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On the weak Freese-Nation property of complete Boolean algebras ☆

Sakaé Fuchino[a,*], Stefan Geschke[b], Saharon Shelah[c,d], Lajos Soukup[e]

[a] *Kitami Institute of Technology, Kitami, Japan*
[b] *II. Mathematisches Institut, Freie Universität Berlin, Germany*
[c] *Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel*
[d] *Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA*
[e] *Institute of Mathematics, Hungarian Academy of Science, Budapest, Hungary*

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Abstract

The following results are proved:

(a) *In a model obtained by adding* $\aleph_2$ *Cohen reals, there is always a c.c.c. complete Boolean algebra without the weak Freese-Nation property.*

(b) *Modulo the consistency strength of a supercompact cardinal, the existence of a c.c.c. complete Boolean algebra without the weak Freese-Nation property is consistent with GCH.*

(c) *If a weak form of* $\square_\mu$ *and* $\mathrm{cof}([\mu]^{\aleph_0}, \subseteq) = \mu^+$ hold for each $\mu > \mathrm{cf}(\mu) = \omega$, *then the weak Freese-Nation property of* $\langle \mathscr{P}(\omega), \subseteq \rangle$ *is equivalent to the weak Freese-Nation property of any of* $\mathbb{C}(\kappa)$ or $\mathbb{R}(\kappa)$ for uncountable κ.

(d) *Modulo the consistency of* $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$, *it is consistent with* GCH *that* $\mathbb{C}(\aleph_\omega)$ *does not have the weak Freese-Nation property and hence the assertion in* (c) *does not hold, and also that adding* $\aleph_\omega$ *Cohen reals destroys the weak Freese-Nation property of* $\langle \mathscr{P}(\omega), \subseteq \rangle$.

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* Corresponding author. Present address: Dept. of Natural Science and Mathematics, College of Engineering, Chubu University, Aichi, Japan.

E-mail addresses: fuchino@math.cs.kitami-it.ac.jp (S. Fuchino), geschke@math.cs.kitami-it.ac.jp (S. Geschke), shelah@math.huji.ac.il (S. Shelah), soukup@math.cs.kitami-it.ac.jp (L. Soukup).

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These results solve all of the problems except Problem 1 in S. Fuchino, L. Soukup, Fundament. Math. 154 (1997) 159–176, and some other problems posed by Geschke.

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1. Introduction

A quasi-ordering $\langle P, \leqslant \rangle$ is said to have *the weak Freese-Nation property* if there is a mapping $f : P \to [P]^{\leqslant \aleph_0}$ such that:

For any $p, q \in P$ with $p \leqslant q$ there is $r \in f(p) \cap f(q)$ suchthat $p \leqslant r \leqslant q$.

A mapping f as above is called a *weak Freese-Nation mapping* on P.

The weak Freese-Nation property was introduced in Chapter 4 of [8] as a weakening of a notion of almost freeness of Boolean algebras.

The weak Freese-Nation property of a quasi-ordering P can be characterized in terms of σ-suborderings: a subordering Q of P is said to be a σ-subordering of P (notation: $Q \leqslant_\sigma P$) if the following holds: for every $p \in P$ there are a countable cofinal subset U of $Q \restriction p$ and a countable coinitial subset V of $Q \uparrow p$.

Proposition 1.1 (see Fuchino et al. [4]). *A quasi-ordering P has the weak Freese-Nation property if and only if $\{Q \in [P]^{\aleph_1} : Q \leqslant_\sigma P\}$ contains a club subset of $[P]^{\aleph_1}$.* □

Note that various notions of almost freeness of Boolean algebras can be also characterized in similar ways in terms of largeness of the family of subalgebras which are "nice" in some sense (e.g. relatively complete subalgebras, free factor etc., see [8]).

In Ref. [4], it was shown that $\langle \mathscr{P}(\omega_1), \subseteq \rangle$ does not have the weak Freese-Nation property. If a complete Boolean algebra B does not have the c.c.c., then $\langle \mathscr{P}(\omega_1), \subseteq \rangle$ can be completely embedded into B. Hence, in this case, B cannot have the weak Freese-Nation property (see Proposition 1.3(1)).

It is easily seen that every quasi-ordering of cardinality $\leqslant \aleph_1$ has the weak Freese-Nation property (see e.g. [4]). It follows that, under CH, $\langle \mathscr{P}(\omega), \subseteq \rangle$ has the weak Freese-Nation property.

Similarly to the situation with $\mathscr{P}(\omega_1)$ and non-c.c.c. complete Boolean algebras, if $\langle \mathscr{P}(\omega), \subseteq \rangle$ does not have the weak Freese-Nation property, then no infinite c.c.c. complete Boolean algebra (and hence no infinite complete Boolean algebra at all) can have the weak Freese-Nation property.

To simplify the formulation of some of the results below, let us say that a model of set-theory is *neat* if $\mathrm{cof}([\mu]^\omega, \subseteq) = \mu^+$ for each $\mu > \mathrm{cf}(\mu) = \omega$ and the very weak variant of $\square_\mu$ introduced in [5] (see the end of Section 4) holds.

In Refs. [4] and [5], it was shown that if CH holds, then every c.c.c. complete Boolean algebra of size $<\aleph_\omega$ has the weak Freese-Nation property; and in a neat model, CH implies that every c.c.c. complete Boolean algebra has the weak Freese-Nation property. The following questions remained unanswered in [5]:

Question 1 (Fuchino and Soukup [5, Problem 5]). *Are the following equivalent?*
(a) $\langle \mathscr{P}(\omega), \subseteq \rangle$ *has the weak Freese-Nation property*;
(b) *every c.c.c. complete Boolean algebra has the weak Freese-Nation property.*

Question 2 (Fuchino and Soukup [5, Problem 2]). *Does* ZFC + GCH *imply that every c.c.c. complete Boolean algebra has the weak Freese-Nation property?*

We give here negative answers to these questions: see Corollary 3.4 for question 1 and Theorem 4.2 for question 2. By the result in [5] mentioned above, we need the consistency strength of some large cardinal to give a negative answer to question 2. Indeed, the ground model V in the negative solution to this question is obtained by starting from a model of ZFC with a supercompact cardinal.

In Ref. [4] it was shown that if CH holds, then adding less than $\aleph_\omega$ many Cohen reals preserve the weak Freese-Nation property of $\langle \mathscr{P}(\omega), \subseteq \rangle$. By Ref. [5], in the generic extension obtained by adding any number of Cohen reals to a neat model satisfying CH, not only $\langle \mathscr{P}(\omega), \subseteq \rangle$ but every tame c.c.c. complete Boolean algebra has the weak Freese-Nation property. Here, letting $P = \mathrm{Fn}(\tau, 2)$ (= the standard p.o. for adding τ Cohen reals), a Boolean algebra B in a P-generic extension is said to be *tame*, if there is a P-name $\dot{\leqslant}$ of a partial ordering of B and a mapping $t : B \to [\tau]^{\aleph_0}$ in V such that, for every $p \in P$ and $x, y \in B$, if $p \Vdash_P$ "$x \mathrel{\dot{\leqslant}} y$", then $p \restriction (t(x) \cup t(y)) \Vdash_P$ "$x \mathrel{\dot{\leqslant}} y$" (we assume here without loss of generality that B is chosen so that its underlying set is a ground model set).

