Arch. Math. Logic 44, 527–560 (2005)
Digital Object Identifier (DOI):
10.1007/s00153-004-0239-x

Mathematical Logic

Saharon Shelah

Middle diamond

Received: 28 February 2003 /
Published online: 10 February 2005 – © Springer-Verlag 2005

Abstract. Under certain cardinal arithmetic assumptions, we prove that for every large enough regular λ cardinal, for many regular $\kappa < \lambda$, many stationary subsets of λ concentrating on cofinality κ has the "middle diamond". In particular, we have the middle diamond on $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$. This is a strong negation of uniformization.

Annotated Content

§0 Introduction
[We state concise versions of the main results: provably in ZFC there are many cases of the middle diamond, for many triples $(\lambda, \Upsilon, \kappa)$ where $\kappa < \lambda$ are regular, $2 \leq \Upsilon < \lambda$ we can give for any $\delta \in S^\lambda_\kappa$ a function f_δ from some subset C_δ of δ of order type κ unbounded in δ into Υ such that any $f : \lambda \to \Upsilon$ extends f_δ for stationarily many δ's. We also review some history.]

§1 Middle Diamond: sufficient conditions
[We define families of relevant sequences $\bar{C} = \langle C_\delta : \delta \in S \rangle$ and properties needed for phrasing and proving our results (see Definition 1.5) and give easy implications; (later in 1.22, 1.25 we define other versions of middle diamond). Then (in 1.10) prove a version of the middle diamond for $(\lambda, \Upsilon, \theta, \kappa)$ guessing only the colour of $f \restriction C_\delta$ when $\lambda = \mathrm{cf}(2^\mu)$, the number of possible colours, θ, is not too large and κ belongs to a not so small set of regular cardinals $\kappa < \mu$ (e.g., every large enough regular $\kappa < \beth_\omega$ if $\beth_\omega \leq \mu$). The demand on κ says 2^μ is not the number of κ-branches of a tree with $< 2^\mu$ nodes. We then prove that a requirement on (μ, θ) is reasonable (see 1.11, 1.9, 1.13). In 1.20 we get the version of middle diamond giving $f \restriction C_\delta$ from the apparently weaker one guessing only the colour of $f \restriction C_\delta$. We quote various needed results and definitions in 1.3, 1.24.]

S. Shelah: The Hebrew University of Jerusalem, Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, Jerusalem 91904, Israel.
e-mail: `shelah@math.huji.ac.it`

Department of Mathematics, Hill Center-Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019 USA.

I would like to thank Alice Leonhardt for the beautiful typing. This research was partially supported by the Israel Science Foundation. Publication 775.

Key words or phrases: Set theory – Normal ideals – Middle Diamond – Weak Diamond

§2 Upgrading to larger cardinals
[Here we try to get such results for "many" large enough regular λ. We succeed to get it for every large enough not strongly inaccessible cardinal. We prove upgrading: i.e., give sufficient conditions for deducing middle diamond on λ_2 from ones on $\lambda_1 < \lambda_2$ (see 2.1) and 2.3 (many disjoint ones), 2.10, 2.12). We deduce (in 2.4) results of the form "if λ is large enough satisfying a weak requirement, then it has middle diamond for quite many κ, θ", using upgrading claims on the results of §1 and generalization of [EK] (see 2.5, 2.6, 2.7). We deal with such theorem more than necessary just for the upgrading.]

§0. Introduction

Using the diamond (discovered by Jensen [J]) we can construct many objects; wonderful but there is a price - we must assume $\mathbf{V} = \mathbf{L}$ or at least instances of G.C.H. (see Gregory [Gre], Shelah [Sh 108]). In fact $\lambda = \mu^+ = 2^\mu > \beth_\omega \Rightarrow \diamondsuit_{\{\delta<\lambda:\mathrm{cf}(\delta)=\kappa\}}$ for every regular large enough $\kappa < \beth_\omega$ (see [Sh 460] and citations there).

Naturally we would like to get the benefit without paying the price. A way to do this is to use the weak diamond (see [DvSh 65], [Sh:f, AP,§1], [Sh 208], [Sh 638] and lately [Sh 755], see on the definable version in [Sh:f, AP,§3], [Sh 576]).

In the diamond principle for a stationary $S \subseteq \lambda$, for $\delta \in S$ we guess $\eta \restriction \delta$ for $\eta \in {}^\lambda 2$.

In the weak diamond, for stationary $S \subseteq \lambda$ for $\delta \in S$ we do not guess $\eta \restriction \delta$ for $\eta \in {}^\lambda 2$, we just guess whether it satisfies a property - is black or white, i.e., the value of $\mathbf{F}(\eta \restriction \delta) \in \{0, 1\}$ for a pre-given $\mathbf{F}$. Many times it is just enough in constructions using diamond to "avoid undesirable company": whatever you will do I will do otherwise. This is, of course, a considerably weaker principle, but for $\aleph_1$ it follows from CH (which does not imply $\diamondsuit_{\aleph_1}$) and even WCH (i.e. $2^{\aleph_0} < 2^{\aleph_1}$). In fact, it holds for many cardinals: $2^\lambda > 2^{<\lambda} = 2^\theta$ suffices. Having $\mathbf{F}$ with two colours (= values) seemed essential. In [Sh 638], we get more colours for $\aleph_1$ if the club filter on ω_1 is $\aleph_2$-saturated but this is a heavy assumption. In [Sh 755], if $\lambda = \lambda^{<\lambda} = 2^\mu$, it is proved that for "many" regular $\kappa < \lambda$, if $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ is stationary then we have the weak diamond for S. Now what does "many" mean? For example, if $\lambda \geq \beth_\omega$, "many" include every large enough regular $\kappa < \beth_\omega$; note that it is for "every stationary S" unlike the situation, e.g. for $\aleph_1$ under CH (see [Sh:f, V]). We get there more cases if the colouring $\mathbf{F}$ on ${}^\delta 2$ depends only on $\eta \restriction C_\delta$, where $C_\delta \subseteq \delta = \sup(C_\delta)$ and C_δ is "small".

Here we get more than two colours and moreover: we guess $\eta \restriction C_\delta$, $C_\delta \subseteq \delta = \sup(C_\delta)$ and so can choose $\langle M_\delta : \delta \in S\rangle$ such that M_δ is a model with universe C_δ and relations only (and e.g. vocabulary $\subseteq \mathcal{H}(\aleph_0)$) such that if M is a model with universe λ and relation only (and vocabulary $\subseteq \mathcal{H}(\aleph_0)$) then for stationarily many $\delta \in S$, $M \restriction C_\delta$ is M_δ. Of course, if $(\exists \mu < \lambda)[2^\mu > \lambda]$ then we cannot have $C_\delta = \delta$ so this is a very reasonable restriction. (It would be even better to have $M_\delta \prec M$ below but we do not know). Also for "most" λ large enough the result holds for many regular κ.

The motivation has been trying to get better principles provably in ZFC.

We use partial squares and $I[\lambda]$, see [Sh 108], [AShS 221], [Sh 237e], [Sh 420, §1] and [DjSh 562]. We prove (in fact this holds for quite "many" stationary $S \subseteq S^\lambda_\kappa$).

Theorem 0.1. *For every μ strong limit of uncountable cofinality, for every regular $\lambda > \mu$ which is not strongly inaccessible, for every large enough regular $\kappa < \mu$ for $\chi < \mu$, S^λ_κ has the χ-middle diamond (called later "λ has the (κ, χ^+)-MD_2-property"), which means:*

$\boxplus_{\lambda,\kappa}$ *for some stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ for every relational vocabulary τ of cardinality $< \chi$ (so τ has no function symbols) we can find $\bar{M} = \langle M_\delta : \delta \in S\rangle$ which is a τ-MD-sequence i.e.*
- *(a) M_δ is a τ-model*
- *(b) the universe of M_δ is an unbounded subset of δ of order type $\chi \times \kappa$ including χ*
- *(c) if M is a τ-model with universe λ <u>then</u> for stationarily many $\delta \in S$ we have $M_\delta \subseteq M$.*

It seems that this is stronger in sufficient measure from the weak diamond to justify this name; there are many properties between the weak diamond and the property from 0.1 and we have to put the line somewhere. We shall use the adjective middle diamond to describe even cases with colouring with more than two values (justified by 1.20).

I thank Todd Eisworth and the referee and Adi Yarden for their helpful comments. Some improvements and elaborations are delayed to [Sh:F611]. This work is continued in [Sh:829]

Theorem 0.2. *In 0.1 we can have also $\bar{f} = \langle f_\delta : \delta \in S\rangle$ where $f_\delta : u_\delta \to \delta$, $u_\delta \subseteq \delta$, $\lambda^{|u_\delta|} = \lambda$ such that:*

(d) if M is a τ-model with universe λ and $f : \lambda \to \lambda$ and $u \subseteq \lambda$, $\lambda^{|u|} = \lambda$, <u>then</u> for stationarily many $\delta \in S$ we have $M_\delta \subseteq M \& (f_\delta = f \restriction u \subseteq f)$ & $u_\delta = u$.

Proof of 0.1 and 0.2. By 2.9(1) if $\lambda > \mu$ is regular not strongly inaccessible, then for every large enough regular $\kappa < \mu$ the following holds (see Definition 1.19(4)):

$(*)_\kappa$ if $\theta < \mu$ then some $(\lambda, \kappa, \kappa^+)$-parameter $\bar{C}$ has the θ-Md property; let $\bar{C} = \langle C_\delta : \delta \in S\rangle$.

Now without loss of generality $\delta \in S \Rightarrow (\chi \times \theta)^\omega | \delta$, and let $C'_\delta = \{\gamma : \gamma < \chi$ or $(\exists \beta \in C_\delta)(\beta \leq \gamma < \beta + \chi)$.

Lastly, let $\theta = 2^{\chi+\kappa+|\tau|}$ and by 1.20 we get the conclusion of 0.1.

As for 0.2, Clause (d) is easy by having a pairwise disjoint sequence of suitable $\langle S_i : i < \lambda\rangle$ which holds by 2.9(2). $\square_{0.1}$ $\square_{0.2}$

Note that having M_δ's is deduced in 1.20 from "$\bar{C}$ has the θ-Md-property" with θ large enough. Note that 0.1 has obvious applications on various constructions, e.g., construction of abelian groups G with $\mathrm{Ext}(G, K) \neq \emptyset$.

Note that in 0.1, 2.9 instead of using "large enough $\kappa < \mu \dots$", μ strong limit and quoting [Sh 460] we can specifically write what we use on κ (and on θ). Also in 0.1, if we omit $\mathrm{cf}(\mu) > \aleph_0$ (but still μ is uncountable) the case λ successor is not affected, and in the case $\mu = \beth_\delta$ when ω^2 divides δ, the statement instead "every large enough κ" can be replaced by "for arbitrarily large regular $\kappa < \mu$", or even: "for arbitrarily large strong limit $\mu' < \mu$, for every large enough regular $\kappa < \mu'$".

A consequence is (see more on this principle in [Sh 667]).

Conclusion 0.3. 1) If $\lambda, \kappa, \chi, S, \langle M_\delta : \delta \in S\rangle$ are as in 0.1 and $\gamma < \kappa \vee \gamma < \chi$, then we can find $\bar{\eta} = \langle \eta_\delta : \delta \in S\rangle$, η_δ is an increasing sequence of length κ of ordinals $< \delta$ with limit δ such that:

$\boxplus$ if $F : \lambda \to \gamma$, then for stationarily many $\delta \in S$ we have $i < \kappa \Rightarrow F(\eta_\delta(2i)) = F(\eta_\delta(2i+1))$.

2) In fact it is enough to find a (λ, κ, χ)-Md-parameter $\langle C_\delta : \delta \in S\rangle$ which has the $(\gamma, 2^{|\gamma|^+})$-MD-property (see Definition 1.6(1)).

Proof. 1) By (2).

2) Let $\tau = \{P_\zeta : \zeta < |\gamma|^+\}$, p_i a monadic predicate. Let $\langle M_\delta : \delta \in S\rangle$ be as in 0.1 for τ. Let $F : \lambda \to \gamma$ and we let M be the following τ-model with universe λ : $P^M_\zeta = \{\alpha : \zeta = \text{pr}(\zeta_0, \zeta_1), \zeta_0 < \zeta_1 < |\gamma|^+$ and $F(\alpha + \zeta_0) = F(\alpha + \zeta_1), \alpha \in M_\delta \Rightarrow \alpha + |\gamma|^+ < \text{Min}(M_\delta \backslash \alpha)$. Without loss of generality $\text{otp}(|M_\delta|) = \kappa, \delta \in S \Rightarrow |\gamma|^+/\delta$. Let $(\eta_\delta(2, i), \eta_\delta(2i+1)) = (\alpha^\delta_i + \zeta^\delta_2, \alpha^\delta_i + \zeta^\delta_2)$ when α^δ_i is the i-th member of $M_\delta, \alpha^\delta_i \in P$.

Note that if a natural quite weak conjecture on pcf holds, then we can replace $\beth_\omega$ by $\aleph_\omega$ (and similar changes) in most places but still need $\lambda > \lambda_1 > \lambda_\delta, 2^{\lambda_1} > 2^{\lambda_0} \geq 2^{\aleph_\omega}$ in the stepping-up lemma; e.g. for every regular large enough λ, for some n, we have middle diamond with λ_0 colours on $S^\lambda_{\aleph_n}$ - is helpful for abelian group theory. (Under weak conditions we get this for $\aleph_1$, see [Sh:F611].

Even without this conjecture it is quite hard to have the negation of this statement as it restricts 2^μ and $\text{cf}(2^\mu)$ for every μ!

Definition 0.4. *1) Let "$S \subseteq \lambda$ has square" mean that λ is regular uncountable, S a stationary subset of λ and: for some club E of λ and S', $S \cap E \subseteq S' \subseteq \lambda$ there is a witness $\bar{C} = \langle C_\delta : \delta \in S'\rangle$ which means:*

(*) $\bar{C} = \langle C_\delta : \delta \in S'\rangle$ has the square property which means

(a) $C_\delta \subseteq \delta$ is closed,
(b) $\delta \in S'$ limit $\Rightarrow \delta = \sup(C_\delta)$,
(c) $\alpha \in C_\delta \Rightarrow C_\alpha = \alpha \cap C_\delta$ and
(d) $\delta \in S \Rightarrow$ otp$(C_\delta) < \delta$.

1A) We say "$S \subseteq \lambda$ has a strict square" if we add
(e) $\delta \in S \cap E \Rightarrow$ otp$(C_\delta) =$ cf(δ).

2) We say "$S \subseteq \lambda$ has a $(\leq \delta^*)$ square" if we replace (d) above by
(d) $\delta \in S \Rightarrow$ otp$(C_\delta) \leq \delta^*$.
If we replace square by $(\leq \delta^*)$-square this means we add otp$(C_\alpha) \leq \delta^*$.

3) We say $(\leq \kappa, \Theta)$-square if above the demand $\alpha \in C_\delta \Rightarrow C_\alpha = \alpha \cap C_\delta$ is restricted to the case cf$(\alpha) \in \Theta$.

4) $S^\lambda_\kappa = \{\delta < \lambda :$ cf$(\delta) = \kappa\}$.

On $I[\lambda]$ see [Sh 420, §1].

Definition 0.5. *1) For λ regular uncountable let $I[\lambda]$ be the family of sets $S \subseteq \lambda$ which have a witness $(E, \bar{\mathcal{P}})$ for $S \in I[\lambda]$, which means*

(*) E is a club of λ, $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$, $\mathcal{P}_\alpha \subseteq \mathcal{P}(\alpha)$, $|\mathcal{P}_\alpha| < \lambda$, and for every $\delta \in E \cap S$ there is an unbounded subset C of δ of order type $< \delta$ such that $\alpha \in C \Rightarrow C \cap \alpha \in \mathcal{P}_\alpha$.

2) Equivalently there is a pair $(E, \bar{a})$, E a club of λ, $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$, $a_\alpha \subseteq \alpha$, $\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta$ and $\delta \in E \cap S \Rightarrow \delta = \sup(a_\delta) >$ otp(a_δ) (or even $\delta = \sup(a_\delta)$, otp$(a_\delta) =$ cf$(\delta) < \delta$). Also $w \log \alpha \in a_\beta \Rightarrow \alpha_\beta$ is a successor ordinal (note we can use $a'_\alpha = \{\beta + 1 : \beta \in a_\alpha\}$ if α is not successor and $a'_\alpha = \{\beta + 1 : \beta \in a_{\alpha-1}\}$ if α is successor).

Notation 0.6. 1) Let $\lambda, \mu, \kappa, \chi, \theta, \sigma$ denote cardinals, Θ a set of cardinals, let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, i, j, \Upsilon$ denote ordinals but $\Upsilon \geq 2$ and $\theta \geq 2$ and δ is a limit ordinal if not said otherwise.

Clearly 1.10 and some results later continue Engełking Karłowic [EK] which continue works on density of some topological spaces, see more history in [Sh 420].

§1. Middle Diamond: Sufficient conditions for cf(2^μ)

Definition 1.1. *For a regular uncountable λ, an ordinal $\Upsilon \geq 2$ and cardinal $\theta \geq 2$*

(1) we say $S \subseteq \lambda$ is (Υ, θ)-small <u>if</u> it is $\mathbf{F}$-(Υ, θ)-small for some $(\lambda, \Upsilon, \theta)$-colouring $\mathbf{F}$, also called a θ-colouring of ${}^{\lambda>}\Upsilon$, that is: a function $\mathbf{F}$ from ${}^{\lambda>}\Upsilon$ to θ where

(2) we say $S \subseteq \lambda$ is $\mathbf{F} - (\Upsilon, \theta)$-small <u>if</u> ($\mathbf{F}$ is a θ-colouring of ${}^{\lambda>}\Upsilon$ and)

$(*)^\theta_{\mathbf{F},S}$ for every $\bar{c} \in {}^S\theta$ for some $\eta \in {}^\lambda\Upsilon$ the set $\{\delta \in S : \mathbf{F}(\eta \restriction \delta) = c_\delta\}$ is not stationary.

(3) Let $D^{md}_{\lambda,\Upsilon,\theta} = \{A \subseteq \lambda : \lambda \setminus A$ is (Υ, θ)-small$\}$, we call it the (Υ, θ)-middle diamond filter on λ. If $\theta = 2$ we call this weak diamond filter and may write $D^{wd}_{\lambda,\Upsilon}$.

(4) We omit Υ if $\Upsilon = \theta$, writing θ instead of (Υ, θ). We may omit θ from "S is $\mathbf{F} - \theta$-small", and $(*)^\theta_{\mathbf{F},S}$ when clear from the context.

Claim 1.2. 1) If $\lambda = \text{cf}(\lambda) > \aleph_0$, $\lambda \geq \Upsilon \geq 2$ and $\lambda \geq \theta$ <u>then</u> $D^{\text{md}}_{\lambda,\Upsilon,\theta}$ is a normal filter on λ.

2) Assume $S \subseteq \lambda$, $2 \leq \Upsilon_\ell \leq \lambda$, $2 \leq \theta \leq \lambda$ for $\ell = 1, 2$. <u>Then</u> S is (Υ_1, θ)-small iff S is (Υ_2, θ)-small [1].

3) In fact $\Upsilon_\ell \in [2, 2^{<\lambda}]$ is enough.

Proof. Straightforward. $\square_{1.2}$

Recall

Definition 1.3. *1) Let $\chi^{<\sigma>} = Min\{|\mathcal{P}| : \mathcal{P} \subseteq [\chi]^\sigma$ and every $A \in [\chi]^\sigma$ is included in the union of $< \sigma$ members of $\mathcal{P}\}$.*

2) Let $\chi^{[\sigma]} = Min\{|\mathcal{P}| : \mathcal{P} \subseteq [\chi]^\sigma$ and every $A \in [\chi]^\sigma$ is equal to the union of $< \sigma$ members of $\mathcal{P}\}$.

---

[1] so Υ seems immaterial, but below when using $\bar{C}$ this is not so (see Definition 1.5)

3) $\chi^{<\sigma>_{tr}} = \sup\{|lim_\sigma(t)| : t$ *is a tree with* $\leq \chi$ *nodes and* σ *levels*$\}$.

4) For J *an ideal on* θ *let* $\mathbf{U}_J(\mu) = Min\{|\mathcal{P}| : \mathcal{P} \subseteq [\mu]^\theta$ *and for every* $f : Dom(J) \to \mu$ *for some* $u \in \mathcal{P}$ *we have* $\{i \in Dom(J) : f(i) \in u\} \neq \emptyset$ *mod* $J\}$; *we say an ideal* J *is an ideal on a set* $Dom(J)$ *of cardinality* θ. *If* $J = [\theta]^{<\theta}$ *we may write* θ *instead of* J.

Remark 1.4. 1) On $\chi^{<\sigma>_{\rm tr}}$ see [Sh 589], on $\chi^{<\sigma>}$ see [Sh 430], on $\chi^{<\sigma>}$, $\chi^{[\sigma]}$ see there and in [Sh 460] but no real knowledge of these references is assumed here.

2) Note that when $\Upsilon = \lambda$ only $\mathbf{F} \restriction \{^\delta\delta : \delta \in S\}$ is relevant in Definition 1.1 because for any $\eta \in {}^\lambda\lambda$ for some club E of λ we have $\delta \in E \Rightarrow \eta \restriction \delta \in {}^\delta\delta$. So we can restrict the colourings to this set.

Below we may usually restrict ourselves to $\Upsilon = \lambda$. But if $\mu < \lambda = \mathrm{cf}(\lambda)$, $2^\mu = 2^\lambda$ even for $\theta = \Upsilon = 2$, the set λ is small (as noted by Uri Abraham, see [DvSh 65]).

