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# THE COFINALITY OF CARDINAL INVARIANTS RELATED TO MEASURE AND CATEGORY

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**Abstract.** We prove that the following are consistent with ZFC:
1. $2^\omega = \aleph_{\omega_1} + K_C = \aleph_{\omega_1} + K_B = K_U = \omega_2$ (for measure and category simultaneously).
2. $2^\omega = \aleph_{\omega_1} = K_C(\mathscr{L}) + K_C(\mathscr{M}) = \omega_2$.
This concludes the discussion about the cofinality of $K_C$.

**§0. Introduction.** In this work we will study the cofinalities of the cardinals associated with the ideal of the measure zero sets of reals and with the ideal of the meager sets. These type of problems are very interesting because they give us the possibility of thinking about the continuum when $2^{\aleph_0}$ is a singular cardinal.

In general these cardinals are defined as follows:

Let $\mathfrak{I}$ be a $\sigma$-ideal of Borel sets of $\mathbb{R}$. Then we define

$$K_A(\mathfrak{I}) = \text{the least } \kappa \text{ such that } (\exists C \in [\mathfrak{I}]^\kappa)(\bigcup C \notin \mathfrak{I}),$$
$$K_B(\mathfrak{I}) = \text{the least } \kappa \text{ such that } (\exists \in [\mathfrak{I}]^\kappa)(\bigcup C = \mathbb{R}),$$
$$K_U(\mathfrak{I}) = \text{the least } \kappa \text{ such that } [\mathbb{R}]^\kappa \setminus \mathfrak{I} \neq \varnothing,$$
$$K_C(\mathfrak{I}) = \text{the least } \kappa \text{ such that } (\exists \mathfrak{F} \in [\mathfrak{I}]^\kappa)(\forall A \in \mathfrak{I})(\exists B \in \mathfrak{F})(A \subseteq B).$$

Usually we drop the letter $\mathfrak{I}$ if it does not lead to any confusion.

The following theorem is part of the folklore.

**0.0.** THEOREM. (a) $K_A \leq K_B \cap K_U \leq K_B \cup K_U \leq K_C$.

(b) $K_A$ *is regular.*

(c) $\operatorname{cof}(K_U) \cap \operatorname{cof}(K_C) > \omega$. ■

**0.1.** THEOREM (D. FREMLIN [F]). $\operatorname{cof}(K_C) \geq K_A$.

PROOF. Suppose $\mathfrak{F} = \bigcup_{\alpha<\kappa} \mathfrak{F}_\alpha$, where $\mathfrak{F}_\alpha \subset \mathfrak{I}, \kappa < \kappa_A$, and $|\mathfrak{F}_\alpha| < \kappa$. Since $\mathfrak{F}_\alpha$ fails to cover $\mathfrak{I}$, there is $A_\alpha \in \mathfrak{I}$ not contained in any member of $\mathfrak{F}_\alpha$. Then $\bigcup_{\alpha<\kappa} A_\alpha \in \mathfrak{I}$ and is not contained in any member of $\mathfrak{F}$. ■

**0.2.** THEOREM [F]. $\operatorname{cof}(K_U) \geq K_A$.

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PROOF. Let $A \in [\mathbb{R}]^{K_U} \setminus \Im$; then

$$A = \bigcup_{\alpha < \operatorname{cof}(K_U)} A_\alpha$$

satisfying for every $\alpha < \operatorname{cof}(K_U)$, $|A_\alpha| < K_U$. Therefore $A_\alpha \in \Im$, and if $\operatorname{cof}(K_U) < K_A$ then $A \in \Im$, a contradiction. ■

Let $\mathscr{L}$ and $\mathscr{M}$ denote ideals of measure zero sets and of meager sets respectively.