These results suggest the following questions posed in [5]:

Question 3 (Fuchino and Soukup [5, Problem 3]). *Assume that $V[G]$ is a model obtained by adding Cohen reals to a model of* ZFC + CH. *Is it true that $\mathscr{P}(\omega)$ has the weak Freese-Nation property in $V[G]$?*

Question 4 (Fuchino and Soukup [5, Problem 4]). *Assume that $V[G]$ is a model obtained by adding $\aleph_2$ Cohen reals to a model of* ZFC + CH. *Is it true that every c.c.c. complete Boolean algebra (not just the tame ones) has the weak Freese-Nation property in $V[G]$?*

The results of the present paper answer these questions in the negative: see Theorem 6.1 for question 3 and Corollary 3.3 for question 4.

By the result in Ref. [5] mentioned above, we need the consistency strength of some large cardinal for a negative solution of Question 3. The ground model V in the negative solution of this problem given in Theorem 6.1 is obtained by starting from a model of ZFC with a large cardinal slightly stronger than a huge cardinal.

After the negative solution of Problem 5 in [5] (listed here as Question 1), the following question still remains:

Problem 1. *For which Boolean algebras B, is the weak Freese-Nation property of B equivalent with the weak Freese-Nation property of* $\langle \mathscr{P}(\omega), \subseteq \rangle$?

The following trivial lemma can also be seen in connection with this problem:

Lemma 1.2. *The following are equivalent*:
(a) $\langle \mathscr{P}(\omega), \subseteq \rangle$ *has the weak Freese-Nation property*;
(b) $\langle \mathscr{P}(\omega), \subseteq^* \rangle$ *has the weak Freese-Nation property*;
(c) $\langle \mathscr{P}(\omega)/fin, \subseteq^* \rangle$ *has the weak Freese-Nation property*;
(d) $\langle {}^{\omega}\omega, \leqslant \rangle$ *has the weak Freese-Nation property*;
(e) $\langle {}^{\omega}\omega, \leqslant^* \rangle$ *has the weak Freese-Nation property.*
(f) $\langle {}^{\omega}\mathbb{R}, \leqslant \rangle$ *has the weak Freese-Nation property.*

Koppelberg [10] pointed out that the weak Freese-Nation property of the Cohen algebra $\mathbb{C}(\omega)$ is equivalent to the weak Freese-Nation property of $\langle \mathscr{P}(\omega), \subseteq \rangle$. In the present paper, we show that it is also equivalent to the weak Freese-Nation property of the measure algebra $\mathbb{R}(\omega)$ (Proposition 5.1) and moreover, in a neat model, also to the weak Freese-Nation property of $\mathbb{C}(\kappa)$ and/or $\mathbb{R}(\kappa)$ for any $\kappa \geqslant \aleph_0$ (Corollary 5.4). Here, we denote by $\mathbb{C}(\kappa)$ and $\mathbb{R}(\kappa)$ the c.c.c. complete Boolean algebras $Borel({}^{\kappa}2)/meager({}^{\kappa}2)$ and $Borel({}^{\kappa}2)/null({}^{\kappa}2)$, respectively. We show that some extra set-theoretic assumptions are really necessary in Corollary 5.4 by constructing a model of GCH and the negation of weak Freese-Nation property of $\mathbb{C}(\aleph_\omega)$ starting from a model of GCH and Chang's conjecture for $\aleph_\omega$.

Assume that $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ is a finite support iteration such that the forcing with $\dot{Q}_\alpha$ just adds a real to V^{P_α}. Then, as Geschke showed in Ref. [7], if this iteration preserves the weak Freese-Nation property of $\mathscr{P}(\omega)$, then in many cases, we can conclude that for stationarily many α the partially ordered set $\dot{Q}_\alpha$ just adds one Cohen real. On the other hand, in any model obtained by adding $\geqslant \aleph_2$ Cohen reals, there is a c.c.c. complete Boolean algebra B without the weak Freese-Nation property (Corollary 3.3). So there is no easy way to blow up the continuum while preserving the weak Freese-Nation property of all c.c.c. complete Boolean algebras. This suggests that the following question is quite a reasonable one:

Problem 2. *Does CH follow from the assumption that every c.c.c. complete Boolean algebra has the weak Freese-Nation property*?

If $\mathbf{b} > \aleph_1$ or if there is an $\aleph_2$-Luzin-gap, then $\langle \mathscr{P}(\omega), \subseteq \rangle$ does not have the weak Freese-Nation property (see [4] and [5]). The following question ([5, Problem 1]) was raised against this background:

Problem 3. *Suppose that $\mathscr{P}(\omega)$ does not have any increasing chain of length $\geqslant \omega_2$ with respect to $\subseteq^*$ and that there is no $\aleph_2$-Luzin gap. Does it follow that $\mathscr{P}(\omega)$ has the weak Freese-Nation property?*

This can be solved negatively using results from [2] and [7]: Let V be a model of CH and $V[G]$ its generic extension by adding many random reals side by side. Using results from [2] we see that, in $V[G]$, there are neither increasing ω_2 chains in $\mathscr{P}(\omega)$ with respect to $\subseteq^*$ nor $\aleph_2$-Luzin gaps. On the other hand Geschke [7] showed that, in $V[G]$, $\langle \mathscr{P}(\omega), \subseteq \rangle$ does not have the weak Freese-Nation property.

Consequences of the weak Freese-Nation property of $\langle \mathscr{P}(\omega), \subseteq \rangle$ were studied in [10] and [6]. In the latter paper it was shown that a set-theoretic universe with the weak Freese-Nation property of $\langle \mathscr{P}(\omega), \subseteq \rangle$ looks quite similar to a Cohen model. In particular, under the weak Freese-Nation property of $\langle \mathscr{P}(\omega), \subseteq \rangle$, all cardinal invariants which appear in [1] take the same value as in a Cohen model with the same size of $2^{\aleph_0}$.

Problem 4. *Find a combinatorial* (Π^1_1) *characterization of the weak Freese-Nation property of $\mathscr{P}(\omega)$.*

The weak Freese-Nation property of a quasi-ordering $\langle P, \leqslant \rangle$ is actually a property of the corresponding partial ordering $\langle \bar{P}, \leqslant \rangle$ obtained as the quotient structure of $\langle P, \leqslant \rangle$ with respect to the equivalence relation "$x \leqslant y \wedge y \leqslant x$": $\langle P, \leqslant \rangle$ has the weak Freese-Nation property if and only if $\langle \overline{P}, \leqslant \rangle$ does (the first author thanks David Fremlin for pointing out this fact).

