

## ON EMBEDDING CERTAIN PARTIAL ORDERS INTO THE P-POINTS UNDER RUDIN-KEISLER AND TUKEY REDUCIBILITY

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**ABSTRACT.** The study of the global structure of ultrafilters on the natural numbers with respect to the quasi-orders of Rudin-Keisler and Rudin-Blass reducibility was initiated in the 1970s by Blass, Keisler, Kunen, and Rudin. In a 1973 paper Blass studied the special class of P-points under the quasi-ordering of Rudin-Keisler reducibility. He asked what partially ordered sets can be embedded into the P-points when the P-points are equipped with this ordering. This question is of most interest under some hypothesis that guarantees the existence of many P-points, such as Martin's axiom for  $\sigma$ -centered posets. In his 1973 paper he showed under this assumption that both  $\omega_1$  and the reals can be embedded. Analogous results were obtained later for the coarser notion of Tukey reducibility. We prove in this paper that Martin's axiom for  $\sigma$ -centered posets implies that the Boolean algebra  $\mathcal{P}(\omega)/\text{FIN}$  equipped with its natural partial order can be embedded into the P-points both under Rudin-Keisler and Tukey reducibility. Consequently, the continuum hypothesis implies that every partial order of size at most continuum embeds into the P-points under both notions of reducibility.

### 1. INTRODUCTION

The analysis of various quasi-orders on the class of all ultrafilters on  $\omega$  provides a great deal of information about the global structure of this class. An early example of such global information was the proof that  $\beta\omega \setminus \omega$  is not homogeneous, obtained through an analysis of what later became known as the Rudin-Frolík order (see [15]). This ordering and the weaker Rudin-Keisler ordering were analyzed in [25] to obtain more information about the topological types in  $\beta\omega \setminus \omega$ . An analysis of the stronger Rudin-Blass order eventually led to the isolation of the principle of near coherence of filters, a principle which postulates a kind of global compatibility between ultrafilters on  $\omega$  and has applications to diverse areas of mathematics (see [4, 5, 8]). Larson's work [19] is a recent application of a slightly stronger principle than near coherence to measure theory. Recall the following definitions.

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**Definition 1.** Let  $\mathcal{F}$  be a filter on a set  $X$  and  $\mathcal{G}$  a filter on a set  $Y$ . We say that  $\mathcal{F}$  is *Rudin-Keisler (RK) reducible to  $\mathcal{G}$*  or *Rudin-Keisler (RK) below  $\mathcal{G}$* , and we write  $\mathcal{F} \leq_{RK} \mathcal{G}$  if there is a map  $f : Y \rightarrow X$  such that for each  $a \subset X$ ,  $a \in \mathcal{F}$  if and only if  $f^{-1}(a) \in \mathcal{G}$ .  $\mathcal{F}$  and  $\mathcal{G}$  are *RK equivalent*, written  $\mathcal{F} \equiv_{RK} \mathcal{G}$ , if  $\mathcal{F} \leq_{RK} \mathcal{G}$  and  $\mathcal{G} \leq_{RK} \mathcal{F}$ .

We say that  $\mathcal{F}$  is *Rudin-Blass (RB) reducible to  $\mathcal{G}$*  or *Rudin-Blass (RB) below  $\mathcal{G}$* , and we write  $\mathcal{F} \leq_{RB} \mathcal{G}$  if there is a finite-to-one map  $f : Y \rightarrow X$  such that for each  $a \subset X$ ,  $a \in \mathcal{F}$  if and only if  $f^{-1}(a) \in \mathcal{G}$ . RB equivalence is defined analogously to RK equivalence.

In this paper we restrict ourselves only to ultrafilters on  $\omega$ . If  $\mathcal{F}$  and  $\mathcal{G}$  are ultrafilters on  $\omega$ , then  $\mathcal{F} \equiv_{RK} \mathcal{G}$  if and only if there is a permutation  $f : \omega \rightarrow \omega$  such that  $\mathcal{F} = \{a \subset \omega : f^{-1}(a) \in \mathcal{G}\}$ . For this reason, ultrafilters that are RK equivalent are sometimes said to be *RK isomorphic*. If  $f : \omega \rightarrow \omega$  is a function such that  $\forall b \in \mathcal{G} [f''b \in \mathcal{F}]$ , then in the case when  $\mathcal{F}$  and  $\mathcal{G}$  are ultrafilters on  $\omega$ ,  $f$  already witnesses that  $\mathcal{F} \leq_{RK} \mathcal{G}$ .

Kunen [16] was the first to construct two ultrafilters  $\mathcal{U}$  and  $\mathcal{V}$  on  $\omega$  such that  $\mathcal{V} \not\leq_{RK} \mathcal{U}$  and  $\mathcal{U} \not\leq_{RK} \mathcal{V}$  using only the axioms of ZFC. In his paper [18], Kunen showed in ZFC that there are  $2^{2^\omega}$  RK incomparable ultrafilters on  $\omega$ . His techniques actually showed in ZFC alone that the class of ultrafilters on  $\omega$  has a fairly complicated structure with respect to the ordering  $\leq_{RK}$ ; for example, this class is  $\mathfrak{c}$ -directed with respect to  $\leq_{RK}$ .

It is also well known that certain special classes of ultrafilters can be characterized using the Rudin-Keisler order. Recall the following notions.

**Definition 2.** An ultrafilter  $\mathcal{U}$  on  $\omega$  is *selective* if for every function  $f : \omega \rightarrow \omega$ , there is a set  $A \in \mathcal{U}$  on which  $f$  is either one-to-one or constant.  $\mathcal{U}$  is called a *P-point* if for every  $f : \omega \rightarrow \omega$ , there is  $A \in \mathcal{U}$  on which  $f$  is finite-to-one or constant.

It is easy to see that an ultrafilter  $\mathcal{U}$  on  $\omega$  is a P-point if and only if for any collection  $\{a_n : n \in \omega\} \subset \mathcal{U}$  there exists  $a \in \mathcal{U}$  such that  $\forall n \in \omega [a \subset^* a_n]$ . Here  $\subset^*$  denotes the relation of containment modulo a finite set:  $a \subset^* b$  if and only if  $a \setminus b$  is finite. Selective ultrafilters are minimal in the Rudin-Keisler ordering, meaning that any ultrafilter that is RK below a selective ultrafilter is RK equivalent to that selective ultrafilter. This minimality in fact characterizes the selective ultrafilters. P-points are minimal in the Rudin-Frolík order. Observe that  $\leq_{RK}$  and  $\leq_{RB}$  coincide for the class of P-points.

Rudin [26] proved in 1956 that P-points exist if the Continuum Hypothesis (CH henceforth) is assumed, and he used this to show that CH implies the non-homogeneity of  $\beta\omega \setminus \omega$ . P-points were also independently considered by several other people in a more model-theoretic context. The question of whether P-points always exist was settled in a landmark paper of Shelah in 1977 (see [27]), where the consistency of their non-existence was proved.

Blass considered the structure of the class of P-points with respect to the Rudin-Keisler order in [3]. As the existence of P-points is independent of ZFC, it makes sense to consider this structure only when some hypothesis that allows us to build P-points with ease is in hand. If this hypothesis is relatively mild and moreover has the status of a “quasi-axiom”, then it may be considered the “right axiom” under which to investigate the class of P-points. In [3], Blass used Martin’s axiom for  $\sigma$ -centered posets. Recall that a subset  $X$  of a forcing notion  $\mathbb{P}$  is *centered* if any

finitely many elements of  $X$  have a lower bound in  $\mathbb{P}$ . A forcing notion  $\mathbb{P}$  is called  $\sigma$ -centered if  $\mathbb{P} = \bigcup_{n \in \omega} \mathbb{P}_n$ , where each  $\mathbb{P}_n$  is centered. *Martin's axiom for  $\sigma$ -centered posets*, denoted  $\text{MA}(\sigma\text{-centered})$ , is the following statement: for every  $\sigma$ -centered poset  $\mathbb{P}$  and every collection  $\mathcal{X}$  of fewer than  $\mathfrak{c} = 2^{\aleph_0}$  many dense subsets of  $\mathbb{P}$ , there is a filter  $G \subset \mathbb{P}$  such that  $\forall D \in \mathcal{X} [G \cap D \neq \emptyset]$ .  $\text{MA}(\sigma\text{-centered})$  is a mild hypothesis; it is implied both by CH and by forcing axioms such as Martin's axiom (MA) and the Proper Forcing Axiom (PFA). It has some status as a “quasi-axiom” because it is a forcing axiom for a class of very well-behaved posets, and last but not least, it allows us to build P-points in a generic manner. For these reasons it is arguable that  $\text{MA}(\sigma\text{-centered})$  is the right axiom under which to study the global structure of the P-points. It should be noted however that  $\text{MA}(\sigma\text{-centered})$  is not the minimal assumption guaranteeing the existence of many P-points. The statement  $\mathfrak{d} = \mathfrak{c}$  is equivalent to the statement that every filter generated by  $< \mathfrak{c}$  elements can be extended to a P-point (see Theorem 4.4.5 of [1]).

We should point out that  $\text{MA}(\sigma\text{-centered})$  is equivalent to the statement that  $\mathfrak{p} = \mathfrak{c}$  (see Theorem 7.12 of [6]). A family  $F \subset [\omega]^\omega$  is said to have the *finite intersection property (FIP)* if for any  $a_0, \dots, a_k \in F$ ,  $a_0 \cap \dots \cap a_k$  is infinite.  $\mathfrak{p}$  is the minimal cardinal  $\kappa$  such that there is a family  $F \subset [\omega]^\omega$  of size  $\kappa$  with the FIP, but for which there is no  $b \in [\omega]^\omega$  such that  $\forall a \in F [b \subset^* a]$ .

Among other results, Blass [3] showed that  $\text{MA}(\sigma\text{-centered})$  implies that both  $\omega_1$  and  $\mathbb{R}$  (the real numbers ordered as usual) can be embedded into the P-points under the Rudin-Keisler ordering. He also proved that there are  $2^{2^\omega}$  many Rudin-Keisler incomparable P-points under  $\text{MA}(\sigma\text{-centered})$ . He posed the following question in his paper.<sup>1</sup>

**Question 3** (Blass, 1973). Assuming  $\text{MA}(\sigma\text{-centered})$ , what partial orders can be embedded into the P-points with respect to the Rudin-Keisler ordering?

Some analogues of Blass' results from [3] were proved much later for the notion of Tukey reducibility of ultrafilters. The general notion of Tukey reducibility between directed quasi-orders arose with the Moore-Smith theory of convergence in topological spaces. We say that a quasi-order  $\langle D, \leq \rangle$  is *directed* if any two members of  $D$  have an upper bound in  $D$ . A set  $X \subset D$  is *unbounded in  $D$*  if it doesn't have an upper bound in  $D$ . A set  $X \subset D$  is said to be *cofinal in  $D$*  if  $\forall y \in D \exists x \in X [y \leq x]$ . Given directed sets  $D$  and  $E$ , a map  $f : D \rightarrow E$  is called a *Tukey map* if the image of every unbounded subset of  $D$  is unbounded in  $E$ . A map  $g : E \rightarrow D$  is called a *convergent map* if the image of every cofinal subset of  $E$  is cofinal in  $D$ . Even though this is not the original definition of a convergent map, it is known to be equivalent to the original definition. It is not difficult to show that there is a Tukey map  $f : D \rightarrow E$  if and only if there is a convergent map  $g : E \rightarrow D$ .

**Definition 4.** We say that  $D$  is *Tukey reducible* to  $E$ , and we write  $D \leq_T E$  if there is a convergent map  $g : E \rightarrow D$ . We say that  $D$  and  $E$  are *Tukey equivalent* or have the same *cofinal type* if both  $D \leq_T E$  and  $E \leq_T D$  hold.

The topological significance of these notions is that if  $D \leq_T E$ , then any  $D$ -net on a topological space contains an  $E$ -subnet.

<sup>1</sup>Question 4 of [3] asks explicitly only about ordinals; but given the other results in that paper, the more general question is implicit.

If  $\mathcal{U}$  is any ultrafilter on  $\omega$ , then  $\langle \mathcal{U}, \supset \rangle$  is a directed set. When ultrafilters are viewed as directed sets in this way, Tukey reducibility is a coarser quasi-order than RK reducibility. In other words, if  $\mathcal{U} \leq_{RK} \mathcal{V}$ , then  $\mathcal{U} \leq_T \mathcal{V}$ . In contrast with Kunen's theorem discussed above, it is unknown whether it is possible to construct two ultrafilters on  $\omega$  that are not Tukey equivalent using only ZFC. For any  $\mathcal{X}, \mathcal{Y} \subset \mathcal{P}(\omega)$ , a map  $\phi : \mathcal{X} \rightarrow \mathcal{Y}$  is said to be *monotone* if  $\forall a, b \in \mathcal{X} [a \subset b \implies \phi(a) \subset \phi(b)]$ , and  $\phi$  is said to be *cofinal in  $\mathcal{Y}$*  if  $\forall b \in \mathcal{Y} \exists a \in \mathcal{X} [\phi(a) \subset b]$ . It is a useful and easy fact that if  $\mathcal{U}$  and  $\mathcal{V}$  are ultrafilters on  $\omega$ , then  $\mathcal{U} \leq_T \mathcal{V}$  if and only if there exists a  $\phi : \mathcal{V} \rightarrow \mathcal{U}$  that is monotone and cofinal in  $\mathcal{U}$ .

The order  $\leq_T$  on the class of ultrafilters and particularly on the class of P-points has been studied recently in Milovich [21, 22], Raghavan and Todorćević [23], Dobrinen and Todorćević [12–14], and Dobrinen, Mijares, and Trujillo [11]. Dobrinen and Todorćević [12] showed that  $\omega_1$  can be embedded into the P-points under the Tukey order, and Raghavan (unpublished) showed the same for  $\mathbb{R}$ . Dobrinen and Todorćević also proved that assuming  $\mathfrak{d} = \mathfrak{u} = \mathfrak{c}$ , there are  $2^{\mathfrak{c}}$  many Tukey incomparable P-points. These results rely on the fact, discovered by Dobrinen and Todorćević [12], that if  $\mathcal{V}$  is a P-point and  $g$  is any monotone cofinal map from  $\mathcal{V}$  to another ultrafilter, then there is an  $X \in \mathcal{V}$  such that  $g$  is continuous when restricted below  $X$ . This map can then be extended to a continuous map  $g^*$  on all subsets of  $\omega$ . As  $2^\omega$  is compact,  $g^*$  is represented by a finitary map on  $2^{<\omega}$ . We will need a consequence of this fact for our construction in this paper, which is stated in Lemma 28.

A recent work of Dobrinen, Mijares, and Trujillo [11] shows that  $\text{MA}(\sigma\text{-centered})$  implies that the structure  $\langle \text{FIN}, \subsetneq \rangle$  embeds into the P-points both under RK and Tukey reducibility. FIN denotes the collection of finite subsets of  $\omega$ . They also show in their paper that the collection of all isomorphism types, partially ordered by embedding, of any finite product of Fraïssé classes of relational structures with the Ramsey property plus another property they call the OPFAP appears as an exact RK structure of P-points, closed under RK reduction. Such an exact structural result is more informative than an embedding result because the embedded structure appears as an initial segment of the P-points and one gets a description of all the ultrafilters that are RK below any given P-point in the embedded structure.

These results motivate us to ask the analogue of Question 3 for the Tukey ordering also. The main aim of this paper is to treat Question 3 as well as its Tukey analogue. We will prove the following theorem.

**Main Theorem.** *Assume  $\text{MA}(\sigma\text{-centered})$ . Then there is a sequence of P-points  $\langle \mathcal{U}_{[a]} : [a] \in \mathcal{P}(\omega)/\text{FIN} \rangle$  such that*

- (1) *if  $a \subset^* b$ , then  $\mathcal{U}_{[a]} \leq_{RK} \mathcal{U}_{[b]}$ ;*
- (2) *if  $b \not\subset^* a$ , then  $\mathcal{U}_{[b]} \not\leq_T \mathcal{U}_{[a]}$ .*

FIN is an ideal in the Boolean algebra  $\mathcal{P}(\omega)$ , and  $\mathcal{P}(\omega)/\text{FIN}$  is the quotient algebra. For each  $a \in \mathcal{P}(\omega)$ ,  $[a]$  denotes the equivalence class of  $a$  in  $\mathcal{P}(\omega)/\text{FIN}$ . Thus the theorem says that  $\mathcal{P}(\omega)/\text{FIN}$  with its natural partial order embeds into the class of P-points with respect to both Rudin-Keisler and Tukey reducibility.