The relations between the corresponding eight cardinals are given succinctly by Cichoń's diagram (see [F]):

$$\begin{array}{ccccccc} K_B(\mathscr{L}) & \to & K_U(\mathscr{M}) & \to & K_C(\mathscr{M}) & \to & K_C(\mathscr{L}) \\ & & \uparrow & & \uparrow & & \\ \uparrow & & \underline{b} & & \underline{d} & & \uparrow \\ & & \uparrow & & \uparrow & & \\ K_A(\mathscr{L}) & \to & K_A(\mathscr{M}) & \to & K_B(\mathscr{M}) & \to & K_U(\mathscr{L}) \end{array}$$

where an arrow ($\to$) between two cardinals indicates "$\leq$" can be proven in ZFC.

In view of 0.0, the following question about the cofinalities of these cardinals appears: Are any of the following consistent?

1. $\operatorname{cof}(K_B) < K_A$,
2. $\operatorname{cof}(K_C) < \operatorname{cof}(K_B)$,
3. $\operatorname{cof}(K_C) < \operatorname{cof}(K_U)$,
4. $\operatorname{cof}(K_C(\mathscr{L})) < \operatorname{cof}(K_C(\mathscr{M}))$,

where $\Im$ is equal to the ideal of measure zero sets (meager sets). In §4 we will give a model for 2 and 3 (for both cases, measure and category) and in §5 we will give a model for 4. Question 1 remains open for the moment. In §1 we will give a result involving measure zero sets. We will explicitly construct a measure zero set associated with a function from $\omega$ to $\omega$, such that if the union of such sets has measure zero then the sets of their respective functions is $\sigma$-bounded.

In §2 we show the parallel result of §1 for category. This result was previously proved by T. Bartoszynski, but we present a new proof.

The results from §§1 and 2 also appear in [F], with a different proof.

In §3 we will give some results involving forcing and finite support iterated forcing. We will introduce the concept of "strong scale", and show that this property is preserved by

(1) $\omega^\omega$-bounding forcing notions (i.e. random reals forcing, Sacks real forcing, etc.),

(2) Cohen real forcing, and

(3) Finite support iteration of forcing notions preserving this property.

In §4 we first add $\aleph_{\omega_1}$ Cohen reals to $L$ and then iterate $\omega_2$ times alternatively Cohen and random reals. We use our previous work in order to show that in this model the following holds, simultaneously for measure and category:

$$K_B = K_U = \aleph_2 \quad \text{and} \quad K_C = \aleph_{\omega_1}.$$

In §5 we add $\aleph_{\omega_1}$ random reals followed by an $\omega_2$-iteration of dominating forcing to get a model where $K_C(\mathscr{M}) = \omega_2$ and $K_C(\mathscr{L}) = \aleph_{\omega_1}$.

The notation used in this article is standard, and a good reference may be [Mi]. The real numbers will be ${}^{\omega}2$ $(=2^{\omega})$, the set of all $\omega$-sequences of zeros and ones. A perfect tree $T \subseteq {}^{\omega>}2$ is a tree satisfying the following condition:

$$\nu \in T \to (\exists \eta \supseteq \nu)(\eta^\frown\langle 0\rangle \in T \wedge \eta^\frown\langle 1\rangle \in T).$$

If $T$ is a tree, then $\bar{T} \subseteq {}^{\omega}2$ is defined by

$$\bar{T} = \{x \in {}^{\omega}2 \colon (\forall n \in \omega)(x \restriction n \in T)\}.$$

If $\eta \in T$, we define $T^{[\eta]}$ by

$$T^{[\eta]} = \{\nu \in T \colon \eta \subseteq \nu \vee \nu \subseteq \eta\}.$$

