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ON POWER-LIKE MODELS FOR HYPERINACCESSIBLE CARDINALS

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§0. Introduction. The main result of this paper is the following transfer theorem: *If T is an elementary theory which has a κ-like model where κ is hyperinaccessible of type ω, then T has a λ-like model for each $\lambda >$* card(T). Helling [3] obtained the same conclusion under the stronger hypothesis that κ is weakly compact. Fuhrken conjectured in [1] that the same conclusion would result if κ were merely inaccessible. (He also showed there the connection with a problem about generalized quantifiers.) Thus, our theorem lies properly between Helling's theorem and Fuhrken's conjecture. In [9] it is shown that this theorem is actually the best possible. This theorem and the other results of this paper were announced by the authors in [10].

In §1 we prove as Theorem 1 a slightly stronger form of the above theorem. This theorem is generalized in §2 to Theorem 2 which concerns theories which permit the omitting of types. The methods used in §§1 and 2 are also applicable to problems regarding Hanf numbers as well as to two-cardinal problems.

0.1. *Notation.* An ordinal number is the set of its predecessors. Ordinals are denoted by α, β, ν, μ; infinite cardinals by κ, λ; finite ordinals by i, j, n. A cardinal κ is strongly inaccessible iff κ is regular and $2^\lambda < \kappa$ whenever $\lambda < \kappa$. We define inductively when a cardinal is *strongly α-inaccessible.* (This notion, which coincides with that of hyperinaccessible of type α of Levy [4], corresponds to the strong Mahlo hierarchy. We hereafter suppress the word "strongly".) The 0-inaccessible cardinals are just the inaccessible cardinals; if $\alpha > 0$, then κ is α-inaccessible iff whenever $\beta < \alpha$, then each closed, cofinal subset of κ contains a β-inaccessible.

We consider only structures $\mathfrak{A}$ for which $<^{\mathfrak{A}}$ is a linear ordering of the universe A of $\mathfrak{A}$. The structure $\mathfrak{A}$ is κ-like iff card$(A) = \kappa$ but every proper initial segment has cardinality $<\kappa$. The structure $\mathfrak{B}$ is an *end-extension* of $\mathfrak{A}$ iff $\mathfrak{A} \subset \mathfrak{B}$ and A is a proper initial segment of B. We denote the similarity type of $\mathfrak{A}$ by $\rho(\mathfrak{A})$, and we say that $\mathfrak{A}$ is a ρ-structure in case $\rho = \rho(\mathfrak{A})$.

0.2. *Relative saturation.* The following is a useful definition introduced by Simpson [13].

DEFINITION 0.1. If $\mathfrak{A} \subset \mathfrak{B}$, then $\mathfrak{A}$ is *relatively saturated* in $\mathfrak{B}$ iff whenever $X \subset A$, card$(X) <$ card(A) and $b \in B$, then there is $a \in A$ such that $(\mathfrak{A}, a, x)_{x \in X} \equiv (\mathfrak{B}, b, x)_{x \in X}$.

Clearly $\mathfrak{A} \prec \mathfrak{B}$ whenever $\mathfrak{A}$ is relatively saturated in $\mathfrak{B}$. We give an easy existence theorem for relatively saturated structures.

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PROPOSITION 0.2. *If* $\mathfrak{A}$ *is a* κ*-like* ρ*-structure where* κ *is* α*-inaccessible and* $\kappa >$ card$(\rho) + \aleph_0$, *then for each* $\beta < \alpha$ *there is a* β*-inaccessible* λ *and a* λ*-like* $\mathfrak{B} \subset \mathfrak{A}$ *such that* $\mathfrak{B}$ *is relatively saturated in* $\mathfrak{A}$ *and* $\mathfrak{A}$ *is an end-extension of* $\mathfrak{B}$.

PROOF. Let X be the set of cardinals λ with the following property: there is a λ-like $\mathfrak{B} \subset \mathfrak{A}$ such that whenever $c \in B$ and $a \in A$, then there is $b \in B$ such that $(\mathfrak{A}, a, x)_{x<c} \equiv (\mathfrak{B}, b, x)_{x<c}$. Clearly X is closed in κ.