Under  $\text{MA}(\sigma\text{-centered})$  every partial order of size less than  $\mathfrak{c}$  as well as every partial order of size at most  $\aleph_1$  can be embedded into  $\mathcal{P}(\omega)/\text{FIN}$  (see exercise 23 in Chapter II of Kunen [17] and also [9]). Baumgartner, Frankiewicz, and Zbierski [2]

proved that it is consistent with  $\text{MA}(\sigma\text{-centered})$  that  $\mathfrak{c}$  is arbitrarily large and every partial order of size at most  $\mathfrak{c}$  embeds into  $\mathcal{P}(\omega)/\text{FIN}$ . This corollary summarizes various consequences of the Main Theorem.<sup>2</sup>

**Corollary 5.** *The following statements are true when the class of  $P$ -points is equipped with either the RK ordering or the Tukey ordering:*

- (1) *Under CH any partial order of size at most  $\mathfrak{c}$  embeds into the  $P$ -points;*
- (2)  *$\text{MA}(\sigma\text{-centered})$  implies that any partial order of size less than  $\mathfrak{c}$  embeds into the  $P$ -points;*
- (3) *for each  $\alpha$  it is consistent with  $\text{MA}(\sigma\text{-centered})$  and  $\mathfrak{c} > \aleph_\alpha$  that any partial order of size at most  $\mathfrak{c}$  embeds into the  $P$ -points.*

Since there are only  $\mathfrak{c}$  many functions from  $\omega$  to  $\omega$  and also only  $\mathfrak{c}$  many continuous functions from  $\mathcal{P}(\omega)$  to  $\mathcal{P}(\omega)$ , any given  $P$ -point can have at most  $\mathfrak{c}$  many ultrafilters below it both with respect to RK and Tukey reducibility. Therefore at least under CH, Corollary 5 is the best possible result for partial orders having a greatest element. However it does not settle which partial orders of size greater than  $\mathfrak{c}$  can consistently be embedded into the  $P$ -points (see Section 3 for further discussion of what remains open).

Theorem 34 is proved using the technique of normed creatures pioneered by Shelah together with some of his coauthors including Goldstern, Kellner, Mildenberger, and Roslanowski. While this method is usually used for getting consistency results in set theory of the reals (see [24]), it is a flexible method that can also be used for carrying out constructions from forcing axioms. The method developed in this paper for building ultrafilters is likely to be applicable to questions that ask whether certain classes of  $P$ -points can be distinguished from each other. For instance, the questions posed at the end of [7] about interval  $P$ -points are likely to be amenable to our methods.

We end this introduction by fixing some notational conventions that will apply to the entire paper.  $A \subset B$  if and only if  $\forall x [x \in A \implies x \in B]$ , so the symbol “ $\subset$ ” does not denote a proper subset. “ $\forall^\infty x \dots$ ” abbreviates the quantifier “for all but finitely many  $x \dots$ ” and “ $\exists^\infty x \dots$ ” stands for “there exist infinitely many  $x$  such that  $\dots$ ”. Given sets  $X$  and  $Y$ ,  $X^Y$  denotes the collection of all functions from  $Y$  to  $X$ . Given a set  $a$ ,  $\mathcal{P}(a)$  denotes the power set of  $a$ .  $[\omega]^\omega$  refers to the collection of all infinite subsets of  $\omega$ , and  $[\omega]^{<\omega}$  is the collection of all finite subsets of  $\omega$ . A filter  $\mathcal{F}$  on  $\omega$  is required to be both *proper*, meaning  $0 \notin \mathcal{F}$ , and *non-principal*, meaning that  $\forall F \in [\omega]^{<\omega} [\omega \setminus F \in \mathcal{F}]$ . Finally  $A \subset^* B$  means  $A \setminus B$  is finite and  $A =^* B$  means  $A \subset^* B$  and  $B \subset^* A$ .

## 2. THE CONSTRUCTION

We will build a set of ultrafilters  $\{\mathcal{U}_A : A \in \mathcal{X}\}$ , where  $\mathcal{X}$  is some set of representatives for  $\mathcal{P}(\omega)/\text{FIN}$ . We will also build a corresponding set of maps in  $\omega^\omega$ ,  $\{\pi_{B,A} : A, B \in \mathcal{X} \wedge A \subset^* B\}$ , ensuring that if  $A \subset^* B$  are any two members of  $\mathcal{X}$ , then  $\pi_{B,A}$  is an RK-map from  $\mathcal{U}_B$  to  $\mathcal{U}_A$ . We first define the notion of a creature needed for the construction and establish its most important properties.

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<sup>2</sup>This corollary was incorrectly stated in an earlier version of this paper. We thank Alan Dow and K. P. Hart for pointing out this error.

**Definition 6.** Let  $A$  be a non-empty finite set. Say that  $u$  is a *creature acting on  $A$*  if  $u$  is a pair of sequences  $\langle\langle u_a : a \subset A \rangle\rangle, \langle\langle \pi_{u,b,a} : a \subset b \subset A \rangle\rangle$  such that the following hold:

- (1) each  $u_a$  is a non-empty finite set;
- (2)  $\pi_{u,b,a} : u_b \rightarrow u_a$  is an onto function;
- (3) if  $a \subset b \subset c$ , then  $\pi_{u,c,a} = \pi_{u,b,a} \circ \pi_{u,c,b}$ .

The collection of all creatures acting on  $A$  is denoted  $\mathcal{CR}(A)$ . Strictly speaking of course  $\mathcal{CR}(A)$  is a proper class, but we may restrict ourselves to the ones in  $H(\omega)$ .

The idea of this definition is that  $u$  acts on the finite bit of information available to it to produce approximations to sets that will end up in various ultrafilters and also approximations to various RK maps. More explicitly, if  $X \in \mathcal{P}(\omega)$  and  $A$  is some appropriately chosen finite subset of  $\omega$ , then  $u_{X \cap A}$  is an approximation to some set in the ultrafilter  $\mathcal{U}_X$ . Similarly if  $X \subset^* Y$  and if  $X \cap A \subset Y \cap A$ , then  $\pi_{u,Y \cap A, X \cap A}$  approximates the RK map  $\pi_{Y,X}$ . In the main construction,  $A$  and  $u_a$  will be subsets of  $\omega$ .

**Definition 7.** For a non-empty finite set  $A$  and  $u \in \mathcal{CR}(A)$ ,  $\Sigma(u)$  denotes the collection of all  $v \in \mathcal{CR}(A)$  such that:

- (1) for each  $a \subset A$ ,  $v_a \subset u_a$ ;
- (2) for each  $a \subset b \subset A$ ,  $\pi_{v,b,a} = \pi_{u,b,a} \upharpoonright v_b$ .

Note that if  $v \in \Sigma(u)$ , then  $\Sigma(v) \subset \Sigma(u)$ .

**Definition 8.** For a non-empty finite set  $A$ , define the *norm* of  $u \in \mathcal{CR}(A)$ , denoted  $\text{nor}(u)$ , as follows. We first define by induction on  $n \in \omega$  the relation  $\text{nor}(u) \geq n$  by the following clauses:

- (1)  $\text{nor}(u) \geq 0$  always holds;
- (2)  $\text{nor}(u) \geq n + 1$  if and only if
  - (a) for each  $a \subset A$ , if  $u_a = u^0 \cup u^1$ , then there exist  $v \in \Sigma(u)$  and  $i \in 2$  such that  $\text{nor}(v) \geq n$  and  $v_a \subset u^i$ ;
  - (b) for any  $a, b \subset A$ , if  $b \not\subset a$ , then for every function  $F : \mathcal{P}(u_a) \rightarrow u_b$ , there exists  $v \in \Sigma(u)$  such that  $\text{nor}(v) \geq n$  and  $F''\mathcal{P}(v_a) \cap v_b = 0$ .

Define  $\text{nor}(u) = \max\{n \in \omega : \text{nor}(u) \geq n\}$ .

It is easily seen that if  $u \in \mathcal{CR}(A)$ ,  $v \in \Sigma(u)$ , and  $\text{nor}(v) \geq n$ , then  $\text{nor}(u) \geq n$  as well. It follows for any  $u \in \mathcal{CR}(A)$  that if  $\text{nor}(u) \geq k$ , then for all  $n \leq k$ ,  $\text{nor}(u) \geq n$ . Because of the requirement that both  $A$  and  $u_a$  be non-empty,  $\text{nor}(u)$  is well defined for every  $u \in \mathcal{CR}(A)$ . To elaborate, if  $k \in \omega$ ,  $u \in \mathcal{CR}(A)$ , and  $\text{nor}(u) \geq k + 1$ , then since  $0, A \subset A$ , and  $A \neq 0$ , clause (2)(b) applies to  $0$  and  $A$ . By definition  $u_A \neq 0$ ; fix  $x_0 \in u_A$ . Define a function  $F : \mathcal{P}(u_0) \rightarrow u_A$  by stipulating that  $F(y) = x_0$ , for every  $y \in \mathcal{P}(u_0)$ . By (2)(b) there exists  $v \in \Sigma(u)$  such that  $\text{nor}(v) \geq k$  and  $F''\mathcal{P}(v_0) \cap v_A = 0$ . Thus  $x_0 \notin v_A$  because  $x_0 \in F''\mathcal{P}(v_0)$ . As  $v_A \neq 0$ , we can choose  $x_1 \in v_A$ . Then  $x_0, x_1 \in u_A$  and  $x_1 \neq x_0$ . So we conclude that  $|u_A| \geq 2$ , if  $\text{nor}(u) \geq k + 1$ . Next, using this fact and clause (2)(a), a straightforward induction on  $k \in \omega$  shows that for any  $u \in \mathcal{CR}(A)$ , if  $\text{nor}(u) \geq k$ , then  $|u_A| \geq k$ . This shows that  $\text{nor}(u)$  is well defined. Clause (2)(a) ensures that we can construct ultrafilters, while clause (2)(b) is needed to ensure that if  $X, Y \in \mathcal{X}$  and  $Y \not\subset^* X$ , then  $\mathcal{U}_Y \not\subseteq_T \mathcal{U}_X$ .

The next lemma, which is a Ramsey type theorem for a finite product of finite sets, is well known. It is a special case of several much more general theorems. It

follows from the work of Llopis and Todorćević [20], and also appears in Chapter 3 of [29]. Far-reaching generalizations of this lemma can be found in [24], [10], and [28]. We give a self-contained proof of the simple special case which we use.

**Lemma 9.** *For each  $n < \omega$ , for each  $0 < l < \omega$ , and for each  $k < l$ , there exists  $0 < i(n, l, k) < \omega$  such that:*

- (1) *for each  $0 < m \leq l$ , if  $\langle F_k : k < m \rangle$  is a sequence of sets such that  $\forall k < m [ |F_k| = i(n+1, l, k) ]$  and if  $\prod_{k < m} F_k = X_0 \cup X_1$ , then there exist  $j \in 2$  and a sequence  $\langle E_k : k < m \rangle$  such that  $\forall k < m [ E_k \subset F_k \wedge |E_k| = i(n, l, k) ]$  and  $\prod_{k < m} E_k \subset X_j$ ;*
- (2)  *$i(n+1, l, k) \geq 2^{x(n, l)} + i(n, l, k)$ , where  $x(n, l) = \prod_{k < l} i(n, l, k)$ .*

*Proof.* We define  $i(n, l, k)$  by induction on  $n \in \omega$ , and for a fixed  $n$  and a fixed  $0 < l < \omega$ , by induction on  $k < l$ . Put  $i(0, l, k) = 1$  for all  $0 < l < \omega$  and  $k < l$ . Fix  $n \in \omega$ . Suppose that  $i(n, l, k)$  is given for all  $0 < l < \omega$  and all  $k < l$ . Fix  $0 < l < \omega$ . We define  $i(n+1, l, k)$  by induction on  $k < l$ . Let  $x(n, l)$  be as in (2) above. Note that  $0 < x(n, l) < \omega$  and that for any  $k < l$ ,  $0 < i(n, l, k) < 2^{x(n, l)} + i(n, l, k) < \omega$ . Now fix  $k < l$  and assume that  $i(n+1, l, k')$  has been defined for all  $k' < k$ . Define  $y(n+1, l, k) = \prod_{k' < k} i(n+1, l, k')$  (when  $k = 0$  this product is taken to be 1)

and let  $z(n+1, l, k) = 2^{y(n+1, l, k)} i(n, l, k)$ . Note that  $0 < z(n+1, l, k) < \omega$ . Put  $i(n+1, l, k) = \max\{z(n+1, l, k), 2^{x(n, l)} + i(n, l, k)\}$ . Thus  $0 < i(n+1, l, k) < \omega$  and  $i(n+1, l, k) \geq 2^{x(n, l)} + i(n, l, k)$  as needed for (2).

To verify (1) fix  $n \in \omega$  and  $0 < l < \omega$ . We induct on  $0 < m \leq l$ . Suppose  $m = 1$  and suppose  $|F_0| = i(n+1, l, 0)$  and suppose that  $F_0 = X_0 \cup X_1$ . Then  $i(n+1, l, 0) \geq 2i(n, l, 0)$ . So there exists  $j \in 2$  and  $E_0 \subset X_j \subset F_0$  such that  $|E_0| = i(n, l, 0)$ , as needed.

Now fix  $0 < m < m+1 \leq l$  and suppose that the required statement holds for  $m$ . Let  $\langle F_k : k < m+1 \rangle$  be a sequence of sets such that  $\forall k < m+1 [ |F_k| = i(n+1, l, k) ]$  and suppose that  $\prod_{k < m+1} F_k = X_0 \cup X_1$ . Let  $\langle \sigma_i : i < y(n+1, l, m) \rangle$  enumerate the members of  $\prod_{k < m} F_k$ . Build a sequence  $\langle E_m^i : -1 \leq i < y(n+1, l, m) \rangle$  such that the following hold:

- (3)  $E_m^{-1} \subset F_m$  and  $\forall -1 \leq i < i+1 < y(n+1, l, m) [ E_m^{i+1} \subset E_m^i ]$ ;
- (4)  $\forall -1 \leq i < y(n+1, l, m) [ |E_m^i| = 2^{y(n+1, l, m) - i - 1} i(n, l, m) ]$ ;
- (5)  $\forall 0 \leq i < y(n+1, l, m) \exists j_i \in 2 \forall x \in E_m^i [ (\sigma_i) \frown \langle x \rangle \in X_{j_i} ]$ .

The sequence is constructed by induction. To start, choose  $E_m^{-1} \subset F_m$  of size equal to  $2^{y(n+1, l, m)} i(n, l, m)$  (possible because  $|F_m| = i(n+1, l, m) \geq 2^{y(n+1, l, m)} i(n, l, m)$ ). Now suppose that  $-1 \leq i < i+1 < y(n+1, l, m)$  and that  $E_m^i$  is given. For each  $j \in 2$  let  $Z_j = \{x \in E_m^i : (\sigma_{i+1}) \frown \langle x \rangle \in X_j\}$ . Then  $E_m^i = Z_0 \cup Z_1$  and so there exist  $E_m^{i+1} \subset E_m^i$  and  $j_{i+1} \in 2$  such that  $|E_m^{i+1}| = 2^{y(n+1, l, m) - i - 2} i(n, l, m)$  and  $E_m^{i+1} \subset Z_{j_{i+1}}$ . It is then clear that  $E_m^{i+1}$  and  $j_{i+1}$  satisfy (3)-(5). This completes the construction of the sequence  $\langle E_m^i : -1 \leq i < y(n+1, l, m) \rangle$ . For  $j \in 2$  define  $Y_j = \{\sigma_i : 0 \leq i < y(n+1, l, m) \wedge j_i = j\}$ . It is clear that

$\prod_{k < m} F_k = Y_0 \cup Y_1$ . So by the inductive hypothesis, there exist  $j \in 2$  and a sequence  $\langle E_k : k < m \rangle$  such that  $\forall k < m [E_k \subset F_k \wedge |E_k| = i(n, l, k)]$  and  $\prod_{k < m} E_k \subset Y_j$ . Now put  $E_m = E_m^{y(n+1, l, m)-1}$ . The sequence  $\langle E_k : k < m+1 \rangle$  and  $j \in 2$  are as needed. This completes the verification of (1) and the proof of the lemma.  $\square$

We use Lemma 9 to show that there exist creatures of arbitrarily high norm. This is an essential step to defining a partial order out of any notion of a creature. In our case each condition of the partial order is an approximation to the final collection of ultrafilters and RK-maps.