If $f, g$ are functions from $\omega$ to $\omega$ then $f \leq^* g$ iff $(\exists n \forall m \geq n)(f(m) \leq g(m))$. If $\eta \in {}^{\omega>}2$ then we define $[\eta] \leq {}^{\omega}2 \cup {}^{\omega>}2$:

$$[\eta] = \{x \in {}^{\omega\geq}2 \colon \eta \subseteq x\}.$$

## §1. Technology for measure.

**1.0.** Definition. Let $T \subseteq 2^{<\omega}$ be a perfect tree. We define the following set $A_n(T) \subseteq \omega$ and the following function $f_T$:

$$A_n(T) = \{k \colon \forall \eta \in ({}^{k-n}2 \cap T)(\exists \nu \in {}^{n}2)(\eta^\frown \nu \notin T)\},$$

(a)
$$f_T(n) = \begin{cases} \sup A_n & \text{if } 0 \neq |A_n(T)| < \aleph_0, \\ 0 & \text{if } 0 = |A_n(T)|, \\ \infty & \text{if } |A_n(T)| = \aleph_0. \end{cases}$$

**1.1.** *Fact.* (a) *If $T_1 \subseteq T_2$ then $f_{T_1}(n) \geq f_{T_2}(n)$.*

(b) *If $\mu(\bar{T}) > 0$ then, for every $n \in \omega$, $f_T(n) < \infty$.*

Proof. (a) Easy.

(b) Assume that $A_n(T)$ is infinite; then it contains an infinite sequence $\langle k_i \colon i < \omega\rangle$ satisfying $\forall^i k_{i+1} \geq k_i + n$. By induction on $i$ we show $\mu(T \cap {}^{k_i}2) \leq (1 - 1/2^n)^i$. This is clear for $i = 0$. For $i \geq 0$ and any $s \in {}^{k_{i+1}-n}2 \cap T$ we have

$$\operatorname{card}(\{t \in {}^{k_{i+1}}2 \colon t \supseteq s, t \in T\}) \leq 2^n - 1;$$

hence

$$\begin{aligned} \mu(T \cap {}^{k_{i+1}}2) &\leq \mu(T \cap {}^{k_{i+1}-n}2) \cdot (1 - 1/2^n) \\ &\leq \mu(T \cap {}^{k_i}2) \cdot (1 - 1/2^n). \end{aligned}$$

The result follows. ■

**1.2.** *Fact. If $g \in \omega^\omega$ and $g(0) = g(1) = g(2) = g(3) = 0$ and, for $n \geq 3$, $g(n+1) > n + 1 + g(n)$, then there exists a perfect tree $T = T_g$ satisfying*

(i) $g = f_{T_g}$,

(ii) $\mu(T_g) > 0$, *and*

(iii) *for every $\eta \in T_g$, if $T = T_g^{[\eta]}$ then $(\forall^\infty n)(f_T(n) = g(n))$.*

Proof. Let $T_g$ be defined by

$$T_g = \{\nu \in {}^{\omega>}2 \colon \text{if } 3 < n < \omega \text{ and } \lg(\nu) \geq g(n) \text{ then } (\exists i)(g(n) - n \leq i \leq g(n) - 1 \wedge \nu(i) = 1)\}.$$

Clearly $T_g$ is a perfect tree. Now we need to show that this tree satisfies (i), (ii) and (iii).

*Checking* (i). Fix $n \in \omega$; then clearly $g(n) \in A_n(T_g)$. It is sufficient to show that if $m > g(n)$ then $m \notin A_n(T_g)$; but this is obvious (it is necessary to consider by cases, $g(n+k) \leq m < g(n+k+1)$).

*Checking* (ii). $x \notin \bar{T}_g$ iff there exists $n \in \omega - 4$ such that $x \restriction [g(n)-n, g(n)) = 0 \restriction [g(n)-n, g(n))$. Hence $\mu(\bar{T}_g) \geq 1 - \sum_{n=4}^{\infty} 1/2^n > 0$.