Definition 1.5. *1) We say* $\bar{C}$ *is a* λ*-Md-parameter* *<u>if</u>:*

(a) λ *is regular uncountable,*

(b) S *is a stationary subset of* λ*,*

(c) $\bar{C} = \langle C_\delta : \delta \in S\rangle$ *and* [2] $C_\delta \subseteq \delta = \sup(C_\delta)$.

1A) We say $\bar{C}$ *is a* (λ, κ, χ)*-Md-parameter* *<u>if</u> in addition* $(\forall\delta \in S)(\forall\alpha \in C_\delta)$ $[cf(\delta) = \kappa \wedge |C_\delta \cap \alpha| < \chi]$.

2) We say that $\mathbf{F}$ *is a* $(\bar{C}, \Upsilon, \theta)$*-colouring if:* $\theta \geq 2$, $\Upsilon \geq 2$, $\bar{C} = \langle C_\delta : \delta \in S\rangle$ *is a* λ*-Md-parameter and* $\mathbf{F}$ *is a function from* ${}^{\lambda>}\Upsilon$ *to* θ *such that: if* $\delta \in S$ *and* $\eta_0, \eta_1 \in {}^\delta\Upsilon$ *and* $\eta_0 \restriction C_\delta = \eta_1 \restriction C_\delta$ *then* $\mathbf{F}(\eta_0) = \mathbf{F}(\eta_1)$.

2A) If $\Upsilon = \theta$ *we may omit it and write* $(\bar{C}, \theta)$*-colouring.*

2B) In part (2) we can replace $\mathbf{F}$ *by* $\bar{F} = \langle F_\delta : \delta \in S\rangle$ *where* $F_\delta : {}^{(C_\delta)}\Upsilon \to \theta$ *such that* $\eta \in {}^\delta\delta \wedge \delta \in S \to \mathbf{F}(\eta) = F_\delta(\eta \restriction C_\delta)$. *If* $\Upsilon = \lambda$ *we may use* $F_\delta : {}^{(C_\delta)}\delta \to \theta$ *(as we can ignore a non-stationary* $S' \subseteq S$*); so abusing notation we may write* $\mathbf{F}(\eta \restriction C_\delta)$. *Similarly if* $cf(\Upsilon) = \lambda$, $\langle\Upsilon_\alpha : \alpha < \lambda\rangle$ *is increasing continuous with limit* Υ *we may restrict ourselves to* ${}^{(C_\delta)}(\Upsilon_\delta)$.

3) Assume $\mathbf{F}$ *is a* $(\bar{C}, \Upsilon, \theta)$*-colouring, where* $\bar{C}$ *is a* λ*-Md-parameter. We say* $\bar{c} \in {}^S\theta$ *(or* $\bar{c} \in {}^\lambda\theta$*) is an* $\mathbf{F}$*-middle diamond sequence, or an* $\mathbf{F}$*-Md-sequence if:*

(*) for every $\eta \in {}^\lambda\Upsilon$*, the set* $\{\delta \in S : \mathbf{F}(\eta \restriction \delta) = c_\delta\}$ *is a stationary subset of* λ*.*

We also may say that $\bar{c}$ *is an* $(\mathbf{F}, S)$*-Md-sequence or an* $(\mathbf{F}, \bar{C})$*-Md-sequence; note that* Υ *can be reconstructed from* $\mathbf{F}$*.*

3A) We say $\bar{c} \in {}^S\theta$ *is a* $D - \mathbf{F}$*-Md-sequence or* $D - (\mathbf{F}, \bar{C})$*-Md-sequence <u>if</u>* D *is a filter on* λ *(no harm to assume that* S *belongs to* D*, or at least* $S \in D^+$*) and (**) for every* $\eta \in {}^\lambda\Upsilon$ *we have*

$$\{\delta \in S : \mathbf{F}(\eta \restriction \delta) = c_\delta\} \neq \emptyset \text{ mod } D.$$

4) We say that $\bar{C}$ *is a good* (λ, λ_1)*-Md-parameter, <u>if</u>* $\bar{C}$ *is a* λ*-Md-parameter and for every* $\alpha < \lambda$ *we have* $\lambda_1 > |\{C_\delta \cap \alpha : \delta \in S \& \alpha \in C_\delta\}|$. *If* $\lambda_1 = \lambda$

[2] we can omit the requirement $\delta = \sup(C_\delta)$ but then we trivially have middle diamond if $\lambda = \lambda^{<\kappa}$ when $\kappa = \cup\{|C_\delta|^+ : \delta \in S\}$

we may write "a good λ-Md-parameter". Similarly for a $(\lambda, \lambda_1, \kappa, \chi)$-good-Md-parameter and a (λ, κ, χ)-good-Md-parameter (i.e. adding that $\bar{C}$ is a (λ, κ, χ)-Md-parameter.

4A) *We say that $\bar{C}$ is a weakly good λ-Md-parameter if it is a λ-Md-parameter and for every $\alpha < \lambda$ we have $\lambda > |\{C_\delta \cap \alpha : \delta \in S \& \alpha \in nacc(C_\delta)\}|$. Similarly we define a weakly good (λ, κ, χ)-Md-parameter and a weakly good $(\lambda, \lambda_1, \kappa, \chi)$-Md-parameter.*

4B) *We say $\bar{C}$ is a good$^+ - \lambda$-Md-parameter if $\bar{C} = \langle C_\delta : \delta \in S\rangle$ is a λ-Md-parameter and $\alpha \in C_{\delta_1} \cap C_{\delta_2} \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$. Similarly we define a good$^+ - (\lambda, \kappa, \chi)$-Md-parameter.*

4C) *We define a weakly good$^+ - \lambda$-Md-parameter or (λ, κ, χ)-Md-parameter analogously, i.e., $\alpha \in nacc(C_{\delta_1}) \cap nacc(C_{\delta_2}) \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$.*

4D) *For a set Θ of regular cardinals we say that $\bar{C} = \langle C_\delta : \delta \in S\rangle$ is Θ-closed if each C_δ is; where a set C of ordinals is Θ-closed if for every ordinal α, $cf(\alpha) \in \Theta \& \alpha = \sup(C \cap \alpha) < \sup(C) \Rightarrow \alpha \in C$.*

4E) *We say $\langle (C_\delta, C'_\delta) : \delta \in S\rangle$ is a good (λ, κ, χ)-Md$_*$ parameter if $\langle C_\delta : \delta \in S\rangle$ is a (λ, κ, χ)-Md-parameter, $C'_\delta \subseteq \delta$, $\sup(C'_\delta) = \delta$ and for every $\alpha < \lambda$ the set $|\{C_\delta \cap \alpha : \alpha \in C'_\delta, \delta \in S\}|$ has cardinality $< \lambda$.*

Definition 1.6. *1) We say that $\bar{C}$ (or $(\lambda, \bar{C})$) has the (D, Υ, θ)-Md-property if*

(a) $\bar{C}$ is a λ-Md-parameter
(b) D is a filter on λ
(c) if $\mathbf{F}$ is a $(\bar{C}, \Upsilon, \theta)$-colouring, then there is a $D - \mathbf{F}$-Md-sequence, see Definition 1.5, part (3A).

We may omit D if it is the club filter on λ.

2) *We say that $\bar{C}$ has the (D, θ)-Md-property as in part (1) only that $\mathbf{F}$ is a $(\bar{C}, \theta, \theta)$-colouring. We may omit D if D is the club filter. We may omit θ if $\theta = 2$.*

3) *We say that $S \subseteq \lambda$ has the (Υ, θ)-Md-property when some $\bar{C} = \langle C_\delta : \delta \in S\rangle$ has, and then we say the θ-Md-property if $\Upsilon = \theta$.*
In Definition 1.5(2),(2A) we replace "colouring" by "colouring$_2$" if we have all $\eta : {}^{\omega>}\lambda \to \Upsilon$ (or $\eta : {}^{\omega>}(C_\delta) \to \Upsilon$, and let "colouring$_1$" be the original notion (so in Definition 1.5(3),(3A) we have new meaning when we use such $\mathbf{F}$'s).

4) *We write Md$_2$-property if we use colouring$_2$ $\mathbf{F}$ and we write Md$_1$-property for the Md-property.*

Remark 1.7. We may consider replacing C_δ by (C_δ, D_δ), D_δ a filter on C_δ, that is $F_\delta(\eta)$ would depend on $(\eta \restriction C_\delta)/D_\delta$ only. For the time being it does not make a real difference.

Claim 1.8. 1) The restrictions in 1.5(2B) are O.K. That is in 1.5(2B), the first sentence is obvious; for the second sentence (if $\Upsilon = \lambda$) recall that for every $\eta \in {}^\lambda\Upsilon$, for some club E of λ, $\delta \in E \Rightarrow \eta \restriction \delta \in {}^\delta\delta$; and similarly in the third sentence.

2) In Definition 1.5(4), if $\lambda_1 > \lambda$ then goodness holds trivially.

3) If $\bar{C} = \langle C_\delta : \delta \in S\rangle$ is a weakly good (λ, κ, χ)-parameter, then for some club E of λ there is $\bar{C}' = \langle C'_\delta : \delta \in S \cap E\rangle$ which is a good$^+$ (λ, κ, χ)-parameter.

4) If $\bar{C} = \langle C_\delta : \delta \in S\rangle$ has the (D, Υ, θ)-Md-property, and $\bar{C}' = \langle C'_\delta : \delta \in S\rangle$, $C'_\delta \subseteq C_\delta$, then also $\bar{C}'$ has the (D, Υ, θ)-Md-property.

Proof. 1) , 2) Easy.

3) Note that the C'_δ's are not required to be a club of δ, see [Sh 420, §1] but we elaborate. Let $\langle A_\beta : \beta < \lambda \rangle$ list without repetition $\{C_\delta \cap \alpha : \delta \in S \text{ and } \alpha \in \operatorname{nacc}(C_\delta)\}$ and be such that $A_\beta = A_\gamma \cap \alpha \Rightarrow A_\beta \in \{A_\varepsilon : \varepsilon \le \gamma\}$. Let $E = \{\delta < \lambda$: if $\beta < \delta$ and $\alpha \in S$ and $\beta \in \operatorname{nacc}(C_\alpha)$ then $C_\alpha \cap \beta \in \{A_\gamma : \gamma < \delta\}\}$, clearly E is a club of λ.

Lastly, for $\delta \in S \cap E$ let $C'_\delta = \{\gamma < \delta$: for some $\alpha \in \operatorname{nacc}(C_\delta)$ we have $C_\delta \cap \alpha = A_\gamma\}$. Now $\langle C'_\delta : \delta \in S \cap E \rangle$ is as required.

4) Should be clear. $\square_{1.8}$

From the following definition the case $\operatorname{Sep}(\mu, \theta)$ is used in the main claim below.

Definition 1.9. *1) $Sep(\chi, \mu, \theta_0, \theta, \Upsilon)$ means that for some $\bar{f}$:*

(a) $\bar{f} = \langle f_\varepsilon : \varepsilon < \chi \rangle$

(b) f_ε is a function from ${}^\mu(\theta_0)$ to θ

(c) for every $\varrho \in {}^\chi\theta$ the set $\{\nu \in {}^\mu(\theta_0)$: for every $\varepsilon < \chi$ we have $f_\varepsilon(\nu) \ne \varrho(\varepsilon)\}$ has cardinality $< \Upsilon$.

2) We may omit θ_0 if $\theta_0 = \theta$. We write $Sep(\mu, \theta)$ if for some $\Upsilon = cf(\Upsilon) \le 2^\mu$ we have $Sep(\mu, \mu, \theta, \theta, \Upsilon)$ and $Sep(< \mu, \theta)$ if for some $\Upsilon = cf(\Upsilon) \le 2^\mu$ and $\sigma < \mu$ we have $Sep(\sigma, \mu, \theta, \theta, \Upsilon)$.

The Main Claim 1.10. Assume

(a) $\lambda = \operatorname{cf}(2^\mu)$, D is a μ^+-complete filter on λ extending the club filter
(b) $\kappa = \operatorname{cf}(\kappa) < \chi \le \lambda$
(c) $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a (λ, κ, χ)-good-Md-parameter, or just weakly good or just $\langle (C_\delta, C'_\delta) : \delta \in S \rangle$ is a good (λ, κ, χ)-Md_*-parameter and $S \in D$
(d) $2^{<\chi} \le 2^\mu$ and $\theta \le \mu$
(e) $\alpha < 2^\mu \Rightarrow |\alpha|^{<\kappa>_{\mathrm{tr}}} < 2^\mu$
(f) $\operatorname{Sep}(\mu, \theta)$.

<u>Then</u> $\bar{C}$ has the $(D, 2^\mu, \theta)$-Md-property (recall that this means that the number of colours is θ not just 2).

We may wonder if clause (f) of the assumption is reasonable; the following Claim gives some sufficient conditions for clause (f) of 1.10 to hold.

Claim 1.11. Clause (f) of 1.10 holds, i.e., $\operatorname{Sep}(\mu, \theta)$ holds, <u>if</u> at least one of the following holds:

(a) $\mu = \mu^\theta$
(b) $\mathbf{U}_\theta(\mu) = \mu + 2^\theta \le \mu$,
(c) $\mathbf{U}_J(\mu) = \mu$ where for some σ we have $J = [\sigma]^{<\theta}, \theta \le \sigma, \sigma^\theta \le \mu$
(d) μ is a strong limit of cofinality $\ne \theta = \operatorname{cf}(\theta) < \mu$
(e) $\mu \ge \beth_\omega(\theta)$.

We shall prove this later.

Remark 1.12. 1) Note that $\lambda = \lambda^{<\lambda} = \chi, \kappa = \operatorname{cf}(\kappa) < \lambda, S = S^\lambda_\kappa, C_\delta = \delta$ is a particular case (of interest) of the assumptions (a)-(b) of 1.10 and that $\langle C_\delta : \delta \in S^\lambda_\kappa \rangle$ is a $(\lambda, \kappa, \lambda)$-good$^+$- Md-parameter.

2) We can replace the requirement "$S \subseteq S^\lambda_\kappa$" (implicit in (c) of 1.10) by (***) [$\delta \in S \Rightarrow \delta$ is a regular cardinal].
Then clause (e) of 1.10 is replaced by "$\delta \in S \Rightarrow \delta^{<\delta>_{\rm tr}} < 2^\mu$".
3) We may consider finding a guessing sequence $\bar{c}$ simultaneously for many colourings. In the notation of [Sh:f, AP,§1] this is the same as having large $\mu(0)$. We may reconsider the problem of λ^+-entangled linear ordering where $\lambda = \lambda^{\aleph_0}$, see [Sh 462].
4) For $\theta = 2$, (i.e. weak diamond) the proof below can be simplified: we can forget $\mathcal{F}$ and choose $\varrho_\delta \in {}^\mu\theta \backslash \{\rho_\eta : \eta \in \mathcal{P}_\delta\}$ such that for some $i < \mu$ the sequence $\langle \varrho_\delta(i) : \delta \in S\rangle$ is a weak diamond sequence for $\mathbf{F}$.
5) Note that the "minimal" natural choices for getting θ-MD property are $\kappa = \aleph_0, \theta = 2^{\aleph_0}, \mu = 2^\theta, \lambda = \mathrm{cf}(2^\mu)$.
6) Note that in the main claim (1.10) if 2^μ is regular, $\Upsilon = 2^\mu$ is allowed.

Proof of 1.10. In clause (c) we can assume $\langle (C_\delta, C'_\delta) : \delta \in S\rangle$ is a good (λ, κ, χ)-Md_*-parameter (in the other cases we choose $C'_\delta = \mathrm{nacc}(C_\delta)$ and use 1.8(3).

Let $\mathbf{F}$ be a given $(\bar{C}, 2^\mu, \theta)$-colouring.

By assumption (f) we have $\mathrm{Sep}(\mu, \theta)$ which means (see Definition 1.9(2)) that for some $\Upsilon = \mathrm{cf}(\Upsilon) \le 2^\mu$ we have $\mathrm{Sep}(\mu, \mu, \theta, \theta, \Upsilon)$.

Let $\mathcal{F} = \{f_\xi : \xi < \mu\}$ where $\mathcal{F}$ exemplify $\mathrm{Sep}(\mu, \mu, \theta, \theta, \Upsilon)$, see Definition 1.9(1) and for $\rho \in {}^\mu\theta$ let $\mathrm{Sol}_\rho =: \{\nu \in {}^\mu\theta : \text{ for every } \varepsilon < \mu \text{ we have } \rho(\varepsilon) \neq f_\varepsilon(\nu)\}$ where Sol stands for solutions, so by clause (c) of the Definition 1.9(1) of Sep it follows that

(*) $\rho \in {}^\mu\theta \Rightarrow |\mathrm{Sol}_\rho| < \Upsilon$.

Let h be an increasing continuous function from λ into 2^μ with unbounded range; if λ is regular, $h = \mathrm{id}_\lambda$ is O.K. Recall that $\lambda = \mathrm{cf}(2^\mu) \ge 2^{<\chi}$. Let cd be a one-to-one function from ${}^\mu(2^\mu)$ onto 2^μ such that

$$\alpha = \mathrm{cd}(\langle \alpha_\varepsilon : \varepsilon < \mu\rangle) \Rightarrow \alpha \ge \sup\{\alpha_\varepsilon : \varepsilon < \mu\}$$

and let the function $\mathrm{cd}_i : 2^\mu \to 2^\mu$ for $i < \mu$ be such that $\mathrm{cd}_i(\mathrm{cd}(\langle \alpha_\varepsilon : \varepsilon < \mu\rangle)) = \alpha_i$.

Let $\mathcal{C} = \{C_\delta \cap \alpha : \delta \in S \text{ and } \alpha \in C'_\delta\}$, (recall that trivially $|\mathcal{C}| \le \lambda \le 2^\mu$) and let $T = \{\eta : \text{for some } C \in \mathcal{C}$ (hence C is of cardinality $< \chi$), we have $\eta \in {}^C(2^\mu)\}$. By assumptions (c) + (d) clearly $|T| = \sum_{C \in \mathcal{C}} 2^{\mu \cdot |C|} \le |\mathcal{C}| \times (2^\mu)^{<\chi} \le 2^\mu$ but $|T| \ge 2^\mu$ hence $|T| = 2^\mu$. So let us list T possibly with repetitions as $\{\eta_\alpha : \alpha < 2^\mu\}$ such that $[\alpha < h(\beta) \Rightarrow \mathrm{Rang}(\eta_\alpha) \subseteq h(\beta)]$ and let $T_{<\alpha} = \{\eta_\beta : \beta < h(\alpha)\}$ for $\alpha < \lambda$. For $\delta \in S$ let $\mathcal{P}_\delta =: \{\eta : \eta$ is a function from C_δ into $h(\delta)$ such that for every $\alpha \in C'_\delta$ we have $\eta \restriction (C_\delta \cap \alpha) \in T_{<\delta}\}$, clearly $\eta \in \mathcal{P}_\delta \Rightarrow \eta \in {}^{(C_\delta)}h(\delta)$. Clearly $\{\eta \restriction (C_\delta \cap \alpha); \eta \in \mathcal{P}_\delta, \alpha \in C'_\delta\}$ is naturally a tree with $\mathcal{P}_\delta$ the set of $\mathrm{otp}(C_\delta)$-branches and $\mathrm{cf}(\mathrm{otp}(C_\delta)) = \kappa$ and set of nodes $\subseteq T_{<\delta}$ hence with $< 2^\mu$ nodes hence by assumption (e)

(**) $|\mathcal{P}_\delta| < 2^\mu$.

For each $\eta \in \mathcal{P}_\delta$ and $\varepsilon < \mu$ we define $\nu_{\eta,\varepsilon} \in {}^{C_\delta}(h(\delta))$ by $\nu_{\eta,\varepsilon}(\alpha) = \mathrm{cd}_\varepsilon(\eta(\alpha))$ for $\alpha \in C_\delta$. Now for $\eta \in \mathcal{P}_\delta$, clearly $\rho_\eta =: \langle \mathbf{F}(\nu_{\eta,\varepsilon}) : \varepsilon < \mu \rangle$ belongs to ${}^\mu\theta$. Clearly $\{\rho_\eta : \eta \in \mathcal{P}_\delta\}$ is a subset of ${}^\mu\theta$ of cardinality $\le |\mathcal{P}_\delta|$ which as said above is $< 2^\mu$. Recall the definition of Sol_ρ from the beginning of the proof and remember $(*)$ above, so $\eta \in \mathcal{P}_\delta \Rightarrow \mathrm{Sol}_{\rho_\eta}$ has cardinality $< \Upsilon = \mathrm{cf}(\Upsilon) \le 2^\mu$. Hence we can find $\varrho^*_\delta \in {}^\mu\theta \setminus \cup\{\mathrm{Sol}_{\rho_\eta} : \eta \in \mathcal{P}_\delta\}$.

[Why? if $\Upsilon < 2^\mu$ then $|\cup\{\mathrm{Sol}_{\rho_\eta} : \eta \in \mathcal{P}_\delta\}| \le \Sigma\{|\mathrm{Sol}_{\rho_\eta}| : \eta \in \mathcal{P}_\delta\} \le \Upsilon \times |\mathcal{P}_\delta| < 2^\mu = \theta^\mu = |{}^\mu\theta|$ and if $\Upsilon = 2^\mu$ then 2^μ is regular, $|\mathcal{P}_\delta| < 2^\mu$, $\eta \in \mathcal{P}_\delta \Rightarrow |\mathrm{Sol}_{\rho_\eta}| < 2^\mu$ and again $|\cup\{\mathrm{Sol}_{\rho_\eta} : \eta \in \mathcal{P}_\delta\}| < 2^\mu = |{}^\mu\theta|$.]