We show that X is cofinal in κ by an easy elementary tower argument. Pick any $\kappa_0 < \kappa$. Let $\langle \mathfrak{B}_i : i < \omega \rangle$ be a sequence of substructures of $\mathfrak{A}$ such that whenever $i < \omega$, then $\kappa_0 \le \text{card}(B_i) < \text{card}(B_{i+1}) < \kappa$, $\{a \in A : \text{for some } b \in B_i, \mathfrak{A} \models a \le b\} \subset B_{i+1}$, and every type realized in $(\mathfrak{A}, x)_{x \in B_i}$ is realized in $(\mathfrak{B}_{i+1}, x)_{x \in B_i}$. Clearly, card$(\bigcup\{B_i : i < \omega\}) \in X - \kappa_0$, so that X is cofinal in κ.

Now for $\beta < \alpha$ let $\lambda \in X$ be β-inaccessible. There is a λ-like $\mathfrak{B}$ such that $\mathfrak{A}$ is an end-extension of $\mathfrak{B}$, and clearly (since λ is inaccessible) this $\mathfrak{B}$ is relatively saturated in $\mathfrak{A}$. ⊣

§1. The transfer theorem for elementary theories. In this section we prove our result for elementary theories by constructing models with built-in elementary end-extensions.

THEOREM 1. *If* T *is a theory such that for each* $n < \omega$ *there is an* n*-inaccessible* κ *and a* κ*-like model of* T, *then, for each* $\lambda >$ card$(T) + \aleph_0$, *there is a* λ*-like model of* T.

We first define a set Γ of sentences. Let ρ be a similarity type. Let $W, R_0, R_1, \cdots$ be new, distinct binary relation symbols. Let $\rho_n = \rho \cup \{W, R_0, \cdots, R_{n-1}\}$ and let $\rho' = \bigcup\{\rho_n : n < \omega\}$. Let Γ be the set of universal closures of the following formulas:

(1) $\forall u \exists v R_i(uv)$, for each $i < \omega$.

(2) $\forall v \exists u \neg R_i(uv)$, for each $i < \omega$.

(3) $\exists u[\forall v(R_i(uv) \leftrightarrow \varphi(x_0, \cdots, x_{n-1}, v)) \vee \forall v(R_i(uv) \leftrightarrow \neg\varphi(x_0, \cdots, x_{n-1}, v))]$, for each $n, i < \omega$ and each $(n+1)$-ary ρ_i-formula.

(4) $\forall u \forall w \exists y \forall v(R_i(uv) \wedge R_i(wv) \leftrightarrow R_i(yv))$, for each $i < \omega$.

(5) $\forall u \exists w \forall v(R_i(uv) \leftrightarrow R_j(wv))$, whenever $i < j < \omega$.

(6) $\exists w \forall v \exists y[y < w \wedge \varphi(x_0, \cdots, x_{n-1}, v, y)] \rightarrow \exists u \exists y \forall v[R_i(uv) \rightarrow \varphi(x_0, \cdots, x_{n-1}, v, y)]$, for each $n, i < \omega$ and each $(n+2)$-ary ρ_i-formula.

(7) $(\exists u \varphi(x_0, \cdots, x_{n-1}, u)) \rightarrow \exists u[\varphi(x_0, \cdots, x_{n-1}, u) \wedge \forall v(\varphi(x_0, \cdots, x_{n-1}, v) \rightarrow W(uv))]$, for each $n < \omega$ and each $(n+1)$-ary ρ'-formula.

(8) $\forall xy(W(xy) \wedge W(yx) \rightarrow y = x)$.

LEMMA 1.1. *Every model of* Γ *has an elementary end-extension.*

PROOF. Let $\mathfrak{A}$ be a model of Γ. For each $i < \omega$, let B_i be the collection of those subsets of A which are parametrically definable in $\mathfrak{A}$ by a ρ_i-formula. Under the usual set-theoretic operations B_i can be considered as a Boolean algebra.

Now let D_i be the collection of subsets X of A which are of the following type: there is some $a \in A$ such that $X = \{x \in A : \mathfrak{A} \models R_i(ax)\}$. Each one of the sentences (1)–(6) induces on D_i a certain property. These properties are respectively as follows:

(1*) $\varnothing \notin D_i$, for each $i < \omega$.