*Checking* (iii). Left to the reader. ■

**1.3.** DEFINITION. Let $g$, $T_g$ satisfy the requirements of 1.2. For every $v \in {}^{w<}2$ we define

$$T_{g,v} = \begin{cases} \{\eta \colon v^\frown\eta \in T_g\} & \text{if } v \in T_g, \\ T_g & \text{if } v \notin T_g. \end{cases} \quad ■$$

**1.4.** *Fact.* $\mu(\bigcup_v \bar{T}_{g,v}) = 1$. ■

**1.5.** DEFINITION. $A_g = {}^{\omega}2 - \bigcup_v \bar{T}_{g,v}$. ■

**1.6.** *Fact. For every $v$ there exists $m(v)$ such that $(\forall^\infty n)(f_{T_{g,v}}(n) \doteq g(n) - m(v))$.*

PROOF. If $v \notin T_g$ then $m(v) = 0$. If $v \in T_g$ then use 1.2(iii). ■

**1.7.** LEMMA. *If $\langle g_\alpha \colon \alpha \in I \rangle$ is a sequence of functions such that for every $\alpha \in I$, $g_\alpha$ satisfies the conditions of* 1.2, *and $\mu(\bigcup_{\alpha \in I} A_{g_\alpha}) = 0$, then there exists $g \colon \omega \to \omega$ such that for every $\alpha \in I$, there exists $n \in \omega$ satisfying $g_\alpha(m) < g(m)$ for every $m \geq n$.*

PROOF. By hypothesis there exists a measure zero set $A$ such that $A_{g_\alpha} \subseteq A$ for every $\alpha \in I$. Therefore there exists perfect trees $T_n \subseteq {}^{\omega<}2$ such that $\mu(\bar{T}_n) = 1 - 1/n$ and $A = {}^{\omega}2 - \bigcup_n \bar{T}_n$. From this we obtain

$${}^{\omega}2 - \bigcup_v \bar{T}_{g_\alpha, v} \subseteq {}^{\omega}2 - \bigcup_n \bar{T}_n;$$

hence $\bigcup T_n \subseteq \bigcup_v \bar{T}_{g_\alpha, v}$ and thus $T_3 \subseteq \bigcup_v \bar{T}_{g_\alpha, v}$, which implies that there exists $v = v(\alpha) \in {}^{\omega<}2$ such that $T_3^{[\eta]} \subseteq T_{g_\alpha, v}$ (if not we can build a branch of $T_3$ which is not in $\bigcup_v \bar{T}_{g_\alpha, v}$).

Therefore $T_3^{[\eta]} \subseteq T_{g_\alpha, v}^{[\eta]}$. By 1.1(a) and 1.6 we have

$$f^*_{T_3^{[\eta]}} \geq g_\alpha - m(v).$$

Now let $f$ dominate $\{f_{T_3^{[\eta]}} + m(v) \colon \eta \in T_3 \text{ and } v \in {}^{\omega<}2\}$. Then $f^* \geq g_\alpha$ for every $\alpha \in I$. (This simplified version of our original proof was suggested by the referee.) ■

**1.8.** REMARK. The reader may obtain easily, using 1.7, the following result of A. Miller: "Additivity of measure implies dominating reals". ■

## §2. Technology for categoricity.

**2.0.** DEFINITION. Let $f \colon \omega \to \omega$ be increasing. We define $A_f \subseteq {}^{\omega}2$ by

$$A_f = \{x \in {}^{\omega}2 \colon (\exists m \forall n \geq m)(|\{i \in f(n) \colon x(i) = 1\}| \geq n)\}.$$

**2.1.** *Claim. $A_f$ is a meager set.* ■

**2.2.** LEMMA. *Let $\langle A_f \colon f \in F \rangle$ be given. Then $\bigcup_{f \in F} A_f$ is a meager set iff there exists $g \colon \omega \to \omega$ such that for every $f \in F$, $f \leq^* g$.*

PROOF. $(\to)$ Suppose that $\bigcup_{f \in F} A_f$ is a meager set. Then there exists $\langle T_i \colon i < \omega \rangle$, a sequence of nowhere dense trees, such that

(i) $T_i \subseteq {}^{\omega<}2$, and

(ii) $\bigcup_{f \in F} A_f \subseteq \bigcup_{n \in \omega} \bar{T}_n$.