Let $\varepsilon < \mu$. Recall that $\varrho^*_\delta \in {}^\mu\theta$ for $\delta \in S$ and f_ε from the list of $\mathcal{F}$ is a function from ${}^\mu\theta$ to θ so $f_\varepsilon(\varrho^*_\delta) < \theta$. Hence we can consider the sequence $\bar{c}^\varepsilon = \langle f_\varepsilon(\varrho^*_\delta) : \delta \in S\rangle \in {}^S\theta$ as a candidate for being an $\mathbf{F}$-Md-sequence. If one of them is, we are done. So assume towards a contradiction that for each $\varepsilon < \mu$ there is a sequence $\eta_\varepsilon \in {}^\lambda(2^\mu)$ that exemplifies the failure of $\bar{c}^\varepsilon$ to be an $\mathbf{F}$-Md-sequence, so there is a $E_\varepsilon \in D$ such that

$\boxtimes_1$ $\delta \in S \cap E_\varepsilon \Rightarrow \mathbf{F}(\eta_\varepsilon \restriction C_\delta) \neq f_\varepsilon(\varrho^*_\delta)$.

Define $\eta^* \in {}^\lambda(2^\mu)$ by $\eta^*(\alpha) = \mathrm{cd}(\langle \eta_\varepsilon(\alpha) : \varepsilon < \mu\rangle)$. Now as λ is regular uncountable it follows by choice of h that $E =: \{\delta < \lambda :$ for every $\alpha < \delta$ we have $\eta^*(\alpha) < h(\delta)$ and if $\delta' \in S, \alpha \in C'_{\delta'} \cap \delta$ and $C'' = C_{\delta'} \cap \alpha$ then $\eta^* \restriction C'' \in T_{<\delta}\}$ is a club of λ (see the choice of T and $T_{<\delta}$, and recall as well that by assumption (c) the sequence $\langle (C_\delta, C'_\delta) : \delta \in S\rangle$ is good, see Definition 1.5(4) + (4D)).

By clause (a) in the assumption of our claim 1.10, the filter D includes the clubs of λ so clearly $E \in D$; also D is μ^+-complete hence $E^* =: \cap\{E_\varepsilon : \varepsilon < \mu\} \cap E$ belongs to D. Now choose $\delta \in E^* \cap S$, fixed for the rest of the proof; clearly $\eta^* \restriction C_\delta \in \mathcal{P}_\delta$; just check the definitions of $\mathcal{P}_\delta$ and E, E^*. Let $\varepsilon < \mu$. Now recall that $\nu_{\eta^* \restriction C_\delta, \varepsilon}$ is the function from C_δ to $h(\delta)$ defined by

$$\nu_{\eta^* \restriction C_\delta, \varepsilon}(\alpha) = \mathrm{cd}_\varepsilon(\eta^*(\alpha)).$$

But by our choice of η^* clearly $\alpha \in C_\delta \Rightarrow \mathrm{cd}_\varepsilon(\eta^*(\alpha)) = \eta_\varepsilon(\alpha)$, so

$$\alpha \in C_\delta \Rightarrow \nu_{\eta^* \restriction C_\delta, \varepsilon}(\alpha) = \eta_\varepsilon(\alpha) \quad \text{so} \quad \nu_{\eta^* \restriction C_\delta, \varepsilon} = \eta_\varepsilon \restriction C_\delta.$$

Hence $\mathbf{F}(\nu_{\eta^* \restriction C_\delta, \varepsilon}) = \mathbf{F}(\eta_\varepsilon \restriction C_\delta)$. As $\delta \in E^* \subseteq E_\varepsilon$ clearly $\mathbf{F}(\eta_\varepsilon \restriction C_\delta) \neq f_\varepsilon(\varrho^*_\delta)$ and as $\eta^* \restriction C_\delta \in \mathcal{P}_\delta$ clearly $\rho_{\eta^* \restriction C_\delta} \in {}^\mu\theta$ is well defined. Now easily $\rho_{\eta^* \restriction C_\delta}(\varepsilon) = \mathbf{F}(\nu_{\eta^* \restriction C_\delta, \varepsilon})$ by the definition of $\rho_{\eta^* \restriction C_\delta}$, so we have $\rho_{\eta^* \restriction C_\delta}(\varepsilon) \neq f_\varepsilon(\varrho^*_\delta)$.

As this holds for every $\varepsilon < \mu$ it follows by the definition of $\mathrm{Sol}_{\rho_{\eta^* \restriction C_\delta}}$ that $\varrho^*_\delta \in \mathrm{Sol}_{\rho_{\eta^* \restriction C_\delta}}$. But $\eta^* \restriction C_\delta \in \mathcal{P}_\delta$ was proved above so $\rho_{\eta^* \restriction C_\delta} \in \{\rho_\eta : \eta \in \mathcal{P}_\delta\}$ hence $\varrho^*_\delta \in \mathrm{Sol}_{\rho_{\eta^* \restriction C_\delta}} \subseteq \cup\{\mathrm{Sol}_{\rho_\nu} : \nu \in \mathcal{P}_\delta\}$ whereas ϱ^*_δ has been chosen outside this set, contradiction. $\square_{1.10}$

We return to looking at Sep (see Definition 1.9) which appears in assumption (f) of 1.10. There are obvious monotonicity properties:

Claim 1.13. If $\mathrm{Sep}(\chi, \mu, \theta_0, \theta_1, \Upsilon)$ holds and $\chi' \ge \chi$, $(\theta'_0)^{\mu'} \le (\theta_0)^\mu$, $\theta'_1 \le \theta_1$ and $\Upsilon' \ge \Upsilon$, then $\mathrm{Sep}(\chi', \mu', \theta'_0, \theta'_1, \Upsilon')$ holds.

Proof. Let h be a one-to-one function from ${}^{\mu'}(\theta'_0)$ into ${}^{\mu}(\theta_0)$. Let $\bar{f} = \langle f_\varepsilon : \varepsilon < \chi \rangle$ exemplify $\text{Sep}(\chi, \mu, \theta_0, \theta_1, \Upsilon)$. For $\varepsilon < \chi'$ we define $f'_\varepsilon : {}^{(\mu')}(\theta'_0) \to \theta'_1$ by: $f'_\varepsilon(\nu) = f_\varepsilon(h(\nu))$ if $\varepsilon < \chi \,\&\, f_\varepsilon(h(\nu)) < \theta'_1$ and $f'_\varepsilon(\nu) = 0$ otherwise. Let $\varrho' \in {}^{(\chi')}(\theta'_1)$ and let $A'_{\varrho'} = \{\nu \in {}^{(\mu')}(\theta'_0)$: for every $\varepsilon < \chi'$ we have $f'_\varepsilon(\nu) \neq \varrho'(\varepsilon)\}$. Let $\varrho =: \varrho' \restriction \chi$ so $\varrho \in {}^{\chi}(\theta'_1) \subseteq {}^{\chi}(\theta_1)$, hence by the choice of $\langle f_\varepsilon : \varepsilon < \chi \rangle$ the set $A_\varrho = \{\nu \in {}^{\mu}(\theta_0)$: for every $\varepsilon < \chi$ we have $f_\varepsilon(\nu) \neq \varrho(\varepsilon)\}$ has cardinality $< \Upsilon$. If $\nu \in A'_{\varrho'}$ then $\nu \in {}^{\mu'}(\theta'_0)$ and $h(\nu) \in {}^{\mu}(\theta_0)$, and for $\varepsilon < \chi'$ we have $f'_\varepsilon(\nu) \neq \varrho'(\varepsilon)$ hence for $\varepsilon < \chi (\leq \chi')$ we have either $f_\varepsilon(h(\nu)) < \theta'_1 \& \varrho(\varepsilon) = \varrho'(\varepsilon) \neq f'_\varepsilon(\nu) = f_\varepsilon(h(\nu))$ or $f_\varepsilon(h(\nu)) \geq \theta'_1 \& \varrho(\varepsilon) = \varrho'(\varepsilon) < \theta'_1$ so $f_\varepsilon(h(\nu)) \neq \varrho(\varepsilon)$. Hence $\nu \in A'_{\varrho'} \Rightarrow h(\nu) \in A_\varrho$ and as h is a one-to-one function we have $|A'_{\varrho'}| \leq |A_\varrho| < \Upsilon \leq \Upsilon'$ and we are done. $\square_{1.13}$

To show that 1.10 has reasonable assumptions we should still prove 1.11.

Proof of 1.11. Our claim gives sufficient conditions for $\text{Sep}(\mu, \theta)$, i.e. $\text{Sep}(\mu, \mu, \theta, \theta, \Upsilon)$ for some $\Upsilon = \text{cf}(\Upsilon) \leq 2^\mu$. Clearly if $\Upsilon < 2^\mu$ we can waive "Υ regular" as Υ^+ can serve as well. Let $\theta_1 = \theta$.

Case 1. $\mu = \mu^\theta$, $\Upsilon = \theta$, $\theta_0 \in [\theta, \mu]$ and we shall prove $\text{Sep}(\mu, \mu, \theta_0, \theta_1, \Upsilon)$. Let

$$\mathcal{F} = \Big\{ f : f \text{ is a function with domain } {}^{\mu}(\theta_0) \text{ and for some } u \in [\mu]^\theta \text{ and sequence } \bar{\rho} = \langle \rho_i : i < \theta \rangle \text{ with no repetition, } \rho_i \in {}^{u}(\theta_0), \text{ we have } (\forall \nu \in {}^{\mu}(\theta_0)[\rho_i \subseteq \nu \Rightarrow f(\nu) = i] \text{ and } (\forall \nu \in {}^{\mu}(\theta_0)[(\bigwedge_{i<\theta} (\rho_i \nsubseteq \nu)) \Rightarrow f(\nu) = 0] \Big\}.$$

We write $f = f^*_{u,\bar{\rho}}$, if $u, \bar{\rho}$ witness that $f \in \mathcal{F}$ as above.

Recalling $\mu = \mu^\theta$, clearly $|\mathcal{F}| = \mu$. Let $\mathcal{F} = \{f_\varepsilon : \varepsilon < \mu\}$ and we let $\bar{f} = \langle f_\varepsilon : \varepsilon < \mu \rangle$. Clearly clauses (a),(b) of Definition 1.9 (with $\mu, \mu, \theta_0, \theta_1 = \theta$ here standing for $\chi, \mu, \theta, \theta_1$ there) holds and let us check clause (c). So suppose $\varrho \in {}^{\mu}\theta$ and let $R = R_\varrho =: \{\nu \in {}^{\mu}(\theta_0)$: for every $\varepsilon < \mu$ we have $f_\varepsilon(\nu) \neq \varrho(\varepsilon)\}$. We have to prove $|R| < \theta$ (as we have chosen $\Upsilon = \theta$).

Towards contradiction, assume that $R \subseteq {}^{\mu}(\theta_0)$ has cardinality $\geq \theta$ and choose $R' \subseteq R$ of cardinality θ. Hence we can find $u \in [\mu]^\theta$ such that $\langle \nu \restriction u : \nu \in R' \rangle$ is without repetitions.

Let $\{\nu_i : i < \theta\}$ list R' without repetitions and let $\rho_i =: \nu_i \restriction u$. Now let $\bar{\rho} = \langle \rho_i : i < \theta \rangle$, so $f^*_{u,\bar{\rho}}$ is well defined and belong to $\mathcal{F}$. Hence for some $\zeta < \mu$ we have $f^*_{u,\bar{\rho}} = f_\zeta$. Now for each $i < \theta$, $\nu_i \in R' \subseteq R$ hence by the definition of R, $(\forall \varepsilon < \mu)(f_\varepsilon(\nu_i) \neq \varrho(\varepsilon))$ in particular for $\varepsilon = \zeta$ we get $f_\zeta(\nu_i) \neq \varrho(\zeta)$. But by the choice of ζ, $f_\zeta(\nu_i) = f^*_{u,\bar{\rho}}(\nu_i)$ and by the definition of $f^*_{u,\bar{\rho}}$ recalling $\nu_i \restriction u = \rho_i$ we have $f^*_{u,\bar{\rho}}(\nu_i) = i$, so $i = f_\zeta(\nu_i) \neq \varrho(\zeta)$. This holds for every $i < \theta$ whereas $\varrho \in {}^{\mu}\theta$, contradiction.

Case 2. $2^\theta \leq \mu$ and $\mathbf{U}_\theta[\mu] = \mu$.

Let $\Upsilon = (2^\theta)^+$ or just $(2^{<\theta})^+$ hence $\Upsilon = \operatorname{cf}(\Upsilon) \leq \mu^+ \leq 2^\mu$.

Let $\{u_i : i < \mu\} \subseteq [\mu]^\theta$ exemplify $\mathbf{U}_\theta[\mu] = \mu$. Define $\mathcal{F}$ as in case 1 except that for notational simplicity $\theta_0 = \theta$ and we restrict ourselves in the definition of $\mathcal{F}$ to $u \in \bigcup\{\mathcal{P}(u_i) : i < \mu\}$. Clearly $|\mathcal{F}| \leq \mu$.

Assume that $\varrho \in {}^\mu\theta$ and $R = R_\varrho \subseteq {}^\mu\theta$ is defined as in Case 1, and toward contradiction assume that $|R| \geq (2^{<\theta})^+$, hence we can find ρ^*, $\langle(\alpha_\zeta, \rho_\zeta) : \zeta < \theta\rangle$ such that:

$$\rho^*, \rho_\zeta \in R$$
$$\rho_\zeta \restriction \{\alpha_\xi : \xi < \zeta\} = \rho^* \restriction \{\alpha_\xi : \xi < \zeta\}$$
$$\rho_\zeta(\alpha_\zeta) \neq \rho^*(\alpha_\zeta).$$

Clearly $\langle\alpha_\zeta : \zeta < \theta\rangle$ is with no repetitions.

So by the choice of $\{u_i : i < \mu\}$ as examplifying $\mathbf{U}_\theta[\mu] = \mu$, i.e., the definition of $\mathbf{U}_\theta[\mu]$, for some $i < \mu$ the set $u_i \cap \{\alpha_\zeta : \zeta < \theta\}$ has cardinality θ, call it u. So $\{\rho \restriction u : \rho \in R\}$ has cardinality $\geq \theta$ and we can continue as in Case 1. If we wish to use $\theta_0 \in [\theta, \mu]$ instead of $\theta_0 = \theta$ above let pr be a pairing function on μ. Without loss of generality each u_i is closed under pr and its inverses, and $f^*_{u,\bar{\rho}}$ is defined iff for some $\varepsilon < \mu$ we have $u \subseteq u_\varepsilon$ and $i < \theta \Rightarrow \operatorname{Rang}(\rho_i) \subseteq u_\varepsilon$. In the end choose i such that $u_i \cap \{\operatorname{pr}(\alpha_\zeta, \operatorname{pr}(\rho_\zeta(\alpha_\zeta), \rho^*(\alpha_\zeta)) : \zeta < \theta\}$ has cardinality θ.

Case 3. $\mathbf{U}_J[\mu] = \mu$ where $J = [\sigma]^{<\theta}$ for σ such that $\sigma^\theta \leq \mu, \sigma \geq \theta$.

Let $\Upsilon = [2^{<\sigma}]^+$.

The proof is similar to Case 2, only find now ρ^* and $\langle(\alpha_\zeta, \rho_\zeta) : \zeta < \sigma\rangle$ and choose $i < \mu$ such that $u_i \cap \{\operatorname{pr}(\alpha_\zeta, \rho_\zeta(\alpha_\zeta)) : \zeta < \sigma\}$ has cardinality θ.

Case 4. $\mu > \theta \neq \operatorname{cf}(\mu)$ and μ is strong limit.

Follows by case 2.

Case 5. $\mu \geq \beth_\omega(\theta)$.

By [Sh 460] we can find a regular $\sigma < \beth_\omega(\theta)$ which is $> \theta$ and such that $\mathbf{U}_\sigma[\mu] = \mu$, so Case 2 applies.

Note that in case 2, we can replace $2^\theta \leq \mu$ by: there is $\mathcal{F} \subseteq {}^\theta\theta$ of cardinality $\leq \mu$ such that for every $A \in [\theta]^\theta$ for some $f \in \mathcal{F}$ we have $\theta = \operatorname{Rang}(f \restriction A)$. The proof is obvious. Similarly in case 3. $\square_{1.11}$

There are some obvious monotonicity properties regarding the Md-property.

Claim 1.14. 1) If $(\lambda, \bar{C})$ has the θ-Md-property (see Definition 1.6(1)) and $\bar{C} = \langle C_\delta : \delta \in S\rangle$ and C^1_δ is an unbounded subset of C_δ for $\delta \in S$ <u>then</u> $(\lambda, \langle C^1_\delta : \delta \in S\rangle)$ has a θ-Md property. We can also increase S and decrease θ; we can deal with the (Υ, θ)-Md-property.

2) Assume $\lambda = \operatorname{cf}(\lambda) > \aleph_0$, $S \subseteq \{\delta : \delta < \lambda, \operatorname{cf}(\delta) = \kappa\}$ is stationary and $\kappa < \chi \leq \lambda$.

<u>Then</u> some $\langle C_\delta : \delta \in S\rangle$ is a (λ, κ, χ)-Md-parameter.

Proof.

1) Obvious.
2) Easy (note: goodness is not required). $\square_{1.14}$

We may consider the following generalization.

Definition 1.15. *1) We say $\bar{C}$ is a λ-Md$_3$-parameter if:*
(a) λ is regular uncountable
(b) S is a stationary subset of λ (consisting of limit ordinals)
(c) $\bar{C} = \langle C_\delta, C'_\delta, C_{\delta,\varepsilon} : \delta \in S, \varepsilon < cf(\delta)\rangle$
(d) $C_\delta \subseteq \delta = \sup(C_\delta)$ and $C'_\delta \subseteq \delta = \sup(C'_\delta)$ and $C_\delta = \cup\{C_{\delta,\varepsilon} : \varepsilon < cf(\delta)\}$
(e) $\zeta < cf(\delta) \Rightarrow \sup\left(\bigcup_{\varepsilon<\zeta} C_{\delta,\varepsilon}\right) < \delta$ and $C_{\delta,\varepsilon}$ increase with ε

2) We say that a λ-Md$_3$-parameter $\bar{C}$ is[3] *good if $\alpha < \lambda \Rightarrow \lambda > |\{C_{\delta,\zeta} : \delta \in S, \zeta < cf(\delta)$ and the ζ-th member of C_δ is $\leq \alpha\}|$.*
3) The (D, Υ, θ)-Md$_3$-property for $\bar{C}$ means the (D, Υ, θ)-Md-property for $\langle C_\delta : \delta \in S\rangle$, similarly for the other variants of this property (so the (D, Υ, θ)-Md$_3$-property for λ for a stationary $S \subseteq S^\lambda_\kappa$ is the same as the (D, Υ, θ)-Md-property).
4) If $\{C_{\delta,\varepsilon} : \varepsilon < cf(\delta)\} = \{C_\delta \cap \alpha : \alpha \in nacc(C_\delta)\}$ we may write just $\langle C_\delta : \delta \in S\rangle$ (so Md$_3$ is like Md in this case).

We may consider the following variant.

Claim 1.16. Assume

(a) λ is regular uncountable, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ is stationary and $\chi < \lambda$
(b) $\mathfrak{B}$ is a model with universe λ such that $A \in [\lambda]^{<\chi} \Rightarrow c\ell_{\mathfrak{B}}(A) \in [\lambda]^{<\chi}$; we may allow infinitary functions provided that
$(*)_{\mathcal{B},\kappa}$ if $\langle A_i : i < \kappa\rangle$ is increasing, $A_i \in [\lambda]^{<\chi}$ then $c\ell_{\mathfrak{B}}\left(\bigcup_{i<\kappa} A_i\right) = \bigcup_{i<\kappa} c\ell_{\mathfrak{B}}(A_i)$
(c) $\bar{C} = \langle C_\delta : \delta \in S\rangle$ is a good $(\lambda, \kappa, \kappa^+)$-MD-parameter
(d) $S^* = \{\delta \in S$: for every $\alpha < \delta, c\ell_{\mathfrak{B}}(\alpha) \subseteq \delta\}$
(e) $C^*_{\delta,\varepsilon} = \alpha_{\delta,\varepsilon} \cap c\ell_{\mathfrak{B}}(C_\delta \cap \alpha_{\delta,\varepsilon})$ for $\delta \in S^*$ and $\varepsilon < \kappa$ where $\alpha_{\delta,\varepsilon} \in C_\delta$ is increasing with ε with limit δ and $C^*_\delta = \cup\{C^*_{\delta,\varepsilon} : \varepsilon < \mathrm{cf}(\delta)\}$, $C'_\delta = \{\alpha_{\delta,\varepsilon} : \varepsilon < \mathrm{cf}(\delta)\}$.

Then $\langle C^*_\delta, C_\delta, C^*_{\delta,\varepsilon} : \delta \in S^*, \varepsilon < \mathrm{cf}(\delta)\rangle$ is a good (S, κ, χ)-Md$_3$-parameter (and $S \setminus S^*$ is non-stationary); if $\chi \leq \kappa$ then $\bigcup_{\varepsilon<\kappa} C^*_{\delta,\varepsilon} = c\ell_{\mathfrak{B}}(C_\delta)$.

Proof. Easy.