The following criteria of the weak Freese-Nation property are used in the later sections. A partial ordering Q is said to be a *retract* of a partial ordering P if there are order preserving mappings $i : Q \to P$ and $j : P \to Q$ such that $j \circ i = id_Q$.

Q is said to be a *σ-subordering of P* (notation: $Q \leqslant_\sigma P$) if, for every $p \in P$, $Q \upharpoonright p = \{q \in Q : q \leqslant p\}$ has a countable cofinal subset and $Q \uparrow p = \{q \in Q : q \geqslant p\}$ has a countable coinitial subset. Note that if C is a complete subalgebra of a complete Boolean algebra B (notation: $C \leqslant_c B$) or a countable union of complete subalgebras of B, then it follows that $C \leqslant_\sigma B$.

Proposition 1.3. (1) (Lemma 2.7 in [4]) *If Q is a retract of P and P has the weak Freese-Nation property then Q has the weak Freese-Nation property.*

(2) *Suppose that Q is a complete Boolean algebra and there is a strictly order-preserving embedding f of Q into P (i.e. f preserves ordering and incomparability). If P has the weak Freese-Nation property then Q also has the weak Freese-Nation property.*

(3) (Lemma 2.3 (a) in [4]) *If $Q \leqslant_\sigma P$ and P has the weak Freese-Nation property, then Q also has the weak Freese-Nation property.*

(4) (Lemma 2.6 in [4]) *If P_α, $\alpha < \delta$ is an increasing sequence of partial orderings with the weak Freese-Nation property such that $P_\alpha \leqslant_\sigma P_{\alpha+1}$ for every $\alpha < \delta$ and $P_\gamma = \bigcup_{\alpha<\gamma} P_\alpha$ for all $\gamma < \delta$ with $\mathrm{cf}(\gamma) > \omega$, then $P = \bigcup_{\alpha<\delta} P_\alpha$ also has the weak Freese-Nation property.*

Proposition 1.3 (2) follows easily form Proposition 1.3 (1): the mapping $g: P \to Q$ defined by $g(p) = \sum\{q \in Q : f(q) \leqslant p\}$ for $p \in P$ witnesses that Q is a retract of P.

2. The partial ordering $P_{\mathscr{S}}$

In this section we introduce a construction of partial orderings $P_{\mathscr{S}}$ and Boolean algebras $B_{\mathscr{S}}$ which will be used in the following Sections 3 and 4. For $S \subseteq \kappa$ and an indexed family $\mathscr{S} = \langle S_\alpha : \alpha \in S \rangle$ of subsets of κ, let

$$P_{\mathscr{S}} = \{x_i : i \in \kappa\} \cup \{y_\alpha : \alpha \in S\},$$

where the x_is and y_αs are pairwise distinct, and let $\leqslant_{\mathscr{S}}$ be the partial ordering on $P_{\mathscr{S}}$ defined by

$$p \leqslant_{\mathscr{S}} q \quad \Leftrightarrow \quad p = q \text{ or}$$
$$p = x_i \text{ and } q = y_\alpha \text{ for some } i \in \kappa \text{ and } \alpha \in S \text{ with } i \in S_\alpha.$$

Let $B_{\mathscr{S}}$ be the Boolean algebra generated freely from $P_{\mathscr{S}}$ except the relation $\leqslant_{\mathscr{S}}$. Note that the identity map on $P_{\mathscr{S}}$ canonically induces a strictly order-preserving embedding of $P_{\mathscr{S}}$ into $B_{\mathscr{S}}$.

Proposition 2.1. *Suppose that* $\mathrm{cf}(\kappa) \geqslant \omega_2$, $S \subseteq \kappa$ *is stationary such that* $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) \geqslant \omega_1$ *and* $\mathscr{S} = \{S_\alpha : \alpha \in S\}$ *is such that* S_α *is a cofinal subset of* α *for each* $\alpha \in S$. *If* $P_{\mathscr{S}}$ *is embedded into a partial ordering* P *by a strictly order-preserving mapping then* P *does not have the weak Freese-Nation property. In particular,* $B_{\mathscr{S}}$ *and its completion do not have the weak Freese-Nation property.*

Proof. Without loss of generality, we may assume that $P_{\mathscr{S}}$ is a subordering of P. Assume to the contrary that there is a weak Freese-Nation mapping $f: P \to [P]^{\leqslant \aleph_0}$. Let

$$C = \{\xi < \kappa : \forall \eta < \xi \forall p \in f(x_\eta)$$
$$[\exists \alpha \in S[x_\eta \leqslant p \leqslant y_\alpha] \to \exists \alpha' \in S \cap \xi[x_\eta \leqslant p \leqslant y_{\alpha'}]]\}.$$

Then C is a club subset of κ. Let $\alpha \in C \cap S$ and let

$$A = \{p \in f(y_\alpha) : \exists \eta \in S_\alpha[p \in f(x_\eta) \wedge x_\eta \leqslant p \leqslant y_\alpha]\}.$$

Since $\alpha \in C$, for each $p \in A$ there is $\alpha_p < \alpha$ such that $p \leqslant y_{\alpha_p}$. Let $\alpha^* = \sup\{\alpha_p : p \in A\}$. Since A is countable we have $\alpha^* < \alpha$. Let $\beta \in S_\alpha \setminus \alpha^*$. Since $x_\beta \leqslant y_\alpha$, there is a $p \in A$ such that $x_\beta \leqslant p \leqslant y_\alpha$. Hence $x_\beta \leqslant y_{\alpha_p}$. But this is impossible since $\alpha_p < \alpha^* \leqslant \beta$. $\square$

3. Cohen models

Consider the following principle:

(**) There is a sequence $\langle S_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$ such that each S_α is a cofinal subset of α and for any pairwise disjoint $\langle x_\beta : \beta < \omega_1\rangle$ with $x_\beta \in [\omega_2]^{<\aleph_0}$ for $\beta < \omega_1$, there are $\beta_0 < \beta_1 < \omega_1$ such that $x_{\beta_0} \cap S_\alpha = \emptyset$ for all $\alpha \in x_{\beta_1} \cap \mathrm{Lim}(\omega_2)$ and that $x_{\beta_1} \cap S_\alpha = \emptyset$ for all $\alpha \in x_{\beta_0} \cap \mathrm{Lim}(\omega_2)$.

Proposition 3.1. *Let $P = \mathrm{Fn}(\omega_2, 2)$. Then $\Vdash_P$ "(**)".*

Proof. Without loss of generality we may assume $P = \mathrm{Fn}(\bigcup_{\alpha \in \mathrm{Lim}(\omega_2)} \alpha \times \{\alpha\}, 2)$. For $\alpha \in \mathrm{Lim}(\omega_2)$, let $\dot{S}_\alpha$ be a P-name such that $\Vdash_P$ "$\dot{S}_\alpha = \{\beta \in \alpha : \dot{g}(\beta, \alpha) = 1\}$", where $\dot{g}$ is the canonical name for the generic function. By genericity, $\Vdash_P$ "$\dot{S}_\alpha$ is cofinal in α" for every $\alpha \in \mathrm{Lim}(\omega_2)$. Let $\dot{S}$ be a P-name such that $\Vdash_P$ "$\dot{S} = \langle \dot{S}_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$".