(2*) $\bigcap D_i = \varnothing$, for each $i < \omega$.

(3*) Each element of B_i or its complement is in D_i.
(4*) Each D_i is closed under finite intersection.
(5*) $D_i \subset D_j$ whenever $i < j < \omega$.
(6*) Each function which is parametrically definable in $\mathfrak{A}$ by a ρ_i-formula and which has a bounded range is constant on some element of D_i.

Conditions (1*)–(4*) imply that $D_i \cap B_i$ is a nonprincipal ultrafilter over the Boolean algebra B_i. Now let $B = \bigcup\{B_i: i < \omega\}$ and $D = \bigcup\{D_i: i < \omega\}$. Clearly B is the set of parametrically definable subsets of $\mathfrak{A}$. Conditions (1*)–(5*) imply that D is a nonprincipal ultrafilter over the Boolean algebra B, and condition (6*) implies that this ultrafilter has a certain degree of completeness. Sentences (7) and (8) guarantee that $\mathfrak{A}$ is adequate in the sense that $\mathfrak{A}$ becomes Skolemized when it is expanded by adjoining all the parametrically definable functions. (For example, if $W^{\mathfrak{A}}$ is a well-ordering of A, then $\mathfrak{A}$ is a model of sentences (7) and (8).)

We are now in a position to take the Skolem ultrapower. Choose any ultrafilter $U \supset D$. Now consider the structure $\mathfrak{A}'$ which is $\mathfrak{A}^A_U$ restricted to the parameterically definable functions of $\mathfrak{A}$. It is routine to verify that $\mathfrak{A} \prec \mathfrak{A}'$ and, of course, $\mathfrak{A} \neq \mathfrak{A}'$ (when $\mathfrak{A}$ is identified in the canonical way with a substructure of $\mathfrak{A}'$). Condition (6*) now assures that $\mathfrak{A}'$ is an end-extension of $\mathfrak{A}$. ⊣

LEMMA 1.2. *If T is a ρ-theory such that for each $n < \omega$ there is an n-inaccessible κ and a κ-like model of T, then $T \cup \Gamma$ is consistent.*

PROOF. By the compactness theorem we may assume that ρ is finite. Furthermore, letting Γ_n be the set of ρ_n-sentences in Γ, it suffices to show that each $T \cup \Gamma_n$ is consistent.

Choose $n \geq 1$. Let $\mathfrak{A}$ be a κ_0-like model of T where κ_0 is n-inaccessible. Expand $\mathfrak{A}$ to $\mathfrak{A}_0$ where $\mathfrak{A}_0$ is a ρ_0-model and $W^{\mathfrak{A}_0}$ is a well-ordering of A. Clearly $\mathfrak{A}_0$ is a model of Γ_0.

By induction we show that for each $i \leq n$ there is a ρ_i-model $\mathfrak{A}_i$ of $T \cup \Gamma_i$ such that $\mathfrak{A}_i$ is κ_i-like for some $(n - i)$-inaccessible κ_i and, in case $i = j + 1$, then $\mathfrak{A}_i$ is a model of the following sentence:

(9_j) $\forall w \exists v \forall u(u < w \to R_j(uv))$.

First we get $\mathfrak{A}_1$. Let $\mathfrak{A}_1' \subset \mathfrak{A}_0$, following Proposition 0.2, be relatively saturated in $\mathfrak{A}_0$ such that $\mathfrak{A}_0$ is an end-extension of $\mathfrak{A}_1'$ and $\mathfrak{A}_1'$ is κ_1-like for some $(n - 1)$-inaccessible κ_1. Choose some $a \in A_0 - A_1'$, and let C be the collection of subsets of A_0 which contain a and which are definable in $\mathfrak{A}_0$ using parameters from A_1'. Then let $f: A_1' \to C$ be a bijection onto C. (This is possible by obvious cardinality considerations.) Expand $\mathfrak{A}_1'$ to a ρ_1-structure $\mathfrak{A}_1$ where

$$\mathfrak{A}_1 \models R_0(cb) \quad \text{iff } b \in f(c).$$

It is clear that $\mathfrak{A}_1$ is a ρ_1-model of $T \cup \Gamma_1$.