Now we define the following function $g' \colon \omega \to \omega$.

$$g'(k) = \inf\left\{ l \colon \text{for every } \eta \in {}^{k}2 \text{ there exists } \nu \in {}^{l}2 \cap [\eta] \text{ such that } [\nu] \cap \bigcup_{j \le k} T_j = \varnothing \right\}.$$

Let $g$ be such that $g(n) = g'(n) + n + 2$.

**2.3.** *Claim. For every $f \in F$, $f \le^* g$.*

*Proof.* If not, fix $f \in F$ and $A \subseteq \omega$ satisfying the following conditions:

(i) $|A| = \aleph_0$.

(ii) If $n \in A$ then $g(n) \le f(n)$.

(iii) If $n \in A$ then $A - (n + 1) \subseteq \omega - (f(n) + 1)$.

Let $A = \{n_k \colon k < \omega\}$, $n_k$ increasing, then we define by induction $\langle \eta_k \colon k < \omega \rangle$ satisfying the following conditions:

(i) $\eta_k \in {}^{f(n_k)}2$.

(ii) $\eta_k \notin \bigcup_{i \le n_k} T_i$.

(iii) For every $f(i) \le n_k$, $|\{j \in f(i) \colon \eta_k(j) = 1\}| \ge i$.

(iv) $\eta_k \subseteq \eta_{k+1}$.

*The induction.* Suppose we have $\eta_k$ satisfying (i), (ii), (iii), and (iv). We define $\eta'_k \in {}^{n_{k+1}}2 \cap [\eta_k]$ and, for every $i \in n_{k+1} - n_k$, $\eta'_k(i) = 1$. Then $\eta'_k$ satisfies (iii) for $f(i) < n_{k+1}$. We define $\eta''_k \in {}^{g'(n_{k+1})}2 \cap [\eta'_k]$ such that

$$\eta''_k \notin \bigcup_{i \le n_{k+1}} T_i.$$

Now we define $\eta_{k+1} \in {}^{f(k_{n+1})}2 \cap [\eta''_k]$ satisfying $\eta_{k+1}(i) = 1$ for every $i \in f(n_{k+1}) - g'(n_{k+1})$. Then clearly $\eta_{k+1}$ satisfies (i), (ii), and (iv); but $g'(n_{k+1}) + n_{k+1} + 2 \le f(n_{k+1})$, and so $\eta_{k+1}$ satisfies also (iii). Let $x = \bigcup_k \eta_k$; then $x \in A_f - \bigcup_{i<\omega} \bar{T}_i$, a contradiction. ■

The other direction is easy. This completes the proof of the lemma. ■

REMARK. This lemma was proved independently by T. Bartoszynski. ■

## §3. Technology for forcing.

**3.0.** DEFINITION. $F = \langle g_i \colon i \in I \rangle$ is a *strong scale* iff $g_i \in {}^{\omega}\omega$ for every $i \in I$, and for every $f \in {}^{\omega}\omega$, $|\{i \in I \colon g_i \le^* f\}| \le \aleph_0$ and $|I| > \aleph_0$. ■

**3.1.** *Fact. If $F \in V$ is a strong scale and $r$ is a random real over $V$, then $V[r] \vDash$ "$F$ is a strong scale".*

*Proof.* Use the ${}^{\omega}\omega$-bounding of random real forcing. ■

**3.2.** *Fact. If $F \in V$ is a strong scale and $r$ is a Cohen real over $V$, then $V[r] \vDash$ "$F$ is a strong scale".*

*Proof. Let* **f** be a Cohen-name for a function from $\omega$ to $\omega$. For every $q \in$ Cohen, we define

$$f_q(n) = \mathrm{Min}\{k \colon q \nVdash \text{"}\mathbf{f}(n) \neq k\text{"}\}.$$