To show that $\dot{S}$ is forced to satisfy the property in (**), let $\langle \dot{x}_\beta : \beta < \omega_1\rangle$ be a P-name of a sequence of pairwise disjoint finite subsets of ω_2. For each $\beta < \omega_1$, let p_β and $x_\beta \in [\omega_2]^{<\aleph_0}$ be such that $p_\beta \Vdash_P$ "$\dot{x}_\beta = x_\beta$". By thinning out the index set ω_1, we may assume without loss of generality that $\mathrm{dom}(p_\beta)$, $\beta < \omega_1$ form a Δ-system with the root d and $p_\beta \restriction d$, $\beta < \omega_1$ are all equal to the same $p \in P$. Since p_β, $\beta < \omega_1$ are then pairwise compatible, x_β, $\beta < \omega_1$ are pairwise disjoint. Further, we may assume that s_β, $\beta < \omega_1$ form a Δ-system with the root s where $s_\beta = \{\gamma : \langle \gamma, \alpha\rangle \in \mathrm{dom}(p_\beta)$ for some $\alpha \in \mathrm{Lim}(\omega_2)\}$. Note that x_β, $\beta < \omega_1$ are pairwise disjoint since p_β's are pairwise compatible.

Let $\beta_0 < \beta_1 < \omega_1$ be such that $x_{\beta_0} \cap s = \emptyset$, $x_{\beta_1} \cap s = \emptyset$, $x_{\beta_0} \cap s_{\beta_1} = \emptyset$ and $x_{\beta_1} \cap s_{\beta_0} = \emptyset$. To take such β_0 and β_1, first fix a β_0 such that $x_{\beta_0} \cap s = \emptyset$. This is possible since s is finite and x_β's are pairwise disjoint. By the same argument we can find infinitely many β's such that $x_\beta \cap (s \cup s_{\beta_0}) = \emptyset$. Now for such β's, since $s_\beta \setminus s$ are pairwise disjoint, there are infinitely may β's among them with $x_{\beta_0} \cap s_\beta = \emptyset$. Let β_1 be one of such β's.

Let

$$\begin{aligned} p^* = {} & p_{\beta_0} \cup p_{\beta_1} \cup \{\langle\langle \beta, \alpha\rangle, 0\rangle : \beta \in x_{\beta_0}, \alpha \in x_{\beta_1} \cap \mathrm{Lim}(\omega_2)\} \\ & \cup \{\langle\langle \beta, \alpha\rangle, 0\rangle : \beta \in x_{\beta_1}, \alpha \in x_{\beta_0} \cap \mathrm{Lim}(\omega_2)\} \end{aligned}$$

Then $p^* \in P$. Clearly $p^* \Vdash_P$ "$\dot{x}_{\beta_0} \cap \dot{S}_\alpha = \emptyset$" for all $\alpha \in \dot{x}_{\beta_1} \cap \mathrm{Lim}(\omega_2)$ and $p^* \Vdash_P$ "$\dot{x}_{\beta_1} \cap \dot{S}_\alpha = \emptyset$" for all $\alpha \in \dot{x}_{\beta_0} \cap \mathrm{Lim}(\omega_2)$. □

Proposition 3.2. *Suppose that $\langle S_\alpha : \alpha \in \mathrm{Lim}(\omega_2)\rangle$ is as in (**). Let $S = \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega_1\}$ and $\mathscr{S} = \langle S_\alpha : \alpha \in S\rangle$. Then $B_{\mathscr{S}}$ satisfies the c.c.c.*

Proof. Otherwise we can find $I_\alpha \in [\omega_2]^{<\aleph_0}$, $J_\alpha \in [S]^{<\aleph_0}$ for $\alpha < \omega_1$ and $t(\alpha, i)$, $u(\alpha, \xi) \in \{+1, -1\}$ for each $i \in I_\alpha$, $\xi \in J_\alpha$ and $\alpha < \omega_1$ such that

$$z_\alpha = \prod_{i \in I_\alpha} t(\alpha, i) x_i \cdot \prod_{\xi \in J_\alpha} u(\alpha, \xi)\, y_\xi, \quad \alpha < \omega_1$$

form a pairwise disjoint family of elements of ${B_{\mathscr{S}}}^+$.

By a Δ-system argument, we may assume that $I_\alpha \cup J_\alpha$, $\alpha < \omega_1$ are pairwise disjoint. Applying $(**)$ to $\langle I_\alpha \cup J_\alpha : \alpha < \omega_1 \rangle$, we find $\beta_0 < \beta_1 < \omega_1$ such that $I_{\beta_0} \cap S_\xi = \emptyset$ for all $\xi \in J_{\beta_1}$ and that $I_{\beta_1} \cap S_\xi = \emptyset$ for all $\xi \in J_{\beta_0}$. By definition of $B_{\mathscr{S}}$, it follows that $z_{\beta_0} \cdot z_{\beta_1} \neq 0$. This is a contradiction. □

Theorem 3.3. *In a Cohen model* (*i.e. any model obtained by adding* $\geqslant \aleph_2$ *Cohen reals*) *there is a c.c.c. complete Boolean algebra B of density* $\aleph_2$ *without the weak Freese-Nation property.*

Proof. By Proposition 3.1, $(**)$ holds in a Cohen model. Hence $B_{\mathscr{S}}$ as in Proposition 3.2 satisfies the c.c.c. By Proposition 2.1, the completion of $B_{\mathscr{S}}$ does not have the weak Freese-Nation property. □

Corollary 3.4. *The weak Freese-Nation property of* $\langle \mathscr{P}(\omega), \subseteq \rangle$ *does not imply the weak Freese-Nation property of all c.c.c. complete Boolean algebras.*

Proof. If we start from a model of CH and add $\aleph_2$ Cohen reals, then $\langle \mathscr{P}(\omega), \subseteq \rangle$ has the weak Freese-Nation property in the resulting model (see e.g. [5]). On the other hand, by Theorem 3.3, there is a c.c.c. complete Boolean algebra without the weak Freese-Nation property in such a model. □

Under CH, every c.c.c. complete Boolean algebra of size $\aleph_2$ has the weak Freese-Nation property ([4]). Hence it follows from the result above that CH implies the negation of the principle $(**)$. This can be also seen directly as follows:

Proposition 3.5. *CH implies* $\neg(**)$.

Proof. Let $\langle S_\alpha : \alpha \in \mathrm{Lim}(\omega_2) \rangle$ be any sequence such that each S_α is a cofinal subset of α for $\alpha \in \mathrm{Lim}(\omega_2)$. To show that $\langle S_\alpha : \alpha \in \mathrm{Lim}(\omega_2) \rangle$ is not as in $(**)$, let χ be sufficiently large and let $M \prec \mathscr{H}(\chi)$ be such that $|M| = \aleph_1$; $\langle S_\alpha : \alpha \in \mathrm{Lim}(\omega_2) \rangle \in M$; $\omega_1 \subseteq M$; $\omega_2 \cap M \in \omega_2$ and, letting $\gamma = \omega_2 \cap M$, $\mathrm{cf}(\gamma) = \omega_1$. By CH—and since $\omega_1 \subseteq M$ and $\mathrm{cf}(\gamma) = \omega_1$, we have $[\gamma]^{\aleph_0} \subseteq M$.