The structure $\mathfrak{A}_1$ is also a model of sentence (9_0). For, suppose $b \in A_1$. For each $c < b$, let X_c be a definable subset of A_0 using parameters from A_1 such that $X_c \cap A_1 = \{x \in A_1: \mathfrak{A}_1 \models R_0(cx)\}$. Then $a \in \bigcap\{X_c: c < b\}$. Then since $\mathfrak{A}_1'$ is relatively saturated in $\mathfrak{A}_0$, we have that $A_1 \cap \bigcap\{X_c: c < b\} \neq \varnothing$.

In general, getting $\mathfrak{A}_{i+1}$, for $i > 0$, is the same, except that more care must be taken in choosing a so as to make certain that $\mathfrak{A}_{i+1}$ is a model of the sentences (5). Suppose we have $\mathfrak{A}_i$, where $0 < i < n$, and we wish to get $\mathfrak{A}_{i+1}$. By the induction hypothesis $\mathfrak{A}_i$ is a model of the sentence (9_{i-1}). Let $\mathfrak{A}'_{i+1} \subset \mathfrak{A}_i$ be relatively saturated in $\mathfrak{A}_i$ such that $\mathfrak{A}_i$ is an end-extension of $\mathfrak{A}'_{i+1}$ and that $\mathfrak{A}'_{i+1}$ is κ_{i+1}-like for some $(n - i - 1)$-inaccessible κ_{i+1}. We want to choose some $a \in A_i - A'_{i+1}$ such that whenever $x \in A'_{i+1}$, then $\mathfrak{A}_i \models R_{i-1}(xa)$. But it is clearly possible to get such an a since $\mathfrak{A}_i$ is a model of sentence (9_{i-1}). As before, we expand $\mathfrak{A}'_{i+1}$ to a ρ_{i+1}-structure $\mathfrak{A}_{i+1}$. Again, as before, $\mathfrak{A}_{i+1}$ is a model of sentence (9_i) and also a model of $T \cup \Gamma_{i+1}$. ⊣

Proof of Theorem 1. By Lemma 1.2, $T \cup \Gamma$ is consistent and hence has a model of cardinality card$(T) + \aleph_0$. Applying Lemma 1.1 to this model λ times (making sure that the cardinality is never increased when taking the elementary end-extension) and using the Tarski-Vaught Union Theorem at limit ordinals, we get a λ-like model of T. ⊣

Remark 1.3. The method of proof of Theorem 1 can also be used to prove Vaught's two-cardinal theorem for cardinals far apart. (See [15].) In fact Vaught himself uses self-extending models in which the distinguished predicate does not enlarge. His proof and set of sentences seem to us less perspicuous since, instead of encoding extensions, he encodes endomorphisms.

Remark 1.4. Suppose that we define an operator O on the class of models of Γ such that $O(\mathfrak{A})$ is the elementary extension described in the proof of Lemma 1.1. Then O is a local uniform extension operator as defined by Gaifman in [2]. Thus, his results can be used to extend Theorem 1 so as to get a λ-like model which is the closure of a set of indiscernibles of length λ just like the model Morley gets in [6]. The results and methods of this section remain valid if ω is considered as being n-inaccessible for all n; we thus obtain a theorem of MacDowell and Specker [5].

By the well-known procedure established by Vaught in [14] the following is an easy corollary of Lemmas 1.1 and 1.2 and of the definition of the set Γ.

Corollary 1.5. *If κ is ω-inaccessible, ρ is recursive, and K is the class of κ-like ρ-structures, then $Th(K)$ is recursively axiomatizable.*

§2. Transfer theorems with type omitting. In this section we prove our result for theories which permit the omitting of types.