Now the result follows from the following

**3.3.** *Claim. For every $i \in I$,*

$$(\forall q \in \text{Cohen})(g_i \not\leq^* f_q) \Rightarrow (\Vdash \text{“} g_i \not\leq^* \mathbf{f}\text{”}). \quad \blacksquare$$

**3.4.** THEOREM. *Suppose $\langle P_\alpha; \mathbf{Q}_\alpha: \alpha < \delta \rangle$ is a finite support iterated forcing notion and $F$ is a strong scale in $V$, and*

(a) *for every $\alpha < \delta$, $P_\alpha \models$ “c.c.c.”, while*

(b) *for every $\alpha < \delta$, $\Vdash_{P_\alpha}$ “$F$ is a strong scale”.*

*Then $\Vdash_{P_\delta}$ is a strong scale”.*

PROOF. Let $\mathbf{f}$ be a $P_\delta$-name for a function from $\omega$ to $\omega$; without loss of generality $\delta = \omega$. For every $i < \omega$ we define a $P_i$-name $\mathbf{f}_i$: For every $G_i \subseteq P_i$ generic over $V$

$$\mathbf{f}_i[G_i](n) = \min\{k: (\exists p \in P_\delta)(p \restriction i \in G_i \text{ and } p \Vdash \mathbf{f}(n) = k\}.$$

Now we have $\Vdash_{P_i}$ “$\mathbf{f}_i \in {}^\omega\omega$”. Therefore

$$(*) \qquad \Vdash_{P_i} \text{“}(\exists J \subseteq I)(|J| = \aleph_0 \text{ and } \alpha \in I - J \rightarrow g_\alpha \not\leq \mathbf{f}_i)\text{”}.$$

Hence there exists a $P_i$-name $\mathbf{J}_i$ witnessing $(*)$. Set $J_i = \{\alpha \in I: \not\Vdash_{P_i} \text{“}\alpha \notin \mathbf{J}_i\text{”}\}$; then clearly $|J_i| \leq \aleph_0$, and $J = \bigcup_{i<\omega} J_i$ is also countable, and for every $\alpha \in I - J$ we have $\Vdash_{P_\delta}$ “$g_\alpha \not\leq \mathbf{f}$”, proving the lemma. $\blacksquare$

## §4. The first model.

**4.0.** LEMMA. *Let $V$ be a model of ZFC. Let $P$ be the forcing notion adding $\aleph_{\omega_1}$ Cohen reals. (Without loss of generality each Cohen real is a function from $\omega$ to $\omega$.) Let $G \subseteq P$ be generic over $V$, and let $\langle g_i: i < \omega_1 \rangle \in V[G]$ be $\omega_1$ Cohen reals from the generic object. Then*

$$V[G] \models \text{“}\langle g_i: i < \omega_1 \rangle \text{ is a strong scale”}.$$

PROOF. Well known. $\blacksquare$

**4.1.** LEMMA. *Let $Q = \langle P_\alpha; \mathbf{Q}_\alpha: \alpha < \omega_2 \rangle$ be a finite support iteration satisfying*

(i) *if $\alpha$ is even then $\Vdash_{P_\alpha}$ “$\mathbf{Q}_\alpha$ is a Cohen real forcing”, and*

(ii) *if $\alpha$ is odd then $\Vdash_{P_\alpha}$ “$\mathbf{Q}_\alpha$ is random real forcing”.*

*Let $P$, $G$ be as in 4.0, and $V \models CH$, and let $H$ be $P_{\omega_2}$-generic over $V[G]$. Then the following facts hold in $V[G][H]$.*

(a) $2^{\aleph_0} = \aleph_{\omega_1}$.

(b) $K_B = K_U = \aleph_2$ (*for measure*).