Now, by induction, choose distinct α^0_β, $\alpha^1_\beta < \gamma$ for $\beta < \omega_1$ such that (1) $\alpha^0_\beta \in S_\gamma$, and (2) $\{\alpha^0_\xi : \xi < \beta\} \subseteq S_{\alpha^1_\beta}$ for all $\beta < \omega_1$. (2) is possible: since $\{\alpha^0_\xi : \xi < \beta\} \subseteq S_\gamma$ and $\{\alpha^0_\xi : \xi < \beta\} \in M$, we have

$$M \models \exists \nu < \omega_2 [\sup\{\alpha^1_\xi : \xi < \beta\} < \nu \wedge \{\alpha^0_\xi : \xi < \beta\} \subseteq S_\nu]$$

by elementarity. Let $x_\beta = \{\alpha^0_\beta, \alpha^1_\beta\}$ for $\beta < \omega_1$. Then there are no $\beta_0 < \beta_1 < \omega_1$ such that $x_{\beta_0} \cap S_\alpha = \emptyset$ for all $\alpha \in x_{\beta_1} \cap \mathrm{Lim}(\omega_2)$. □

4. The weak Freese-Nation property of c.c.c. complete Boolean algebras under GCH

In Ref. [5] it is proved that, assuming CH and a weak form of the square principle at singular cardinals of cofinality ω, every c.c.c. complete Boolean algebra has the weak Freese-Nation property. In this section we show that even GCH does not suffice for this result.

Hajnal et al. [9] showed that, starting from a model with a supercompact cardinal, a model of GCH and the following principle can be constructed:

(***) There are a stationary $S \subseteq \{\alpha < \omega_{\omega+1} : \mathrm{cf}(\alpha) = \omega_1\}$ and a family $\mathscr{S} = \langle S_\alpha : \alpha \in S \rangle$ such that each S_α is a cofinal subset of α of ordertype ω_1 and that, for all distinct $\alpha, \beta \in S$, $S_\alpha \cap S_\beta$ is finite.

Proposition 4.1. *Suppose that* $\mathscr{S} = \langle S_\alpha : \alpha \in S \rangle$ *is as in* $(***)$. *Then* $B_{\mathscr{S}}$ *satisfies the c.c.c.*

Proof. Otherwise we can find $I_\alpha \in [\omega_{\omega+1}]^{<\aleph_0}$, $J_\alpha \in [S]^{<\aleph_0}$, $\alpha < \omega_1$ and $t(\alpha, i)$, $u(\alpha, \xi) \in \{+1, -1\}$ for each $\alpha < \omega_1$, $i \in I_\alpha$ and $\xi \in J_\alpha$ such that

$$z_\alpha = \prod_{i \in I_\alpha} t(\alpha, i) x_i \cdot \prod_{\xi \in J_\alpha} u(\alpha, \xi)\, y_\xi, \quad \alpha < \omega_1$$

form a pairwise disjoint family of elements of $B^+_{\mathscr{S}}$.

By a Δ-system argument, we may assume that $I_\alpha \cup J_\alpha$, $\alpha < \omega_1$ are pairwise disjoint and each I_α has the same size, say n.

For $\alpha < \beta < \omega_1$, since $z_\alpha \cdot z_\beta = 0$, either (I) there is $\eta \in J_\alpha$ such that $I_\beta \cap S_\eta \neq \emptyset$ or else (II) there is $\xi \in J_\beta$ such that $I_\alpha \cap S_\xi \neq \emptyset$. If (I) holds then let us say that $\langle \alpha, \beta \rangle$ is of type (I).

Now, one of the following two cases holds. We show that both of them lead to a contradiction.

Case 1: There is an infinite subset S of ω such that for every $\beta \in \omega_1 \setminus \omega$, $\{k \in S : \langle k, \beta \rangle$ is of type (I)$\}$ is cofinite in S. In this case, by thinning out the index set ω_1, we may assume that, for any $k \in \omega$ and $\beta \in \omega_1 \setminus \omega$, $\langle k, \beta \rangle$ is of type (1). For all $\beta \in \omega_1 \setminus \omega$, since $|I_\beta| = n$, there are $0 \leqslant i^0(\beta) < i^1(\beta) < n + 1$ such that $I^*_\beta = I_\beta \cap S_{\alpha^0(\beta)} \cap S_{\alpha^1(\beta)} \neq \emptyset$ for some $\alpha^k(\beta) \in J_{i^k(\beta)}$ for $k = 0, 1$ by the pigeonhole principle. Hence we can find an infinite $X \subseteq \omega_1 \setminus \omega$, $0 \leqslant i^0 < i^1 < n + 1$ and $\alpha^0, \alpha^1 \in \omega_1$, $\alpha^0 \neq \alpha^1$ such that $\alpha^0(\beta) = \alpha^0$ and $\alpha^1(\beta) = \alpha^1$ for all $\beta \in X$. But then $\bigcup_{\beta \in X} I^*_\beta \subseteq S_{i^0} \cap S_{i^1}$. Since the set on the left side is infinite as a disjoint union of pairwise disjoint non-empty sets, this is a contradiction to $(***)$.

Case 2: For any infinite subset $S \subseteq \omega$, there is $\beta \in \omega_1 \setminus \omega$ such that for infinitely many $k \in S$, $\langle k, \beta \rangle$ is not of type (1). In this case, by thinning out the first ω elements of the index set ω_1, we may assume that for each $k \in \omega$, there is $\xi(k) \in J_\omega$ such that $I_k \cap S_{\xi(k)} \neq \emptyset$. Note that J_ω is finite. So by thinning out further the first ω elements of the index set ω_1, we may assume that there is $\xi_0 \in J_\omega$ such that $I_k \cap S_{\xi_0} \neq \emptyset$ for every $k < \omega$. Similarly we may also assume that there are $\xi_i \in J_{\omega+i}$ for $1 \leqslant i \leqslant n$ such that

$I_k \cap S_{\xi_i} \neq \emptyset$ for every $k < \omega$. Then for each $k < \omega$, we can find $i^0(k) < i^1(k) \leqslant n$ such that $I_k^* = I_k \cap S_{\xi_{i^0(k)}} \cap S_{\xi_{i^1(k)}} \neq \emptyset$ by the pigeonhole principle. Since there are only $n(n-1)/2$ possibilities of $i^0(k) < i^1(k) \leqslant n$, there are $i^0 < i^1 \leqslant n$ and an infinite set $X \subseteq \omega$ such that for every $k \in X$, $i^0(k) = i^0$ and $i^1(k) = i^1$. It follows that $S_{\xi_{i^0}} \cap S_{\xi_{i^1}} \supseteq \bigcup_{k \in X} I_k^*$. As $S_{\xi_{i^0}} \cap S_{\xi_{i^1}}$ is finite, while $\bigcup_{k \in X} I_k^*$ is infinite as a union of infinitely many pairwise disjoint non-empty sets, this is a contradiction. □