First we define an ordinal which in a sense measures the descriptive power of a language. Let $\delta(\kappa_0, \kappa_1)$ be the first ordinal δ with the following property: if T is a ρ-theory and $\mathcal{S}$ is a set of ρ-types such that card$(\rho) \le \kappa_0$, card$(\mathcal{S}) \le \kappa_1$, and for each $\nu < \delta$, the theory T has a model $\mathfrak{A}$ which omits $\mathcal{S}$ and in which $\prec^{\mathfrak{A}}$ is a well-ordering of length at least ν, then T has a model $\mathfrak{B}$ omitting $\mathcal{S}$ in which $\prec^{\mathfrak{B}}$ is not a well-ordering. ($\mathfrak{A}$ omits $\mathcal{S}$ iff it omits each type in $\mathcal{S}$.) For each κ_0 and κ_1 the ordinal $\delta(\kappa_0, \kappa_1)$ exists. Bounds for $\delta(\kappa_0, \kappa_1)$ and other relevant information can be found in [7], [11] and [12, §5].

Theorem 2. *Let T be a ρ-theory and $\mathcal{S}$ a set of ρ-types where* card$(\rho) \le \kappa_0$ *and* card$(\mathcal{S}) \le \kappa_1$. *Suppose, for each $\nu < \delta(\kappa_0, \kappa_1)$, there is a ν-inaccessible $\kappa > \kappa_0$ and a κ-like model of T omitting $\mathcal{S}$. Then for each $\lambda > \kappa_0$ there is a λ-like model of T omitting $\mathcal{S}$.*

We prove this theorem using the method of model construction of Ehrenfeucht and Mostowski. We first need a combinatorial result using partitions. A partition C of X is a set of nonempty, pairwise disjoint sets such that $\bigcup C = X$. If $x, y \in E$ for

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some $E \in C$, then x is C-equivalent to y. The set of finite subsets of X is denoted by $S_\omega(X)$. We denote by $f[X]$ the set $\{f(x): x \in X\}$.

DEFINITION 2.1. $M \lhd N$ iff M and N are finite sets of ordinals such that $N = \{\nu \in M: \mu < \nu\}$ for some $\mu \in M$.

DEFINITION 2.2. C is a *partition system* of κ iff for each $\nu < \kappa$, C_ν is a partition of $S_\omega(\kappa)$ such that $\text{card}(C_\nu) < \kappa$.

DEFINITION 2.3. A function $F: S_\omega(\alpha) \to \kappa$ is *C-homogeneous* iff C is a partition system of κ and whenever $M \lhd M_0 \lhd \cdots \lhd M_{n-1}$ and $M \lhd N_0 \lhd \cdots \lhd N_{n-1}$, where $M \in S_\omega(\alpha)$, then $F[\{M_0, \cdots, M_{n-1}\}]$ is $C_{F(M)}$-equivalent to $F[\{N_0, \cdots, N_{n-1}\}]$ and $F(M) < F(M_0) < \cdots < F(M_{n-1})$.

LEMMA 2.4. *If κ is α-inaccessible, $\kappa > \alpha$, and C is a partition system of κ, then there is a C-homogeneous function $F: S_\omega(\alpha) \to \kappa$.*

PROOF. We give a proof by induction on α. For $\alpha = 0$ the lemma is obviously trivial. So assume $\alpha > 0$ and the lemma is valid for all ordinals less than α.

Consider the structure $\mathfrak{A} = (\kappa, G_1, G_2, \cdots)$, where $G_i: \kappa^{i+1} \to \kappa$ such that whenevery $\nu, \nu_0, \cdots, \nu_{i-1}, \mu_0, \cdots, \mu_{i-1} < \kappa$, then

$$G_i(\nu, \nu_0, \cdots, \nu_{i-1}) = G_i(\nu, \mu_0, \cdots, \mu_{i-1})$$

iff $\{\nu_0, \cdots, \nu_{i-1}\}$ is C_ν-equivalent to $\{\mu_0, \cdots, \mu_{i-1}\}$. By Proposition 0.2, for each $\beta < \alpha$, let $\mathfrak{A}_\beta$ and λ_β be such that $\mathfrak{A}_\beta$ is λ_β-like, $\mathfrak{A}_\beta$ is relatively saturated in $\mathfrak{A}$, $\mathfrak{A}$ is an end-extension of $\mathfrak{A}_\beta$, λ_β is β-inaccessible and $\lambda_\beta > \beta$. Choose λ such that $\lambda_\beta < \lambda < \kappa$ for each $\beta < \alpha$. (Remember that $\alpha < \kappa$.) Then, because of the relative saturation, it is clear that we can find functions $f^\beta: \lambda_\beta \to \lambda_\beta$ such that for each $\beta < \alpha$ and each $\nu < \lambda_\beta$, the elements $f^\beta(\nu)$ and λ realize the same type in $(\mathfrak{A}, f^\beta(\mu))_{\mu<\nu}$.