**Corollary 10.** *Let  $A$  be a non-empty finite set and  $l = 2^{|A|}$ . Suppose  $\langle s_k : k < l \rangle$  is an enumeration of all the subsets of  $A$  such that if  $k' < k$ , then  $s_k \not\subset s_{k'}$ . For each  $a \subset A$ , let  $D_a$  denote  $\{k < l : s_k \subset a\}$ . For each  $n \in \omega$ , if  $\langle F_k : k < l \rangle$  is any sequence of sets such that  $\forall k < l [|F_k| = i(n, l, k)]$ , then  $u = \langle \langle u_a : a \subset A \rangle, \langle \pi_{u, b, a} : a \subset b \subset A \rangle \rangle$ , where  $u_a = \prod \{F_k : k \in D_a\}$  and  $\pi_{u, b, a}(s) = s \upharpoonright D_a$ , is a member of  $\mathcal{CR}(A)$  and has norm at least  $n$ .*

*Proof.* Since  $i(n, l, k)$  is always at least 1,  $u$  as defined above is always a member of  $\mathcal{CR}(A)$  with  $\text{nor}(u) \geq 0$  regardless of what  $n$  is. So the claim holds for  $n = 0$ . We assume that the claim is true for some  $n \in \omega$  and check it for  $n+1$ . Indeed let  $\langle F_k : k < l \rangle$  be any sequence of sets with  $|F_k| = i(n+1, l, k)$  and let  $u$  be defined as above from  $\langle F_k : k < l \rangle$ . Suppose that  $a \subset A$  and that  $u_a = u^0 \cup u^1$ . Then  $\prod \{F_k : k < l\} = X_0 \cup X_1$ , where  $X_j = \{s \in X : s \upharpoonright D_a \in u^j\}$ . By (1) of Lemma 9 applied with  $m = l$ , there exist a sequence  $\langle E_k : k < l \rangle$  and a  $j \in 2$  such that  $E_k \subset F_k$ ,  $|E_k| = i(n, l, k)$ , and  $\prod \{E_k : k < l\} \subset X_j$ . Now if  $v$  is defined from the sequence  $\langle E_k : k < l \rangle$  in the same manner in which  $u$  is defined from the sequence  $\langle F_k : k < l \rangle$  in the statement of Corollary 10, then by the inductive hypothesis  $v \in \mathcal{CR}(A)$  and  $\text{nor}(v) \geq n$ . Moreover it is clear that  $v \in \Sigma(u)$  and that  $v_a \subset u^j$ . So this checks clause (2)(a) of Definition 8.

For clause (2)(b), fix  $a, b \subset A$  with  $b \not\subset a$ . Let  $F : \mathcal{P}(u_a) \rightarrow u_b$  be any function. For each  $k < l$ , let  $G_k \subset F_k$  with  $|G_k| = i(n, l, k)$ . This is possible to do because by (2) of Lemma 9,  $\forall k < l [|F_k| = i(n+1, l, k) \geq 2^{x(n, l)} + i(n, l, k) \geq i(n, l, k)]$ , where  $x(n, l)$  is defined as there. Note that  $D_b \setminus D_a \neq \emptyset$ . Fix  $k_0 \in D_b \setminus D_a$ . Let  $e = \prod \{G_k : k \in D_a\}$  and let  $o = \prod \{G_k : k \in l\}$ . Since  $|G_k| = i(n, l, k)$ ,  $|o| = \prod_{k < l} i(n, l, k) = x(n, l)$ , and so  $|\mathcal{P}(o)| = 2^{x(n, l)}$ . Let  $M = \{s(k_0) : s \in F''\mathcal{P}(e)\}$ . Then  $M \subset F_{k_0}$  and  $|M| \leq |\mathcal{P}(e)| \leq |\mathcal{P}(o)| = 2^{x(n, l)}$ . There exists  $E_{k_0} \subset F_{k_0}$  such that  $|E_{k_0}| = i(n, l, k_0)$  and  $E_{k_0} \cap M = \emptyset$  because  $|F_{k_0}| \geq 2^{x(n, l)} + i(n, l, k_0)$ . For all  $k \in l \setminus \{k_0\}$ , let  $E_k = G_k$ . Then  $\langle E_k : k < l \rangle$  is a sequence of sets such that  $\forall k < l [E_k \subset F_k \wedge |E_k| = i(n, l, k)]$ . So by the inductive hypothesis if  $v$  is defined as above from  $\langle E_k : k < l \rangle$ , then  $v \in \mathcal{CR}(A)$  and  $\text{nor}(v) \geq n$ . Moreover  $v \in \Sigma(u)$ . We check that  $F''\mathcal{P}(v_a) \cap v_b = \emptyset$ . Since  $k_0 \notin D_a$ ,  $\forall k \in D_a [E_k = G_k]$ . Therefore  $v_a = e$ . So if  $s \in F''\mathcal{P}(v_a) \cap v_b$ , then  $s(k_0) \in M$ . On the other hand by the definition of  $v_b$ ,  $s(k_0) \in E_{k_0}$ . Hence  $M \cap E_{k_0} \neq \emptyset$ , contradicting the choice of  $E_{k_0}$ . Therefore  $F''\mathcal{P}(v_a) \cap v_b = \emptyset$ . This concludes the verification of clause (2)(b) of Definition 8 and the proof that  $\text{nor}(u) \geq n+1$ .  $\square$



One of the main features of the final construction will be that creatures will be allowed to “shift” their scene of action. In fact, we will want to perform this shifting operation infinitely often. The following two lemmas ensure that the two main features of a creature  $u$ , namely  $\text{nor}(u)$  and  $\Sigma(u)$ , are preserved while shifting.

**Definition 11.** Let  $A$  and  $B$  be non-empty finite sets and suppose  $h : B \rightarrow A$  is an onto function. Let  $u$  be a creature acting on  $B$ . Define  $h[u] = v = \langle \langle v_a : a \subset A \rangle, \langle \pi_{v,a^*,a} : a \subset a^* \subset A \rangle \rangle$  by the following clauses:

- (1) for all  $a \subset A$ ,  $v_a = u_{h^{-1}(a)}$ ;
- (2) for all  $a \subset a^* \subset A$ ,  $\pi_{v,a^*,a} = \pi_{u,h^{-1}(a^*),h^{-1}(a)}$ .

**Lemma 12.** Let  $A$ ,  $B$ ,  $h$ ,  $u$ , and  $v = h[u]$  be as in Definition 11. Then  $v$  is a creature acting on  $A$ . Moreover, for any  $w \in \Sigma(u)$ ,  $h[w] \in \Sigma(v)$ .

*Proof.* For any  $a \subset A$ ,  $h^{-1}(a) \subset B$ , and so  $v_a = u_{h^{-1}(a)}$  is a non-empty finite set. Similarly if  $a \subset a^* \subset A$ , then  $h^{-1}(a) \subset h^{-1}(a^*) \subset B$ , and so  $\pi_{v,a^*,a} = \pi_{u,h^{-1}(a^*),h^{-1}(a)}$  is an onto map from  $v_{a^*} = u_{h^{-1}(a^*)}$  to  $u_{h^{-1}(a)} = v_a$ . Also if  $a \subset a^* \subset a^{**} \subset A$ , then  $h^{-1}(a) \subset h^{-1}(a^*) \subset h^{-1}(a^{**}) \subset B$ , and so  $\pi_{v,a^{**},a} = \pi_{u,h^{-1}(a^{**}),h^{-1}(a)} = \pi_{u,h^{-1}(a^{**}),h^{-1}(a^*)} \circ \pi_{u,h^{-1}(a^*),h^{-1}(a)} = \pi_{v,a^{**},a^*} \circ \pi_{v,a^*,a}$ . Thus  $v$  is a creature acting on  $A$ .

Next, suppose that  $w \in \Sigma(u)$ . By the above  $h[w]$  is a creature acting on  $A$ . If  $a \subset A$ , then  $(h[w])_a = w_{h^{-1}(a)} \subset u_{h^{-1}(a)} = v_a$ . Likewise, if  $a \subset a^* \subset A$ , then  $\pi_{h[w],a^*,a} = \pi_{w,h^{-1}(a^*),h^{-1}(a)} = \pi_{u,h^{-1}(a^*),h^{-1}(a)} \upharpoonright w_{h^{-1}(a^*)} = \pi_{v,a^*,a} \upharpoonright (h[w])_{a^*}$ . Thus  $h[w] \in \Sigma(v)$ .  $\square$

**Lemma 13.** Let  $A$ ,  $B$ ,  $h$ ,  $u$ , and  $v$  be as in Definition 11. For each  $n \in \omega$ , if  $\text{nor}(u) \geq n$ , then  $\text{nor}(v) \geq n$ .

*Proof.* The proof is by induction on  $n$ . For  $n = 0$ , by Lemma 12  $v$  is a creature acting on  $A$ , and so  $\text{nor}(v) \geq 0$ . Assume that it holds for  $n$  and suppose  $\text{nor}(u) \geq n + 1$ . We first check clause (2)(a) of Definition 8. Let  $a \subset A$  and suppose that  $v_a = v^0 \cup v^1$ . Then  $h^{-1}(a) \subset B$  and  $v_a = u_{h^{-1}(a)} = v^0 \cup v^1$ . So there exists  $w \in \Sigma(u)$  with  $\text{nor}(w) \geq n$  and  $i \in 2$  such that  $w_{h^{-1}(a)} \subset v^i$ . By Lemma 12  $h[w] \in \Sigma(v)$  and by the induction hypothesis  $\text{nor}(h[w]) \geq n$ . Also  $(h[w])_a = w_{h^{-1}(a)} \subset v^i$ . This checks clause (2)(a) of Definition 8.

For clause (2)(b), fix  $a, a^* \subset A$  and suppose that  $a^* \not\subset a$ . Let  $F : \mathcal{P}(v_a) \rightarrow v_{a^*}$ . We have  $h^{-1}(a), h^{-1}(a^*) \subset B$ . Since  $h$  is onto,  $h^{-1}(a^* \setminus a) = h^{-1}(a^*) \setminus h^{-1}(a) \neq \emptyset$ . So  $h^{-1}(a^*) \not\subset h^{-1}(a)$ . Also  $F : \mathcal{P}(u_{h^{-1}(a)}) \rightarrow u_{h^{-1}(a^*)}$ . As  $\text{nor}(u) \geq n + 1$ , we can find  $w \in \Sigma(u)$  with  $\text{nor}(w) \geq n$  such that  $F''\mathcal{P}(w_{h^{-1}(a)}) \cap w_{h^{-1}(a^*)} = \emptyset$ . By Lemma 12  $h[w] \in \Sigma(v)$  and by the inductive hypothesis  $\text{nor}(h[w]) \geq n$ . Also  $(h[w])_a = w_{h^{-1}(a)}$  and  $(h[w])_{a^*} = w_{h^{-1}(a^*)}$ . Therefore  $F''\mathcal{P}((h[w])_a) \cap (h[w])_{a^*} = \emptyset$ . This checks that  $\text{nor}(v) \geq n + 1$  and concludes the proof.  $\square$

We are now ready to define the forcing poset which we use. We define a version of the poset that makes sense even in the absence of  $\text{MA}(\sigma\text{-centered})$ , though  $\text{MA}(\sigma\text{-centered})$  is needed for the various density arguments.

**Definition 14.** We say that  $q$  is a *standard sequence* if  $q$  is a pair  $\langle I_q, U_q \rangle$  such that:

- (1)  $I_q = \langle I_{q,n} : n \in \omega \rangle$  is a sequence of non-empty finite subsets of  $\omega$  such that  $\forall n \in \omega [\max(I_{q,n}) < \min(I_{q,n+1})]$ ;

- (2)  $U_q = \langle u^{q,n} : n \in \omega \rangle$  is a sequence such that for each  $n \in \omega$ ,  $u^{q,n}$  is a creature acting on  $I_{q,n}$ ; if  $a \subset b \subset I_{q,n}$ , then  $\pi_{u^{q,n},b,a}$  will be denoted  $\pi_{q,b,a}$ ;
- (3) for each  $n \in \omega$  and  $a \subset I_{q,n}$ ,  $u_a^{q,n} \subset \omega$ ;
- (4)  $\text{nor}(u^{q,n}) < \text{nor}(u^{q,n+1})$ , and for all  $a \subset I_{q,n}$  and all  $b \subset I_{q,n+1}$ ,  $\max(u_a^{q,n}) < \min(u_b^{q,n+1})$ .

$\mathbb{Q}$  denotes the set of all standard sequences.

There are several natural partial orderings that can be defined on  $\mathbb{Q}$ . However, we will not be using any ordering on  $\mathbb{Q}$  in our construction.

**Definition 15.**  $p$  is called a 0-condition if  $p = \langle \mathcal{A}_p, \mathcal{C}_p, \mathcal{D}_p \rangle$  where:

- (1)  $\mathcal{A}_p \subset \mathcal{P}(\omega)$ ,  $0, \omega \in \mathcal{A}_p$ ,  $|\mathcal{A}_p| < \mathfrak{c}$ , and  $\forall A, B \in \mathcal{A}_p [A \neq B \implies A \neq^* B]$ ;
- (2)  $\mathcal{D}_p = \langle \mathcal{D}_{p,A} : A \in \mathcal{A}_p \rangle$  is a sequence of non-principal filters on  $\omega$  with the property that for each  $A \in \mathcal{A}_p$  there exists a family  $\mathcal{F}_{p,A} \subset \mathcal{D}_{p,A}$  with  $|\mathcal{F}_{p,A}| < \mathfrak{c}$  such that  $\forall X \in \mathcal{D}_{p,A} \exists Y \in \mathcal{F}_{p,A} [Y \subset X]$ ;
- (3)  $\mathcal{C}_p = \langle \pi_{p,B,A} : A, B \in \mathcal{A}_p \wedge A \subset^* B \rangle$  is a sequence of elements of  $\omega^\omega$ ;
- (4) for all  $A, B \in \mathcal{A}_p$ , if  $A \subset^* B$ , then  $\forall X \in \mathcal{D}_{p,B} [\pi''_{p,B,A} X \in \mathcal{D}_{p,A}]$ .

$\mathbb{P}_0 = \{p : p \text{ is a 0-condition}\}$ . Define an ordering on  $\mathbb{P}_0$  as follows. For any  $p_0, p_1 \in \mathbb{P}_0$ ,  $p_1 \leq p_0$  if and only if  $\mathcal{A}_{p_1} \supset \mathcal{A}_{p_0}$ ,  $\forall A, B \in \mathcal{A}_{p_0} [A \subset^* B \implies \pi_{p_1,B,A} = \pi_{p_0,B,A}]$ , and  $\forall A \in \mathcal{A}_{p_0} [\mathcal{D}_{p_1,A} \supset \mathcal{D}_{p_0,A}]$ .

**Definition 16.** Let  $p \in \mathbb{P}_0$  and  $q \in \mathbb{Q}$ . We say that  $q$  induces  $p$  if the following hold:

- (1) let  $\mathcal{B}$  denote the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $\mathcal{A}_p$ ; then for every infinite member  $A$  of  $\mathcal{B}$ ,  $\forall^\infty n \in \omega [|A \cap I_{q,n}| < |A \cap I_{q,n+1}|]$ ;
- (2) for each  $A \in \mathcal{A}_p$  and each  $X \in \mathcal{D}_{p,A}$ ,  $\forall^\infty n \in \omega [u_{A \cap I_{q,n}}^{q,n} \subset X]$ ;
- (3) for each  $A, B \in \mathcal{A}_p$  with  $A \subset^* B$  the following holds:

$$\forall^\infty n \in \omega \left[ \pi_{p,B,A} \upharpoonright u_{B \cap I_{q,n}}^{q,n} = \pi_{q,B \cap I_{q,n}, A \cap I_{q,n}} \right].$$

Note that if  $p, p' \in \mathbb{P}_0$ ,  $p \leq p'$ ,  $q \in \mathbb{Q}$ , and  $q$  induces  $p$ , then  $q$  also induces  $p'$ . Note also that if  $q$  induces  $p$ , then each of the filters  $\mathcal{D}_{p,A}$  is contained in a countably generated filter. Thus if  $\mathfrak{p} < \mathfrak{c}$ , then there are many 0-conditions  $p$  with the property that no  $p'$  induced by any member of  $\mathbb{Q}$  can be  $\leq p$ . Additionally, if  $p$  is any 0-condition such that  $\pi_{p,B,A}$  is a 1-1 function for some distinct  $A, B \in \mathcal{A}_p$  satisfying  $A \subset^* B$ , then no extension of  $p$  can be induced by any member of  $\mathbb{Q}$ .

The next lemma shows that whenever a  $p$  is induced by some  $q$ , then there is a strongest  $p_0 \leq p$  with the same  $\mathcal{A}_p$  and  $\mathcal{C}_p$  induced by this same  $q$ .

**Lemma 17.** Let  $p \in \mathbb{P}_0$  and suppose  $q \in \mathbb{Q}$  induces  $p$ . Define  $p_0 = \langle \mathcal{A}_{p_0}, \mathcal{C}_{p_0}, \mathcal{D}_{p_0} \rangle$ , where  $\mathcal{A}_{p_0} = \mathcal{A}_p$ ,  $\mathcal{D}_{p_0} = \langle \mathcal{D}_{p_0,A} : A \in \mathcal{A}_{p_0} \rangle$ , where

$$\mathcal{D}_{p_0,A} = \left\{ a \subset \omega : \left( \bigcup_{n \in \omega} u_{A \cap I_{q,n}}^{q,n} \right) \subset^* a \right\},$$

and  $\mathcal{C}_{p_0} = \mathcal{C}_p$ . Then  $p_0 \in \mathbb{P}_0$ ,  $p_0 \leq p$ , and  $q$  induces  $p_0$ .