(c) $K_B = K_U = \aleph_2$ (*for categoricity*).

(d) $K_C = \aleph_{\omega_1}$ (*for measure*).

(e) $K_C = \aleph_{\omega_1}$ (*for categoricity*).

PROOF. (a) Easy. (b) Use the $\omega_2$ Cohen reals of $P_{\omega_2}$ for $K_B$ and the $\omega_2$ random reals for $K_U$. (c) Use the $\omega_2$ random reals of $P_{\omega_2}$ for $K_B$ and the $\omega_2$ Cohen reals for $K_U$. (d) This clearly follows from §§1 and 3 and $2^{\aleph_0} = \aleph_{\omega_1}$ (see (d)). (e) Clearly $K_C \leq \aleph_{\omega_1}$. If $K_C < \aleph_{\omega_2}$, then there exists $\langle g_i: i < \omega_1 \rangle \subseteq G$ such that $\bigcup_{i<\omega_1} A_{g_i}$ is meager; but this implies that $\langle g_i: i < \omega_1 \rangle$ is not a strong scale (by §2), a contradiction with §3. $\blacksquare$

**4.2.** REMARK. As the referee remarked, (d) follows from (e), and for (e) it is (by Cichoń's diagram) enough to prove $d \geq \aleph_{\omega_1}$, by 4.0. $\blacksquare$

**§5. The second model.** In this section we will construct a model in which $K_C(\mathscr{L}) = \aleph_{\omega_1}$ and $K_C(\mathscr{M}) = \omega_2$.

We use the following two theorems characterizing cardinals $K_C(\mathscr{L})$ and $K_C(\mathscr{M})$ which were proved independently by A. Miller, D. Fremlin, and T. Bartoszynski, and probably many others.

**5.1.** THEOREM. *$K_C(\mathscr{L})$ is the smallest cardinal $k$ such that there exists a family $\{I_n^\alpha; \alpha < k, n < \omega\}$ satisfying the following conditions:*

a) $I_n^\alpha \subseteq \omega$, $|I_n^\alpha| \leq n$ *for* $\alpha < k$, $n < \omega$.

b) $\forall f \in \omega^\omega \exists \alpha < k\, \forall n\, f(n) \in I_n^\alpha(n)$. ■

The proof of this theorem is implicit in [Ba1] and [F].

**5.2.** THEOREM. $K_C(\mathscr{M}) = \min\{K_U(\mathscr{M}), d\}$, *where $d$ is the cardinality of the smallest family of functions in $\omega^\omega$ which dominates $\omega^\omega$.* (*See* [Ba2].) ■

Let $V$ be a model of $ZFC$ and $\mathbb{B}$ a notion of forcing which adds $\aleph_{\omega_1}$ random reals. Let $\mathbb{D}_{\omega_2}$ be the finite support iteration of $\sigma$-centered notion of forcing which adds dominating reals. Let $G$ be generic for $\mathbb{B} * \mathbb{D}_{\omega_2}$. Then the following is true in $V[G]$:

*Claim* 1. $K_C(\mathscr{M}) = \omega_2$.

*Proof*. Obviously in $V[G]$ there is a family of functions in $\omega^\omega$ of cardinality $\omega_2$ which dominates $\omega^\omega$. There is also a nonmeager set of cardinality $\omega_2$, since the set of generic Cohen reals added by $\mathbb{C}_{\omega_2}$ is an $\omega_2$-Luzin set. This shows that $K_C(\mathscr{M}) \leq \omega_2$.

To show that $K_C(\mathscr{M}) \geq \omega_2$ it is enough to notice that every subset of $\mathbb{R}$ of cardinality $\omega_1$ in $V[G]$ is meager. ■

*Claim* 2. $K_C(\mathscr{L}) = \aleph_{\omega_1}$.

*Proof*. Suppose not. Then by 5.1 there exists $\lambda < \aleph_{\omega_1}$ and a family $\{I_n^\alpha : \alpha < \lambda\}$ satisfying conditions a) and b) of 5.1.