Now for each $\beta < \alpha$ we define a partition system C^β of λ_β as follows: if $M, N \in S_\omega(\lambda_\beta)$ and $\nu < \lambda_\beta$, then M and N are C^β_ν-equivalent iff $f^\beta[M] \cup \{\lambda\}$ is $C_{f(\nu)}$-equivalent to $f^\beta[N] \cup \{\lambda\}$. By the induction hypothesis there are C^β-homogeneous functions $G^\beta: S_\omega(\beta) \to \lambda_\beta$. Now define $F: S_\omega(\alpha) \to \kappa$ according to

$$\begin{aligned} F(M) &= \lambda && \text{if } M = \varnothing, \\ &= G^\beta(M - \{\beta\}) && \text{if } \max(M) = \beta. \end{aligned}$$

One readily checks that F is C-homogeneous. ⊣

REMARK 2.5. It is easily seen that the lemma can be slightly improved so as to get a C-homogeneous function $F: S_\omega(3 + \alpha) \to \kappa$. As shown in [8] this is the best possible result.

PROOF OF THEOREM 2. We can assume without loss of generality that T is a Skolemized theory. For each $\nu < \delta(\kappa_0, \kappa_1)$ let $\mathfrak{A}_\nu$ be a κ-like model of T omitting $\mathscr{S}$ for some ν-inaccessible $\kappa > \kappa_0$. We expand each of the structures $\mathfrak{A}_\nu$ to $\mathfrak{A}'_\nu = (\mathfrak{A}_\nu, U, H, \prec, X, C, F)$ so that the following hold:

(1) U is a subset of A_ν which is well-ordered by $<$ with length ν. (We can assume in fact that $U = \nu$.)

(2) H is a binary relation which indexes $S_\omega(A_\nu)$ in the following sense: for each $M \in S_\omega(A_\nu)$ there is a unique $a \in A_\nu$ such that $M = M_a = \{b \in A_\nu: \mathfrak{A}_\nu \models H(ab)\}$.

(3) $\prec$ is a partial ordering such that $a \prec b$ iff $M_a, M_b \subset U$ and $M_a \lhd M_b$. (Hence, the partial order type of $\prec$ is the same as that of $S_\omega(\nu)$.)

(4) X is a well-ordered cofinal subset of A_ν such that if $a_0, \cdots, a_{n-1} \le x < y$, where $x, y \in X$, and t is an n-ary ρ-term, then $t(a_0, \cdots, a_{n-1}) < y$.

(5) C is a "partition system" of A_ν. More precisely, C is a binary function such that

(i) for each $a \in A_\nu$, the set $\{C(ab): b \in A_\nu\}$ is bounded in A_ν;

(ii) if $C(ab) = C(ac)$, then, for some n, $M_b = \{b_0, \cdots, b_{n-1}\}$, $M_c = \{c_0, \cdots, c_{n-1}\}$, where $b_0 < \cdots < b_{n-1}$, $c_0 < \cdots < c_{n-1}$ and $\langle b_0, \cdots, b_{n-1}\rangle$ and $\langle c_0, \cdots, c_{n-1}\rangle$ realize the same type in the structure $(\mathfrak{A}_\nu, x)_{x \le a}$.

(6) F is a function mapping the field of $\prec$ into X which is C-homogeneous (in the obvious sense).

It is clear that each $\mathfrak{A}_\nu$ can be expanded to such an $\mathfrak{A}'_\nu$. Condition (6) requires the use of Lemma 2.4.

Now there is a model $\mathfrak{B}'$ of all sentences true in each of the $\mathfrak{A}'_\nu$ such that $\mathfrak{B}'$ omits $\mathscr{S}$ and $U^{\mathfrak{B}'}$ is not well-ordered. We can find an infinite decreasing sequence $u_0 > u_1 > \cdots$ of elements of U. Let c_n be such that $U_{c_n} = \{u_0, \cdots, u_{n-1}\}$. Then we have that $c_0 \succ c_1 \succ c_2 \succ \cdots$. Let $Y = \{F(c_n): n < \omega\}$.