Claim 1.17. 1) Assume $\bar{C} = \langle C_\delta, C_{\delta,\varepsilon} : \delta \in S, \varepsilon < \mathrm{cf}(\delta)\rangle$ is a good (λ, κ, χ)-Md$_3$-parameter. Then claim 1.10 holds (i.e. we get there the (D, θ)-Md$_3$-property).
2) In part (1) hence in 1.10, we can weaken clause (e) to

[3] if $\langle C_\delta : \delta \in S\rangle$ is a good (λ, κ, χ)-Md-parameter $C'_\delta \subseteq \delta = \sup(C'_\delta)$, $\mathrm{otp}(C'_\delta) = \kappa$ and we let $C_{\delta,\varepsilon} = \{\alpha \in C_\delta : \mathrm{otp}(\alpha \cap C'_\delta) \leq \varepsilon + 1\}$ for $\varepsilon < \kappa$ then $\langle C_\delta, C_{\delta,\varepsilon} : \delta \in S, \varepsilon < \kappa\rangle$ is a λ-Md$_1$-parameter

$(e)'$ (if $\lambda = 2^\mu$ we can use $h = \mathrm{id}_\lambda$): there is $h : \lambda \to 2^\mu$ increasing continuous with limit 2^μ and there is a set $T = \{\eta_\alpha : \alpha < 2^\mu\}$ and function $g : \lambda \to \lambda$ such that if $\eta \in {}^\lambda(2^\mu)$ then for stationary many $\delta \in S$ we have

(a) $\eta \restriction C_\delta \in \{\eta_\alpha : \alpha < h(g(\delta))\}$

(b) $\zeta < \mathrm{cf}(\delta) \Rightarrow \eta \restriction \left(\bigcup_{\varepsilon<\zeta} C_{\delta,\varepsilon}\right) \in \{\eta_\beta : \beta < \delta\}$.

Proof. Same proof as 1.10.

Let $C_{1,\delta} = \bigcup\{C_{\delta,\varepsilon} : \varepsilon < \mathrm{cf}(\delta)\}$ and $C'_{1,\delta} = C_\delta$. Now we can apply the main claim to $\langle (C_{1,\delta}, C'_{1,\delta}) : \delta \in S\rangle$. $\square_{1.17}$

Claim 1.18. In the proof of 1.10 instead of $f : \lambda \to \Upsilon$ and $f_\delta : C_\delta \to \theta$ we can use $f : [\lambda]^{<\aleph_0} \to \Upsilon$ and $f_\delta : [C_\delta]^{<\aleph_0} \to \theta$ (so get the θ-MD_ℓ-property for $\ell = 0, 1, 2$).

Proof. Let cd_* be a one-to-one function from ${}^{\omega>}\lambda$ into λ and without loss of generality $\alpha_\ell \le \mathrm{cd}_*(\langle \alpha_0, \alpha_1, \dots, \alpha_{n-1}\rangle) < \mathrm{Max}\{(\omega)\} \cup \{(\alpha_{\pi(0)} + |\alpha_{\pi(0)}|) + \cdots + (\alpha_{\pi(n-1)} + |\alpha_{\pi(n-1)}|) : \pi$ is a permutation of $n\}$. We are given $\bar{C} = \langle C_\delta : \delta \in S\rangle$, without loss of generality every $\delta \in S$ is infinite and divisible by (δ).

Let $C_{1,\delta} = \{\mathrm{cd}_*(\eta) : \eta \in {}^{\omega>}(C_\delta)\}$ and $C'_{1,\delta} = \mathrm{nacc}(C_\delta)$. $\square_{1.18}$

Definition 1.19. *1) For a set A of ordinals and an ordinal α let $A * \alpha = \{\beta : \beta < \alpha$ or for some $\gamma \in A$ we have $\beta \in [\gamma, \gamma + \alpha)\}$.*

2) The pair $(\lambda, \bar{C})$ or just $\bar{C}$ has the (χ, θ)-MD_0-property (or use MD instead of MD_0) <u>if</u> we can find $\langle f_\delta : \delta \in S\rangle$ such that:

(a) f_δ is a function from $C'_\delta = C_\delta * \alpha^*_\delta$ to θ for some $\alpha^*_\delta < \chi$

(b) if f is a function from λ to θ and $\alpha < \chi$ is non-zero then for stationary many $\delta \in S$ we have $f_\delta \subseteq f$ and $\alpha^*_\delta = \alpha$.

If we omit χ we mean $\chi = 2$ so $\alpha^*_\delta = 1$ for every $\delta \in S$.

3) (D, χ, θ)-MD_2-property means: if τ is a relational vocabulary of cardinality $< \theta$, <u>then</u> we can find $\langle M_\delta : \delta \in S\rangle$ such that:

(a) M_δ is a τ-model with universe $C'_\delta = C_\delta * \alpha^*_\delta$ for some $\alpha^*_\delta < \chi$

(b) if M is a τ-model with universe λ and $\alpha < \chi$ <u>then</u> the set $A_{M,\alpha} = \{\delta \in S : M \restriction C_\delta = M_\delta$ and $\alpha^*_\delta = \alpha\}$ is stationary and even belongs to D^+.

If we omit χ we mean $\chi = 2$ so $\alpha^*_\delta = 1$ for every $\delta \in S$. Similarly with D.

4) We say (λ, S) has the (χ, θ)-MD_ℓ-property if for some $\bar{C} = \langle C_\delta : \delta \in S\rangle$ the pair $(\lambda, \bar{C})$ has the (χ, θ)-MD_ℓ-property. We say that λ has the (κ, χ, θ)-property if $(\lambda, S^\lambda_\kappa)$ has the (χ, θ)-MD_ℓ-property.

The following claim shows the close affinity of the properties stated in theorem 0.1 in the introduction and the θ-Md-property with which we deal in this section here. This claim is needed to deduce Theorem 0.1 from 1.10.

Claim 1.20. 1) Assume $(\lambda, \bar{C})$ has the θ-Md-property where $\bar{C} = \langle C_\delta : \delta \in S\rangle$ is a (λ, κ, χ)-Md-parameter. If $\kappa < \chi, \theta = \theta^{<\chi}$ <u>then</u> $(\lambda, \bar{C})$ has the (χ, θ)-MD_0-property.

2) Assume in addition that $\bar{C}$ is a good$^+$ (λ, κ, χ)-Md-parameter <u>then</u> $(\lambda, \bar{C})$ has the (D, χ, θ)-MD_2-property.

3) In part (2) we can allow τ to have function symbols but have:
(a)$'$ M_δ is a τ-model with universe $\supseteq C'_\delta = C_\delta * \alpha^*_\delta$ for some $\alpha^*_\delta < \chi$
(b)$'$ if M is a τ-model with universe λ and $\alpha < \lambda$ <u>then</u> the following set belongs to D^+
$A'_{M,\alpha} = \{\delta \in S : \alpha^*_\delta = \alpha$ and $(M \restriction c\ell_M(C'_\delta), c)_{c \in C'_\delta} \cong (M_\delta, c)_{c\in C'_\delta}\}$.
4) In part (2) we can replace good$^+$ by good <u>if</u> we add
$\boxdot$ each C_δ is closed under pr, where pr is a fixed pairing function on λ.

Proof. The proof consists of a translation of one version of middle diamond to another (note: "τ is a relational" is a strong restriction).

1) First fix $\alpha^* < \chi$ and deal with $\langle C''_\delta : \delta \in S\rangle, C''_\delta = C'_\delta \backslash [0, \alpha^*)$ recalling $C'_\delta = C_\delta * \alpha^*$. Let cd be a one-to-one function from ${}^{\chi >}\theta$ onto θ. Define $\langle F_\delta : \delta \in S\rangle$ as follows: if $f \in {}^{(C_\delta)}\theta$ then $F_\delta(f) = \mathrm{cd}(\langle f(\beta_{\delta,\varepsilon}) : \varepsilon < \mathrm{otp}(C_\delta)\rangle)$ where $\langle \beta_{\delta,\varepsilon} : \varepsilon < \mathrm{otp}(C_\delta)\rangle$ list C_δ in increasing order so F_δ is a function from ${}^{(C_\delta)}\theta$ into θ. As we assume "$(\lambda, \bar{C})$ has the θ-Md-property", we can apply its definition, hence for $\bar{F} = \langle F_\delta : \delta \in S\rangle$ there is a diamond sequence $\langle c_\delta : \delta \in S\rangle$.

We now define $f^*_\delta : C''_\delta \to \theta$ as follows: if β_0 is a member of C_δ and $\alpha \in [\beta_0, \beta_0 + \alpha^*)$ and $\alpha < \mathrm{Min}(C''_\delta \backslash (\beta_0 + 1))$ and $c_\delta = \mathrm{cd}(\langle \xi_{\delta,\varepsilon} : \varepsilon < \mathrm{otp}(C_\delta)\rangle$ and $\xi_{\delta,\varepsilon} = \mathrm{cd}(\langle \zeta_{\delta,\varepsilon,i} : i < \alpha^*\rangle$ then $f^*_\delta(\alpha) = \zeta_{\delta,\mathrm{otp}(C_\delta \cap \beta_0),\alpha-\beta_0}$; otherwise $f^*_\delta(\alpha) = 0$. So $f^*_\delta : C''_\delta \to \theta$. Why is $\langle f^*_\delta : \delta \in S\rangle$ as required for $\langle C''_\delta : \delta \in S\rangle$ (and α^*)? Assume that $f : \lambda \to \theta$, so we can define $f' : \lambda \to \theta$ by $f'(\alpha) = \mathrm{cd}(\langle f(\alpha + \varepsilon) : \varepsilon < \alpha^*\rangle)$, hence for stationarily many $\delta \in S$ we have $c_\delta = F_\delta(f' \restriction C_\delta)$. Now we check that $f \restriction C''_\delta$ is equal to f^*_δ defined above.

Let $\xi_{\delta,\varepsilon,i} = f'(\beta_{\delta,\varepsilon})$ and $\zeta_{\delta,\varepsilon+i} = f(\beta_{\delta,\varepsilon} + i)$ for $i < \alpha^*, \varepsilon < \mathrm{otp}(C_\delta)$ so $\xi_{\delta,\varepsilon} = f'(\beta_{\delta,\varepsilon}) = \mathrm{cd}(\langle \zeta_{\delta,\varepsilon,i} : i < \alpha^*\rangle), c_\delta = F_\delta(f' \restriction C_\delta) = \mathrm{cd}(\langle \xi_{\delta,\varepsilon} : \varepsilon < \mathrm{otp}(C_\delta)\rangle$. Hence by the definition of f^*_δ we have: if $\beta = \beta_{\delta,\varepsilon} \in C_\delta, \alpha \in [\beta, \beta + \varepsilon)$ and $\alpha < \mathrm{Min}(C_\delta \backslash (\beta + 1))$ then $f^*_\delta(\alpha) = \zeta_{\delta,\varepsilon,\alpha-\beta} = f(\beta_{\delta,\varepsilon} + (\alpha - \beta)) = f(\beta + (\alpha - \beta)) = f(\alpha)$ as required.

To have "every $\alpha^* < \chi$" (and guess also $f \restriction (C'_\delta \backslash C''_\delta) \subseteq f \restriction [0, \alpha^*)$) we just have to partition S to $\langle S_\alpha : \alpha < \chi\rangle$ such that each $\bar{C} \restriction S_\alpha$ has the θ-Md-property which we can do by 1.23(3) below (or work a little more above, i.e., $f'(\alpha) = \mathrm{cd}(\langle f(\alpha + \varepsilon) : \varepsilon < \alpha_\delta\rangle\hat{}\langle f(\varepsilon) : \varepsilon < \alpha_\delta\rangle))$.

2), 3), 4) Left to the reader. $\square_{1.20}$

Claim 1.21. 1) Assume $\lambda = \mathrm{cf}(\lambda) > \kappa = \mathrm{cf}(\kappa)$, $S \subseteq S^\lambda_\kappa$ and
(a) $\bar{C}^\ell$ is a λ-Md-parameter for $\ell = 1, 2$
(b) $\bar{C}^1 = \langle C^1_\delta : \delta \in S\rangle$ has the (Υ, θ)-Md-property
(c) $A_\alpha \in [\alpha]^{<\chi} \& \Upsilon^{|A_\alpha|} = \Upsilon$ for $\alpha < \lambda$ and $\delta \in S \Rightarrow C^2_\delta = \cup\{A_\alpha : \alpha \in C^1_\delta\}$.
<u>Then</u> $\bar{C}^2$ has the (Υ, θ)-Md-property.
2) If $\bar{C}$ is a $(\lambda, \kappa, \chi^+)$-Md-parameter with the (Υ, θ)-Md-property, $\Upsilon^\chi = \Upsilon$ and $C'_\delta = \{\alpha + i : \alpha \in C_\delta \vee \alpha = 0$ and $i < \chi\}$, <u>then</u> $\bar{C}' = \langle C'_\delta : \delta \in \mathrm{Dom}(\bar{C})\rangle$ is a $(\lambda, \kappa, \chi^+)$-Md-parameter with the θ-Md-property.

Proof. Easy by coding (as in the proof of 1.20).

Definition 1.22. *Assume $\lambda > \kappa$ are regular and $\bar{C} = \langle C_\delta : \delta \in S\rangle$ is a (λ, κ, χ)-Md-parameter with the (Υ, θ)-Md-property. Let $ID_{\lambda,\kappa,\chi,\Upsilon,\theta}(\bar{C}) = \{A \subseteq \lambda : \bar{C} \restriction (S \cap A)$ does not have the (Υ, θ)-Md-property$\}$. If $\Upsilon = \theta$ we may omit it.*

Claim 1.23. Let $\bar{C}, \lambda, \kappa, \chi, \Upsilon, \theta$ be as above in Definition 1.22 and $J = \mathrm{ID}_{\lambda,\kappa,\chi,\Upsilon,\theta}(\bar{C})$.

1) J is an ideal on λ such that $\lambda \backslash S \in J$, $\lambda \notin J$ if $\Upsilon \geq \aleph_0$.
2) J is λ-complete and even normal if $\Upsilon \geq \lambda$ (if $\aleph_0 \leq \Upsilon < \lambda$ it is still Υ^+-complete).
3) If $S \subseteq \lambda$ and $S \neq \emptyset$ mod J, then for any $\bar{F}^i = \langle F^i_\delta : \delta \in S \rangle$, $F^i_\delta : {}^{(C_\delta)}\theta \to \theta$, λ can be partitioned to θ subsets each having a $\bar{F}^i$-Md-sequence.

Proof. 1) Let cd: $\Upsilon \times \Upsilon \to \Upsilon$ be one-to-one onto and $\mathrm{cd}_\ell : \Upsilon \to \Upsilon$ be such that $\mathrm{cd}_\ell(\mathrm{cd}(\langle \alpha_0, \alpha_\ell \rangle)) = \alpha_\ell$. Clearly J is a subfamily of $\mathcal{P}(\lambda)$ and it is closed under subsets, so we need just to prove that it is closed under union of two. So suppose $A = A_1 \cup A_2$, $A_\ell \in J$ for $\ell = 1, 2$. As J is closed under subsets, without loss of generality $A \cap A_2 = \emptyset$. Let the sequence $\bar{F}^\ell = \langle F^\ell_\delta : \delta \in S \cap A_\ell \rangle$ exemplify $A_\ell \in J$. Now we define $\bar{F} = \langle F_\delta : \delta \in S \cap A \rangle$ as follows:

⊛ if $\delta \in S \cap A$ and $f \in {}^{(C_\delta)}\Upsilon$ then:
$\delta \in A_\ell \Rightarrow F_\delta(f) = F^\ell_\delta(\mathrm{cd}_\ell \circ f)$, i.e., $F_\delta(f) = F^\ell_\delta(f^\ell)$ where $\alpha \in C_\delta \Rightarrow f^\ell(\alpha) = \mathrm{cd}_\ell(f(\alpha))$.

2) Straight as the sequence $\langle F_\delta : \delta \in S \rangle$ is not required to be nicely definable.
3) Guessing $\theta \times \theta$ is the same as guessing θ. $\square_{1.23}$

Well, are there good $(\lambda, \kappa, \kappa^+)$-parameters? Some version of this is crucial for having interesting consequences of 1.10, but we shall have to quote (on $I[\lambda]$ see Definition 0.5; see on it [Sh 420, §1]).

Claim 1.24. 1) If S is a stationary subset of a regular cardinal λ and $S \in I[\lambda]$, see Definition 0.5, and $(\forall \delta \in S)[\mathrm{cf}(\delta) = \kappa]$, then

(α) for some club E of λ, there is a good$^+$ $(S \cap E, \kappa, \kappa^+)$-Md-parameter. (Of course, we can replace the κ^+ by any $\chi \leq \lambda$) (recall that we do not required the C_δ's to be a closed subset of δ)

(β) for some club E of λ there is a weakly good$^+$ $(S \cap E, \kappa, \kappa^+)$-Md-parameter consisting [4] of clubs (this is of interest when $\kappa > \aleph_0$)

(γ) for some club E of λ there is a good $(S \cap E, \kappa, \kappa^+)$-Md-parameter $\bar{C} = \langle C_\delta : \delta \in S \cap E \rangle$ such that

(i) $\bar{C}$ is weakly good$^+$, i.e., $\alpha \in \mathrm{nacc}(C_{\delta_1}) \cap \mathrm{nacc} C_{\delta_2}) \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$

(ii) $\bar{C}$ is good, i.e., for each $\alpha < \lambda$ we have $\lambda > |\{\alpha \cap C_\delta : \delta \in S, \alpha \in C_\delta\}|$

(iii) $\bar{C}$ is Θ_λ-closed that is each C_δ is Θ_λ-closed which means $\alpha = \sup(C_\delta \cap \alpha) \& \mathrm{cf}(\alpha) \in \Theta_\lambda \Rightarrow \alpha \in C_\delta$ where $\Theta_\lambda = \{\sigma : \sigma = \mathrm{cf}(\sigma)$ and $\alpha < \lambda \Rightarrow |\alpha|^{<\sigma>} < \lambda\}$.

2) If $\kappa = \mathrm{cf}(\kappa)$, $\kappa^+ < \lambda = \mathrm{cf}(\lambda)$, then there is a stationary $S \in I[\lambda]$ with $(\forall \delta \in S)[\mathrm{cf}(\delta) = \kappa]$.
3) In part (1), moreover, we can find a good $(S \cap E, \kappa, \kappa^+)$-Md-parameter and a λ-complete filter D on λ such that: for every club E' of λ, the set $\{\delta \in S \cap E : C_\delta \subseteq E' \cap E\}$ belongs to D.

Proof. 1) By the definition of $I[\lambda]$ see Definition 0.4 and 1.5 above (see more in [Sh 420, §1]) and for clause (γ) the definition of $|\alpha|^{<\sigma>_{\mathrm{tr}}}$.

[4] this means: the parameter $\bar{C} = \langle C_\delta : \delta \in S \cap E \rangle$, C_δ a club of δ

2) By [Sh 420, §1].
3) By [Sh:g, III]. $\square_{1.24}$

Definition 1.25. *Let $\bar{C}$ be a (λ, κ, χ)-Md-parameter*

(1) For a family $\mathcal{F}$ of $\bar{C}$-colourings and $\mathcal{P} \subseteq {}^{\lambda}2$, let $id_{\bar{C},\mathcal{F},\mathcal{P}}$ be

$$\Big\{W \subseteq \lambda : \text{ for some } \mathbf{F} \in \mathcal{F} \text{ for every } \bar{c} \in \mathcal{P} \text{ for some } \eta \in {}^{\lambda}\lambda \text{ the set } \{\delta \in W \cap S : \mathbf{F}(\eta \restriction C_\delta) = c_\delta\} \text{ is a non stationary subset of } \lambda\Big\}$$

(2) If $\mathcal{F}$ is the family of all $\bar{C}$-colouring we may omit it. If we write Def instead of $\mathcal{F}$ this mean as in [Sh 576, §1] (not used so the reader can ignore this).

We can strengthen 1.5(2) as follows.

Definition 1.26. *We say the λ-colouring $\mathbf{F}$ is (S, κ, χ)-Md-good if:*

(a) $S \subseteq \{\delta < \lambda : cf(\delta) = \kappa\}$ is stationary
(b) we can find E and $\langle C_\delta : \delta \in S \cap E\rangle$ such that
- *(α) E is a club of λ*
- *(β) C_δ is an unbounded subset of δ, $|C_\delta| < \chi$*
- *(γ) if $\rho, \rho' \in {}^{\delta}\delta$, $\delta \in S \cap E$, and $\rho' \restriction C_\delta = \rho \restriction C_\delta$ then $\mathbf{F}(\rho') = \mathbf{F}(\rho)$*
- *(δ) for every $\alpha < \lambda$ we have $\lambda > |\{C_\delta \cap \alpha : \delta \in S \cap E \text{ and } \alpha \in C_\delta\}|$*
- *(ϵ) $\delta \in S \Rightarrow |\delta|^{<cf(\delta)>_{tr}} < \lambda$ or just $\delta \in S \Rightarrow \lambda > |\{C : C \subseteq \delta$ is unbounded in δ and for every $\alpha \in C$ for some $\gamma \in S$ we have $C \cap \alpha = C_\gamma \cap \alpha$ and $\alpha \in C_\gamma\}|$.*

We can sum up (relying again on some quotations).