We will find a function $f \in \omega^\omega$ such that $f(n) \leq 2^n$ for all $n \in \omega$, and such that $\forall \alpha < \lambda \exists^\infty n f(n) \notin I_n^\alpha$, which will give a contradiction.

Define $\tilde{I}_n^\alpha = I_n^\alpha \cap 2^n$ for $n < \omega$, $\alpha < \lambda$.

The following is well known.

*Claim. For every $f \in \omega^\omega \cap V[G]$ and $g \in V[G \cap B] \cap \omega^\omega$, if $\forall n f(n) < g(n)$ then there exists $h \in V[G \cap B] \cap \omega^\omega$ such that $\exists^\infty n f(n) = h(n)$.*

*Proof*. This is due to the fact that the forcing $\mathbb{D}$ is $\sigma$-centered. ■

Let $f_\alpha(n) = \tilde{I}_n^\alpha$ for $n < \omega$, $\alpha < \lambda$.

Functions $f_\alpha$ for $\alpha < \lambda$ correspond to bounded functions in $\omega^\omega$. Therefore the above claim is applicable and we get, for every $\alpha < \lambda$, $\{\mathfrak{I}_n^\alpha : n \in \omega\} \in V[G \cap \mathbb{B}]$ such that

$$(*) \qquad \exists^\infty n \mathfrak{I}_n^\alpha = \tilde{I}_n^\alpha.$$

Without losing generality we can assume that $\mathfrak{I}_n^\alpha \subseteq 2^n$ and $|\mathfrak{I}_n^\alpha| \leq n$ for all $\alpha < \lambda$, $n < \omega$. This is because if $\mathfrak{I}_n^\alpha$ does not satisfy one of those conditions we are sure that $\mathfrak{I}_n^\alpha \neq \tilde{I}_n^\alpha$, so we can change it without affecting (*). Now let $X = \prod_{n=1}^\infty 2^n$ and $\mu = \prod_{n=1}^\infty \mu_n$, where $\mu_n$ is a measure on $2^n$ such that $\mu_n(\{i\}) = 1/2^n$ for $i \in 2^n$. Notice that the space $(X, \mu)$ is essentially the same as the Cantor set with Lebesgue measure. Let $\mathscr{U}_\alpha = \{x \in X : \forall^\infty n x(n) \notin \mathfrak{I}_n^\alpha\}$ for $\alpha < \lambda$. Since

$$\mathscr{U}_\alpha = \bigcup_{n \in \omega} \bigcap_{m \geq n} \{x \in X : x(m) \notin \mathfrak{I}_m^\alpha\}$$

we have

$$\mu(\mathcal{U}_\alpha) \geq \prod_{m \geq n} \left(\frac{2^m - m}{2^m}\right) \xrightarrow[n\to\infty]{} 1 \quad \text{for } \alpha < \lambda.$$

Therefore $V[G \cap \mathbf{B}] \models \bigcap_{\alpha<\lambda} \mathcal{U}_\alpha \neq \emptyset$, since $V[G \cap \mathbf{B}] \models K_B(\mathcal{L}) = \aleph_{\omega_1}$ ($\aleph_{\omega_1}$ random reals are added).

Let $x \in \bigcap_{\alpha<\lambda} \mathcal{U}_\alpha$. It means that $\forall \alpha < \lambda\ \forall^\infty n\ x(n) \notin \mathfrak{I}_n^\alpha$, and by (*)

$$\forall \alpha < \lambda\ \exists^\infty n\ x(n) \notin \tilde{I}_n^\alpha.$$

Thus by the definition of $\tilde{I}_n^\alpha$, $\alpha < \lambda$, we get $\forall \alpha < \lambda\ \exists^\infty n\ x(n) \notin I_n^\alpha$, which finishes the proof. ■

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