*Proof.* The only clause in Definition 15 that is not obvious is (4). To check it, fix  $A, B \in \mathcal{A}_{p_0}$  with  $A \subset^* B$ . Since  $\pi_{p_0,B,A} = \pi_{p,B,A}$ , it suffices to show that  $\pi''_{p,B,A} \left( \bigcup_{n \in \omega} u_{B \cap I_{q,n}}^{q,n} \right) \in \mathcal{D}_{p_0,A}$ . For all but finitely many  $n \in \omega$ ,  $A \cap I_{q,n} \subset B \cap I_{q,n}$

and  $\pi_{p,B,A} \upharpoonright u_{B \cap I_{q,n}}^{q,n} = \pi_{q,B \cap I_{q,n}, A \cap I_{q,n}}$  because  $q$  induces  $p$  and  $A \subset^* B$ . Let  $n \in \omega$  be arbitrary such that these two things hold. Then

$$\pi_{q,B \cap I_{q,n}, A \cap I_{q,n}} : u_{B \cap I_{q,n}}^{q,n} \rightarrow u_{A \cap I_{q,n}}^{q,n}$$

is an onto function. Therefore  $\pi''_{p,B,A} u_{B \cap I_{q,n}}^{q,n} = u_{A \cap I_{q,n}}^{q,n}$ . Thus we have shown that  $\forall^\infty n \in \omega \left[ u_{A \cap I_{q,n}}^{q,n} = \pi''_{p,B,A} u_{B \cap I_{q,n}}^{q,n} \right]$ , which implies  $\pi''_{p,B,A} \left( \bigcup_{n \in \omega} u_{B \cap I_{q,n}}^{q,n} \right) \in \mathcal{D}_{p_0,A}$ .

Checking that  $p_0 \leq p$  and that  $q$  induces  $p_0$  is straightforward.  $\square$

**Definition 18.** We say that a 0-condition  $p$  is *finitary* if  $|\mathcal{A}_p| < \omega$  and  $\forall A \in \mathcal{A}_p \exists \mathcal{F}_{p,A} \subset \mathcal{D}_{p,A} [|\mathcal{F}_{p,A}| \leq \omega \wedge \forall X \in \mathcal{D}_{p,A} \exists Y \in \mathcal{F}_{p,A} [Y \subset X]]$ . A 0-condition  $p$  is called a *1-condition* if every finitary  $p' \in \mathbb{P}_0$  that satisfies  $p \leq p'$  is induced by some  $q \in \mathbb{Q}$ . Let  $\mathbb{P}_1 = \{p \in \mathbb{P}_0 : p \text{ is a 1-condition}\}$ . We partially order  $\mathbb{P}_1$  by the same ordering  $\leq$  as  $\mathbb{P}_0$ .

If  $p, p' \in \mathbb{P}_0$ ,  $p$  is finitary, and  $p' \geq p$ , then  $p'$  is not necessarily finitary. The next lemma shows that there are 1-conditions. This essentially comes down to showing that there is a finitary  $p \in \mathbb{P}_0$  which is induced by some  $q \in \mathbb{Q}$ .

**Lemma 19.**  $\mathbb{P}_1$  is non-empty.

*Proof.* Let  $\mathcal{A}_p = \{0, \omega\}$ . Define  $i_0 = 0$  and  $i_{n+1} = 2^{n+1}$  for all  $n \in \omega$ . Let  $I_n = [i_n, i_{n+1})$  and find a sequence  $U = \langle u^n : n \in \omega \rangle$  satisfying clauses (2)-(4) of Definition 14 with respect to  $I = \langle I_n : n \in \omega \rangle$  using Corollary 10. Then  $q = \langle I, U \rangle \in \mathbb{Q}$ . Let  $A_0 = \bigcup_{n \in \omega} u_0^n$  and let  $A_\omega = \bigcup_{n \in \omega} u_{I_n}^n$ . Both of these sets are infinite subsets of  $\omega$ . Let  $\mathcal{D}_{p,0} = \{a \subset \omega : A_0 \subset^* a\}$  and  $\mathcal{D}_{p,\omega} = \{a \subset \omega : A_\omega \subset^* a\}$ . Let  $\mathcal{D}_p = \langle \mathcal{D}_{p,A} : A = 0 \vee A = \omega \rangle$ . Define  $\pi_{p,\omega,0}, \pi_{p,0,0}, \pi_{p,\omega,\omega} \in \omega^\omega$  as follows. Fix  $k \in \omega$ . If  $k \in A_\omega$ , then  $\pi_{p,\omega,0}(k) = \pi_{u^n, I_n, 0}(k)$  and  $\pi_{p,\omega,\omega}(k) = \pi_{u^n, I_n, I_n}(k)$ , where  $n$  is the unique member of  $\omega$  such that  $k \in u_{I_n}^n$ ; if  $k \notin A_\omega$ , then  $\pi_{p,\omega,0}(k) = 0 = \pi_{p,\omega,\omega}(k)$ ; if  $k \in A_0$ , then let  $\pi_{p,0,0}(k) = \pi_{u^n, 0,0}(k)$ , where  $n$  is the unique member of  $\omega$  such that  $k \in u_0^n$ ; if  $k \notin A_0$ , then put  $\pi_{p,0,0}(k) = 0$ . Let  $\mathcal{C}_p = \langle \pi_{p,B,A} : A, B \in \mathcal{A}_p \wedge A \subset^* B \rangle$ . Let  $p = \langle \mathcal{A}_p, \mathcal{C}_p, \mathcal{D}_p \rangle$ . It is easy to check that  $p \in \mathbb{P}_0$  and that  $q$  induces  $p$ . So  $q$  also induces any  $p' \in \mathbb{P}_0$  with  $p \leq p'$ . Thus  $p \in \mathbb{P}_1$ .  $\square$

$\mathbb{P}_1$  is the poset that will be used in the construction. As mentioned earlier, MA( $\sigma$ -centered) is not needed for the definition of  $\mathbb{P}_1$  or to prove that it is non-empty, although it will be needed to prove most of its properties. The first of these properties, proved in the next lemma, shows that for each  $p \in \mathbb{P}_1$  there is a single standard sequence that induces the entire condition.

**Lemma 20** (Representation Lemma). *Assume MA( $\sigma$ -centered). Every  $p \in \mathbb{P}_1$  is induced by some  $q \in \mathbb{Q}$ .*

*Proof.* Fix  $p \in \mathbb{P}_1$ . For each  $A \in \mathcal{A}_p$  choose  $\mathcal{F}_{p,A} \subset \mathcal{D}_{p,A}$  as in (2) of Definition 15. Define a partial order  $\mathbb{R}$  as follows. A condition  $r \in \mathbb{R}$  if and only if  $r = \langle f_r, g_r, F_r, \Phi_r \rangle$  where:

- (1)  $\langle f_r, g_r \rangle$  is an initial segment of some standard sequence—that is, there exist  $n_r \in \omega$  and a standard sequence  $\langle I, U \rangle$  such that  $f_r = I \upharpoonright n_r$  and  $g_r = U \upharpoonright n_r$ ;
- (2)  $F_r$  is a finite subset of  $\mathcal{A}_p$ ;
- (3)  $\Phi_r$  is a function with domain  $F_r$  such that  $\forall A \in F_r [\Phi_r(A) \in \mathcal{D}_{p,A}]$ .

Partially order  $\mathbb{R}$  by stipulating that  $s \preceq r$  if and only if

- (4)  $f_s \supset f_r$ ,  $g_s \supset g_r$ ,  $F_s \supset F_r$ , and  $\forall A \in F_r [\Phi_s(A) \subset \Phi_r(A)]$ ;
- (5) if  $\mathcal{B}_r$  is the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $F_r$ , then for every  $B \in \mathcal{B}_r$ ,  $\forall n \in n_s \setminus n_r [f_s(n) \cap B \neq 0$  iff  $B$  is infinite];
- (6) for every infinite  $B \in \mathcal{B}_r$ ,

$$\forall n \in n_s [n + 1 \in n_s \setminus n_r \implies |B \cap f_s(n)| < |B \cap f_s(n + 1)|];$$

- (7) for each  $A \in F_r$ ,  $\forall n \in n_s \setminus n_r [(g_s(n))_{(A \cap f_s(n))} \subset \Phi_r(A)]$ ;

- (8) for each  $A, B \in F_r$ , if  $A \subset^* B$ , then

$$\forall n \in n_s \setminus n_r [\pi_{p,B,A} \upharpoonright ((g_s(n))_{B \cap f_s(n)}) = \pi_{g_s(n), B \cap f_s(n), A \cap f_s(n)}].$$

It is easily checked that  $\langle \mathbb{R}, \preceq \rangle$  is a  $\sigma$ -centered poset. It is also easy to check that for each  $A \in \mathcal{A}_p$  and each  $Y \in \mathcal{F}_{p,A}$ ,  $R_{A,Y} = \{s \in \mathbb{R} : A \in F_s \wedge \Phi_s(A) \subset Y\}$  is dense in  $\mathbb{R}$ . Now check the following claim.

**Claim 21.** For each  $n \in \omega$ ,  $R_n = \{s \in \mathbb{R} : n < n_s\}$  is dense in  $\mathbb{R}$ .

*Proof.* The proof is by induction on  $n$ . Fix  $n$  and suppose the claim is true for all  $m < n$ . Let  $r \in \mathbb{R}$ . By the inductive hypothesis, we may assume that  $n \leq n_r$ . If  $n < n_r$ , then there is nothing to do, so we assume  $n = n_r$  and define  $s$  so that  $n_s = n + 1$ . Also  $0, \omega \in \mathcal{A}_p$ . So we may assume that  $\{0, \omega\} \subset F_r$ . Let  $\mathcal{B}_r$  be the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $F_r$ . This is finite. So we can find a finite, non-empty set  $f_s(n) \subset \omega$  such that:

- (9) for any finite  $B \in \mathcal{B}_r$ ,  $B \cap f_s(n) = 0$ ;
- (10) for any infinite  $B \in \mathcal{B}_r$ ,  $B \cap f_s(n) \neq 0$ ;
- (11) if  $n > 0$ , then  $\min(f_s(n)) > \max(f_r(n - 1))$ , and for any infinite  $B \in \mathcal{B}_r$ ,  $|f_r(n - 1) \cap B| < |f_s(n) \cap B|$ .

Now we will define a finitary  $p_0 \in \mathbb{P}_0$  with  $p \leq p_0$ . The idea will be to find a  $q_0 \in \mathbb{Q}$  that induces  $p_0$ , and then we will appropriately choose an  $m \in \omega$  and a function  $h$  so that  $g_s(n)$  can be defined to be the shift of  $u^{q_0, m}$  by  $h$ . To ensure that this definition will result in an extension of  $r$ ,  $q_0$  must know about  $F_r$  and  $\Phi_r$ . Let  $\mathcal{A}_{p_0} = F_r$ . We define by induction on  $n \in \omega$  sequences  $\bar{X}_n = \langle X_{A,n} : A \in \mathcal{A}_{p_0} \rangle$  such that  $\forall n \in \omega \forall A \in \mathcal{A}_{p_0} [X_{A,n} \in \mathcal{D}_{p,A} \wedge X_{A,n+1} \subset X_{A,n}]$ . Define  $X_{A,0} = \Phi_r(A)$ , for all  $A \in \mathcal{A}_{p_0}$ . Suppose that  $\bar{X}_n$  having the required properties is given for some  $n \in \omega$ . For each  $A \in \mathcal{A}_{p_0}$ , define  $X_{A,n+1} = X_{A,n} \cap (\bigcap \{ \pi''_{p,B,A} X_{B,n} : B \in \mathcal{A}_{p_0} \wedge A \subset^* B \})$ . It is easy to see that  $\bar{X}_{n+1}$  has the required properties. This completes the definition of the  $\bar{X}_n$ . Now define  $\mathcal{D}_{p_0,A} = \{a \subset \omega : \exists n \in \omega [X_{A,n} \subset^* a]\}$ , for each  $A \in \mathcal{A}_{p_0}$ . Note  $\forall A \in \mathcal{A}_{p_0} \forall n \in \omega [X_{A,n} \in \mathcal{D}_{p_0,A}]$ . Let  $\mathcal{D}_{p_0} = \langle \mathcal{D}_{p_0,A} : A \in \mathcal{A}_{p_0} \rangle$ . Finally, for any  $A, B \in \mathcal{A}_{p_0}$  with  $A \subset^* B$ , let  $\pi_{p_0,B,A} = \pi_{p,B,A}$  and let  $\mathcal{C}_{p_0} = \langle \pi_{p_0,B,A} : B, A \in \mathcal{A}_{p_0} \wedge A \subset^* B \rangle$ . Then  $p_0 = \langle \mathcal{A}_{p_0}, \mathcal{C}_{p_0}, \mathcal{D}_{p_0} \rangle$  is in  $\mathbb{P}_0$ ,  $p \leq p_0$ , and  $p_0$  is finitary. Since  $p \in \mathbb{P}_1$ , we can fix  $q_0 \in \mathbb{Q}$  inducing  $p_0$ . Since  $\mathcal{A}_{p_0}$  and  $\mathcal{B}_r$  are both finite, it is possible to find  $m \in \omega$  such that:

- (12) for each  $A \in \mathcal{B}_r$ ,  $I_{q_0,m} \cap A \neq 0$  if and only if  $A$  is infinite; moreover for every infinite  $A \in \mathcal{B}_r$ ,  $|A \cap I_{q_0,m}| \geq |A \cap f_s(n)|$ ;
- (13) for each  $A \in \mathcal{A}_{p_0}$ ,  $u_{A \cap I_{q_0,m}}^{q_0, m} \subset X_{A,0}$ ;
- (14) for all  $A, B \in \mathcal{A}_{p_0}$  with  $A \subset^* B$ ,  $\pi_{p_0,B,A} \upharpoonright u_{B \cap I_{q_0,m}}^{q_0, m} = \pi_{q_0, B \cap I_{q_0,m}, A \cap I_{q_0,m}}$ ;
- (15) if  $n > 0$ , then  $\text{nor}(u^{q_0, m}) > \text{nor}(g_r(n - 1))$ , and for every  $a \subset f_r(n - 1)$  and every  $b \subset I_{q_0,m}$ ,  $\min(u_b^{q_0, m}) > \max((g_r(n - 1))_a)$ .

Let  $\{A_0, \dots, A_l\}$  enumerate the members of  $\mathcal{A}_{p_0}$ . For each  $\sigma \in 2^{l+1}$  define  $b_\sigma = (\bigcap \{A_i : \sigma(i) = 0\}) \cap (\bigcap \{\omega \setminus A_i : \sigma(i) = 1\})$  (in this definition  $\bigcap 0 = \omega$ ). Let  $T = \{\sigma \in 2^{l+1} : b_\sigma \text{ is infinite}\}$ . Because of (9), (10), and (12),  $f_s(n) = \bigcup_{\sigma \in T} (b_\sigma \cap f_s(n))$  and  $I_{q_0, m} = \bigcup_{\sigma \in T} (b_\sigma \cap I_{q_0, m})$ . Also if  $\sigma \neq \tau$ , then  $b_\sigma \cap b_\tau = 0$ , and if  $\sigma \in T$ , then  $|b_\sigma \cap I_{q_0, m}| \geq |b_\sigma \cap f_s(n)| \neq 0$ . Therefore there is an onto map  $h : I_{q_0, m} \rightarrow f_s(n)$  such that  $\forall \sigma \in T [h^{-1}(b_\sigma \cap f_s(n)) = b_\sigma \cap I_{q_0, m}]$ . Let  $g_s(n) = h[u^{q_0, m}]$ . Then by Lemmas 12 and 13,  $g_s(n)$  is a creature acting on  $f_s(n)$ , and if  $n > 0$ , then  $\text{nor}(g_s(n)) > \text{nor}(g_r(n-1))$ . Also if  $a \subset f_s(n)$ , then  $(g_s(n))_a = u_{h^{-1}(a)}^{q_0, m} \subset \omega$ , and if  $n > 0$ , then for any  $x \subset f_r(n-1)$ ,  $\max((g_r(n-1))_x) < \min((g_s(n))_a)$ . So if we define  $n_s = n + 1$ ,  $f_s = f_r \widehat{\setminus} \langle f_s(n) \rangle$ ,  $g_s = g_r \widehat{\setminus} \langle g_s(n) \rangle$ ,  $F_s = F_r$ , and  $\Phi_r = \Phi_s$ , then  $s \in \mathbb{R}$ . We check that  $s \preceq r$ . Clause (4) is obvious and clause (5) follows from (9) and (10). Since  $n_s \setminus n_r = \{n\}$ , clause (6) just amounts to the second part of clause (11).