Theorem 4.2. *It is consistent with GCH* (*modulo the consistency strength of a supercompact cardinal*) *that there is a c.c.c. complete Boolean algebra of size* $\aleph_{\omega+1}$ *without the weak Freese-Nation property.*

Proof. Let $\mathscr{S}$ be a family as in $(***)$. By Proposition 4.1, $B_{\mathscr{S}}$ satisfies the c.c.c. Hence the completion B of $B_{\mathscr{S}}$ is a c.c.c. complete Boolean algebra of size $\aleph_{\omega+1}$. By Proposition 2.1, B does not have the weak Freese-Nation property. □

The following weak form of the square principle was introduced in Ref. [5]. $\square^{***}_{\aleph_1,\mu}$ is the following assertion: there exists a sequence $\langle C_\alpha : \alpha < \mu^+ \rangle$ and a club set $D \subseteq \mu^+$ such that for $\alpha \in D$ with $\mathrm{cf}(\alpha) \geqslant \omega_1$

(y1) $C_\alpha \subseteq \alpha$, C_α is unbounded in α;

(y2) $[\alpha]^{<\omega_1} \cap \{C_{\alpha'} : \alpha' < \alpha\}$ dominates $[C_\alpha]^{<\omega_1}$ (with respect to $\subseteq$).

It can be easily seen that $\square^{***}_{\aleph_1,\mu}$ follows from the very weak square principle for μ by Foreman and Magidor [3] (see Ref. [5]).

Proposition 4.3. $\square^{***}_{\aleph_1,\mu}$ *together with CH implies the negation of* $(***)$.

Proof. In Ref. [5] it is proved that under CH and $\square^{***}_{\aleph_1,\aleph_\omega}$, every c.c.c. complete Boolean algebra of cardinality $\aleph_{\omega+1}$ has the weak Freese-Nation property. Hence the assertion follows from the proof of Theorem 4.2. □

By Proposition 4.3, we see that the consistency strength of some large cardinal is involved in GCH + $(***)$.

5. The weak Freese-Nation property of c.c.c. complete Boolean algebras under very weak square principles

In this section, we turn to the question for which c.c.c. complete Boolean algebras B the weak Freese-Nation property of B is equivalent to the weak Freese-Nation property of $\langle \mathscr{P}(\omega), \subseteq \rangle$. Lemma 1.2 was already an easy observation in ZFC in this direction. Koppelberg pointed out in Ref. [10] that the Cohen algebra $\mathbb{C}(\omega)$ is such a Boolean algebra.

Note that $\langle \mathscr{P}(\omega), \subseteq \rangle$ can be embedded in every complete Boolean algebra. Hence it follows from Proposition 1.3(2) that, if $\langle \mathscr{P}(\omega), \subseteq \rangle$ does not have the weak

Freese-Nation property then no complete Boolean algebra can have the weak Freese-Nation property.

Proposition 5.1. *The measure algebra $\mathbb{R}(\omega)$ has the weak Freese-Nation property if and only if $\langle\mathscr{P}(\omega),\subseteq\rangle$ has the weak Freese-Nation property.*

Proof. By the remark above and by Proposition 1.3(2) and Lemma 1.2, it is enough to find a strictly order-preserving embedding of $\mathbb{R}(\omega)$ into $\langle {}^{\omega}\mathbb{R}, \leqslant\rangle$. We may replace ω by the countable set $Clop({}^{\omega}2)$ where $Clop({}^{\omega}2)$ denotes the clopen sets of the Cantor space ${}^{\omega}2$.

We define $e:\mathbb{R}(\omega)\rightarrow {}^{Clop({}^{\omega}2)}\mathbb{R}$ by taking $e(a)(c)=\mu(a\cap c)$ for $c\in Clop({}^{\omega}2)$. Then clearly e is an order-preserving map of $\mathbb{R}(\omega)$ into ${}^{Clop({}^{\omega}2)}\mathbb{R}$.

To show that e is strictly order-preserving, assume that $\mu(a\backslash b)>0$. Then there is $c\in Clop({}^{\omega}2)$ such that $\mu((a\backslash b)\cap c)>\mu(c)/2$. Then $e(a)(c)>\mu(c)/2>e(b)(c)$, so $e(a)\nleqslant e(b)$. $\square$

In a similar way, we can also show that $\mathbb{R}(\omega)$ is a retract of $\mathscr{P}(\omega)$ as a partially ordered set though it is known that $\mathbb{R}(\omega)$ is *not* a retract of $\mathscr{P}(\omega)$ as a Boolean algebra: the mapping $e:\mathbb{R}(\omega)\rightarrow\mathscr{P}(Clop({}^{\omega}2)\times\mathbb{Q})$ defined by $e(c)=\{\langle a,q\rangle : a\in Clop({}^{\omega}2), q<\mu(a\cap c)\}$ is easily seen to be a strictly order preserving embedding.

In general, the weak Freese-Nation property of $\mathbb{C}(\kappa)$ or that of $\mathbb{R}(\kappa)$ for arbitrary $\kappa\geqslant\omega$ is not equivalent with the weak Freese-Nation property of $\langle\mathscr{P}(\omega),\subseteq\rangle$. In the next section we shall give a model where $\langle\mathscr{P}(\omega),\subseteq\rangle$ has the weak Freese-Nation property while $\mathbb{C}(\aleph_\omega)$ (and hence also $\mathbb{R}(\aleph_\omega)$) does not.

However the equivalence does hold if $\kappa<\aleph_\omega$ or some consequences of $\neg 0^{\#}$ are available. To prove this, we need the following instance of Theorem 7 in [5]:

Theorem 5.2 (Theorem 7 in [5] for $\kappa=\aleph_1$). *Suppose that $\mu>\mathrm{cf}(\mu)=\omega$, $\mathrm{cf}([\lambda]^{\aleph_0},\subseteq)=\lambda$ for cofinally many $\lambda<\mu$ and $\square^{***}_{\aleph_1,\mu}$ holds. Then for any sufficiently large regular χ and $x\in\mathscr{H}(\chi)$, there is a matrix $\langle M_{\alpha,i}:\alpha<\mu^+, i<\omega\rangle$ such that*