Clearly, by condition (5), Y is an infinite set of indiscernibles in the structure $\mathfrak{B} = \mathfrak{B}' \restriction \rho$. Furthermore,

(I) $\mathfrak{B} \models \forall v_0 \cdots v_{n-1}(v_0 \le x \wedge \cdots \wedge v_{n-1} \le x \rightarrow t(v_0, \cdots, v_{n-1}) < y)$ whenever $x < y$ are elements of Y and t is an n-ary ρ-term;

(II) $$\mathfrak{B} \models \forall v_0 \cdots v_{n-1}[(v_0 \le x \wedge \cdots \wedge v_{n-1} \le x) \rightarrow (\varphi(v_0, \cdots, v_{n-1}, x_0, \cdots, x_{m-1}) \leftrightarrow \varphi(v_0, \cdots, v_{n-1}, y_0, \cdots, y_{m-1}))],$$

where $x < x_0 < \cdots < x_{m-1}$, $x < y_0 < \cdots < y_{m-1}$ are elements of Y and φ is an $(m+n)$-ary ρ-formula.

Condition (I) follows readily from (4), whereas (II) follows from (5).

We now employ the familiar technique of stretching indiscernibles. Choose any $\lambda > \kappa_0$. Let $\mathfrak{A}$ be a structure which is its Skolem hull over a set Z of indiscernibles of length λ, where Z has the same elementary properties in $\mathfrak{A}$ as the set Y does in $\mathfrak{B}$. It is well-known that $\mathfrak{A}$ is a model of T which omits $\mathscr{S}$. We show that $\mathfrak{A}$ is λ-like.

It follows from (I) that Z is cofinal in $\mathfrak{A}$. Thus, we need only show that each element z of Z has fewer than λ predecessors in A. A typical predecessor of z is given by $t(z_0, \cdots, z_{n-1}, z, x_0, \cdots, x_{m-1})$ where $z_0 < \cdots < z_{n-1} < z < x_0 < \cdots < x_{m-1}$ are elements of Z and t is some $(n+m+1)$-ary ρ-term. But then, by (II),

$$t(z_0, \cdots, z_{n-1}, z, y_0, \cdots, y_{m-1}) = t(z_0, \cdots, z_{n-1}, z, x_0, \cdots, x_{m-1})$$

whenever $z < y_0 < \cdots < y_{m-1}$ are elements of Z. Thus the number of predecessors of z in A is easily seen to be at most card$(\{x \in Z: x < z\}) + \kappa_0 < \lambda$. The proof of the theorem is complete. ⊣

REMARK 2.6. The method of the proof of Theorem 2 also suffices to prove the following: *If σ is a sentence such that for each recursive ν there is a ν-inaccessible κ and a κ-like ω-model of σ, then for each uncountable λ there is a λ-like ω-model of σ.* This corresponds to a similar result of Morley on the Hanf number of a single sentence of ω-logic. Other such theorems can also be similarly proved as is done in [8].

REMARK 2.7. The method of the proof of Theorem 2 can also be used to give the following characterization of Hanf numbers: *Let T be a ρ-theory and $\mathscr{S}$ a set of ρ-types where* card$(\rho) \le \kappa_0$ *and* card$(\mathscr{S}) \le \kappa_1$. *Suppose, for each $\nu < \delta(\kappa_0, \kappa_1)$, there* *is a model of T omitting $\mathscr{S}$ of cardinality at least $\beth_\nu$. Then for each $\lambda \geq \kappa_0$ there is a model of T omitting $\mathscr{S}$ of cardinality λ.* This result was first proved by M. and V. Morley, but their approach (see [7]) is different. See also Shelah [12, §5].

REMARK 2.8. Theorem 2, as well as those theorems mentioned in the preceding two remarks, can also be proved using the method of self-extending models as in the proof of Theorem 1.

REMARK 2.9. Theorem 2 is a best possible result. For a proof of this consult the Main Theorem of [9].

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