta : \mathrm{cf}(\delta) = \theta\}$ belongs. Then in the game $GM^*_{\omega+1}(D)$ (see Definition 8 above) the second player does not have a winning strategy.*
2) Assume λ is regular larger than $|\theta|^+$, θ an ordinal, D is a normal filter on λ to which a set S belongs, and for S there is a $(\leq \theta)$-square (as defined in Definition 7 above) (or just every $S \subseteq \lambda$, $S \neq \emptyset \operatorname{mod} D$ has a subset S' for which there is a $(\leq \theta)$-square, $S' \neq \emptyset \operatorname{mod} D$).

Then in the game $GM^*_{\omega+1}(D)$ (see Definition 8 above), the second player does not have a winning strategy.

Proof. Part (1) follows form part (2) as the assumption of part (2) follows by [Sh 365, 2.14] (or [Sh 351, Th. 4.1]). So we concentrate on proving part (2).

So let $\langle C_\alpha : \alpha \in S^+ \rangle$ be as in Definition 8. So without loss of generality $S^+ \in D$. We divide $\{\delta : \delta < \lambda, \mathrm{cf}(\delta) = \aleph_0\}$ to $|\theta|^+$ stationary sets $\langle T_i : i < |\theta|^+ \rangle$. As D is a normal ideal on λ, $|\theta|^+ < \lambda$, clearly for each stationary subset S' of S which is D-positive there are $S^* \subseteq S'$ which is D-positive and ordinal $j^* < |\theta|^+$ such that for every $\alpha \in S^*$ we have: $C_\alpha \cup \{\alpha\}$ is disjoint to $T_{j(*)}$.

Now suppose the second player has a winning strategy in $\mathrm{GM}^*_{\omega+1}(D)$ which we call Sty. We can choose by induction on $n < \omega$ a sequence $\langle A_\rho, B_\rho, \beta_\rho : \rho \in {}^n\lambda \rangle$ such that

1. for every $\rho \in {}^n\lambda$ the sequence $\langle A_{\rho \restriction k}, B_{\rho \restriction k} : k \leq n \rangle$ is an initial segment of a play of the game in which the second player uses his winning strategy Sty
2. for some $j < |\theta|^+$, for every $\alpha \in A_{\langle\rangle}$ we have $C_\alpha \cup \{\alpha\}$ is disjoint to T_j.
3. $\beta_\rho \in S^+$ and for every $\rho \in {}^n\lambda$ and $\alpha \in A_\rho$ we have $\beta_\rho \in C_\alpha$.
4. for $\rho \in {}^n\lambda$ we have: β_ρ is larger than sup range (ρ).

There is no problem to carry the definition (for clause (3) remember D is a normal filter on λ); now let $E \stackrel{\text{def}}{=} \{\delta < \lambda : \text{for every } \rho \in {}^{\omega>}\delta \text{ we have } \beta_\rho < \delta\}$; clearly E is a club of λ hence there is an ordinal $\delta \in E \cap T_j$; so choose an increasing ω-sequence ρ of ordinals $< \delta$ with limit δ; look at $\langle A_{\rho \restriction k}, B_{\rho \restriction k} : k < \omega \rangle$ which is an initial segment of a play of the game in which the second player uses his winning strategy Sty. Let now $B = \cap\{B_{\rho \restriction k} : k < \omega\}$; if $\sup(B) > \delta$ (which holds if $B \neq \emptyset \operatorname{mod} D$), $\alpha \in B \setminus (\delta+1)$ then for every n, $\beta_{\rho \restriction n} \in C_\alpha$. Note: as $\rho \in {}^\omega\delta$, and $\delta \in E$ clearly $\beta_{\rho \restriction n} < \delta$; so $\delta \geq \bigcup_{n<\omega} \beta_{\rho \restriction n}$; as $\beta_{\rho \restriction n} \geq \sup \mathrm{range}(\rho \restriction n)$ necessarily $\delta \leq \bigcup_{n<\omega} \beta_{\rho \restriction n}$ so equality holds. Hence also $\delta = (\cup_{n<\omega} \beta_{\rho \restriction n}) \in C_\alpha$ (as $\alpha > \delta = \bigcup_{n<\omega} \beta_{\rho \restriction n}$). So $\delta \in C_\alpha$ but $\delta \in T_j$ whereas $\alpha \in B_{\langle\rangle}$, contradiction. So B is a subset of $\delta + 1$, contradicting to “Sty is a winning strategy”. □

9A Remark. This continues the argument that e.g. not for every stationary $S \subseteq \{\delta < \aleph_3 : \mathrm{cf}(\delta) = \aleph_0\}$, there is a club E of $\aleph_3$ such that $\delta \in E$ & if $(\delta) = \aleph_2 \Rightarrow S \cap \delta$ stationary in δ (find pairwise disjoint $S_i \subseteq \{\delta < \aleph_3 : \mathrm{cf}(\delta) = \aleph_0\}$, for $i < \aleph_3$, if for S_i we have E_i, choose $\delta \in \bigcap_{i<\aleph_2} E_i$ of cofinality $\aleph_2$.

9B Remark. In Claim 9, even if we weaken the demand on player $\prod$ to guarantee only $\bigcap_{n<\omega} B_n \neq \emptyset$ he still does not have a winning strategy. In the choice of the A_ρ, B_ρ (for $p \in {}^n\lambda$) we can add

5) if $\alpha < \beta < \lambda$ then $A_{\rho\char`\^\langle\alpha\rangle} \cap A_{\rho\char`\^\langle\beta\rangle} = \emptyset$

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and redefine E as

$$\{\delta < \lambda : \quad \delta \text{ a limit ordinal and } \rho \in {}^{\omega>}\delta \Rightarrow \beta_\rho < \delta \text{ and for every } \rho \in {}^{\omega>}\delta \text{ and } \beta < \delta \text{ for some } \gamma \text{ we have } \beta < \beta_{\rho\hat{}\langle\delta\rangle} \& A_{\rho\hat{}\langle\delta\rangle} \cap \beta \neq \emptyset\}.$$

So in the proof above we still can choose, by induction on n, $\rho \restriction {}^n\lambda$ such that:

$\delta \notin A_{\rho\restriction 1}$(possible as $\rho \restriction 1 = \langle 0\rangle$ or $\rho \restriction 1 = \langle 1\rangle$ is as required) and $A_{\rho\restriction(n+1)} \cap \rho(n) = \emptyset$ and $\rho(n) > \alpha_n$, where $\langle\alpha_n : n < \omega)$ is a prechosen ω-sequence of ordinals $< \delta$ with limit δ.

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