(1) *$M_{\alpha,i}\prec\mathscr{H}(\chi)$, $x\in M_{\alpha,i}$, $\omega_1\subseteq M_{\alpha,i}$ and $|M_{\alpha,i}|<\mu$ for all $\alpha<\mu^+$ and $i<\omega$;*
(2) *$\langle M_{\alpha,i}:i<\omega\rangle$ is an increasing sequence for each $\alpha<\mu^+$;*
(3) *If $\alpha<\mu^+$ and $\mathrm{cf}(\alpha)\geqslant\omega_1$, then there is an $i^*<\omega$ such that for every $i^*\leqslant i<\omega$, $[M_{\alpha,i}]^{\aleph_0}\cap M_{\alpha,i}$ is cofinal in $\langle[M_{\alpha,i}]^{\aleph_0},\subseteq\rangle$;*
(4) *Let $M_\alpha=\bigcup_{i<\omega}M_{\alpha,i}$ for $\alpha<\mu^+$. Then $M_\alpha\prec\mathscr{H}(\chi)$ (by (1) and (2)). Moreover $\langle M_\alpha:\alpha<\mu^+\rangle$ is continuously increasing and $\mu^+\subseteq\bigcup_{\alpha<\mu^+}M_\alpha$.*

For a complete Boolean algebra B and $X\subseteq B$, let us denote by $\langle X\rangle^{\mathrm{cm}}_B$ the complete subalgebra of B generated completely by X. The following theorem shows that in a neat model (in the sence of Section 1) if every countably generated c.c.c. Boolean algebra has the weak Freese-Nation property, then every c.c.c. Boolean algebra (without the restriction on the size of its set of generators) has the weak Freese-Nation property.

Theorem 5.3. *Let λ be a cardinal. Suppose that for every $\mu<\lambda$ with $\mu>\mathrm{cf}(\mu)=\omega$, we have*:
(i) $\mathrm{cf}([\mu]^{\aleph_0},\subseteq)=\mu^+$;
(ii) $\Box^{***}_{\aleph_1,\mu}$.
Then for any c.c.c. complete Boolean algebra B with a set of complete generators of size $<\lambda$, B has the weak Freese-Nation property if and only if every complete subalgebra of B generated completely by a countable subset of B has the weak Freese-Nation property.

Proof. The "only if" part of the theorem follows from Proposition 1.3(3). The "if" part of the theorem is proved by induction on the minimal cardinality of a subset X of B completely generating B.

If X is countable, then there is nothing to prove. Let $\mu=|X|<\lambda$ and suppose that we have the theorem for any c.c.c. complete Boolean algebra with a set of complete generators of size $<\mu$.

If $\mathrm{cf}(\mu)>\omega$, let $X=\{x_\alpha:\alpha<\mu\}$ and $B_\beta=\langle\{x_\alpha:\alpha<\beta\}\rangle^{\mathrm{cm}}_B$ for $\beta<\mu$. B_β is a complete subalgebra of B and hence $B_\beta\leqslant_\sigma B$ for all $\beta<\mu$. By induction hypothesis, every B_β, $\beta<\mu$ has the weak Freese-Nation property. By the c.c.c. of B, we have $B_\gamma=\bigcup_{\beta<\gamma}B_\beta$ for all limit $\gamma<\mu$ with $\mathrm{cf}(\gamma)>\omega$ and $B=\bigcup_{\beta<\mu}B_\beta$. By Proposition 1.3(4), it follows that B has the weak Freese-Nation property.

If $\mathrm{cf}(\mu)=\omega$, then there is $\langle M_{\alpha,i}:\alpha<\mu^+,i<\omega\rangle$ as in the previous theorem for $x=\langle B,X\rangle$.

For $\alpha<\mu^+$ and $i<\omega$, let $B_{\alpha,i}=\langle B\cap M_{\alpha,i}\rangle^{\mathrm{cm}}_B$ and $B_\alpha=\bigcup_{i<\omega}B_{\alpha,i}$.

Claim 5.3.1. *For every $\alpha<\mu^+$, B_α has the weak Freese-Nation property and $B_\alpha\leqslant_\sigma B$.*

Proof. For every $i<\omega$, $B_{\alpha,i}$ has the weak Freese-Nation property by induction hypothesis. Since $B_{\alpha,i}$ is a complete subalgebra of B for every $i<\omega$ it follows that $B_\alpha=\bigcup_{i<\omega}B_{\alpha,i}\leqslant_\sigma B$. Also by Proposition 1.3(4) it follows that B_α has the weak Freese-Nation property. □

Claim 5.3.2. *If $\gamma<\mu^+$ and $\mathrm{cf}(\gamma)>\omega$, then $B_\gamma=\bigcup_{\alpha<\gamma}B_\alpha$.*

Proof. Suppose that $a\in B_\gamma$. Then, by the c.c.c. of B, there is an $i<\omega$ and $s\in[B\cap M_{\gamma,i}]^{\aleph_0}$ such that $a\in\langle s\rangle^{\mathrm{cm}}_B$. By (3) in Theorem 5.2, we may assume that $s\in M_{\gamma,i}$. By (4), there is $\alpha<\gamma$ and $j<\omega$ such that $s\in M_{\alpha,j}$. It follows that $s\subseteq M_{\alpha,j}$ and hence $a\in B_{\alpha,j}\subseteq B_\alpha$. □

Claim 5.3.3. $B=\bigcup_{\alpha<\mu^+}B_\alpha$.

Proof. By the last statement of (4) in Theorem 5.2 and (i), $[X]^{\aleph_0}\cap\bigcup_{\alpha<\mu^+}M_\alpha$ is cofinal in $\langle[X]^{\aleph_0},\subseteq\rangle$.

Suppose now that $a\in B$. Then by the c.c.c. of B, there is a countable $s\in[X]^{\aleph_0}$ such that $a\in\langle s\rangle^{\mathrm{cm}}_B$. By the remark above, we may assume that $s\in\bigcup_{\alpha<\mu^+}M_\alpha$, say $s\in M_{\alpha^*,i^*}$ for some $\alpha^*<\mu^+$ and $i^*<\omega$. Then $s\subseteq B\cap M_{\alpha^*,i^*}$ and hence $a\in B_{\alpha^*,i^*}\subseteq B_{\alpha^*}$. □

Now by Theorem 1.3(4) and the claims above, it follows that B has the weak Freese-Nation property. □

Corollary 5.4. *Let* λ *be as in Theorem* 5.3 *and* $\omega \leqslant \kappa < \lambda$. *Then the following are equivalent*:
(a) $\langle \mathscr{P}(\omega), \subseteq \rangle$ *has the weak Freese-Nation property*;
(b) $\mathbb{C}(\kappa)$ *has the weak Freese-Nation property*;
(c) $\mathbb{R}(\kappa)$ *has the weak Freese-Nation property.*

Proof. $(1) \Rightarrow (0)$ and $(2) \Rightarrow (0)$ follow from Proposition 1.3(3).

For $(0) \Rightarrow (2)$ assume that $\langle \mathscr{P}(\omega), \subseteq \rangle$ has the weak Freese-Nation property. Since $\mathbb{R}(\kappa)$ has the set $Clop({}^{\kappa}2)$ of complete generators of cardinality κ, it is enough by Theorem 5.3 to show that every countably generated subalgebra of $\mathbb{R}(\kappa)$ has the weak Freese-Nation property. Let A be such an algebra. Then there is $X \in [\kappa]^{\aleph_0}$ such that A is a complete subalgebra of $\mathbb{R}(X)$. Since $\mathbb{R}(X)$ has the weak Freese-Nation property by Proposition 5.1, it follows by Proposition 1.3(3) that A also has the weak Freese-Nation property.