Conclusion 1.27. If $\lambda = \mathrm{cf}(2^\mu)$ and we let $\Theta =: \{\theta : \theta = \mathrm{cf}(\theta)$ and $(\forall \alpha < 2^\mu)(|\alpha|^{<\theta>_{\mathrm{tr}}} < 2^\mu)\}$ then

(a) Θ is "large" e.g. contains every large enough $\theta \in$ Reg $\cap \beth_\omega$ if $\beth_\omega < \lambda$
(b) if $\theta \in \Theta_\lambda \wedge \theta^+ < \lambda$ then there is a stationary $S \in I[\lambda]$ such that $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \theta$
(c) if θ, S are as in clause (b) above then for some club E of λ
- (i) there is a weakly good $(\lambda, \kappa, \kappa^+)$-Md-parameter $\bar{C} = \langle C_\delta : \delta \in S \cap E\rangle$
- (ii) there is a good $(\lambda, \kappa, \kappa^+)$-Md-parameter $\langle C_\delta : \delta \in E \cap S\rangle$
- (iii) there is a good$^+$ $(\lambda, \kappa, \kappa^+)$-Md-parameter $\bar{C} = \langle C_\delta : \delta \in E \cap S\rangle$

(d) for $\theta, S, \bar{C}$ as in clause (b), (c) above and $\Upsilon \le 2^\mu$, if $\mathbf{F}$ is a $(\bar{C}, \Upsilon, \theta)$-colouring and $D = D_\lambda$ (or D is a μ^+-complete filter on λ extending D_λ), and lastly $S \in D^+$ then there is a $\mathbf{F}$-middle diamond sequence for $\bar{C}$ and there is an $D - \mathbf{F}$-Md-sequence; see Definition 1.5(3),(3A)
(e) if $\theta, S, \bar{C}$ as in clauses (b), (c): $\Upsilon \le 2^\mu$, then $\bar{C}$ has the (Υ, θ)-property

(f) for θ, S as in clause (b) above $\Upsilon \le 2^\mu$, if $\theta > \aleph_0$ $\bar{C}' = \langle C'_\delta : \delta \in S\rangle$, C_δ a stationary subset of δ for each δ then we can find $\bar{C}'' = \langle C''_\delta : \delta \in E \cap S\rangle$ such that C''_δ is a club of δ for each δ and $\langle C'_\delta \cap C''_\delta : \delta \in S\rangle$ has the (Υ, θ)-Md-property (really $\bar{C}''$ does not depend on $\bar{C}'$)

(g) in clause (f) if $\theta = \theta^\sigma$, $\sigma \ge \kappa$, $2^\chi \le \theta$ we can conclude also that $\langle C'_\delta \cap C''_\delta : \delta \in S\rangle$ has the (χ, σ)-MD$_2$-property.

Proof. Clause (a) holds by [Sh 460], clause (b), (c) holds by 1.24, clause (d) by the major claim 1.10 and clause (e) is a rephrasing of clause (d), and lastly for clause (f) use above case (iii) of clause (c) and use the obvious monotonicity (and $C'_\delta \cap C''_\delta$ is unbounded in δ being stationary). $\square_{1.27}$

Note some open questions.

Question 1.28. 1) Assume λ is a weakly inaccessible which is not strong limit (i.e. a limit regular uncountable cardinal but not strong limit) and $\langle 2^\theta : \theta < \lambda\rangle$ is not eventually constant.

Does λ have the weak diamond? Other consequences? In this case $\mathrm{cf}(2^{<\lambda}) = \lambda$ as $\langle 2^\theta : \theta < \lambda\rangle$ witness it hence $2^\lambda > 2^{<\lambda} =: \Sigma\{2^\theta : \theta < \lambda\}$. If $\lambda = 2^{<\lambda}$ then $\lambda = \lambda^{<\lambda}$ and such λ's are investigated in [Sh 755] (for weak diamond on many stationary sets).

2) Consider generalizing [Sh:f, AP, §3] to our present context. Also we may consider $\langle c_\eta : \eta \in \lambda^{+n} \times \lambda^{+(n-3)} \times \ldots \times \lambda\rangle$ where $c_\eta \in \theta$.

Claim 1.29. Assume

$\circledast_{\mu,\theta}$ (i) $\theta \le \mu$
(ii) if $\mathcal{F} \subseteq {}^\mu\theta$ has cardinality $< 2^\mu$ then there is $\rho \in {}^\mu\theta$ such that $(\forall f \in \mathcal{F})(\exists \varepsilon < \mu)(\rho(\varepsilon) = f(\varepsilon))$.

1) $\mathrm{Sep}(\mu, \mu, \theta, 2^\mu)$ hence if 2^μ is a regular cardinal then $\mathrm{Sep}(\mu, \theta)$ holds so clause (f) of 1.10 is satisfied.
2) Even if 2^μ is singular still in Claim 1.10 we can replace clause (f) in the assumption by $\circledast_{\mu,\theta}$.

Remark. On relatives of $\circledast_{\mu,\theta}$ see $\mathfrak{e}_\kappa$, see [MPSh 713].

Proof. 1) Clearly ${}^\mu\theta$ has cardinality 2^μ, so let $\langle \nu_\alpha : \alpha < 2^\mu\rangle$ be a list without repetitions of ${}^\mu\theta$. For $\alpha \le 2^\mu$, let $\Gamma_\alpha = \{\nu_\beta : \beta < \alpha\}$ and we choose $\bar{f}^\alpha$ by induction on $\alpha \le 2^\mu$ such that

(a) $\bar{f}^\alpha = \langle f^\alpha_\varepsilon : \varepsilon < \mu\rangle$
(b) f^α_ε is a function from Γ_α to θ
(c) f^α_ε is increasing continuous in α, i.e., $\beta < \alpha \Rightarrow f^\beta_\varepsilon = f^\alpha_\varepsilon \restriction \mathrm{Dom}(f^\beta_\varepsilon)$
(d) for every $\gamma < \beta < \alpha$ the set $\{\varepsilon < \mu : f^\alpha_\varepsilon(\nu_\beta) = \nu_\gamma(\varepsilon)\}$ is not empty.

If we succeed, then we let $f_\varepsilon = f^{2^\mu}_\varepsilon = \bigcup\{f^\alpha_\varepsilon : \alpha < 2^\mu\}$ for $\varepsilon < \mu$ and $\langle f_\varepsilon : \varepsilon < \mu\rangle$ is clearly as required.

For clause (d) it is enough to consider the case $\alpha = \beta + 1$, as the non trivial case in the induction is $\alpha = \beta + 1$. We can choose $\langle f^{\beta+1}_\varepsilon(\nu_\beta) : \varepsilon < \mu\rangle \in {}^\mu\theta$ such that it holds by the assumption $(*)$.

2) In the proof of 1.10, now $\rho \in {}^{\mu}\theta \Rightarrow |\mathrm{Sol}_\rho| < 2^\mu$ does not suffice. However, before choosing $\langle f_\varepsilon : \varepsilon < \mu\rangle$ we are given $\mathbf{F}$ and we can choose cd, $\langle \mathrm{cd}_\varepsilon : \varepsilon < \mu\rangle$ and define T, $\langle \eta_\alpha : \alpha < 2^\mu\rangle$, $T_{<\alpha}$ and $\langle \mathcal{P}_\delta : \delta \in S\rangle$. Note that $\alpha < \lambda \Rightarrow |h(\alpha)|^{<\kappa>_{\mathrm{tr}}} < 2^\mu$ hence $\alpha < \lambda \Rightarrow |\cup\{\mathcal{P}_\delta : \delta \in S \cap \alpha\}| < 2^\mu$. We can also choose $\langle \nu_{\eta,\varepsilon} : \eta \in \mathcal{P}_\delta, \varepsilon < \mu\rangle$ and define $\rho_\eta \in {}^{\mu}\theta$ for $\eta \in \mathcal{P}_\delta, \delta \in S$.
Now we can phrase our demands on $\langle f_\varepsilon : \varepsilon < \mu\rangle$, which is in fact weaker than the full generality of $\mathrm{Sep}(\mu, \theta)$; it is
$\boxdot$ ${}^{\mu}\theta \neq \{\nu \in {}^{\mu}\theta$: for some $\eta \in \mathcal{P}_\delta$ we have $(\forall \varepsilon < \mu)(f_\varepsilon(\nu) \neq \rho_\eta(\varepsilon))\}$.
For this it is enough that for the list $\langle \nu_\alpha : \alpha < 2^\mu\rangle$ chosen in the proof of part (1), for each $\delta \in S$, for some $\gamma < 2^\mu$ we have $\{\rho_\eta : \eta \in \mathcal{P}_\delta\} \subseteq \{\nu_\alpha : \alpha < \gamma\}$ which is easy to obtain. $\square_{1.29}$

Observation 1.30. If $\mu \geq \aleph_0, \theta = 2$, then $\circledast_{\mu,\theta}$ holds.

Proof. If $\mathcal{F} \subseteq {}^{\mu}\theta$, $|\mathcal{F}| < 2^\mu$, let $\nu \in {}^{\mu}\theta \backslash \mathcal{F}$ and let $\rho = \langle 1 - \nu(\varepsilon) : \varepsilon < \mu\rangle$. $\square_{1.30}$

§2. Upgrading to larger cardinals

In §1 we get the θ-middle diamond property for regular cardinals λ of the form $\mathrm{cf}(2^\mu)$ on "most" $S \subseteq S^\lambda_\kappa$ for "many" $\kappa = \mathrm{cf}(\kappa)$ (and θ). We would like to prove Theorem 0.1. For this we first give an upgrading, which is a stepping up claim (for MD_ℓ) given as 2.1 below. Then we prove that it applies to λ_2 (for quite a few λ_1), when λ_2 is a successor of regular, then when λ_2 is successor of singular and lastly when λ_2 is weakly inaccessible (not strong limit). We can upgrade each of the variants from §1, but we shall concentrate on the one from 1.5(5A) because of Claim 1.20.

The main result of this section is claim 2.9.
Question. What occurs to λ_2 strongly inaccessible, does it have to have some Md-property?

Possibly combining §1 and §2 we may eliminate the $\lambda_2 > 2^{\lambda_1 + \Upsilon^{<\kappa} + \theta}$ in 2.1, but does not seem urgent now.

Upgrading Claim 2.1. 1) Assume
 (*a*) $\lambda_1 < \lambda_2$ are regular cardinals such that $\lambda_2 > 2^{\lambda_1 + \Upsilon^{<\chi} + \theta}$ (this last condition can be waived for "nice" colourings, see Definition 2.2 below)
 (*b*) for some stationary $S_1 \subseteq S^{\lambda_1}_\kappa$, some sequence $\bar{C} = \langle C^1_\delta : \delta \in S_1\rangle$ is a $(\lambda_1, \kappa, \chi)$-Md-parameter which has a $(D_{\lambda_1}, \Upsilon, \theta)$-Md-property
 (*c*) $\lambda_2 > \lambda_1^+$, λ_2 is a successor of regular.
 Then for some stationary $S_2 \subseteq S^{\lambda_2}_\kappa$, there is $\bar{C} = \langle C_\delta : \delta \in S_2\rangle$, a good [5] $(\lambda_2, \kappa, \chi)$-Md-parameters which has the $(D_{\lambda_2}, \Upsilon, \theta)$-Md-property.
2) We can in part (1) replace clause (c) by any of the following clauses
 $(c)'$ there is a $\leq \lambda_1$ square on some stationary $S \subseteq S^{\lambda_2}_{\lambda_1}$, (see Definition 0.4(1A); this holds if $\lambda_2 = \mu^+$, $\mu^{\lambda_1} = \mu$)
 $(c)''$ λ_2 has a (λ_1, S_1)-square (where S_1 is from clause (b)) which means:

[5] the goodness is a bonus

$\boxplus^{\lambda_2}_{\lambda_1,S_1}$ $\lambda_1 < \lambda_2$ are regular, $S_1 \subseteq \lambda_1$ is stationary and there are $S_2 \subseteq S^{\lambda_2}_{\lambda_1}$ stationary and $\bar{C}^2 = \langle C^2_\delta : \delta \in S_2 \rangle$ satisfying $C^2_\delta \subseteq \delta = \sup(C^2_\delta)$, C^2_δ is closed, $\mathrm{otp}(C^2_\delta) = \lambda_1$ and if $\alpha \in C^2_{\delta_1} \cap C^2_{\delta_2}$, $\{\delta_1, \delta_2\} \subseteq S_2$ and $\alpha = \sup(C^2_{\delta_1} \cap \alpha)$ and $\mathrm{otp}(\alpha \cap C^2_{\delta_2}) \in S_1$, <u>then</u> $\alpha \cap C^2_{\delta_2} = \alpha \cap C^2_{\delta_1}$.

$(c)'''$ λ_2 has a $(\lambda_1, \bar{C}_1)$-square $\bar{C}_2$ (where $\bar{C}_1$ is from clause (b)) which means:

$\boxplus^{\lambda_2}_{\lambda_1,\bar{C}_1,\bar{C}_2}$ (α) $\bar{C}_2 = \langle C^2_\delta : \delta \in S_2 \rangle$

(β) $S_2 \subseteq S^{\lambda_2}_{\lambda_1}$ is stationary

(γ) C^2_δ is a closed unbounded subset of δ of order type λ_1

(δ) if $\alpha \in C^2_{\delta_1} \cap C^2_{\delta_2}$ (so $\{\delta_1, \delta_2\} \subseteq S_2$) and otp $(\alpha \cap C^2_{\delta_1}) \in S_1$, and $\alpha = \sup(C^2_{\delta_1} \cap \alpha)$ <u>then</u> $\mathrm{otp}(\alpha \cap C^2_{\delta_1}) = \mathrm{otp}(\alpha \cap C^2_{\delta_2})$, call it γ and $\{\beta \in C^2_{\delta_1} : \mathrm{otp}(\beta \cap C^2_{\delta_1}) \in C^1_\gamma\} = \{\beta \in C^2_{\delta_2} : \mathrm{otp}(\beta \cap C^2_{\delta_2}) \in C^1_\gamma\}$.

(3) Assume $\ell \in \{0, 2\}$

(a) $\lambda_1 < \lambda_2$ are regular uncountable

(b) some $(\lambda_1, \kappa, \chi)$-Md-parameter has the θ-MD_ℓ-property

(c) at least one of the variants of clause (c) above holds.

<u>Then</u> some $(\lambda_2, \kappa, \chi)$-$\mathrm{Md}_\ell$-parameter has the θ-MD_ℓ-property.

Definition 2.2. *We say that a (Υ, θ)-colouring $\mathbf{F} = \langle F_\delta : \delta \in S \rangle$ for a λ-Md-parameter $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is nice when:*

⊛ *if $\delta_1, \delta_2 \in S$ and $otp(C_{\delta_1}) = otp(C_{\delta_2})$ and $\eta_1 \in {}^{(C_{\delta_1})}\Upsilon$, $\eta_2 \in {}^{(C_{\delta_2})}\Upsilon$ and $\alpha_1 \in C_{\delta_1} \wedge \alpha_2 \in C_{\delta_2} \wedge otp(C_{\delta_1} \cap \alpha_1) = otp(C_{\delta_2} \cap \alpha_2) \Rightarrow \eta_1(\alpha_1) = \eta_2(\alpha_2)$, <u>then</u> $F_{\delta_1}(\eta_1) = F_{\delta_2}(\eta_2)$.*

Proof. 1) If clause (c) holds then clause $(c)'$ of part (2) holds by [Sh 351, §4], so it is enough to prove part (2).

2) Trivially clause $(c)'$ implies clause $(c)''$ and clause $(c)''$ implies clause $(c)'''$; so assume $(c)'''$; and let $\bar{C}^2 = \langle C^2_\delta : \delta \in S_2 \rangle$ be as there.
Let $\bar{h} = \langle h_\delta : \delta \in S_2 \rangle$ be such that h_δ is an increasing continuous function from λ_1 onto C^2_δ recalling clause (γ), i.e., C^2_δ is a closed unbounded subset of δ of order type λ_1. Let $S = \{h_\delta(i) : \delta \in S_2$ and $i \in S_1\}$ and if $\alpha \in S, \alpha = h_\delta(i), \delta \in S_2$ and $i \in S_1$ then we let $C^*_\alpha = \{h_\delta(j) : j \in C^1_i\}$; and let $\bar{C}^* = \langle C^*_\alpha : \alpha \in S \rangle$. Why is C^*_α (hence $\bar{C}^*$) well defined? We shall check that if $\alpha \in S, \alpha = h_{\delta_\ell}(i_\ell), \delta_\ell \in S_2, i_\ell \in S$ for $\ell = 1, 2$ then $\{h_{\delta_1}(j) : j \in C^1_{i_1}\} = \{h_{\delta_2}(j) : j \in C^1_{i_2}\}$; this holds by clause (δ) of the assumption $(c)'''$. We should check the conditions for "$\bar{C}^*$ is a good $(\lambda_2, \kappa, \chi)$-Md-parameter with the $(D_{\lambda_2}, \Upsilon, \theta)$-Md-property.
The main point is: assume that $\mathbf{F}$ is a $(\bar{C}^*, \Upsilon, \theta)$-colouring, find a $\bar{c}$ as required. For each $\delta \in S_2$ we define a $(\bar{C}^1, \Upsilon, \theta)$-colouring $\mathbf{F}^\delta$ by $\mathbf{F}^\delta(\eta) = \mathbf{F}(\eta \circ h^{-1}_\delta)$ that is for $i \in S_1$ and $\eta \in {}^{C^1_i}\Upsilon$ we define $\mathbf{F}^\delta(\eta)$ as $\mathbf{F}(\nu)$ where $\nu \in {}^{C^*_{h_\delta(i)}}\Upsilon$, where ν is defined by $\forall \beta \in C^*_{h_\delta(i)}$, we let $\nu(\beta) = (h^{-1}_\delta(\beta))$. As [6] $\lambda_2 > 2^{\lambda_1 + |\Upsilon|^{<\chi} + \theta}$ because $\theta^{\Sigma\{|\Upsilon|^{|C^1_\delta|} : \delta \in S_1\}} \leq 2^{\theta \lambda_1 \Sigma\{|\Upsilon|^{|\alpha|} : \alpha < \chi\}} = 2^{\lambda_1 + |\Upsilon|^{<\chi} + \theta}$, for some stationary $S^*_2 \subseteq S_2$ we have $\delta \in S^*_2 \Rightarrow \mathbf{F}^\delta = \mathbf{F}^*$, so $\mathbf{F}^*$ is a $(\bar{C}^1, \Upsilon, \theta)$-colouring, hence it has a good $(D_{\lambda_1}, \Upsilon, \theta)$-Md-sequence $\bar{c}^1 = \langle c^1_i : i \in S_1 \rangle$.

[6] only place this is used, if $\mathbf{F}$ is nicely defined this is not necessary

We define $\bar{c}^* = \langle c^*_\alpha : \alpha \in S\rangle$ by $\delta \in S_2 \wedge i \in S_1 \Rightarrow c^*_{h_\delta(i)} = c^1_i$. By clause (δ) of $(c)'''$ it is well defined. Now for any $\eta \in {}^{(\lambda_2)}\Upsilon$ and club E_2 of λ_2, we can find $\delta \in S^*_2 \cap \operatorname{acc}(E_2)$, hence $E_1 = \{\alpha < \lambda_1 : h_\delta(\alpha) \in E_2\}$ is a club of λ_1 and let $\eta_1 \in {}^{(\lambda_1)}\Upsilon$ be $\eta_1(\alpha) = \eta(h_\delta(\alpha))$, so by the choice of $\bar{c}^1$ for some $\delta_1 \in S_1 \cap E_1, c^1_{\delta_1} = \mathbf{F}^*(\eta_1 \restriction C^1_{\delta_1})$ and let $\delta_2 = h_\delta(\delta_1)$, so $\delta_2 \in E_2$ and $c^*_{\delta_2} = \delta = \mathbf{F}^*(\eta_1 \restriction C^1_{\delta_1}) = \mathbf{F}^\delta(\eta_1 \restriction C^1_{\delta_1}) = \mathbf{F}(\eta \restriction C^*_{\delta_2})$, so we are done.

3) Similar only easier (or see the proof of 2.10). $\square_{2.1}$

Claim 2.3. We can strengthen the conclusion of 2.1(1) and 2.1(2) to: we can find a sequence $\langle S_{2,i} : i < \lambda_2\rangle$ of pairwise disjoint stationary subsets of λ_2 and for each $i < \lambda_2$ a sequence $\langle C^i_\delta : \delta \in S_{2,i}\rangle$ which is a $(\lambda_2, \kappa, \chi)$-Md-parameter having the $(D_{\lambda_2}, \Upsilon, \theta)$-Md-property.

Proof. By the proof of 2.1, without loss of generality clause $(c)''$ of 2.1(2) and clauses (a), (b) of 2.1(1) hold. For some $\zeta < \lambda_1$ for unboundedly many $i < \lambda_2$ the set $S_{2,i} =: \{\delta \in S_2 : i$ is the ζ-th member of $C^2_\delta\}$ is stationary. $\square_{2.3}$

Conclusion 2.4. 1) If μ is a strong limit, λ is a successor cardinal, $\lambda = \operatorname{cf}(\lambda) > 2^\mu$, moreover $\lambda > 2^{2^\mu}$ (superfluous for nice colourings, e.g., for MD_ℓ, $\ell \in \{0, 2\}$), <u>then</u> for every large enough regular $\kappa < \mu$, some good$^+$ $(\lambda, \kappa, \kappa^+)$-Md-parameter $\bar{C}$ has the $(2^\mu, \theta)$-Md-property for every $\theta < \mu$.

2) We can add: $\bar{C}$ has the θ-MD_ℓ-property for $\ell = \{0, 2\}$.

3) We can find a sequence $\langle S_i : i < \lambda\rangle$ of pairwise disjoint stationary subsets of S^λ_κ such that for each $i < \lambda$ there is $\bar{C}^i = \langle C^i_\delta : \delta \in S\rangle$ as in parts (1) and (2).

Proof. 1) Let $\kappa_0 < \mu$ be large enough such that $[\theta \in \{\lambda, 2^\mu\} \& \alpha < \theta \& \kappa_0 \leq \kappa = \operatorname{cf}(\kappa) < \mu] \Rightarrow |\alpha|^{[\kappa]} < \theta$, (see Definition 1.3, exists by [Sh 460]).

By 1.10, for any regular $\kappa \in [\kappa_0, \mu)$ and $\theta < \mu$ we have the result for $\operatorname{cf}(2^\mu)$ in place of λ. If λ is a successor of regular we can apply 2.1(1), i.e., for clause (c) there is a square as required by [Sh 351, §4]. In the case that λ is a successor of a singular cardinal, by 2.7 below, clause $(c)''$ of 2.1(2) holds for $\lambda_2 = \lambda, \lambda_1 = \operatorname{cf}(2^\mu)$ so we can apply 2.1.