In order to check (7) and (8), we first make a preliminary observation. For each  $0 \leq i \leq l$ , put  $T_i = \{\sigma \in T : \sigma(i) = 0\}$ . Because of (9), (10), and (12)  $A_i \cap f_s(n) = \bigcup \{b_\sigma \cap f_s(n) : \sigma \in T_i\}$  and  $A_i \cap I_{q_0, m} = \bigcup \{b_\sigma \cap I_{q_0, m} : \sigma \in T_i\}$ . Therefore for any  $0 \leq i \leq l$ ,  $h^{-1}(A_i \cap f_s(n)) = \bigcup \{h^{-1}(b_\sigma \cap f_s(n)) : \sigma \in T_i\} = \bigcup \{b_\sigma \cap I_{q_0, m} : \sigma \in T_i\} = A_i \cap I_{q_0, m}$ . With this observation in hand, let us check (7) and (8). Take any  $A \in F_r = \mathcal{A}_{p_0}$ . There is  $0 \leq i \leq l$  such that  $A = A_i$  and  $(g_s(n))_{(A_i \cap f_s(n))} = u_{h^{-1}(A_i \cap f_s(n))}^{q_0, m} = u_{A_i \cap I_{q_0, m}}^{q_0, m} \subset X_{A_i, 0} = \Phi_r(A_i)$ , as needed for (7). For (8), fix  $A, B \in F_r = \mathcal{A}_{p_0}$  with  $A \subset^* B$ . Then for some  $0 \leq i, j \leq l$ ,  $A = A_i$  and  $B = A_j$ . Observe that  $A_i \setminus A_j$  is a finite member of  $\mathcal{B}_r$  because  $A_i \subset^* A_j$ . Therefore by (9)  $(A_i \setminus A_j) \cap f_s(n) = 0$ , and  $A_i \cap f_s(n) \subset A_j \cap f_s(n)$ . Therefore  $\pi_{g_s(n), A_j \cap f_s(n), A_i \cap f_s(n)}$  is defined and is equal to  $\pi_{u^{q_0, m}, h^{-1}(A_j \cap f_s(n)), h^{-1}(A_i \cap f_s(n))}$ . So  $\pi_{p, A_j, A_i} \uparrow \left( (g_s(n))_{A_j \cap f_s(n)} \right) = \pi_{p_0, A_j, A_i} \uparrow \left( u_{h^{-1}(A_j \cap f_s(n))}^{q_0, m} \right) = \pi_{p_0, A_j, A_i} \uparrow \left( u_{A_j \cap I_{q_0, m}}^{q_0, m} \right) = \pi_{u^{q_0, m}, A_j \cap I_{q_0, m}, A_i \cap I_{q_0, m}} = \pi_{u^{q_0, m}, h^{-1}(A_j \cap f_s(n)), h^{-1}(A_i \cap f_s(n))} = \pi_{g_s(n), A_j \cap f_s(n), A_i \cap f_s(n)}$ , which is exactly what is needed. This checks  $s \preceq r$  and completes the proof of the claim.  $\square$

Using MA( $\sigma$ -centered) we can find a filter  $G \subset \mathbb{R}$  that meets every member of  $\{R_{A, Y} : A \in \mathcal{A}_p \wedge Y \in \mathcal{F}_{p, A}\} \cup \{R_n : n \in \omega\}$  (recall that  $\mathfrak{c}$  is regular under MA( $\sigma$ -centered)). Let  $I = \bigcup \{f_r : r \in G\}$  and  $U = \bigcup \{g_r : r \in G\}$ . Then it is clear that  $q = \langle I, U \rangle \in \mathbb{Q}$ . We check that  $q$  induces  $p$ . We first verify clause (1) of Definition 16. Let  $\mathcal{B}$  be the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $\mathcal{A}_p$ . Take  $A \in \mathcal{B}$ . Then there exist  $A_0, \dots, A_l \in \mathcal{A}_p$  such that  $A \in \mathcal{B}_0$ , where  $\mathcal{B}_0$  is the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $\{A_0, \dots, A_l\}$ . For each  $0 \leq i \leq l$ ,  $\mathcal{F}_{p, A_i}$  is non-empty. Choosing  $Y_i \in \mathcal{F}_{p, A_i}$ ,  $R_{A_i, Y_i}$  is a dense open set met by  $G$ . So there is  $r \in G \cap \left( \bigcap_{i \leq l} R_{A_i, Y_i} \right)$ . Then  $A \in \mathcal{B}_r$ . For any  $n \geq n_r$  there is  $t \in G$  such that  $t \preceq r$  and  $n + 1 < n_t$ . Then if  $A$  is infinite, since  $n + 1 \in n_t \setminus n_r$ , by (6), we have  $|A \cap I_n| = |A \cap f_t(n)| < |A \cap f_t(n+1)| = |A \cap I_{n+1}|$ . Thus if  $A$  is infinite, then for all  $n \geq n_r$   $[|A \cap I_n| < |A \cap I_{n+1}|]$ , as needed for clause (1) of Definition 16. Next, we check clause (2) of Definition 16. Take  $A \in \mathcal{A}_p$  and  $X \in \mathcal{D}_{p, A}$ . Choose  $Y \in \mathcal{F}_{p, A}$  with  $Y \subset X$ . Again there is  $r \in G \cap R_{A, Y}$ . Fix  $n \geq n_r$ . There is  $t \in G$  such that  $t \preceq r$  and  $n < n_t$ . Since  $n \in n_t \setminus n_r$ , by (7),  $u_{A \cap I_{q, n}}^{q, n} = (g_t(n))_{A \cap f_t(n)} \subset \Phi_r(A) \subset Y \subset X$ . So  $\forall^\infty n \in \omega [u_{A \cap I_{q, n}}^{q, n} \subset X]$ , as needed for clause (2). Finally, we check clause (3) of Definition 16. Take  $A, B \in \mathcal{A}_p$  with  $A \subset^* B$ .  $\mathcal{F}_{p, A}$  and  $\mathcal{F}_{p, B}$  are non-empty. Take  $Y_0 \in \mathcal{F}_{p, A}$  and  $Y_1 \in \mathcal{F}_{p, B}$ . Since

$R_{A,Y_0}$  and  $R_{B,Y_1}$  are dense open sets met by  $G$ , we can find  $r \in G \cap R_{A,Y_0} \cap R_{B,Y_1}$ . Then  $A, B \in F_r$  and  $n_r \in \omega$ . Fix  $n \geq n_r$ . Then there is  $t \in G$  such that  $t \preceq r$  and  $n < n_t$ . Since  $n \in n_t \setminus n_r$ , by (8),  $\pi_{p,B,A} \upharpoonright (u_{B \cap I_{q,n}}^{q,n}) = \pi_{p,B,A} \upharpoonright ((g_t(n))_{B \cap I_{q,n}}) = \pi_{g_t(n), B \cap I_{q,n}, A \cap I_{q,n}} = \pi_{q, B \cap I_{q,n}, A \cap I_{q,n}}$ . Therefore

$$\forall^\infty n \in \omega \left[ \pi_{p,B,A} \upharpoonright (u_{B \cap I_{q,n}}^{q,n}) = \pi_{q, B \cap I_{q,n}, A \cap I_{q,n}} \right].$$

This is what is needed for clause (3), which completes the verification that  $q$  induces  $p$  and hence also the proof of the lemma.  $\square$

**Lemma 22.** *Assume MA( $\sigma$ -centered). For every  $C \in \mathcal{P}(\omega)$ ,  $\{p \in \mathbb{P}_1 : \exists C^* \in \mathcal{A}_p [C =^* C^*]\}$  is dense in  $\mathbb{P}_1$ .*

*Proof.* Fix  $p \in \mathbb{P}_1$ . If  $\exists A \in \mathcal{A}_p [A =^* C]$ , then there is nothing to do. So assume that  $\forall A \in \mathcal{A}_p [A \neq^* C]$ . Since  $0, \omega \in \mathcal{A}_p$  this implies that both  $C$  and  $\omega \setminus C$  are infinite. Let  $\mathcal{B}$  denote the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $\mathcal{A}_p$ . For each  $A \in \mathcal{A}_p$  choose a family  $\mathcal{F}_{p,A} \subset \mathcal{D}_{p,A}$  as in (2) of Definition 15. Let  $\mathbb{R}$  be the poset defined in the proof of Lemma 20 (with respect to the fixed condition  $p$ ). Let  $\preceq$  also be as in the proof of Lemma 20. We define a stronger ordering on  $\mathbb{R}$ . For  $r, s \in \mathbb{R}$ ,  $s \triangleleft r$  if and only if  $s \preceq r$  and

- (1) let  $\mathcal{B}_r^+$  denote the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $F_r \cup \{C\}$ ; then for any  $A \in \mathcal{B}_r^+$ ,  $\forall n \in n_s \setminus n_r [A \cap f_s(n) \neq 0$  iff  $A$  is infinite];
- (2) for each infinite  $A \in \mathcal{B}_r^+$ ,

$$\forall n \in n_s [n + 1 \in n_s \setminus n_r \implies |A \cap f_s(n)| < |A \cap f_s(n + 1)|].$$

Then it is easy to check that  $\langle \mathbb{R}, \triangleleft \rangle$  is a  $\sigma$ -centered poset. Moreover for each  $A \in \mathcal{A}_p$  and  $Y \in \mathcal{F}_{p,A}$  let  $R_{A,Y} = \{s \in \mathbb{R} : A \in F_s \wedge \Phi_s(A) \subset Y\}$ ; then it is easy to check that  $R_{A,Y}$  is dense open in  $\langle \mathbb{R}, \triangleleft \rangle$ . Now we check the following claim.

**Claim 23.** *For each  $n \in \omega$ ,  $R_n = \{s \in \mathbb{R} : n < n_s\}$  is dense open in  $\langle \mathbb{R}, \triangleleft \rangle$ .*

*Proof.* It is easy to check that  $R_n$  is open in  $\langle \mathbb{R}, \triangleleft \rangle$ . The proof that it is dense is by induction on  $n$ . Fix  $n$  and suppose that the claim holds for all  $m < n$ . Take  $r \in \mathbb{R}$ . By the inductive hypothesis and by the openness of the  $R_m$  for  $m < n$ , we may assume that  $n \leq n_r$ . If  $n < n_r$ , then there is nothing to do. So we assume  $n = n_r$  and we will define  $s$  so that  $n_s = n + 1$ . Also  $0, \omega \in \mathcal{A}_p$  and  $\mathcal{F}_{p,0}$  and  $\mathcal{F}_{p,\omega}$  are non-empty. If  $Y_0 \in \mathcal{F}_{p,0}$  and  $Y_1 \in \mathcal{F}_{p,\omega}$ , then  $R_{0,Y_0}$  and  $R_{\omega,Y_1}$  are dense open in  $\langle \mathbb{R}, \triangleleft \rangle$ , and so we may assume that  $0, \omega \in F_r$ . Since  $\mathcal{B}_r^+$  is finite, we can find a finite non-empty  $f_s(n) \subset \omega$  such that:

- (3) for every finite  $A \in \mathcal{B}_r^+$ ,  $A \cap f_s(n) = 0$ ;
- (4) for every infinite  $A \in \mathcal{B}_r^+$ ,  $A \cap f_s(n) \neq 0$ ;
- (5) if  $n > 0$ , then  $\min(f_s(n)) > \max(f_r(n - 1))$  and for every infinite  $A \in \mathcal{B}_r^+$ ,  $|f_s(n) \cap A| > |f_r(n - 1) \cap A|$ .

By the Representation Lemma fix  $q \in \mathbb{Q}$  that induces  $p$ . Let  $\mathcal{B}_r$  be the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $F_r$ . As  $\mathcal{B}_r$  is a finite subset of  $\mathcal{B}$  and  $F_r$  is a finite subset of  $\mathcal{A}_p$ , we can find  $m \in \omega$  such that the following hold:

- (6) for each finite  $A \in \mathcal{B}_r$ ,  $A \cap I_{q,m} = 0$ ; for each infinite  $A \in \mathcal{B}_r$ ,  $A \cap I_{q,m} \neq 0$ ; moreover for each infinite  $A \in \mathcal{B}_r$ ,  $|A \cap I_{q,m}| \geq 2|A \cap f_s(n)|$ ;
- (7) for each  $A \in F_r$ ,  $u_{A \cap I_{q,m}}^{q,m} \subset \Phi_r(A)$ ;
- (8) for each  $A, B \in F_r$ , if  $A \subset^* B$ , then  $\pi_{p,B,A} \upharpoonright u_{B \cap I_{q,m}}^{q,m} = \pi_{q, B \cap I_{q,m}, A \cap I_{q,m}}$ ;

- (9) if  $n > 0$ , then  $\text{nor}(u_b^{q,m}) > \text{nor}(g_r(n-1))$ , and for each  $a \subset f_r(n-1)$  and each  $b \subset I_{q,m}$ ,  $\min(u_b^{q,m}) > \max((g_r(n-1))_a)$ .

Let  $\{A_0, \dots, A_{l+1}\}$  enumerate the elements of  $F_r \cup \{C\}$ , with  $\{A_0, \dots, A_l\}$  being an enumeration of  $F_r$  and  $A_{l+1} = C$ . For each  $\sigma \in 2^{l+2}$  define the set  $b_\sigma = (\bigcap \{A_i : \sigma(i) = 0\}) \cap (\bigcap \{\omega \setminus A_i : \sigma(i) = 1\})$  (in this definition  $\bigcap 0 = \omega$ ). It is clear that each  $b_\sigma \in \mathcal{B}_r^+$ . For each  $\tau \in 2^{l+1}$  define  $b_\tau = (\bigcap \{A_i : \tau(i) = 0\}) \cap (\bigcap \{\omega \setminus A_i : \tau(i) = 1\})$ . Note that each  $b_\tau \in \mathcal{B}_r$ . Let  $T = \{\sigma \in 2^{l+2} : b_\sigma \text{ is infinite}\}$  and let  $S = \{\tau \in 2^{l+1} : b_\tau \text{ is infinite}\}$ . If  $\sigma \in T$ , then  $\sigma \upharpoonright l+1 \in S$ . Also if  $\tau \in S$ , then at least one of  $\tau \frown \langle 0 \rangle$  or  $\tau \frown \langle 1 \rangle$  is in  $T$ . For each  $\tau \in S$ , by (6),  $|b_\tau \cap I_{q,m}| \geq 2|b_\tau \cap f_s(n)|$ . So we can find disjoint sets  $b_\tau^0$  and  $b_\tau^1$  such that  $|b_\tau^0| \geq |b_\tau \cap f_s(n)|$ ,  $|b_\tau^1| \geq |b_\tau \cap f_s(n)|$ , and  $b_\tau \cap I_{q,m} = b_\tau^0 \cup b_\tau^1$ . For each  $\sigma \in T$  define a set  $c_\sigma$  as follows. If both  $(\sigma \upharpoonright l+1) \frown \langle 0 \rangle$  and  $(\sigma \upharpoonright l+1) \frown \langle 1 \rangle$  are members of  $T$ , then  $c_\sigma = b_{(\sigma \upharpoonright l+1)}^{\sigma(l+1)}$ . Otherwise  $c_\sigma = b_{(\sigma \upharpoonright l+1)} \cap I_{q,m}$ . It is easy to check that  $f_s(n) = \bigcup_{\sigma \in T} (b_\sigma \cap f_s(n))$  and that  $I_{q,m} = \bigcup_{\sigma \in T} (c_\sigma \cap I_{q,m})$ . Also for each  $\sigma, \sigma' \in T$ , if  $\sigma \neq \sigma'$ , then  $c_\sigma \cap c_{\sigma'} = 0$  and  $b_\sigma \cap b_{\sigma'} = 0$ . Moreover for each  $\sigma \in T$ ,  $|b_\sigma \cap f_s(n)| \leq |c_\sigma \cap I_{q,m}|$  and  $b_\sigma \cap f_s(n) \neq 0$ . So there is an onto map  $h : I_{q,m} \rightarrow f_s(n)$  such that  $\forall \sigma \in T [h^{-1}(b_\sigma \cap f_s(n)) = c_\sigma \cap I_{q,m}]$ . For each  $0 \leq i \leq l$ , let  $T_i = \{\sigma \in T : \sigma(i) = 0\}$ . It is easy to check that for each  $0 \leq i \leq l$ ,  $A_i \cap f_s(n) = \bigcup_{\sigma \in T_i} (b_\sigma \cap f_s(n))$  and  $A_i \cap I_{q,m} = \bigcup_{\sigma \in T_i} (c_\sigma \cap I_{q,m})$ . Therefore for any  $0 \leq i \leq l$ ,  $h^{-1}(A_i \cap f_s(n)) = \bigcup_{\sigma \in T_i} (h^{-1}(b_\sigma \cap f_s(n))) = \bigcup_{\sigma \in T_i} (c_\sigma \cap I_{q,m}) = A_i \cap I_{q,m}$ . Define  $g_s(n) = h[u_b^{q,m}]$ . Then  $g_s(n)$  is a creature acting on  $f_s(n)$ , and if  $n > 0$ , then  $\text{nor}(g_s(n)) > \text{nor}(g_r(n-1))$ . Also if  $a \subset f_s(n)$ ,  $(g_s(n))_a = u_{h^{-1}(a)}^{q,m} \subset \omega$  such that if  $n > 0$ , then for all  $x \subset f_r(n-1)$ ,  $\max((g_r(n-1))_x) < \min((g_s(n))_a)$ . Therefore if we let  $n_s = n+1$ ,  $f_s = f_r \frown \langle f_s(n) \rangle$ ,  $g_s = g_r \frown \langle g_s(n) \rangle$ ,  $F_s = F_r$ , and  $\Phi_s = \Phi_r$ , then  $s = \langle f_s, g_s, F_s, \Phi_s \rangle$  is a member of  $\mathbb{R}$ . We check that  $s \leq r$ . Clause (1) follows from (3) and (4), while (2) is a consequence of (5). Next, to see that  $s \leq r$ , note that (4) of Lemma 20 is obvious, while (5) of the proof of Lemma 20 follows from (1). (6) of the proof of Lemma 20 is by (2). Next, take  $A \in F_r$ . Then  $A = A_i$  for some  $0 \leq i \leq l$ . So by (7)  $(g_s(n))_{(A \cap f_s(n))} = u_{h^{-1}(A \cap f_s(n))}^{q,m} = u_{A \cap I_{q,m}}^{q,m} \subset \Phi_r(A)$ . Finally take  $A, B \in F_r$  and suppose  $A \subset^* B$ . Note that  $A \setminus B$  is a finite member of  $\mathcal{B}_r$ . So  $f_s(n) \cap (A \setminus B) = 0$ . Hence  $A \cap f_s(n) \subset B \cap f_s(n) \subset f_s(n)$ . Therefore  $\pi_{g_s(n), B \cap f_s(n), A \cap f_s(n)}$  is defined and is equal to  $\pi_{q, h^{-1}(B \cap f_s(n)), h^{-1}(A \cap f_s(n))}$ , which in turn equals  $\pi_{q, B \cap I_{q,m}, A \cap I_{q,m}}$ . By (8),  $\pi_{q, B \cap I_{q,m}, A \cap I_{q,m}} = \pi_{p, B, A} \upharpoonright u_{B \cap I_{q,m}}^{q,m} = \pi_{p, B, A} \upharpoonright ((g_s(n))_{B \cap f_s(n)})$  because  $(g_s(n))_{B \cap f_s(n)} = u_{h^{-1}(B \cap f_s(n))}^{q,m} = u_{B \cap I_{q,m}}^{q,m}$ . This concludes the verification that  $s \leq r$  and hence the proof of the claim.  $\square$