2) Follows by 1.20.

3) In the proof of parts (1) + (2) above, instead of 2.1 we use 2.3. $\square_{2.4}$

Below we generalize Engełking Karłowic theorem [EK] (which says that if $\mu = \mu^\kappa, \lambda = 2^\mu$ then there are $f_\varepsilon : \lambda \to \mu$ for $\varepsilon < \mu$ such that every partial function from λ to μ with domain of cardinality $\leq \kappa$, is extended by f_ε for some $\varepsilon < \mu$); we do more than is necessary for the middle diamond.

Claim 2.5. 1) Assume

(a) $\mu = \operatorname{cov}(\mu, \theta^+, \theta^+, \kappa^+)$

(b) $2^\theta \leq \mu$

(c) $\mu < \lambda \leq 2^\mu$

<u>Then</u> there is $\bar{F}$ such that:

(α) $\bar{F} = \langle F_\varepsilon : \varepsilon < \mu\rangle$

(β) each F_ε is a function from λ to μ

$(\gamma)^-$ if $u \in [\lambda]^{\leq\theta}$ then we can find $\bar{E}$ such that

(i) $\bar{E} = \langle E_i : i < \kappa\rangle$ is a sequence of equivalence relations on u

(ii) $\alpha \in u \Rightarrow \{\alpha\} = \cap\{\alpha/E_i : i < \kappa\}$

(iii) if $f : u \to \mu$ respects some E_i (i.e. $\alpha E_i \beta \Rightarrow f(\alpha) = f(\beta)$) then we can find a partition $\langle u_\xi : \xi < \kappa\rangle$ of u such that $\xi < \kappa \Rightarrow f \restriction u_\xi \subseteq F_\varepsilon$ for some $\varepsilon < \mu$.

2) In part (1) we can strengthen clause $(\gamma)^-$ to (γ) below if $(d)_\ell$ for some $\ell \in \{1, \dots, 5\}$ below holds

(γ) if $u \in [\lambda]^{\leq\theta}$ and $f : u \to \mu$, then we can find $\bar{u} = \langle u_i : i < \kappa\rangle$ such that

(i) $u = \cup\{u_i : i < \kappa\}$ and

(ii) for every $i < \kappa$ for some $\varepsilon < \mu$ we have $f \restriction u_i \subseteq F_\varepsilon$

$(d)_1$ there is a sequence $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ of pairwise distinct functions from μ to μ such that: if $u \in [\lambda]^{\leq\theta}$ then we can find sequences $\bar{u} = \langle u_i : i < \kappa\rangle$ and $\langle B_i : i < \kappa\rangle$ such that $u = \cup\{u_i : i < \kappa\}$, $B_i \in [\mu]^{<\aleph_0}$ for $i < \kappa$ and for each $i < \kappa$ the sequence $\langle f_\alpha \restriction B_i : \alpha \in u_i\rangle$ is with no repetitions

$(d)_2$ there is a sequence $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ of functions from κ to μ and an ideal J on κ such that $\bar{f}$ is θ^+-free modulo J, i.e., (θ^+, J)-free (which means that for $u \in [\lambda]^{\leq\theta}$, we can find a sequence $\bar{a} = \langle a_\alpha : \alpha \in u\rangle$ of subsets of κ such that $a_\alpha = \kappa$ mod J and $i \in a_\alpha \cap a_\beta \& \alpha \neq \beta \Rightarrow f_\alpha(i) \neq f_\beta(i)$)

$(d)_3$ like $(d)_2$ but $a_\alpha \in J^+$ that is $a_\alpha \notin J$

$(d)_4$ $(\forall\alpha < \mu)(|\alpha|^\theta < \mu)$ and $\lambda < \mathrm{pp}_J(\mu)$ for some ideal J on κ

$(d)_5$ $\mu = \mu^{\aleph_0} < \lambda < \mu^\kappa$.

3) If we weaken the conclusion of part (2), replacing (γ) by

$(\gamma)'''_{\theta,\kappa}$ for every club E of λ for some closed $u \subseteq E$ of order type θ there is $\langle u_i : i < \kappa\rangle$ such that $u_i \subseteq u$ and $\zeta \in u \& \mathrm{cf}(\zeta) > \kappa \Rightarrow \zeta \in \cup\{u_i : i < \kappa\}$ and $(\forall i < \kappa)(\exists\varepsilon < \mu)[f \restriction u_i \subseteq F_\varepsilon)$

then we can weaken the assumption in a parallel way, e.g.

$(d)^-_{1,\theta,\kappa}$ there is $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$, $f_\alpha \in {}^\mu\mu$ such that for any club E of λ we can find a closed $e \subseteq E$ of order type $\theta + 1$, and $\bar{u} = \langle u_i : i < \kappa\rangle$, $\bar{B} = \langle B_i : i < \kappa\rangle$ such that $B_i \in [\mu_1] < \aleph_0$, $u_i \subseteq \theta$ and $\zeta \in u \& \mathrm{cf}(\zeta) \geq \kappa \Rightarrow \zeta \in \cup\{u_i : i < \kappa\}$ and for each i, $\langle f_\alpha \restriction B_i : \alpha \in u_i\rangle$ is with no repetitions (and in $(d)_3$ it becomes easy).

Remark. We can replace λ (as a domain) by $[\lambda]^2$, etc.

Proof. 1) Let $\mathcal{A}$ be a family of $\leq \mu$ subsets of μ each of cardinality $\leq \theta$ such that any $A \in [\mu]^{\leq\theta}$ is included in the union of $\leq \kappa$ many of them; it exists by assumption (a). By assumption (b) without loss of generality $A \in \mathcal{A} \& B \subseteq A \Rightarrow B \in \mathcal{A}$ so we replace "included" by "equal" and also without loss of generality $A_1 \in \mathcal{A} \& A_2 \in \mathcal{A} \Rightarrow A_1 \cup A_2 \in \mathcal{A}$. Let pr be a pairing function on μ. As $\mu < \lambda \leq 2^\mu$ there is a sequence $\langle f^*_\zeta : \zeta < \lambda\rangle$ of pairwise distinct functions from μ to $\{0, 1\}$.

Let

$$\begin{aligned}\mathcal{Y} = \{(A, B, \bar{g}, \bar{\beta}) : B \in \mathcal{A}, &\ \bar{g} = \langle g_i : i < \theta\rangle \\ &g_i \in {}^B2 \text{ and } i < j < \theta \Rightarrow g_i \neq g_j \\ &\text{and } \bar{\beta} = \langle \beta_i : i < \theta\rangle \text{ satisfies } \{\beta_i : i < \theta\} \subseteq A \in \mathcal{A}\}.\end{aligned}$$

Clearly $|\mathcal{Y}| \le \mu \times 2^\theta = \mu$. For each $x = (A, \bar{g}, \bar{\beta}) \in \mathcal{Y}$ we define a function $F_x : \lambda \to \mu$ as follows:

$F_x(\zeta)$ is β_i if $g_i \subseteq f^*_\zeta$ and is 0 if there is no such i (clearly we cannot have $i_0 \ne i_1 < \theta \& g_{i_0} \subseteq f^*_\zeta \& g_{i_1} \subseteq f^*_\zeta$).

Let $\bar{F} = \langle F_\varepsilon : \varepsilon < \mu \rangle$ list $\{F_x : x \in \mathcal{Y}\}$, clearly $\bar{F}$ satisfied demand $(\alpha) + (\beta)$. As for demand $(\gamma)^-$ let $u \in [\lambda]^{\le\theta}$ be given. For any $\zeta_1 \ne \zeta_2 \in u$ let $\alpha = \alpha(\zeta_1, \zeta_2) < \mu$ be minimal such that $f^*_{\zeta_1}(\alpha) \ne f^*_{\zeta_2}(\alpha)$, exists as $f^*_{\zeta_2} \ne f^*_{\zeta_1} \in {}^\mu 2$ and let $B = \{\alpha(\zeta_1, \zeta_2) : \zeta_1 < \zeta_2$ are from $u\}$. By the choice of $\mathcal{A}$ we can find pairwise disjoint $B_\xi \in \mathcal{A}$ for $\xi < \kappa$ such that $B = \bigcup\{B_\xi : \xi < \kappa\}$. Lastly, we define $\bar{E} = \langle E_\xi : \xi < \kappa \rangle$ by

⊛ for $\zeta_1, \zeta_2 \in u$, $\zeta_1 E_\xi \zeta_2$ iff $f^*_{\zeta_1} \restriction B_\xi = f^*_{\zeta_2} \restriction B_\xi$.

Assume that $f : u \to \mu$ respects E_ξ, then we can find non empty $A_\xi \in \mathcal{A}$ for $\xi < \kappa$ such that $\mathrm{Rang}(f) = \bigcup\limits_{\xi<\kappa} A_\xi$. Now for any pair $(\xi_1, \xi_2) \in \kappa \times \kappa$, let $u_{\xi_1} = \mathrm{Min}\{\zeta \in u : f(\zeta) \in A_{\xi_1}\}$, let $\langle \zeta_i : i < \theta \rangle$ list u_{ξ_2}, let x_{ξ_1,ξ_2} be $(A_{\xi_2}, B_{\xi_1} \cup B_{\xi_2}, \langle f^*_{\zeta_i} \restriction B_{\xi_1} : i < \theta \rangle, \langle f(\zeta_i) : i < \theta \rangle)$. Now check.

2) Naturally the proof splits to five cases, but first let $\mathcal{A}$ be as in the proof of part (1), pr(..., ...) be a pairing function on μ, let $\langle f^*_\zeta : \zeta < \lambda \rangle$ be a sequence of functions from μ to μ (chosen separately in each case). Let

$$\begin{aligned} \mathcal{X} = \{x : x = (A, \bar{g}, \bar{\beta}) \text{ satisfies} \\ &(i) \quad A \in \mathcal{A} \\ &(ii) \quad \bar{g} = \langle g_i : i < \theta \rangle, \\ &(iii) \quad \text{each } g_i \text{ is a partial function from } A \text{ to } A \\ &(iv) \quad \bar{\beta} = \langle \beta_i : i \le \theta \rangle \text{ and } i \le \theta \Rightarrow \beta_i \in A\}. \end{aligned}$$

Clearly $|\mathcal{X}| \le \mu$ and for each $x = (A, \bar{g}, \bar{\beta}) \in \mathcal{X}$ we define $F_x : \lambda \to \mu$ by

$$F_x(\zeta) = \beta_i \text{ \underline{iff} } i = \mathbf{i}_x(\zeta) = \mathrm{Min}\{\varepsilon \le \theta : \text{ if } \varepsilon < \theta \text{ then } g_\varepsilon \subseteq f^*_\zeta\}.$$

So let $u \in [\lambda]^{\le\theta}$ and $f : u \to \mu$ be given and let $\langle v_i : i < \kappa \rangle$ be such that $u = \bigcup\{v_i : i < \kappa\}$ and $\langle A_i : i < \kappa \rangle$ be such that $i < \kappa \Rightarrow A_i \in \mathcal{A}$ and $\mathrm{Rang}(f^{v_i}) \subseteq A_i$.

Case 1. Clause $(d)_1$ holds.

We choose $f^*_\zeta = f_\zeta$ for every $\zeta < \lambda$ where f_ζ is from clause $(d)_1$. Let $\langle (u_i, B_i) : i < \kappa \rangle$ be as guaranteed to exist for u in clause $(d)_1$. For each $i, j < \kappa$ such that $u_i \cap v_j \ne \emptyset$ and let $\langle \zeta^{i,j}_\varepsilon : \varepsilon < \theta \rangle$ list $u_i \cap v_j$ (possibly with repetition) and let $x_{i,j} = (A^*_{i,j}, \langle g^{i,j}_\varepsilon : \varepsilon < \theta \rangle, \langle \beta^{i,j}_\varepsilon : \varepsilon < \theta \rangle)$ where

(a) $A^*_{i,j} = B_i \cup A_j$

(b) $g^{i,j}_\varepsilon = f^*_{\zeta^{i,j}_\varepsilon} \restriction B_i$

(c) $\beta^{i,j}_\varepsilon = f(\zeta^{i,j}_\varepsilon)$.

We can check that

(α) $x_{i,j} \in \mathcal{X}$ and
(β) $\bigcup\{u_i \cap v_j : i, j < \kappa\} = u$
(γ) $F_{x_{i,j}} \supseteq f \restriction (u_i \cap v_j)$.

[Why? Let $\zeta(*) \in u_i \cap v_j$ so $\zeta(*) = \zeta^{i,j}_{\varepsilon(*)}$ for some $\varepsilon(*) < \theta$ so $g^{i,j}_{\varepsilon(*)} = f^*_{\zeta^{i,j}_{\varepsilon(*)}} \restriction B_i$ by clause (b) above hence $\varepsilon(*) \in \{\varepsilon \subseteq \theta$: if $\varepsilon < \theta$ then $g_\varepsilon \subseteq f^*_{\zeta(*)}\}$. But $\langle f^*_\zeta \restriction B_i : \zeta \in u_i\rangle$ is a sequence without repetitions, so

$$\varepsilon < \varepsilon(*) \Rightarrow \zeta^{i,j}_\varepsilon \in u_i \cap \zeta \Rightarrow g^{i,j}_\varepsilon = f^*_{\zeta^{i,j}_\varepsilon} \restriction B_i$$
$$\neq f^*_{\zeta^{i,j}_{\varepsilon(*)}} = g^{i,j}_{\varepsilon(*)}$$

(the middle inequality holds by the choice of $\langle (u_i, B_i) : i < \kappa\rangle$ above, i.e., by clause $(d)_1$).

So $\mathbf{i}_{x_{i,j}}(\zeta(*)) = \varepsilon(*)$ so $F_{x_{i,j}}(\zeta(*)) = \beta^{i,j}_{\varepsilon(*)} = f(\zeta^{i,j}_{\varepsilon(*)})$, the last equality holds by clause (c). So we are done.

Case 2. Clause $(d)_2$ holds.
Follows from Case 3.

Case 3. $(d)_3$ holds.
Let f'_ζ be any function from μ to μ extending f_ζ. It suffices to show that $\langle f'_\zeta : \zeta < \lambda\rangle$ satisfies clause $(d)_1$. So let $u \in [\lambda]^{\leq\theta}$, so by $(d)_3$ we can find a sequence $\bar{a} = \langle a_\zeta : \zeta \in u\rangle$ of subsets of κ such that $a_\alpha \notin J$ and

$\circledast$ $\zeta_1 \neq \zeta_2 \in u \& i \in a_{\zeta_1} \cap a_{\zeta_2} \Rightarrow f_\alpha(i) \neq f_\beta(i)$.

Let $B_i = \{i\} \in [\mu]^{<\aleph_0}$ and $u_i = \{\zeta \in u : i \in a_\zeta\}$, clearly they are as required.

Case 4. Clause $(d)_4$ holds.
This implies Clause $(d)_3$ by [Sh:g, II,1.3,p.46+1.4](3),p.50 (or see [Sh 506]).

Case 5. $(d)_5$.
Let $\mu_1 = \text{Min}\{\partial : \partial^\kappa > \lambda\}$, so clearly $\mu_1 \leq \mu$. As $\kappa \leq \theta, 2^\theta \leq \mu$ we have $2^\kappa < \mu_1$, hence $\kappa_1 = \text{cf}(\mu_1)$ is $\leq \kappa$ and $(\forall \alpha < \mu_1)(|\alpha|^\kappa < \mu_1)$. By $(d)_5$ we have $\kappa_1 > \aleph_0$. So by [Sh:g, VIII], $\text{pp}_{J^{\text{bd}}_\kappa}(\mu_1) \geq (\mu_1)^\kappa = \mu^\kappa > \lambda$ so Clause $(d)_4$ holds. $\square_{2.5}$

Claim 2.6. 1) Assume
(a) $\mu = \text{cov}(\mu, \theta^+, \theta^+, \kappa^+)$ so $\mu > \theta \geq \kappa$
(b) $2^\theta \leq \mu$
(c) $\mu < \lambda = \text{cf}(\lambda) \leq \mu^\kappa$
(d) $\theta = \text{cf}(\theta)$, $S \subseteq S^\lambda_\theta$ is stationary and $S \in I[\lambda]$
(e) $T \subseteq \theta$ and $[\alpha < \lambda \& \delta \in T \Rightarrow |\alpha|^{<\text{cf}(\delta)>_{\text{tr}}} < \lambda]$.
<u>Then</u> there is $\bar{F}$ such that
(α) $\bar{F} = \langle F_\varepsilon : \varepsilon < \mu\rangle$
(β) F_ε is a function from λ to μ
(γ) for a club of $\delta \in S$, there is an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \theta\rangle$ with limit δ such that:

(∗) for every function f from $\{\alpha_\varepsilon : \varepsilon \in T\}$ to μ we can find a sequence $\langle u_i : i < \kappa\rangle$ of subsets of T with union T such that $(\forall i < \kappa)(\exists \varepsilon < \mu)(f \restriction \{\alpha_j : j \in u_i\} \subseteq F_\varepsilon)$.

Proof. 1) Let $\bar{a} = \langle a_\alpha : \alpha < \lambda\rangle$ and E_0 as in 0.5(2) witness $S \in I[\lambda]$. Let $\bar{M} = \langle M_\alpha : \alpha < \lambda\rangle$ be an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$, each of cardinality $< \lambda$ satisfying $\bar{M} \restriction (\alpha + 1) \in M_{\alpha+1}$, $\|M_0\| = \mu$, $M_{1+\alpha} \cap \lambda \in \lambda$ and $\{\lambda, \theta, T, S, \bar{a}, E_0\} \cup (\mu + 1) \subseteq M_0$. Let $E = \{\delta < \lambda : M_\delta \cap \lambda = \delta\}$, clearly a club of λ and $E \subseteq E_0$ as $E_0 \in M_0$.
Now we choose by induction on $\alpha < \lambda$ a function f_α from κ to μ such that:
$(*)_1$ if $b \in [\alpha]^{\leq\theta} \& b \in M_{\alpha+1}$ then
$\mathrm{Rang}(f_\alpha) \nsubseteq \cup\{\mathrm{Rang}(f_\beta) : \beta \in b\}$
$(*)_2$ f_α is the $<^*_\chi$-first object satisfying $(*)_1$.
This is clearly possible as $\kappa \leq 2^\theta \leq \mu < \lambda \leq \mu^\kappa$.
For every $i < \kappa$ and $u \in M_0 \cap [\mu]^{\leq\theta}$ and $g : u \to \mu$ from M_0, let $F_{g,i}$ be the unique $F_{g,i} : \lambda \to \mu$ and it satisfies: if $f_\alpha(i) \in \mathrm{Dom}(g)$ then $F_{g,i}(\alpha) = g(f_\alpha(i))$. [Why $F_{g,i} \in M_0$? As $\mu + 1 \subseteq M_0$ and $u, g \in M_0$.]
Let $\langle (g_\varepsilon, i_\varepsilon) : \varepsilon < \mu\rangle$ list the set of all such pairs (g, i) and let $F_\varepsilon = F_{g_\varepsilon, i_\varepsilon}$. We shall show that $\langle F_\varepsilon : \varepsilon < \mu\rangle$ is as required; so clauses (α) + (β) hold trivially.
Now if $\delta \in \mathrm{acc}(E) \cap S \subseteq S \cap E_0$ so $a_\delta \subseteq \delta = \sup(a_\delta)$, $\mathrm{otp}(a_\delta) = \mathrm{cf}(\delta) = \theta$ and let $\langle \alpha_\varepsilon : \varepsilon < \theta\rangle$ list the closure in δ of a_δ in the increasing order. So for some club C of θ consisting of limit ordinals we have $\varepsilon \in C \Rightarrow \alpha_\varepsilon \in E$. Let $\Theta = \{\mathrm{cf}(\delta) : \delta \in T\}$, now suppose $\varepsilon \in C$ is a limit ordinal with cofinality $\in \Theta$. So for every $\zeta < \varepsilon, \alpha_{\zeta+1} \in a_{\alpha_\varepsilon}$ so, as $\bar{a} \in M_0 \prec M_{\alpha_\varepsilon} \& \alpha_{\zeta+1} \in M_{\alpha_\varepsilon}$ clearly $a_{\alpha_{\zeta+1}} \in M_{\alpha_\varepsilon}$ but (see Definition 0.5(2)) $c\ell(a_{\alpha_{\zeta+1}}) = \{\alpha_\varepsilon : \varepsilon \leq \zeta\}$ hence $\{\alpha_\varepsilon : \varepsilon \leq \zeta\} \in M_{\alpha_\varepsilon}$. Now the tree $(\{\nu : \nu \in M_{\alpha_\varepsilon}$ is a sequence of ordinals of length $< \delta\}, \trianglelefteq)$ is a tree which belongs to $M_{\alpha_\varepsilon+1}$, has ε levels and $\|M_{\alpha_\varepsilon}\| < \lambda$ nodes so by assumption (e) it has $< \lambda$ ε-branches. As this set (of ε-branches) belongs to $M_{\alpha_\varepsilon+1}$, every ε-branch of this tree belongs to $M_{\alpha_\varepsilon+1}$ but $\langle \alpha_\zeta : \zeta < \varepsilon\rangle$ is (essentially) such a branch; so together $\varepsilon \in C \wedge \mathrm{cf}(\varepsilon) \in \Theta \Rightarrow \langle \alpha_\zeta : \zeta < \varepsilon\rangle \in M_{\alpha_\varepsilon+1}$.
Let

$$v_i = \{\varepsilon \in C : \mathrm{cf}(\varepsilon) \in \Theta \text{ and } f_{\alpha_\varepsilon}(i) \notin \cup\{\mathrm{Rang}(f_\beta) : \beta \in \{\alpha_\zeta : \zeta < \varepsilon\}\}.$$

So $\langle v_i : i < \kappa\rangle$ is a sequence of subsets of C with union $\supseteq T$. For each $i < \kappa$ by assumption (a) there is $\langle v_{i,j} : j < \kappa\rangle$ be a sequence of subsets of v_i with union v_i such that for every $j < \kappa \{f_{\alpha_\varepsilon}(i) : \varepsilon \in v_{i,j}\} \subseteq v^+_{i,j} \in M_0$.
Let h be a function from $\{\alpha_\varepsilon : \varepsilon \in T\}$ to μ. So by assumption (a) we can find a sequence $\langle w_j : j < \kappa\rangle$ of subsets of μ each of cardinality $\leq \theta$ such that $\mathrm{Rang}(h) \subseteq \bigcup_{j<\kappa} w_j$ and $\{w_j : j < \kappa\} \subseteq M_0$. Let $u_{i,j_1,j_2} = \{\varepsilon : \varepsilon \in v_i, f_{\alpha_\varepsilon}(i) \in v^+_{i,j_1}$ and $h(\alpha_\varepsilon) \in w_{j_2}\}$. Hence for $i, j_1, j_2 < \kappa$ there is $g = g_{i,j_1,j_2} : \{f_{\alpha_\varepsilon}(i) : \varepsilon \in u_{i,j_1,j_2}\} \to \mu$ satisfying $\varepsilon \in u_{i,j_1,j_2} \Rightarrow g(f_{\alpha_\varepsilon}(i)) = h(\alpha_\varepsilon)$, see the choice of $v_i, v_{i,j_1}, w_{j_2}, u_{i,j_1,j_2}$. Now as $v_i \subseteq \theta$, $2^\theta \leq \mu$, $\mu + 1 \subseteq M_0$ and $v^+_{i,j_1}, w_{j_2} \in$

M_0 are subsets of μ of cardinality $\leq \theta$ clearly the set of partial functions from v^+_{i,j_1} to w_{j_2} belongs to M_0 and has cardinality $\leq^{|u_{i,j}|} |w_j| \leq \theta^\theta = 2^\theta \leq \mu$ hence is included in M_0. So $g_{i,j}$ belongs to M_0 for every $i, j < \kappa$. Also easily $F_{g_{i,j,i}}$ extends $h \restriction (\{\alpha_\varepsilon : \varepsilon \in u_{i,j}\})$. So $\langle \{\alpha_\varepsilon : \varepsilon \in u_{i,j}\} : i, j < \kappa\rangle$ and $\langle \alpha_\varepsilon : \varepsilon \in C\rangle$ are as required (modulo some renaming). But $\mathrm{Dom}(g_{i,j}) = \{f_{\alpha_\varepsilon}(i) : \varepsilon \in u_{i,j}\} \subseteq w_j$. $\square_{2.6}$

Compare the following with [Sh 237e].