Let  $G \subset \mathbb{R}$  be a filter meeting all the dense open sets in  $\{R_n : n \in \omega\} \cup \{R_{A,Y} : A \in \mathcal{A}_p \wedge Y \in \mathcal{F}_{p,A}\}$ . Let  $I = \bigcup_{r \in G} f_r$  and  $U = \bigcup_{r \in G} g_r$ , and let  $q_0 = \langle I, U \rangle$ . Then  $q_0 \in \mathbb{Q}$ . Let  $\mathcal{A}_{p_0} = \mathcal{A}_p \cup \{C\}$ . Then  $\mathcal{A}_p \subset \mathcal{A}_{p_0} \subset \mathcal{P}(\omega)$ ,  $|\mathcal{A}_{p_0}| < \mathfrak{c}$ , and  $\forall A, B \in \mathcal{A}_{p_0} [A \neq B \implies A \neq^* B]$ . Let  $\mathcal{B}_0$  be the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $\mathcal{A}_{p_0}$ . Let  $A$  be an infinite member of  $\mathcal{B}_0$ . There is a finite set  $F \subset \mathcal{A}_p$  such that  $A$  is in the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $F \cup \{C\}$ . Fix  $r \in G$  such that  $F \subset F_r$ . Then  $A$  is an infinite member of  $\mathcal{B}_r^+$ . For any  $n \geq n_r$ ,  $|A \cap I_{q_0, n}| < |A \cap I_{q_0, n+1}|$  because of (2). Therefore, for any infinite  $A \in \mathcal{B}_0$ ,  $\forall^\infty n \in \omega [|A \cap I_{q_0, n}| < |A \cap I_{q_0, n+1}|]$ . It is also easy to see that  $q_0$  induces  $p$ . Now for each  $A \in \mathcal{A}_{p_0}$ , let  $X_A = \bigcup_{n \in \omega} u_{I_{q_0, n} \cap A}^{q_0, n}$  and let  $\mathcal{D}_{p_0, A} = \{a \subset \omega : X_A \subset^* a\}$ . Put  $\mathcal{D}_{p_0} = \langle \mathcal{D}_{p_0, A} : A \in \mathcal{A}_{p_0} \rangle$ . For  $A, B \in \mathcal{A}_{p_0}$  with  $A \subset^* B$ , if  $A, B \in \mathcal{A}_p$ , then define

$\pi_{p_0,B,A} = \pi_{p,B,A}$ . If either  $A$  or  $B$  belongs to  $\mathcal{A}_{p_0} \setminus \mathcal{A}_p$ , then define  $\pi_{p_0,B,A} : \omega \rightarrow \omega$  as follows. Given  $k \in \omega$ , if  $k \in X_B$ , then there is a unique  $n \in \omega$  such that  $k \in u_{I_{q_0,n}^{q_0,n} \cap B}^{q_0,n}$ . If  $A \cap I_{q_0,n} \subset B \cap I_{q_0,n}$ , then  $\pi_{p_0,B,A}(k) = \pi_{q_0,B \cap I_{q_0,n}, A \cap I_{q_0,n}}(k)$ . If either  $A \cap I_{q_0,n} \not\subset B \cap I_{q_0,n}$  or if  $k \notin X_B$ , then put  $\pi_{p_0,B,A}(k) = 0$ . Let  $\mathcal{C}_{p_0} = \langle \pi_{p_0,B,A} : A, B \in \mathcal{A}_{p_0} \wedge A \subset^* B \rangle$  and let  $p_0 = \langle \mathcal{A}_{p_0}, \mathcal{C}_{p_0}, \mathcal{D}_{p_0} \rangle$ . Then it is not hard to see that  $p_0 \in \mathbb{P}_0$ ,  $p_0 \leq p$ , and that  $q_0$  induces  $p_0$ . Hence  $q_0$  also induces any  $p_1 \in \mathbb{P}_0$  with  $p_0 \leq p_1$ . So  $p_0 \in \mathbb{P}_1$  and  $p_0 \leq p$ . As  $C \in \mathcal{A}_{p_0}$ , this concludes the proof of the lemma.  $\square$

*Remark 24.* We now make some simple observations that will be useful for the remaining part of the proof. Suppose  $q \in \mathbb{Q}$ . Suppose  $\langle k_n : n \in \omega \rangle \subset \omega$  is a sequence such that  $\forall n \in \omega [k_n < k_{n+1}]$ . For each  $n \in \omega$ , put  $I_{q_0,n} = I_{q,k_n}$ . Suppose also that for each  $n \in \omega$ , we are given  $u^{q_0,n} \in \Sigma(u^{q,k_n})$  in such a way that for all  $n \in \omega$ ,  $\text{nor}(u^{q_0,n}) < \text{nor}(u^{q_0,n+1})$ . Then if we let  $I_{q_0} = \langle I_{q_0,n} : n \in \omega \rangle$ ,  $U_{q_0} = \langle u^{q_0,n} : n \in \omega \rangle$ , and  $q_0 = \langle I_{q_0}, U_{q_0} \rangle$ , we have  $q_0 \in \mathbb{Q}$ . Moreover, if  $p \in \mathbb{P}_0$  and  $q$  induces  $p$ , then  $q_0$  also induces  $p$ . We can now define  $p_0$  using  $p$  and  $q_0$  as follows. Put  $\mathcal{A}_{p_0} = \mathcal{A}_p$ . For each  $A \in \mathcal{A}_{p_0}$ , let  $X_A = \bigcup_{n \in \omega} u_{I_{q_0,n}^{q_0,n} \cap A}^{q_0,n}$  and let  $\mathcal{D}_{p_0,A} = \{a \subset \omega : X_A \subset^* a\}$ . Put  $\mathcal{D}_{p_0} = \langle \mathcal{D}_{p_0,A} : A \in \mathcal{A}_{p_0} \rangle$ . Define  $\mathcal{C}_{p_0} = \mathcal{C}_p$  and  $p_0 = \langle \mathcal{A}_{p_0}, \mathcal{C}_{p_0}, \mathcal{D}_{p_0} \rangle$ . Then  $p_0 \in \mathbb{P}_0$ ,  $p_0 \leq p$ , and  $q_0$  induces  $p_0$ . Therefore,  $q_0$  also induces any  $p_1 \in \mathbb{P}_0$  with  $p_0 \leq p_1$ . Hence  $p_0 \in \mathbb{P}_1$ .

**Lemma 25.** *Suppose  $p \in \mathbb{P}_1$  and  $A \in \mathcal{A}_p$ . Let  $b \subset \omega$ . There exists  $p_0 \in \mathbb{P}_1$ ,  $p_0 \leq p$  such that either  $b \in \mathcal{D}_{p_0,A}$  or  $\omega \setminus b \in \mathcal{D}_{p_0,A}$ .*

*Proof.* Let  $b_0 = b$  and  $b_1 = \omega \setminus b$ . By the Representation Lemma fix  $q \in \mathbb{Q}$  that induces  $p$ . Fix  $n \geq 1$ . Then  $\text{nor}(u^{q,n}) \geq (n-1) + 1$ . We have that  $u_{A \cap I_{q,n}}^{q,n} = \left( u_{A \cap I_{q,n}}^{q,n} \cap b_0 \right) \cup \left( u_{A \cap I_{q,n}}^{q,n} \cap b_1 \right)$ . So there exist  $j_n \in 2$  and  $v^n \in \Sigma(u^{q,n})$  such that  $\text{nor}(v^n) \geq n-1$  and  $v_{A \cap I_{q,n}}^{q,n} \subset u_{A \cap I_{q,n}}^{q,n} \cap b_{j_n}$ . Clearly, there is  $j \in 2$  such that  $\{n \geq 1 : j_n = j\}$  is infinite. So it is possible to find a sequence  $\langle k_n : n \in \omega \rangle \subset \omega$  such that for each  $n \in \omega$ ,  $k_n \geq 1$ ,  $j_{k_n} = j$ ,  $k_n < k_{n+1}$ , and  $\text{nor}(v^{k_{n+1}}) > \text{nor}(v^{k_n})$ . For each  $n \in \omega$ , let  $u^{q_0,n} = v^{k_n} \in \Sigma(u^{q,k_n})$ . Also  $\text{nor}(u^{q_0,n}) < \text{nor}(u^{q_0,n+1})$  holds for all  $n \in \omega$ . Therefore if  $q_0$  and  $p_0$  are defined as in Remark 24, then  $p_0 \in \mathbb{P}_1$  and  $p_0 \leq p$ . Moreover note that for each  $n \in \omega$ ,  $u_{I_{q_0,n}^{q_0,n} \cap A}^{q_0,n} = v_{I_{q,k_n}^{k_n} \cap A}^{k_n} \subset \left( u_{A \cap I_{q,k_n}}^{q,k_n} \cap b_{j_{k_n}} \right) \subset b_j$ . Hence  $X_A \subset b_j$ , whence  $b_j \in \mathcal{D}_{p_0,A}$ , as needed.  $\square$

We would like to be able to kill unwanted Tukey maps. That is, if  $p \in \mathbb{P}_1$ ,  $A, B \in \mathcal{A}_p$ ,  $B \not\subset^* A$ , and  $\phi$  is a map that could potentially witness  $\mathcal{U}_B \leq_T \mathcal{U}_A$ , then we would like to extend  $p$  in such a way that  $\phi$  can no longer witness this unwanted Tukey reduction. This requires Lemma 28, which is a consequence of Theorem 20 in the paper of Dobrinen and Todorcevic [12]. According to Theorem 20 of [12], if  $\phi$  is any monotone cofinal map from a P-point  $\mathcal{V}$  to an ultrafilter  $\mathcal{U}$ , then there is some  $X \in \mathcal{V}$  such that  $\phi$  is continuous when restricted below  $X$ . This  $\phi$  can then be extended to a continuous map  $\phi^*$  on all subsets of  $\omega$ . It follows from Dobrinen and Todorcevic's proof in [12] that for each  $k < \omega$  and  $Y \in \mathcal{V}$ , if  $k \in \phi^*(Y)$ , then  $k \in \phi^*(Z)$ , for every  $Z \in \mathcal{V}$  which end-extends  $Y \cap (k+1)$ . Take  $P$  to be the set of all  $Y \cap (k(Y)+1)$ , where  $k(Y)$  is the minimal  $k$  such that  $\phi^*(Y) \cap (k+1) \neq \emptyset$ , and let  $f(Y \cap (k(Y)+1)) = \min(\phi^*(Y))$ . It follows from the proof in [12] that this



value  $f(Y \cap (k(Y) + 1))$  must be  $k(Y)$ , since  $\mathcal{V}$  is non-principal. This produces the  $P$  and  $f$  claimed to exist in Lemma 28.<sup>3</sup>

We will give a different game-theoretic proof of Lemma 28 below, which does not rely on the methods of [12].

**Definition 26.** Let  $\mathcal{U}$  be an ultrafilter on  $\omega$ . The  $P$ -point game on  $\mathcal{U}$  is a two player game in which Players I and II alternatively choose sets  $a_n$  and  $s_n$  respectively, where  $a_n \in \mathcal{U}$  and  $s_n \in [a_n]^{<\omega}$ . Together they construct the sequence

$$a_0, s_0, a_1, s_1, \dots$$

Player I wins if and only if  $\bigcup_{n \in \omega} s_n \notin \mathcal{U}$ .

A proof of the following useful characterization of P-points in terms of the P-point game can be found in Bartoszyński and Judah [1].

**Theorem 27.** An ultrafilter  $\mathcal{U}$  is a P-point if and only if Player I does not have a winning strategy in the P-point game on  $\mathcal{U}$ .

**Lemma 28** (Dobrinen and Todorćević [12]). Suppose  $\mathcal{V}$  is a P-point and  $\mathcal{U}$  is any ultrafilter. Suppose  $\phi : \mathcal{V} \rightarrow \mathcal{U}$  is monotone and cofinal in  $\mathcal{U}$ . Then there exist  $P \subset [\omega]^{<\omega} \setminus \{0\}$  and  $f : P \rightarrow \omega$  such that the following hold:

- (1)  $\forall s, t \in P [s \subset t \implies s = t]$ ;
- (2)  $f$  is finite-to-one;
- (3)  $\forall a \in \mathcal{V} \forall b \in \mathcal{U} \exists s \in P [s \subset a \wedge f(s) \in b]$ .

*Proof.* Define  $\psi : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$  by  $\psi(x) = \bigcap \{\phi(a) : a \in \mathcal{V} \wedge x \subset a\}$ , for all  $x \in \mathcal{P}(\omega)$ . Note that  $\psi$  is monotone. Also  $\psi(0) = 0$ . To see this, suppose for a contradiction that  $k \in \psi(0)$ . Then  $\omega \setminus \{k\} \in \mathcal{U}$ . Take  $a \in \mathcal{V}$  such that  $\phi(a) \subset \omega \setminus \{k\}$ . However since  $k \in \psi(0)$ ,  $k \in \phi(a)$ , a contradiction. Now we define a strategy for Player I in the P-point game (on  $\mathcal{V}$ ) as follows. He first plays  $a_0 = \omega$ . Given  $n \in \omega$  and a partial play  $a_0, s_0, \dots, a_n, s_n$ , he considers  $\mathcal{P}(\bigcup_{i \leq n} s_i)$ . For each  $s \in \mathcal{P}(\bigcup_{i \leq n} s_i)$ , if  $n \notin \psi(s)$ , then he chooses  $a_{n,s} \in \mathcal{V}$  such that  $s \subset a_{n,s}$  and yet  $n \notin \phi(a_{n,s})$ . He plays

$$a_{n+1} = (a_n \setminus l_n) \cap \left( \bigcap \{a_{n,s} : s \in \mathcal{P}(\bigcup_{i \leq n} s_i) \wedge n \notin \psi(s)\} \right),$$

where  $l_n = \sup\{k + 1 : k \in \bigcup_{i \leq n} s_i\} \in \omega$  (in this definition of  $a_{n+1}$ ,  $\bigcap 0$  is taken to be  $\omega$ ). Since this is not a winning strategy for Player I, there is a run  $a_0, s_0, \dots, a_n, s_n, \dots$  of the P-point game in which he implements this strategy and loses. So  $b = \bigcup_{n \in \omega} s_n \in \mathcal{V}$ . Note that by the definition of the strategy,  $\forall n \in \omega [a_{n+1} \subset a_n]$ . Also since  $s_{n+1} \subset a_{n+1}$ , if  $k \in s_n$  and  $k' \in s_{n+1}$ , then  $k < k'$ . Let  $P = \{t \in [b]^{<\omega} : \psi(t) \neq 0 \wedge \forall s \subsetneq t [\psi(s) = 0]\}$ . Since  $\psi(0) = 0$ ,  $P \subset [\omega]^{<\omega} \setminus \{0\}$ . It is clear that  $P$  satisfies (1) by definition. Define  $f : P \rightarrow \omega$  by  $f(t) = \min(\psi(t))$ , for all  $t \in P$ . Now we claim the following.