Claim 2.7. 1) Assume

(*a*) $\lambda = \mu^+$ and $\kappa = \mathrm{cf}(\mu) < \mu$ and $\aleph_0 < \theta = \mathrm{cf}(\theta) < \mu$

(*b*) (*i*) $\Theta = \{\sigma : \aleph_0 \leq \sigma = \mathrm{cf}(\sigma) < \mu$ and $\mu^{<\sigma>_{\mathrm{tr}}} = \mu, \sigma \neq \kappa\}$

(*ii*) $T_\theta = \{\delta < \theta : \mathrm{cf}(\delta) \in \Theta\}$,

(*iii*) $T^+_\theta = T_\theta \cup \{0\} \cup \{\zeta + 1 : \zeta < \theta\}$,

(*iv*) $T^-_\theta = T^+_\theta \backslash T_\theta$

(*c*) $\mathbf{P} \subseteq \{\bar{A} : \bar{A}$ a sequence of length $\leq \mu$ of subsets of θ each included in T^+_θ and including $T^-_\theta\}$

(*d*) $\mathcal{P} \subseteq [\mu]^{\leq\theta}$ has cardinality $\leq \mu$

(*e*) if $\alpha_\varepsilon < \mu$ for $\varepsilon < \theta$ and $\langle w^*_i : i < \kappa\rangle$ a sequence of subsets of T^+_θ and including T^-_θ <u>then</u> for some $\bar{A} \in \mathbf{P}$ we have:

(i) for some $\zeta < \ell g(\bar{A})$ and $j < \kappa$, $A_\zeta \subseteq w^*_j$

(ii) $\zeta < \ell g(\bar{A}) \& j < \kappa \& A_\zeta \subseteq w^*_j \Rightarrow \{\alpha_\varepsilon : \varepsilon \in A_\zeta\} \in \mathcal{P}$.

<u>Then</u> we can find $(\bar{C}_i, A^*_i, S_i)$ for $i < \mu$ such that

(α) $\{0\} \cup \{\zeta + 1 : \zeta < \theta\} \subseteq A^*_i \subseteq \theta$

(β) $\bar{C}_i = \langle C^i_\delta : \delta \in S_i\rangle$; $S_i \subseteq \lambda$ and $[\mathrm{otp}(\alpha \cap C^i_\delta) \in A^*_i \& C^i_\delta \subseteq \delta \& \alpha \in C^i_\delta \Rightarrow \alpha \in S_i \& C^i_\alpha = \alpha \cap C^i_\delta]$

(γ) $S^*_i = \{\delta \in S_i : \mathrm{cf}(\delta) = \theta\}$ is stationary

(δ) for every club E of λ there are $\delta^* \in E$ and a sequence $\langle w^*_i : i < \kappa\rangle$ of subsets of T^+_θ including T^-_θ with union T^+_θ such that for each $j < \kappa$ there is $\bar{A} \in \mathbf{P}$ satisfying: $(\forall^\varepsilon < \ell g(\bar{A}))(A^\varepsilon \subseteq w^*_j \rightarrow (\exists i < \mu)[\delta^* \in S^*_i \& A^*_i = A^\varepsilon])$

(ε) if $B \in I[\lambda]$ is stationary and $B \subseteq S^\lambda_\theta$ then we can demand $B \backslash \cup \{S^*_i : i < \mu\}$ is not stationary and in clause (δ) any $\delta \in B$ satisfies the requirement on δ^*.

2) Assume, (a), (b) and $\alpha < \mu \Rightarrow |\alpha|^\theta < \mu$, $T'_\theta = \{\delta < \theta : \mathrm{cf}(\delta) \neq \kappa\}$ and we let $\mathbf{P} = \{\bar{A} : \bar{A} = \langle A_i : i < \kappa\rangle$ is a sequence of sets with union $T^+_\theta\}$. <u>Then</u> $T'_\theta = T^+_\theta$ and for some $\mathcal{P}$ all the assumptions hence the conclusion of part (1) holds.

3) Assume μ_* is strong limit $< \mu, \kappa = \mathrm{cf}(\mu) < \mu, \lambda = \mu^+$, <u>then</u> for some $\theta_0 < \mu_*$, for any regular $\theta \in (\theta_0, \mu_*)$ the set T_θ include $\{\delta < \theta : \mathrm{cf}(\delta) \geq \theta_1$ or δ nonlimit$\}$ and for $\mathbf{P} = \{\bar{A} : \bar{A}$ is a sequence of $< \theta_0$ subsets of θ with union T_θ such that $\{A_\zeta : \zeta < \ell g(\bar{A})\}$ is closed under finite unions$\}$.

<u>Then</u> for some $\mathcal{P}$ the assumptions hence the conclusion of part (1) holds.

4) In (1) (and (2), (3), if $\bar{A} = \langle A_\varepsilon : \varepsilon < \varepsilon(*)\rangle \in \mathbf{P}$ we can replace it by $\bar{A}' = \langle A'_\varepsilon : \varepsilon < \varepsilon(*)\rangle$, $A'_\varepsilon = A_\varepsilon \cup \{\delta < \theta : \mathrm{cf}(\delta) > \aleph_0$ and $A_\varepsilon \cap \delta$ is stationary in $\delta\}$ and still obtain an element of $\mathbf{P}$.

Remark 2.8. 1) If λ is (weakly) inaccessible we have weaker results: letting $\Theta = \{\sigma : \sigma = \mathrm{cf}(\sigma)$ and $\alpha < \lambda \Rightarrow |\alpha|^{<\sigma>_{\mathrm{tr}}} < \lambda\}$, there is $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle, \mathcal{P}_\alpha \subseteq [\alpha]^{<\theta}, |\mathcal{P}_\alpha| < \lambda$ such that for regular $\theta < \lambda$ for every stationarily many $\delta \in S^\lambda_\theta$ for some increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \alpha \rangle$ with limit $\delta, [\varepsilon < \theta \& \ \mathrm{cf}(\varepsilon) \in \Theta_\lambda \Rightarrow \{\alpha_\zeta : \zeta < \varepsilon\} \in \mathcal{P}_{\alpha_\varepsilon}]$ (see [Sh 420, §1]).

2) Can we restrict ourselves to: $\langle w^*_j : j < \kappa \rangle$ is increasing continuous in part (1) and similarly in (2).

It seems very reasonable (if $\lambda = \mu^+, \kappa = \mathrm{cf}(\mu) < \mu$) that we can find $\langle \lambda_i : i < \kappa \rangle$, an increasing sequence of regular cardinals $< \mu$ with limit μ such that $\lambda^+ = \mathrm{tcf}(\prod_{i<\ell} \lambda_i, \sigma \leq_{J^{\mathrm{bd}}_\kappa})$ and $j < \kappa \Rightarrow \lambda_j > \max \mathrm{pcf}\{\lambda_i : i < j\}$. Why?

This is trivial if $\kappa = \aleph_0$ and if $\kappa > \aleph_0$, it folows by the weak hypothesis (see [Sh 410]), whose negation is not known to be consistent and which has large consistency strength.

3) If there is $\langle \lambda_i : i < \kappa \rangle$ is as above then in part (1) of 2.7 we can strengthen clauses (δ), (ε) in the conclusion. By demanding that "$\langle w^*_j : j < \kappa \rangle$" is increasing continuous. Why? Choosing $\langle f_\alpha : \alpha < \lambda \rangle$ we add

$\boxdot$ if $f_{\alpha_1}(i_1) = f_{\alpha_2}(i_2)$ then $i_1 = i_2 \& f_{\alpha_1} \restriction i_1 = f_{\alpha_2} \restriction t_2$.
so there is a sequence $\langle \mathbf{g}_{\mathbf{i}:\mathbf{i}} < \kappa \rangle, \mathbf{g}_\mathbf{i} : \mu \to \mu$ and $\mathbf{g}_{\mathbf{i}_1}(f_{\alpha_1}(i_1)) = f_{\alpha_1}(i_0)$ when $i_0 < i_1 < \kappa, \alpha < \lambda$ and a function $\mathbf{i} : \mu \to \kappa$ such that $\mathbf{i}(f_\alpha(i)) = i$.

Now in the end of the proof we replace $\alpha'_\varepsilon = \mathrm{cd}(j, f_{\alpha_\varepsilon}(j), \xi(\alpha_\varepsilon))$ (so fixing j) by $\alpha'_\varepsilon = \mathrm{cd}(\xi)$.

Proof. There is B as in clause (ε), that is $B \in I[\lambda]$ and $B \subseteq S^\lambda_\theta$ is a stationary subset of λ, see Claim 1.24(2). So let B be such a set; clearly omitting from B a nonstationary subset of λ is O.K. as if $B' \subseteq B$, $B \backslash B'$ is non-stationary and the demands in clause (ε) for B' are exemplified by the sequence $\langle \bar{C}_i, A^*_i, S^*_i) : i < \mu \rangle$ then the same sequence exemplifies it for B.

We can find $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ witnessing $B \in I[\lambda]$ see Definition 0.5(2), so (possibly omitting from B a nonstationary set) we have: $\alpha \in B \Rightarrow a_\alpha \subseteq \alpha = \sup(a_\alpha) \& \theta = \mathrm{otp}(a_\alpha)$ and $[\beta \in a_\alpha \Rightarrow a_\beta = \beta \cap a_\alpha]$ and $[\alpha \in \lambda \backslash B \Rightarrow a_\alpha \subseteq \alpha \& \ \mathrm{otp}(a_\alpha) < \theta]$.

Let $\chi = \lambda^+$ and $\bar{M} = \langle M_\alpha : \alpha < \lambda \rangle$ be an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in, <^*_\chi)$ to each of which $\{\bar{a}, \mu, \mathbf{P}, \mathcal{P}\}$ belongs, and satisfying $\bar{M} \restriction (\alpha + 1) \in M_{\alpha+1}, \mu \subseteq M_\alpha$ and $\|M_\alpha\| < \lambda$.

As $\lambda = \mu^+$ clearly $\alpha < \lambda \Rightarrow \|M_\alpha\| = \mu$. For any $\alpha < \lambda$ let $\langle b_{\alpha,\zeta} : \zeta < \mu \rangle$ list $M_{\alpha+1} \cap [\alpha]^{<\theta}$.

Let $\langle \lambda_i : i < \kappa \rangle$ be an increasing sequence of regular cardinals with limit $\mu, \lambda_i > \theta$ with $\lambda = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa})$ and let $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ be an increasing sequence in $(\prod_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa})$ cofinal in it and obeying $\bar{a}$ (i.e. $i < \kappa \& \beta \in a_\alpha \Rightarrow f_\beta(i) < f_\alpha(i))$ and $\zeta < \lambda_i \wedge i < j < \kappa \wedge \beta \in b_{\alpha,\zeta} \Rightarrow f_\beta(j) < f_\alpha(j)$ (there is such $\langle \lambda_i : i < \kappa \rangle$ by [Sh:g, II]); there is such $\bar{f}$ by [Sh:g, I]. For notational help later without loss of generality each $f_\alpha(i)$ is divisible by θ.

Now for any $x = (h_1, h_2, A, i) = (h_1^x, h_2^x, A^x, i^x)$ where h_1 is a partial function from μ to θ, h_2 a function from A to μ where $T_\theta^- \subseteq A \subseteq T_\theta^+$ and $i < \kappa$, we define

$$S_x = \Big\{\alpha < \lambda : \text{we have } \mathrm{cf}(\alpha) \neq \theta,\ f_\alpha(i) \in \mathrm{Dom}(h_1) \text{ and } \zeta =: h_1(f_\alpha(i)) \in A \text{ satisfies}$$
$$(i)\quad b_{\alpha,h_2(\zeta)} \text{ is a subset of } \alpha \text{ of order type } \zeta,$$
$$(ii)\quad b_{\alpha,h_2(\zeta)} \text{ is unbounded in } \alpha \text{ if } \zeta \text{ is a limit ordinal}$$
$$(iii)\quad \text{if } \varepsilon \in A \cap \zeta \text{ and } \beta \in b_{\alpha,h_2(\zeta)} \text{ and } \varepsilon = \mathrm{otp}(b_{\alpha,h_2(\zeta)} \cap \beta)$$
$$\text{then } \varepsilon = h_1(f_\beta(i)) \& \beta \cap b_{\alpha,h_2(\zeta)} = b_{\beta,h_2(\varepsilon)}\Big\}$$

and for $\alpha \in S_x$ let $C_\alpha^x = b_{\alpha,h_2^x(h_1^x(f_\alpha(i^x)))}$.

For x as above let $S_x^* = \{\delta < \lambda : \mathrm{cf}(\delta) = \theta$ and there is an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \theta\rangle$ with limit δ such that $\varepsilon \in A \Rightarrow \alpha_\varepsilon \in S_x \& C_{\alpha_\varepsilon}^x = \{\alpha_\zeta : \zeta < \varepsilon\}\}$. Now for any such x, for $\delta \in S_x^*$ choose $\langle \alpha_\varepsilon : \varepsilon < \theta\rangle$ witnessing it and let $C_\delta^x = \bigcup\{C_{\alpha_\varepsilon}^x : \varepsilon \in A\}$; note that $S_x \cap S_x^* = \emptyset$ because $[\alpha \in S_x \Rightarrow \mathrm{cf}(\alpha) \neq \theta]$, $[\alpha \in S_x^* \Rightarrow \mathrm{cf}(\alpha) = \theta]$.

Let $\langle x_\varepsilon : \varepsilon < \varepsilon^* \leq \mu\rangle$ list the set of $x \in M_0$ which are as above and S_x^* is stationary, and for each $\varepsilon < \varepsilon^*$, let $S_\varepsilon =: S_{x_\varepsilon} \cup S_{x_\varepsilon}^*$, $\bar{C}_\varepsilon =: \langle C_\alpha^{x_\varepsilon} : \alpha \in S_\varepsilon\rangle$, $A_\varepsilon =: A^{x_\varepsilon}$.

We have to check the five clauses of 2.7(1):

Clause (α): See the demand on x above.

Clause (β): Holds by clause (iii) in the definition of "$\alpha \in S_x''$ and of C_α^x for $\alpha \in S_x$ and similarly for $\alpha \in S_x^*$.

Clause (γ): Holds by the choice of $\langle x_\varepsilon : \varepsilon < \varepsilon^*\rangle$.

Clause $(\delta) + (\varepsilon)$: Let $\delta \in B$. Let $\langle \alpha_\zeta : \zeta \leq \theta\rangle$ list the closure of a_δ in the increasing order. By the choice of $\bar{a}$, a_δ is a set of non-limit ordinals hence $\mathrm{cf}(\alpha_\zeta) = \theta \Leftrightarrow \zeta = \theta$ and $\alpha_\theta = \delta$.

Now for every $\zeta \in T_\theta^+$, for some $\xi(\zeta) < \mu$ we have $\{\alpha_\varepsilon : \varepsilon < \zeta\} = b_{\zeta,\xi(\zeta)}$ ($\xi(\zeta)$ exists as $\mu^{<\mathrm{cf}(\zeta)>_{\mathrm{tr}}} = \mu$ by the definition of T_θ) and let $j(\zeta) = \mathrm{Min}\{j < \kappa : f_{\alpha_\zeta}(i) > f_{\alpha_\varepsilon}(i)$ for every non limit $\varepsilon < \zeta$ in $[j, \kappa]\}$.

For each $j < \kappa$ let $w_j^* = \{\zeta : \zeta \in T_\theta^-$ or $\zeta \in T_\theta \& j(\zeta) = j\}$ and apply clause (e) of the assumptions to the sequence $\langle \alpha'_\zeta : \zeta < \theta\rangle$ where $\alpha'_\zeta = \mathrm{cd}(\zeta, j(\zeta), f_{\alpha_\zeta}(j_\zeta), \xi(\alpha_\zeta))$, cd a coding function on μ so (by (e)) we can find $\bar{A} \in \mathbf{P}$ satisfying clauses (i) and (ii) of assumption (e). So by (i) of (e) there are $\varepsilon(*) < \ell g(\bar{A})$ and $j(*) < \kappa$ such that $A_{\varepsilon(*)} \subseteq w_{j(*)}^*$.

For any pair (i, j) such that $i < \ell g(\bar{A})$, $j < \kappa$ and $A_i \subseteq w_j^*$ we shall define $x = x_{\varepsilon,j}$. By (ii) of (e) we have $\{\alpha'_\varepsilon : \varepsilon \in A_\varepsilon\} \in \mathcal{P}$.

We now define $x = (h_1^x, h_2^x, A^x, i^x)$ as follows: we let $i^x = j$, $A^x = A_\varepsilon$, $\mathrm{Dom}(h_1^x) = \{f_{\alpha_\zeta}(j) : \zeta \in A^x\}$, $h_1^x(f_{\alpha_\zeta}(j)) = \zeta$, $\mathrm{Dom}(h_2) = A^x$, $h_2(\zeta) = \xi(\zeta)$. Easily $x = x_{\varepsilon,j} \in M_0$ (as $\mathcal{P} \in M_0$, $|\mathcal{P}| \leq \mu$, $\mu + 1 \subseteq M_0$) and $\alpha_\theta \in S_x^*$, so we are done.

2) Let $\mathcal{P} = \{A : A$ is a bounded subset of μ of cardinality $\leq \theta\}$. Clearly $|\mathcal{P}| \leq \Sigma\{|\alpha|^\theta : \alpha < \mu\} \leq \Sigma\{|\alpha| : \alpha < \mu\} = \mu$, so clause (d) holds.

Now if $\sigma = \mathrm{cf}(\sigma) < \theta$ and $\sigma \neq \kappa$ then $\mu^{<\sigma>_{\mathrm{tr}}} = \mu$. [Why? For completeness if T is a tree with set of nodes $\subseteq \mu$ and σ levels, then very σ-branch of T includes a $<_T$-unbounded subset which is bounded as a subset of μ hence belongs to $\mathcal{P}$, and clearly the branch is determined by any $<_T$-unbounded subset hence the number of σ-branches of T is $\leq |\mathcal{P}| \leq \mu$.] So $T'_\theta = T_\theta$ and the non-obvious clause to check is (e). So assume $\alpha_\varepsilon < \mu$ for $\varepsilon < \theta$ and $\langle w^*_i : i < \kappa\rangle$ is a sequence of subsets of T^+_θ. Let $\mu = \Sigma\{\mu_i : i < \kappa\}, \mu_i < \mu$ and let $\bar{A} = \langle A_i : i < \kappa\rangle$ list the family $\{\{\alpha_\varepsilon : \varepsilon \in w^*_i \text{ and } \alpha_\varepsilon < \mu_j\} : i, j < \kappa\}$.

3), 4) Left to the reader. $\square_{2.7}$

Claim 2.9. 1) If λ is regular and not strongly inaccessible and $\mu < \lambda$ is a strong limit singular of uncountable cofinality, <u>then</u> for every large enough regular $\kappa < \mu$, for every $\Upsilon, \theta < \mu$, some $(\lambda, \kappa, \kappa^+)$-parameter $\bar{C}$ has the (Υ, θ)-Md-property (see Definition 1.19(4)); in fact, $\bar{C}$ is a sequence of clubs.