**Claim 29.** For any  $n \in \omega$  and any  $c \in \mathcal{V}$ , if  $c \subset b$  and  $n \in \phi(c)$ , then  $n \in \psi\left(c \cap \left(\bigcup_{i \leq n} s_i\right)\right)$ .

*Proof.* Suppose not. Let  $s = c \cap \left(\bigcup_{i \leq n} s_i\right)$ . Since  $n \notin \psi(s)$ ,  $a_{n,s}$  exists and  $a_{n+1} \subset a_{n,s}$ . Moreover, for any  $m \geq n + 1$ ,  $s_m \subset a_m \subset a_{n+1} \subset a_{n,s}$ . Therefore,

<sup>3</sup>We thank the anonymous referee for providing us this proof sketch.

$c = c \cap b = \bigcup_{m \in \omega} (c \cap s_m) = s \cup \left( \bigcup_{m \geq n+1} (c \cap s_m) \right) \subset a_{n,s}$ . Hence  $\phi(c) \subset \phi(a_{n,s})$ , whence  $n \notin \phi(c)$ .  $\square$

Both (2) and (3) easily follow from Claim 29. For (2), fix  $n \in \omega$  and suppose  $t \in P$  is such that  $f(t) = n$ . Then  $n \in \psi(t)$ . Consider  $c = t \cup \left( \bigcup_{m \geq n+1} s_m \right)$ . It is clear that  $c \in \mathcal{V}$ ,  $t \subset c$ , and  $c \subset b$ . So  $n \in \phi(c)$ . So by Claim 29,  $n \in \psi \left( c \cap \left( \bigcup_{m \leq n} s_m \right) \right) = \psi \left( t \cap \left( \bigcup_{m \leq n} s_m \right) \right)$ . Since  $t \in P$ , this implies that  $t \cap \left( \bigcup_{m \leq n} s_m \right) = t$ . Thus  $f^{-1}(\{n\}) \subset \mathcal{P} \left( \bigcup_{m \leq n} s_m \right)$ , which is finite.

Next for (3), fix  $c \in \mathcal{V}$  and  $d \in \mathcal{U}$ . Let  $e \in \mathcal{V}$  be such that  $\phi(e) \subset d$ . Then  $b \cap c \cap e \in \mathcal{V}$ ,  $\phi(b \cap c \cap e) \in \mathcal{U}$ . So  $\phi(b \cap c \cap e) \neq 0$ . If  $n \in \phi(b \cap c \cap e)$ , then  $n \in \psi(u)$ , where  $u = (b \cap c \cap e) \cap \left( \bigcup_{m \leq n} s_m \right)$ . Thus  $\psi(u) \neq 0$ , and we may find  $t \subset u$  that is  $\subset$ -minimal w.r.t. the property that  $\psi(t) \neq 0$ . Then  $t \in P$  and  $t \subset u \subset b \cap c \cap e \subset c$ , and  $f(t) \in \psi(t)$ . Since  $t \subset e$  and  $e \in \mathcal{V}$ ,  $f(t) \in \phi(e) \subset d$ , as needed.  $\square$

**Lemma 30.** *Assume MA( $\sigma$  - centered). Suppose  $p \in \mathbb{P}_1$ ; suppose  $A, B \in \mathcal{A}_p$  with  $B \not\dot{\subset} A$ ; and suppose that  $P \subset [\omega]^{<\omega} \setminus \{0\}$  and  $f : P \rightarrow \omega$  satisfy (1)-(2) of Lemma 28. Then there exists  $p_0 \in \mathbb{P}_1$  such that  $p_0 \leq p$  and there exist sets  $X \in \mathcal{D}_{p_0, A}$  and  $Y \in \mathcal{D}_{p_0, B}$  such that  $\forall s \in P [s \subset X \implies f(s) \notin Y]$ .*

*Proof.* Fix  $q \in \mathbb{Q}$  that induces  $p$ . There is an  $m \in \omega$  such that

$$\forall n \geq m [|(B \setminus A) \cap I_{q,n}| < |(B \setminus A) \cap I_{q,n+1}|]$$

because  $B \setminus A$  is an infinite member of the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $\mathcal{A}_p$ . For each  $n \in \omega$ , consider  $\bigcup_{k \leq n} u_{I_{q,k} \cap A}^{q,k}$ . This is a finite subset of  $\omega$ . So  $l(n) = \sup \left\{ f(s) : s \in P \wedge s \subset \bigcup_{k \leq n} u_{I_{q,k} \cap A}^{q,k} \right\} < \omega$ . Similarly  $\bigcup_{k \leq n} u_{I_{q,k} \cap B}^{q,k}$  is a finite subset of  $\omega$ . By (2) of Lemma 28, for each  $i \in \omega$ ,  $\bigcup (f^{-1}(\{i\}))$  is a finite subset of  $\omega$ . So  $l^+(n) = \sup \left( \bigcup \left\{ \bigcup (f^{-1}(\{i\})) : i \in \bigcup_{k \leq n} u_{I_{q,k} \cap B}^{q,k} \right\} \right) < \omega$ . Build two sequences  $\langle k_n : n \in \omega \rangle$  and  $\langle u^{q_0, n} : n \in \omega \rangle$  such that for each  $n \in \omega$ :

- (1)  $k_n \in \omega$  and  $u^{q_0, n} \in \Sigma(u^{q, k_n})$ ;
- (2)  $\forall j < n [k_j < k_n \text{ and } \text{nor}(u^{q_0, j}) < \text{nor}(u^{q_0, n})]$ ;
- (3) for any  $s \subset \left( \bigcup_{j < n} u_{I_{q, k_j} \cap A}^{q_0, j} \right)$  and any  $t \subset \left( u_{I_{q, k_n} \cap B}^{q_0, n} \right)$ , if  $s \cup t \in P$ , then  $f(s \cup t) \notin u_{I_{q, k_n} \cap B}^{q_0, n}$ ;
- (4)  $\forall j < n [l(k_j) < \min \left( u_{I_{q, k_n} \cap B}^{q_0, n} \right) \text{ and } l^+(k_j) < \min \left( u_{I_{q, k_n} \cap A}^{q_0, n} \right)]$ .

Suppose for a moment that such a sequence can be built. Let  $q_0$  and  $p_0$  be defined as in Remark 24. Then  $p_0 \in \mathbb{P}_1$ ,  $p_0 \leq p$ ,  $q_0 \in \mathbb{Q}$ , and  $q_0$  induces  $p_0$ . Let  $X_A = \bigcup_{n \in \omega} u_{I_{q, k_n} \cap A}^{q_0, n}$  and  $X_B = \bigcup_{n \in \omega} u_{I_{q, k_n} \cap B}^{q_0, n}$ . Note that  $X_A \in \mathcal{D}_{p_0, A}$  and  $X_B \in \mathcal{D}_{p_0, B}$ . Suppose towards a contradiction that there exists  $s^* \in P$  such that  $s^* \subset X_A$  and  $f(s^*) \in X_B$ . As  $s^*$  is a non-empty finite subset of  $\omega$ ,  $\max(s^*)$  exists and there exists a unique  $n \in \omega$  such that  $\max(s^*) \in u_{I_{q, k_n} \cap A}^{q_0, n}$ . Then  $s^* = s \cup t$ , where  $s = s^* \cap \left( \bigcup_{j < n} u_{I_{q, k_j} \cap A}^{q_0, j} \right)$  and  $t = s^* \cap u_{I_{q, k_n} \cap A}^{q_0, n}$ . By clause (3),  $f(s^*) \notin u_{I_{q, k_n} \cap B}^{q_0, n}$ . By the definition of  $l(k_n)$ ,  $f(s^*) \leq l(k_n)$ . So by clause (4),  $\forall n^* > n [f(s^*) \notin u_{I_{q, k_{n^*}} \cap B}^{q_0, n^*}]$ . So it must be that  $f(s^*) \in u_{I_{q, k_j} \cap B}^{q_0, j}$  for some  $j < n$ . But then  $\max(s^*) \leq l^+(k_j)$ ,

contradicting clause (4). Therefore there is no  $s^* \in P$  such that  $s^* \subset X_A$  and  $f(s^*) \in X_B$ . Hence  $p_0$  is as required.

To build the sequences  $\langle k_n : n \in \omega \rangle$  and  $\langle u^{q_0, n} : n \in \omega \rangle$  proceed as follows. Fix  $n \in \omega$  and suppose that  $\langle k_j : j < n \rangle$  and  $\langle u^{q_0, j} : j < n \rangle$  are given. Let  $M = \{m\} \cup \{k_j : j < n\} \cup \{\text{nor}(u^{q_0, j}) + 1 : j < n\} \cup \{l(k_j) : j < n\} \cup \{l^+(k_j) : j < n\}$ . Note that  $M$  is a finite non-empty subset of  $\omega$ , and put  $k = \max(M) < \omega$ . Next, let  $x = \left( \bigcup_{j < n} u_{I_{q, k_j} \cap A}^{q_0, j} \right)$ , and note that  $x$  is also a finite set. Put  $k_n = k + 2^{|x|} < \omega$ . Note that  $k_n > k \geq m$ . Therefore  $(B \setminus A) \cap I_{q, k_n} \neq \emptyset$ . So  $B \cap I_{q, k_n} \not\subset A \cap I_{q, k_n}$ . Also  $\text{nor}(u^{q, k_n}) \geq k_n = k + 2^{|x|}$ . Let  $\langle s_i : i < 2^{|x|} \rangle$  enumerate all subsets of  $x$ . Now build a sequence  $\langle v^i : i < 2^{|x|} \rangle$  such that for each  $i < 2^{|x|}$ :

- (5)  $v^i \in \Sigma(u^{q, k_n})$  and  $\text{nor}(v^i) \geq k + 2^{|x|} - i - 1$ ;
- (6)  $\forall i^* < i [v^i \in \Sigma(v^{i^*})]$ ;
- (7) for any  $t \subset v_{I_{q, k_n} \cap A}^i$ , if  $s_i \cup t \in P$ , then  $f(s_i \cup t) \notin v_{I_{q, k_n} \cap B}^i$ .

This sequence is constructed by induction on  $i < 2^{|x|}$ . Fix  $i < 2^{|x|}$  and suppose that  $v^{i^*}$  is given for all  $i^* < i$ . If  $i > 0$ , let  $v = v^{i-1}$ , and if  $i = 0$ , then let  $v = u^{q, k_n}$ . In either case  $v \in \mathcal{CR}(I_{q, k_n})$  and  $\text{nor}(v) \geq (k + 2^{|x|} - i - 1) + 1$ . Now  $v_{I_{q, k_n} \cap B}$  is a non-empty set. Fix  $z_0 \in v_{I_{q, k_n} \cap B}$ . Define a function  $F : \mathcal{P}(v_{I_{q, k_n} \cap A}) \rightarrow v_{I_{q, k_n} \cap B}$  as follows. Given  $t \in \mathcal{P}(v_{I_{q, k_n} \cap A})$ , if  $s_i \cup t \in P$  and  $f(s_i \cup t) \in v_{I_{q, k_n} \cap B}$ , then let  $F(t) = f(s_i \cup t)$ . Otherwise let  $F(t) = z_0$ . There exists  $v^i \in \Sigma(v)$  with  $\text{nor}(v^i) \geq k + 2^{|x|} - i - 1$  such that  $F''\mathcal{P}(v_{I_{q, k_n} \cap A}) \cap v_{I_{q, k_n} \cap B}^i = \emptyset$ . It is clear that  $v^i$  is as needed.

Now let  $i = 2^{|x|} - 1 < 2^{|x|}$  and define  $u^{q_0, n} = v^i$ . By (5),  $v^i \in \Sigma(u^{q, k_n})$ , and so (1) is satisfied. For (2) note that  $\text{nor}(v^i) \geq k + 2^{|x|} - i - 1 = k \geq \text{nor}(u^{q_0, j}) + 1 > \text{nor}(u^{q_0, j})$ , for all  $j < n$ . Next, to check (3) fix  $s \subset \left( \bigcup_{j < n} u_{I_{q, k_j} \cap A}^{q_0, j} \right) = x$  and  $t \subset \left( u_{I_{q, k_n} \cap A}^{q_0, n} \right)$ . Suppose  $s \cup t \in P$ . Then  $s = s_{i^*}$  for some  $i^* \leq i$ . It follows from (6) that  $u_{I_{q, k_n} \cap A}^{q_0, n} \subset v_{I_{q, k_n} \cap A}^{i^*}$  and  $u_{I_{q, k_n} \cap B}^{q_0, n} \subset v_{I_{q, k_n} \cap B}^{i^*}$ . So by (7) applied to  $i^*$  we have that  $f(s \cup t) \notin u_{I_{q, k_n} \cap B}^{q_0, n}$ . Finally, for (4) note that  $u_{I_{q, k_n} \cap A}^{q_0, n} \subset u_{I_{q, k_n} \cap A}^{q, k_n}$  and  $u_{I_{q, k_n} \cap B}^{q_0, n} \subset u_{I_{q, k_n} \cap B}^{q, k_n}$ . So  $\min(u_{I_{q, k_n} \cap A}^{q_0, n}) \geq \min(u_{I_{q, k_n} \cap A}^{q, k_n}) \geq k_n > k \geq l^+(k_j)$ , for all  $j < n$ , and  $\min(u_{I_{q, k_n} \cap B}^{q_0, n}) \geq \min(u_{I_{q, k_n} \cap B}^{q, k_n}) \geq k_n > k \geq l(k_j)$ , for all  $j < n$ . Thus  $u^{q_0, n}$  and  $k_n$  are as required.  $\square$

The following lemma is easy to check and tells us what to do at limit stages of the final inductive construction. We leave the proof to the reader.

**Lemma 31.** *Assume MA( $\sigma$ -centered). Let  $\delta < \mathfrak{c}$  be a limit ordinal. Suppose  $\langle p_\alpha : \alpha < \delta \rangle$  is a sequence of conditions in  $\mathbb{P}_0$  such that  $\forall \alpha \leq \beta < \delta [p_\beta \leq p_\alpha]$ . Define  $\mathcal{A}_{p_\delta} = \bigcup_{\alpha < \delta} \mathcal{A}_{p_\alpha}$ . For any  $A \in \mathcal{A}_{p_\delta}$  let  $\alpha_A = \min\{\alpha < \delta : A \in \mathcal{A}_{p_\alpha}\}$ . For  $A \in \mathcal{A}_{p_\delta}$  define  $\mathcal{D}_{p_\delta, A} = \bigcup_{\alpha_A \leq \alpha < \delta} \mathcal{D}_{p_\alpha, A}$ , and define  $\mathcal{D}_{p_\delta} = \langle \mathcal{D}_{p_\delta, A} : A \in \mathcal{A}_{p_\delta} \rangle$ . Given  $A, B \in \mathcal{A}_{p_\delta}$  with  $A \subset^* B$ , let  $\alpha_{A, B} = \max\{\alpha_A, \alpha_B\}$ , and define  $\pi_{p_\delta, B, A} = \pi_{p_{\alpha_{A, B}}, B, A}$ . Define  $\mathcal{C}_{p_\delta} = \langle \pi_{p_\delta, B, A} : A, B \in \mathcal{A}_{p_\delta} \wedge A \subset^* B \rangle$ . Finally define  $p_\delta = \langle \mathcal{A}_{p_\delta}, \mathcal{C}_{p_\delta}, \mathcal{D}_{p_\delta} \rangle$ . Then  $p_\delta \in \mathbb{P}_0$  and  $\forall \alpha < \delta [p_\delta \leq p_\alpha]$ .*

**Lemma 32.** *Assume MA( $\sigma$ -centered). Let  $\delta < \mathfrak{c}$  be a limit ordinal with  $\text{cf}(\delta) = \omega$ . Suppose  $\langle p_\alpha : \alpha < \delta \rangle$  is a sequence of conditions in  $\mathbb{P}_1$  such that  $\forall \alpha \leq \beta < \delta [p_\beta \leq p_\alpha]$ . Suppose  $p_\delta \in \mathbb{P}_0$  is defined as in Lemma 31. Then  $p_\delta \in \mathbb{P}_1$ .*