2) Moreover, we can find such $\bar{C}^\alpha$ for $\alpha < \lambda$ satisfying: $\langle \mathrm{Dom}(\bar{C}^\alpha) : \alpha < \lambda\rangle$ are pairwise disjoint.

3) Moreover for every large enough regular $\kappa < \mu$ for every $\chi, \sigma < \mu$ some $(\lambda, \kappa, \kappa^+)$-parameter has the (χ, σ)-Md_ℓ-parameter for $\ell = 0, 1, 2$.

Proof. 1) Let $\lambda_1 < \lambda$ be such that $\lambda \leq 2^{\lambda_1}$ and let $\kappa_0 < \mu$ be large enough such that

$(*)_1(i)$ if λ is not a successor of regular, <u>then</u> $\alpha < \lambda \Rightarrow \mathrm{cov}(|\alpha|, \mu, \mu, \kappa_0) < \lambda$

(ii) if λ is inaccessible, then $\mathrm{cov}(\lambda_1, \mu, \mu, \kappa_0) = \lambda_1$.

(Exist by [Sh 460].)

Note that λ_1 is needed only if λ is (weakly) inaccessible.

Let $\mu_* \in (\kappa_0, \mu)$ be strong limit; this is possible as μ is a strong limit of uncountable cofinality. Let $\kappa_1 = \kappa_1[\mu_*] < \mu_*$ be $> \kappa_0$ and such that $\alpha < 2^{\mu_*} \Rightarrow \mathrm{cov}(|\alpha|, \mu, \mu, \kappa_1) < 2^{\mu_*}$; (such κ_1 exists by [Sh 460]).

Let $\kappa \in \mathrm{Reg} \cap \mu_* \backslash \kappa_1$ and we shall prove the result for κ and every $\Upsilon, \theta < \mu_*$, so without loss of generality $\Upsilon = \theta$. By Fodor's Lemma on μ this is enough.

Let $\lambda_* = \mathrm{cf}(2^{\mu_*})$. By 1.24(1) for every stationary $S \subseteq \{\delta < \lambda_* : \mathrm{cf}(\delta) = \kappa\}$ from $I[\lambda_*]$ (and there are such, see 1.24(2)) there are $\bar{C}^S = \langle C^S_\delta : \delta \in S\rangle$, D such that $\bar{C}^S$ is a weakly good$^+$ $(\lambda_*, \kappa, \kappa^+)$-Md-parameter, D is a λ_*-complete filter on λ_* containing the clubs of λ_* and by 1.10 + 1.11 $S \in D^+$, (e.g., $D = \mathcal{D}_\lambda$), the pair $(\bar{C}^S, D)$ has the θ-Md-property.

For the case λ is (weakly) inaccessible we add "each C_δ is a club of δ" . This is permissible as we can use clause (β) of 1.24(1). Fix such $S_1 \subseteq S^{\lambda_*}_\kappa$ and $\bar{C} = \langle C_\delta : \delta \in S_1\rangle = \bar{C}^{S_1}$.

Note that we actually have

$\boxtimes(a)$ if $S' \subseteq S_1$ is stationary, $C'_\delta \subseteq C_\delta \subseteq \delta = \sup(C'_\delta)$ for $\delta \in S'$ then $\bar{C}' = \langle C'_\delta : \delta \in S'\rangle$ is a weakly good$^+$ $(\lambda_*, \kappa, \kappa^+)$-Md-parameter and $\bar{C}'$ has the (D, θ)-Md-property

(b) if $\lambda_* = \cup\{A_i : i < i(*) < \kappa\}$ then for some $i, j < i(*)$, $\bar{C}'$ is a good $(\lambda_*, \kappa, \kappa^+)$-Md-parameter and $\bar{C}'$ has the (D, θ)-Md-property where $S' =: \{\delta \in S_1 : \delta \in A_j \& \delta = \sup(\mathrm{nacc}(C_\delta) \cap A_i)\}$ and for $\delta \in S'$ we let $C'_\delta =: \mathrm{nacc}(C_\delta) \cap A_j$.

[Why? Let $S'_{i,j} = \{\delta \in S_1 : \delta \in A_j$ and $\delta = \sup(\mathrm{nacc}(C_\delta) \cap A_i)\}$. Clearly $\langle S_{i,j} : i, j < i(*)\rangle$ is a sequence of subsets of S_1 with union S_1. By Claim 1.23 for some $i, j < i(*)$ the sequence $\bar{C} \restriction S_{i,j}$ has the (D, θ)-Md-property. By monotonicity, i.e., Claim 1.14(1), the sequence $\langle \mathrm{nacc}(C_\delta) \cap A_i : \delta \in S_{i,j}\rangle$ has the (D, θ)-Md-property.]

Now

Case 1. λ is the successor of regulars.

We apply 2.1(1) with (λ, λ_*) here standing for (λ_2, λ_1) there; there is suitable partial square (in ZFC by [Sh 351, §4]). Note that we do not "lose" any $\kappa < \mu$ because of "the lifting to" λ.

Case 2. λ is the successor of a singular.

The proof is similar only now we use 2.7(3) to get the appropriate partial square using $\boxtimes$ to derive the $\bar{C}'$ used as input for it (recalling that each sequence $\bar{A} \in \mathbf{P}$ is closed under the union of two!)

Case 3. λ is (weakly) inaccessible.

Really the proof in this case covers the other cases (but λ_1 in this case contributes to increasing κ_0). This time we use 2.6 + 2.10 which is proved below.

Let S_2 be a stationary subset of $S^\lambda_{\lambda_*}$ from $I[\lambda]$, it exists, see 1.24(2). Let $\langle F_\varepsilon : \varepsilon < \lambda_1\rangle$ be as in 2.6 with $(\lambda, \lambda_1, \lambda_*, \kappa)$ here standing for $(\lambda, \mu, \theta, \kappa)$ there. Let g^* be a function from λ_1 onto $\mathcal{H}_{\leq\kappa}(\lambda_*)$. Possibly replacing S_2 by its intersection with some club of λ, there is a good λ_*-Md-parameter $\bar{C}^2 = \langle C^2_\delta : \delta \in S_2\rangle$. Let $\bar{e} = \langle e_\alpha : \alpha \in S^\lambda_\kappa\rangle$, e_α a club of α of order type κ. Clearly

$(*)$ for some $\varepsilon < \lambda_1$ for any club E of λ for some increasing continuous sequence $\langle \alpha_i : i \leq \lambda_*\rangle$ of member of E the set $B_\varepsilon(\langle \alpha_i : i \leq \lambda_*\rangle) = \{i \in S_1 : g^*(F_\varepsilon(\alpha_i)) = \{(i, \zeta, \mathrm{otp}(\alpha_\zeta \cap C^2_{\alpha_{\lambda_*}}), \mathrm{otp}(\alpha_\zeta \cap e_{\alpha_i})) : \zeta < i$ and $\alpha_\zeta \in e_{\alpha_i}\}\}$ is a stationary subset of λ_*

[Why? For every club E we can choose $\alpha_i \in E$ $(i \leq \lambda_*)$ increasing continuous so for some ε the set $B_\varepsilon(\langle \alpha_i : i \leq \lambda_*\rangle)$ is a stationary subset of λ_*. As the number of possible ε is $\lambda_1 < \lambda = \mathrm{cf}(\lambda)$, there is ε as above.]

For this ε let

(i) $S^*_3 = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa$ and $g^*(F_\varepsilon(\alpha))$ has the right form$\}$
(ii) $\delta \in S^*_3$ let $C^3_\delta = \{\beta \in e_\delta : (i, \zeta, j, \mathrm{otp}(\beta \cap e_\delta)) \in g_*(F_\varepsilon(\delta))\}$
(iii) let $F' : \lambda \to \lambda_*$ be defined by $F'(\alpha) = \zeta$ if:
$\zeta = \zeta_1$ for some $(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in g_*(F_\varepsilon(\alpha))$.

We apply 2.10.

2) , 3) Similarly using Claim 2.3 when using Claim 2.4. $\square_{2.9}$

We now state another version of the upgrading of middle diamonds.

Claim 2.10. 1) Assume

(a) $\lambda_1 < \lambda_2$ are regular cardinals (*b*) $S_1 \subseteq S_\kappa^{\lambda_1}$ is stationary, $\bar{C}^1 = \langle C^1_\delta : \delta \in S_1 \rangle$ is a $(\lambda_1, \kappa, \chi)$-Md-parameter which has the $(D_{\lambda_1}, \Upsilon, \theta)$-Md-property, D_{λ_1} is a filter on λ_1 to which S_1 belongs

(*c*) $S_2 \subseteq S_\kappa^{\lambda_2}$ is stationary and $\bar{C}^2 = \langle C^2_\delta : \delta \in S_2 \rangle$ is a $(\lambda_2, \kappa, \chi)$-Md-parameter

(*d*) $F : \lambda_2 \to \lambda_1$ satisfies: for every club E of λ_2 there is an increasing continuous function h from λ_1 into λ_2 with $\sup(\mathrm{Rang}(h)) \in \mathrm{acc}(E)$ such that $\{\delta \in S_1 : h(\delta) \in S_2$ and $C^2_{h(\delta)} \subseteq \{h(\alpha) : \alpha \in C^1_\delta$ and $F(h(\alpha)) = \alpha\}$ and $F(h(\delta)) = \delta\}$ belongs to $D^+_{\lambda_1}$

(*e*) $2^{2^{\lambda_1}} < \lambda_2$.

Then $\bar{C}^2$ has the (Υ, θ)-Md-property.

2) Assume that in part (1) we omit assumption (e) and replace the Md-property by MD_ℓ-property where $\ell \in \{0, 2\}$.
Then $\bar{C}^2$ has the (Υ, θ)-MD_ℓ-property.

3) In part (1) if we omit assumption (e) and strengthen clause (d) by demanding $\{\delta \in S_1 : h(\delta) \in S_2$ and $C^2_{h(\delta)} = \{h(\alpha) : \alpha \in C^1_\delta$ and $F(h(\alpha)) = \alpha\}$ and $F(h(\delta)) = \delta\}$ belongs to $D^+_{\lambda_1}$, then we can still conclude that: for every nice $(\bar{C}^2, \Upsilon, \theta)$-colouring $\mathbf{F}$ (see Definition 2.2) there is an $\mathbf{F}$-middle diamond sequence.

Remark. 1) Is clause (d) of the assumption reasonable? Later in [Sh:F611, 4.5] we prove that it holds under reasonable conditions.
2) This + 2.12 are formalized elsewhere.

Proof of 2.10. Similar to 2.1.

First, assume $\ell = 2$ (so we are in part (2)) and let τ be a relational vocabulary of cardinality $< \Upsilon$ and by clause (b) of the assumption there is a sequence $\bar{M}^1 = \langle M^1_\delta : \delta \in S_1 \rangle$, M^1_δ a τ-model with universe C^1_δ such that

$\circledast_1$ if M is a τ-model with universe λ_1, then the set of $\delta \in S_1$ for which $M^1_\delta = M \restriction C^1_\delta$ is $\neq \emptyset \bmod D_{\lambda_1}$.

Now for every $\delta \in S_2$ if $F(\delta) \in S_1$ and $\{F(\alpha) : \alpha \in C^2_\delta\} \subseteq C^1_{F(\delta)}$ is an unbounded subset of $F(\delta)$, then let M^2_δ be the unique τ-model with universe C^2_δ such that $F \restriction C^2_\delta$ is an isomorphism from M^2_δ onto $M^1_{F(\delta)} \restriction \{F(\alpha) : \alpha \in C^2_\delta\}$. If $\delta \in S_2$ does not satisfy the requirement above let M^2_δ be any τ-model with universe C^2_δ.

Now it is easy to check that $\bar{M}^2 = \langle M^2_\delta : \delta \in S_2 \rangle$ is as required.

Second, assume $\ell = 1$ (and we are in part (1)) and let $\mathbf{F}^2$ be a $(\bar{C}^2, \Upsilon, \theta)$ colouring. For each $(\bar{C}^1, \Upsilon, \theta)$-colouring $\mathbf{F}^1$ we ask:

Question 2.11. Do we have, for every club E of λ_2 an increasing continuous function $h : \lambda_1 \to \lambda_2$ such that

(*i*) $\sup(\mathrm{Rang}(h)) \in \mathrm{acc}(E)$

(*ii*) the following set is a stationary subset of λ_1
$\{\delta \in S_1 : h(\delta) \in S_2$ and $C^2_{h(\delta)} \subseteq \{h(\alpha) : \alpha \in C^1_\delta\}$ and $F(h(\delta)) = \delta$ and $(\forall \eta_1 \in {}^{(C^1_\delta)}\Upsilon)(\forall \eta_2 \in {}^{(C^2_{h(\delta)})}\Upsilon)[(\forall \alpha \in C^1_\delta)(\eta_1(\alpha) = \eta_2(h(\alpha))) \Rightarrow \mathbf{F}^1(\eta_1) = \mathbf{F}^2(\eta_2)\}$.

If for some $\mathbf{F}^1$ the answer is yes, we proceed as in the proof for $\ell = 2$. Otherwise, for each such $\mathbf{F}^1$ there is a club $E[\mathbf{F}^1]$ exemplifying the negative answer. Let $E = \bigcap\{E[\mathbf{F}^1] : \mathbf{F}^1$ is a $(\bar{C}^1, \Upsilon, \theta)$-colouring$\}$. As there are $\leq 2^{2^{\lambda_1}}$ such functions $\mathbf{F}^1$ clearly E is a club of λ_2 and using (d) of the assumption, there is $h : \lambda_1 \to \lambda_2$ as there. Now we can define $\mathbf{F}^1$ by h, $\mathbf{F}^2$ naturally and get a contradiction.

The case $\ell = 0$ and part (3) is left to the reader. $\square_{2.10}$

Trying to apply 2.10, clause (d) of the assumption looks the most serious. By the following if each C^1_δ is a stationary subset of δ (so $\kappa > \aleph_0$), it helps.

Claim 2.12. In 2.10, if $2^{\lambda_1} < \lambda_2$, and we replace (d) by $(d)^- + (f)$ below, then we can conclude that for some $\bar{C}^3 = \langle C^3_\delta : \delta \in S\rangle$ has the θ-MD$_\ell$-property where $C^3_\delta \in D_\delta$

(f) $\bar{D} = \langle D'_\delta : \delta \in S_2\rangle$, D'_δ is a filter on C^2_δ containing the co-bounded subsets of C^2_δ (or just $D_\delta \subseteq \mathcal{P}(C^2_\delta)\backslash\{A \subseteq C^2_\delta : \sup(A) < \delta\}$

$(d)^-$ $F : \lambda_2 \to \lambda_1$ satisfies: for every club E of λ for some increasing continuous function h from λ_1 to λ_2 with $\sup(\mathrm{Rang}(h)) \in \mathrm{acc}(E)$ the set $\{\delta \in S_1 : h(\delta) \in S_2$ and $\{h(\alpha) : \alpha \in C^1_\alpha\} \cap C^2_\delta \in D'_\delta\}$ belong to D_{λ_1}.

Proof. Let $\mathcal{C} = \{\bar{e} : \bar{e} = \langle e_\delta : \delta \in S_1\rangle, e_\delta$ is a subset of $C^2_\delta\}$.

So $|\mathcal{C}| \leq 2^{\lambda_1} < \lambda$ and for each $\bar{e} \in E$ we define $\bar{C}^{\bar{e}} = \langle C^{\bar{e}}_\delta : \delta \in S^{\bar{e}}_2\rangle$ by $S^{\bar{e}}_\delta = \{\delta \in S_2 : C^{\bar{e}}_\delta \in D_\delta\}$ where $C^{\bar{e}}_\delta = \{\alpha \in C^2_\delta : \mathrm{otp}(C^2_\delta \cap \alpha) \in e_{F(\delta)}\}$. If for some $\bar{e} \in \mathcal{C}$, $\bar{C}^{\bar{e}}$ is as required. $\square_{2.12}$

Glossary

§0 Introduction

Theorem 0.1: Many cases of MD-diamond.

Theorem 0.2: With division to λ.

Conclusion 0.3: Connection to the principle from [Sh 667].

Definition 0.4: Having (partial) squares.

§2 Middle Diamond: sufficient conditions

Definition 1.1: Definition of S being $\mathbf{F} - \theta$-small and the normal filter.

Claim 1.2: The filter is normal and Υ does not matter.

Definition 1.3: Recalling pcf definitions.

Remark 1.4:

Definition 1.5: The main notions are defined:

$\bar{C}$ is λ-Md-parameter, (λ, κ, χ)-Md-parameter,
$\mathbf{F}$ is a $(\bar{C}, \mu, \theta)$-coloring,
$\bar{c}$ is a $(D, \mathbf{F})$-middle diamond sequence,
$\bar{C}$ is good and $\bar{C}$ has the (D, μ, θ)-Md-property or the
(D, μ, θ)-Md-property, also (good$^+$-Md-property and Md$_\ell$.

Definition 1.6:

Claim 1.8: Easy implications.

Major Claim 1.10: Sufficient conditions for $\bar{C}$ having the (D, θ)-Md-properties (D a filter on λ, $\lambda = \mathrm{cf}(2^\mu)$).

Claim 1.11: E.g., $\mu = \mu^\theta$ implies $\mathrm{Sep}(\mu, \theta)$, one of the assumptions of 1.10.

References

[AShS 221] Abraham, U., Shelah, S., Solovay, R.M.: Squares with diamonds and Souslin trees with special squares. Fundamenta Mathematicae **127**, 133–162 (1987)

[DvSh 65] Devlin, K.J., Shelah, S.: A weak version of ◇ which follows from $2^{\aleph_0} < 2^{\aleph_1}$. Israel J. Math. **29**, 239–247 (1978)

[DjSh 562] Džamonja, M., Shelah, S.: On squares, outside guessing of clubs and $I_{<f}[\lambda]$. Fundamenta Mathematicae **148**, 165–198 (1995) math.LO/9510216

[EK] Engelking, R., Karłowicz, M.: Some theorems of set theory and their topological consequences. Fundamenta Math. **57**, 275–285 (1965)

[Gre] Gregory, J.: Higher Souslin trees and the generalized continuum hypothesis. J. Symbolic Logic **41** (3), 663–671 (1976)

[J] Jech, T.: Set theory. Academic Press, New York, 1978

[MPSh 713] Matet, P., Péan, C., Shelah, S.: Cofinality of normal ideals on $P_\kappa(\lambda)$, I. submitted, Archieve for Mathematical Logic

[Sh 638] Shelah, S.: More on Weak Diamond. The East-West Journal of Mathematics. Accepted math.LO/9807180

[Sh 108] Shelah, S.: On successors of singular cardinals. In: Logic Colloquium '78 (Mons, 1978), volume 97 of Stud. Logic Foundations Math, North-Holland, Amsterdam-New York, 1979, pp. 357–380

[Sh 208] Shelah, S.: More on the weak diamond. Annals of Pure and Applied Logic, **28**, 315–318 (1985)

[Sh 237e] Shelah, S.: Remarks on squares. In: Around classification theory of models, volume 1182 of Lecture Notes in Mathematics, Springer, Berlin, 1986, pp. 276–279

[Sh 351] Shelah, S.: Reflecting stationary sets and successors of singular cardinals. Arch. Math. Logic **31**, 25–53 (1991)

[Sh 420] Shelah, S.: Advances in Cardinal Arithmetic. In: Finite and Infinite Combinatorics in Sets and Logic, N.W. Sauer et al. (eds.), Kluwer Academic Publishers, 1993, pp. 355–383

[Sh 410] Shelah, S.: More on Cardinal Arithmetic. Arch. Math. Logic **32**, 399–428 (1993)

[Sh:g] Shelah, S.: Cardinal Arithmetic. Volume 29 of Oxford Logic Guides. Oxford University Press, 1994

[Sh 430] Shelah, S.: Further cardinal arithmetic. Israel J. Math. **95**, 61–114 (1996) math.LO/9610226

[Sh 462] Shelah, S.: σ-entangled linear orders and narrowness of products of Boolean algebras. Fundamenta Mathematicae **153**, 199–275 (1997) math.LO/9609216

[Sh 506] Shelah, S.: The pcf-theorem revisited. In: The Mathematics of Paul Erdős, II, Graham, Nešetřil, (eds.), volume 14 of Algorithms and Combinatorics, Springer, 1997, pp. 420–459 math.LO/9502233

[Sh:f] Shelah, S.: Proper and improper forcing. Perspectives in Mathematical Logic, Springer, 1998

[Sh 589] Shelah, S.: Applications of PCF theory. J. Symbolic Logic **65**, 1624–1674 (2000)

[Sh 460] Shelah, S.: The Generalized Continuum Hypothesis revisited. Israel J. Math. **116**, 285–321 (2000) math.LO/9809200

[Sh 576] Shelah, S.: Categoricity of an abstract elementary class in two successive cardinals. Israel J. Math. **126**, 29–128 (2001) math.LO/9805146

[Sh 755] Shelah, S.: Weak Diamond. Mathematica Japonica, **55**, 531–538 (2002) math.LO/0107207

[Sh 667] Shelah, S.: Successor of singulars: combinatorics and not collapsing cardinals $\leq \kappa$ in $(< \kappa)$-support iterations. Israel J. Math. **134**, 127–155 (2003) math.LO/9808140

[Sh:F611] Shelah, S.: Around Very Weak Diamonds

[Sh:829] Shelah, S.: More on the revised GCH and the middle diamond, Proc. of Conference in Jerusalem 2004, In honour of Azriel Levy, Annals of Pure and Applied Logic, submitted