*Proof.* Take a finitary  $p' \in \mathbb{P}_0$  with  $p_\delta \leq p'$ . For each  $A \in \mathcal{A}_{p'}$  let  $\alpha_A$  be defined as in Lemma 31. Since  $p'$  is finitary,  $\mathcal{A}_{p'}$  is finite and for each  $A \in \mathcal{A}_{p'}$ , there exists a non-empty and countable  $\mathcal{F}_{p',A} \subset \mathcal{D}_{p',A}$  such that  $\forall X \in \mathcal{D}_{p',A} \exists Y \in \mathcal{F}_{p',A} [Y \subset X]$ ; let  $\{Y_{A,n} : n \in \omega\}$  enumerate  $\mathcal{F}_{p',A}$ . For each  $A \in \mathcal{A}_{p'}$  and  $n \in \omega$  choose  $\alpha_A \leq \alpha_{A,n} < \delta$  such that  $Y_{A,n} \in \mathcal{D}_{p_{\alpha_A},A}$ . Find a strictly increasing cofinal sequence  $\langle \alpha_n : n \in \omega \rangle$  of elements of  $\delta$  such that  $\mathcal{A}_{p'} \subset \mathcal{A}_{p_{\alpha_0}}$  and  $\forall A \in \mathcal{A}_{p'} \forall i < n [\alpha_{A,i} < \alpha_n]$ . Define a standard sequence  $q$  as follows. Fix  $n \in \omega$  and suppose that  $I_{q,m}$  and  $u^{q,m}$  are given for all  $m < n$ . Choose  $q_n \in \mathbb{Q}$  inducing  $p_{\alpha_n}$ . We now define six collections of natural numbers as follows. First, let  $\mathcal{B}_{p'}$  denote the Boolean subalgebra of  $\mathcal{P}(\omega)$  generated by  $\mathcal{A}_{p'}$ . If  $A$  is an infinite member of  $\mathcal{B}_{p'}$ , then there exists  $k_A \in \omega$  such that  $\forall k \geq k_A [|A \cap I_{q_n,k}| < |A \cap I_{q_n,k+1}|]$ . Define  $\sup\{k_A + |I_{q,m} \cap A| + 1 : m < n\} = l_A$ . Second, say  $A \in \mathcal{A}_{p'}$  and  $i < n$ . Then there exists  $l_{A,i} \in \omega$  such that  $\forall k \geq l_{A,i} [u_{A \cap I_{q_n,k}}^{q_n,k} \subset Y_{A,i}]$ . Third, say  $A, B \in \mathcal{A}_{p'}$  with  $A \subset^* B$ . Then there exists  $l_{A,B} \in \omega$  such that  $\forall k \geq l_{A,B} [\pi_{p_{\alpha_n},B,A} \upharpoonright u_{B \cap I_{q_n,k}}^{q_n,k} = \pi_{q_n,B \cap I_{q_n,k}, A \cap I_{q_n,k}}]$ . Observe that since  $p_\delta \leq p_{\alpha_n}$  and  $p_\delta \leq p'$ ,  $\pi_{p_{\alpha_n},B,A} = \pi_{p',B,A}$ . Fourth, define  $l_0 = \sup\{\max(I_{q,m}) + 1 : m < n\}$ . Fifth, let  $\sup\{\text{nor}(u^{q,m}) + 1 : m < n\} = l_1$ . Sixth, define  $l_2 = \sup\{\max(u_a^{q,m}) + 1 : m < n \wedge a \in \mathcal{P}(I_{q,m})\}$ . Now consider  $M = \{l_A : A \in \mathcal{B}_{p'} \wedge A \text{ is infinite}\} \cup \{l_{A,i} : A \in \mathcal{A}_{p'} \wedge i < n\} \cup \{l_{A,B} : A, B \in \mathcal{A}_{p'} \wedge A \subset^* B\} \cup \{l_0, l_1, l_2\}$ .  $M$  is a finite non-empty subset of  $\omega$ . Let  $l = \max(M)$ . Then  $l \in \omega$ . Put  $I_{q,n} = I_{q_n,l}$  and  $u^{q,n} = u^{q_n,l}$ . This completes the definition of  $q$ . It is easy to see that  $q \in \mathbb{Q}$  and that  $q$  induces  $p'$ . Therefore  $p_\delta \in \mathbb{P}_1$ .  $\square$

**Lemma 33.** *Assume MA( $\sigma$ -centered). Let  $\delta < \mathfrak{c}$  be a limit ordinal with  $\text{cf}(\delta) > \omega$ . Suppose  $\langle p_\alpha : \alpha < \delta \rangle$  is a sequence of conditions in  $\mathbb{P}_1$  such that  $\forall \alpha \leq \beta < \delta [p_\beta \leq p_\alpha]$ . Suppose  $p_\delta \in \mathbb{P}_0$  is defined as in Lemma 31. Then  $p_\delta \in \mathbb{P}_1$ .*

*Proof.* Take a finitary  $p' \in \mathbb{P}_0$  with  $p_\delta \leq p'$ . Since  $\text{cf}(\delta) > \omega$ , there is  $\alpha < \delta$  such that  $p_\alpha \leq p'$ . There is a  $q \in \mathbb{Q}$  such that  $q$  induces  $p_\alpha$ . This  $q$  also induces  $p'$ . Hence  $p_\delta \in \mathbb{P}_1$ .  $\square$

We are now ready to prove the main theorem. Recall that  $\mathcal{P}(\omega)/\text{FIN}$  is the collection of equivalence classes  $\{[A] : A \in \mathcal{P}(\omega)\}$ , where  $[A] = \{B \in \mathcal{P}(\omega) : A =^* B\}$ . The natural partial order on it is defined by  $[A] \leq [B]$  if and only if  $A \subset^* B$ . We construct a set  $\mathcal{X}$  of representatives for the equivalence classes in  $\mathcal{P}(\omega)/\text{FIN}$  and a sequence of P-points  $\langle \mathcal{U}_A : A \in \mathcal{X} \rangle$  satisfying (3) and (4) of Theorem 34 below. Use this to define a map  $\varphi$  from  $\mathcal{P}(\omega)/\text{FIN}$  into the class of P-points as follows. Given  $[C] \in \mathcal{P}(\omega)/\text{FIN}$ , let  $A$  be the unique member of  $\mathcal{X} \cap [C]$ , and define  $\varphi([C]) = \mathcal{U}_A$ . By clauses (3) and (4) of Theorem 34 and by the fact that  $\mathcal{U}_A \leq_{RK} \mathcal{U}_B$  implies  $\mathcal{U}_A \leq_T \mathcal{U}_B$ ,  $[C] \leq [D]$  if and only if  $\varphi([C]) \leq_{RK} \varphi([D])$  if and only if  $\varphi([C]) \leq_T \varphi([D])$ . This implies that  $\varphi$  is 1-1 because  $[C] \leq [D]$  and  $[D] \leq [C]$  imply that  $[C] = [D]$ . Thus  $\varphi$  is an embedding of  $\mathcal{P}(\omega)/\text{FIN}$  into the class of P-points with respect to both  $\leq_{RK}$  and  $\leq_T$ .

**Theorem 34.** *Assume MA( $\sigma$ -centered). There exists a set  $\mathcal{X} \subset \mathcal{P}(\omega)$  and a sequence  $\langle \mathcal{U}_A : A \in \mathcal{X} \rangle$  such that the following hold:*

- (1)  $\forall A, B \in \mathcal{X} [A \neq B \implies A \neq^* B]$  and  $\forall C \in \mathcal{P}(\omega) \exists A \in \mathcal{X} [C =^* A]$ ;
- (2) for each  $A \in \mathcal{X}$ ,  $\mathcal{U}_A$  is a P-point;
- (3)  $\forall A, B \in \mathcal{X} [A \subset^* B \implies \mathcal{U}_A \leq_{RK} \mathcal{U}_B]$ ;
- (4)  $\forall A, B \in \mathcal{X} [B \not\subset^* A \implies \mathcal{U}_B \not\leq_T \mathcal{U}_A]$ .

*Proof.* Let  $\mathfrak{c} = T_0 \cup T_1 \cup T_2 \cup T_3$  be a partition of  $\mathfrak{c}$  into four disjoint pieces each of size  $\mathfrak{c}$ . Let  $\langle A_\alpha : \alpha \in T_0 \rangle$  be an enumeration of  $\mathcal{P}(\omega)$ . Let  $\langle \langle A_\alpha, X_\alpha \rangle : \alpha \in T_1 \rangle$  enumerate  $\mathcal{P}(\omega) \times \mathcal{P}(\omega)$  in such a way that each element of  $\mathcal{P}(\omega) \times \mathcal{P}(\omega)$  occurs  $\mathfrak{c}$  times on the list. Let

$$\mathcal{T} = \{ \langle P, f \rangle : P \subset [\omega]^{<\omega} \setminus \{0\} \text{ and } f : P \rightarrow \omega \text{ satisfy (1)-(2) of Lemma 28} \}.$$

Let  $\langle \langle A_\alpha, B_\alpha, P_\alpha, f_\alpha \rangle : \alpha \in T_2 \rangle$  enumerate  $\mathcal{P}(\omega) \times \mathcal{P}(\omega) \times \mathcal{T}$  in such a way that every element of  $\mathcal{P}(\omega) \times \mathcal{P}(\omega) \times \mathcal{T}$  occurs  $\mathfrak{c}$  times on the list. Build a decreasing sequence  $\langle p_\alpha : \alpha < \mathfrak{c} \rangle$  of conditions in  $\mathbb{P}_1$  by induction as follows. Since  $\mathbb{P}_1$  is non-empty choose an arbitrary  $p_0 \in \mathbb{P}_1$ . If  $\delta < \mathfrak{c}$  is a limit ordinal, then by Lemmas 32 and 33 there is a  $p_\delta \in \mathbb{P}_1$  such that  $\forall \alpha < \delta [p_\delta \leq p_\alpha]$ . Now suppose  $\delta = \alpha + 1$ . If  $\alpha \in T_0$ , then use Lemma 22 to find  $p_\delta \in \mathbb{P}_1$  such that  $p_\delta \leq p_\alpha$  and  $\exists C \in \mathcal{A}_{p_\delta} [A_\alpha =^* C]$ . If  $\alpha \in T_1$  and  $A_\alpha \in \mathcal{A}_{p_\alpha}$ , then use Lemma 25 to find  $p_\delta \in \mathbb{P}_1$  such that  $p_\delta \leq p_\alpha$  and either  $X_\alpha \in \mathcal{D}_{p_\delta, A_\alpha}$  or  $\omega \setminus X_\alpha \in \mathcal{D}_{p_\delta, A_\alpha}$ . If  $A_\alpha \notin \mathcal{A}_{p_\alpha}$ , then let  $p_\delta = p_\alpha$ . Next, suppose  $\alpha \in T_2$ ,  $A_\alpha, B_\alpha \in \mathcal{A}_{p_\alpha}$ , and that  $B_\alpha \not\leq^* A_\alpha$ . Use Lemma 30 to find  $p_\delta \in \mathbb{P}_1$  such that  $p_\delta \leq p_\alpha$  and there exist  $X_\alpha \in \mathcal{D}_{p_\delta, A_\alpha}$  and  $Y_\alpha \in \mathcal{D}_{p_\delta, B_\alpha}$  such that  $\forall s \in P_\alpha [s \subset X_\alpha \implies f_\alpha(s) \notin Y_\alpha]$ . If  $\alpha \in T_2$ , but the other conditions are not satisfied, then let  $p_\delta = p_\alpha$ . Finally if  $\alpha \in T_3$ , then use Lemma 17 to find  $p_\delta \in \mathbb{P}_1$  such that  $p_\delta \leq p_\alpha$  and  $\forall A \in \mathcal{A}_{p_\delta} \exists Y_{A,\alpha} \in \mathcal{D}_{p_\delta, A} \forall X \in \mathcal{D}_{p_\delta, A} [Y_{A,\alpha} \subset^* X]$ . This concludes the construction of  $\langle p_\alpha : \alpha < \mathfrak{c} \rangle$ .

Now define  $\mathcal{X} = \bigcup_{\alpha < \mathfrak{c}} \mathcal{A}_{p_\alpha}$ . To see that the first part of (1) holds, let  $A, B \in \mathcal{X}$  and suppose that  $A \neq B$ . Then there is  $\alpha < \mathfrak{c}$  such that  $A, B \in \mathcal{A}_{p_\alpha}$ , and since  $p_\alpha$  is a 0-condition,  $A \neq^* B$ . For the second part of (1), if  $A \in \mathcal{P}(\omega)$ , then there exists  $\alpha \in T_0$  such that  $A = A_\alpha$ , and so there is a  $C \in \mathcal{A}_{p_{(\alpha+1)}} \subset \mathcal{X}$  with  $A_\alpha =^* C$ . Thus (1) is verified. Next, for any  $A \in \mathcal{X}$ , let  $\alpha_A = \min\{\alpha < \mathfrak{c} : A \in \mathcal{A}_{p_\alpha}\}$ . Define  $\mathcal{U}_A = \bigcup_{\alpha_A \leq \alpha < \mathfrak{c}} \mathcal{D}_{p_\alpha, A}$ . To check that  $\mathcal{U}_A$  is an ultrafilter, fix  $X \in \mathcal{P}(\omega)$ . There is an  $\alpha \in T_1$  such that  $\alpha_A \leq \alpha$  and  $\langle A_\alpha, X_\alpha \rangle = \langle A, X \rangle$ . Then by construction, either  $X$  or  $\omega \setminus X$  belongs to  $\mathcal{D}_{p_{(\alpha+1)}, A}$ , and hence to  $\mathcal{U}_A$ . To see that  $\mathcal{U}_A$  is a P-point, let  $\{X_n : n \in \omega\} \subset \mathcal{U}_A$ . There exists  $\alpha \in T_3$  such that  $\alpha_A \leq \alpha$  and  $\{X_n : n \in \omega\} \subset \mathcal{D}_{p_\alpha, A}$ . Then there exists  $Y_{A,\alpha} \in \mathcal{D}_{p_{(\alpha+1)}, A} \subset \mathcal{U}_A$  such that  $\forall n \in \omega [Y_{A,\alpha} \subset^* X_n]$ . This verifies (2). Next, say  $A, B \in \mathcal{X}$  with  $A \subset^* B$ . Let  $\alpha_{A,B} = \max\{\alpha_A, \alpha_B\} < \mathfrak{c}$ . Define  $\pi_{B,A} = \pi_{p_{\alpha_{A,B}}, B, A} \in \omega^\omega$ . To check (3), suppose  $X \in \mathcal{U}_B$ . There is an  $\alpha \geq \alpha_{A,B}$  such that  $X \in \mathcal{D}_{p_\alpha, B}$ . Since  $p_\alpha \leq p_{\alpha_{A,B}}$ ,  $\pi_{p_\alpha, B, A} = \pi_{p_{\alpha_{A,B}}, B, A} = \pi_{B,A}$ , and since  $p_\alpha$  is a 0-condition,  $\pi_{p_\alpha, B, A}'' X \in \mathcal{D}_{p_\alpha, A} \subset \mathcal{U}_A$ . So  $\pi_{B,A}'' X \in \mathcal{U}_A$ . This implies that  $\mathcal{U}_A \leq_{RK} \mathcal{U}_B$ , verifying (3). Finally suppose  $A, B \in \mathcal{X}$  and that  $B \not\leq^* A$ . Suppose for a contradiction that  $\mathcal{U}_B \leq_T \mathcal{U}_A$ . Applying Lemma 28 with  $\mathcal{V} = \mathcal{U}_A$  and  $\mathcal{U} = \mathcal{U}_B$  we can find  $P \subset [\omega]^{<\omega} \setminus \{0\}$  and  $f : P \rightarrow \omega$  satisfying (1)-(3) of Lemma 28. There exists  $\alpha \in T_2$  such that  $\alpha_{A,B} \leq \alpha$  and  $A_\alpha = A, B_\alpha = B, P_\alpha = P$ , and  $f_\alpha = f$ . Let  $\delta = \alpha + 1$ . Then by construction there exist  $X_\alpha \in \mathcal{D}_{p_\delta, A} \subset \mathcal{U}_A = \mathcal{V}$  and  $Y_\alpha \in \mathcal{D}_{p_\delta, B} \subset \mathcal{U}_B = \mathcal{U}$  such that  $\forall s \in P [s \subset X_\alpha \implies f(s) \notin Y_\alpha]$ , contradicting (3) of Lemma 28. This concludes the verification of (4) and the proof of the theorem.  $\square$

### 3. REMARKS AND OPEN QUESTIONS

Under MA( $\sigma$ -centered) there are  $2^\mathfrak{c}$  P-points. Our results here leave open the question of which partial orders of size greater than  $\mathfrak{c}$  can be embedded into the P-points. As pointed out in the introduction, each P-point can have at most  $\mathfrak{c}$  predecessors with respect to  $\leq_{RK}$  and also with respect to  $\leq_T$ .

**Definition 35.** A partial order  $\langle X, < \rangle$  is said to be *locally of size  $\mathfrak{c}$*  if for each  $x \in X$ ,  $|\{x' \in X : x' \leq x\}| \leq \mathfrak{c}$ .

**Question 36.** Suppose  $\text{MA}(\sigma\text{-centered})$  holds. Let  $\langle X, < \rangle$  be a partial order of size at most  $2^{\mathfrak{c}}$  that is locally of size  $\mathfrak{c}$ . Does  $\langle X, < \rangle$  embed into the class of P-points with respect to both the Rudin-Keisler and Tukey orders?

A positive answer to Question 36 will give a complete solution to Blass' Question 3. It would say that anything that could possibly embed into the P-points does. Some partial answers toward Question 36 include the result of Dobrinen and Todorćevic [12] that assuming  $\mathfrak{d} = \mathfrak{u} = \mathfrak{c}$ , there are  $2^{\mathfrak{c}}$  many Tukey incomparable P-points, and their analogous result for selective ultrafilters assuming  $\text{cov}(\mathcal{M}) = \mathfrak{c}$ . However a general solution may require new ideas.

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