The proof of $(0) \Rightarrow (1)$ is similar. □

Note that the conditions on λ in Theorem 5.3 hold vacuously for $\lambda = \aleph_\omega$. Hence we obtain the following as a special case of the corollary above:

Corollary 5.5. *The following are equivalent* (*in* ZFC):
(a) $\langle \mathscr{P}(\omega), \subseteq \rangle$ *has the weak Freese-Nation property*;
(b) $\mathbb{C}(\aleph_n)$ *has the weak Freese-Nation property for some/all* $n \in \omega$;
(c) $\mathbb{R}(\aleph_n)$ *has the weak Freese-Nation property for some/all* $n \in \omega$.

6. Chang's conjecture

In this section, we give the negative answer to Problem 3 mentioned in the introduction (Theorem 6.1). We also show that Corollary 5.5 in the previous section is an optimal result in ZFC (Theorem 6.2).

Theorem 6.1. *Suppose that* V_0 *is a transitive model of* ZFC *such that*

$$V_0 \models \mathrm{GCH} + (\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0).$$

Let P *be a c.c.c. partial ordering in* V_0 *of cardinality* $\aleph_1$ *adding a dominating real. Let* $\eta \in {}^{\omega}\omega$ *be a dominating real over* V_0 *generically added by* P *and let* $V_1 = V_0[\eta]$. *Note that* $V_1 \models \mathrm{GCH}$. *In* V_1 *let* $Q = \mathrm{Fn}(\aleph_\omega, \omega)$ *and let* $\dot{Q}$ *be a corresponding* P*-name. Then we have*:

$$V_1 \models \Vdash_Q \text{“}\langle \mathscr{P}(\omega), \subseteq \rangle \textit{ does not have the weak Freese-Nation property”}.$$

Proof. In the proofs of this and the next theorem, we shall denote by a dotted symbol a name of an element in a generic extension. By the same symbol without the dot, we denote the corresponding element in a fixed generic extension. Without further mention, we shall identify $P * \dot{Q}$ names with the corresponding Q-name in V_1 and vice versa.

Now toward a contradiction, assume that there is a Q-name $\dot{F}$ in V_1 such that

$$(\otimes) \qquad V_1 \models \Vdash_Q \text{“}\dot{F} \text{ is a weak Freese-Nation mapping on } \langle {}^{\omega}\omega, \leqslant^* \rangle\text{”}.$$

Let $\dot{\varphi}$ be a $P * \dot{Q}$-name of the function $\aleph_\omega \to \omega$ generically added by Q over V_1. Let $V_2 = V_1[\varphi]$. By GCH, we can find in V_0 a scale $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ in $\langle \prod_{n \in \omega} \aleph_n, \leqslant^* \rangle$. Without loss of generality we may assume that for every $\alpha < \aleph_{\omega+1}$ and $n \in \omega$, $f_\alpha(n) \in \aleph_n \backslash \aleph_{n-1}$ (where we set $\aleph_{-1} = \emptyset$). For each $\alpha \in \aleph_{\omega+1}$, let $g_\alpha : \omega \to \omega$ be defined by

$$g_\alpha = \varphi \circ f_\alpha.$$

Let χ be sufficiently large and let $N \prec \langle \mathscr{H}(\chi), \in \rangle$ be such that N contains everything we need in the course of the proof—in particular $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle \in N$, $|\aleph_\omega \cap N| = \aleph_0$ and $otp(\aleph_{\omega+1} \cap N) = \omega_1$. The last two conditions are possible by the assumption of $V_0 \models (\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$.

In V_0, let $\{\xi_{n,k} : k \in \omega\}$ be an enumeration of $(\aleph_n \backslash \aleph_{n-1}) \cap N$ for each $n \in \omega$. Here again, we use the convention that $\aleph_{-1} = \emptyset$. Let $\dot{h}^*$ be a $P * \dot{Q}$-name of an element of ${}^{\omega}\omega$ such that

$$\Vdash_{P*\dot{Q}} \text{“}\dot{h}^*(n) = \max\{\dot{\varphi}(\xi_{n,k}) : k \leqslant \dot{\eta}(n)\} \quad \text{for all } n \in \omega\text{”}.$$

Claim 6.1.1. *For every* $\alpha \in \aleph_{\omega+1} \cap N$, $\Vdash_{P*\dot{Q}}$ “$\dot{g}_\alpha \leqslant^* \dot{h}^*$”.

Proof. Since $\alpha \in N$ we have $f_\alpha \in N$. Let $e_\alpha : \omega \to \omega$ be the function in V_0 such that $f_\alpha(n) = \xi_{n,e_\alpha(n)}$ for all $n \in \omega$. Since η is dominating, there is $n^* \in \omega$ such that

$$V_1 \models e_\alpha \restriction (\omega \backslash n^*) \leqslant \eta \restriction (\omega \backslash n^*).$$

By definition of $\dot{h}^*$, it follows that

$$V_2 \models g_\alpha(n) = \varphi \circ f_\alpha(n) = \varphi(\xi_{n,e_\alpha(n)}) \leqslant h^*(n)$$

for all $n \geqslant n^*$. $\square$

Let $N_0 = N$, $N_1 = N_0[\eta]$ and $N_2 = N_1[\varphi]$. Then we have $N_2 \prec \mathscr{H}(\chi)[\eta][\varphi]$.

Let $\dot{h}_l \in N_0$, $l \in \omega$ be $P * \dot{Q}$-names such that

$$\Vdash_{P*\dot{Q}} \text{“}\{\dot{h}_l : l \in \omega\} = \dot{F}(\dot{h}^*) \cap \dot{N}_2\text{”}.$$

For $l \in \omega$, let $S_l \in [\aleph_\omega]^{\aleph_0} \cap N_0$ be such that, regarding $\dot{h}_l$ as a P-name of $\dot{Q}$-name,

$$V_0 \models \Vdash_P \text{“}\dot{h}_l \text{ is a } \mathrm{Fn}(S_l, \omega)\text{-name”}.$$

This is possible since P has the c.c.c., $\Vdash_P$ "$\dot{Q}$ has the c.c.c." and by the elementarity of N_0. For each $l\in\omega$, let $s_l\in\prod_{n\in\omega}\aleph_n\cap N_0$ be defined (in N_0) by $s_l(0)=0$ and

$$s_l(m)=\sup S_l\cap\aleph_m$$

for $m\in\omega\setminus\{0\}$. Since $\langle f_\alpha:\alpha<\aleph_{\omega+1}\rangle$ was taken to be a scale on $\langle\prod_{n\in\omega}\aleph_n,\leqslant^*\rangle$, there is $\alpha_l\in\aleph_{\omega+1}\cap N_0$ for each $l\in\omega$ such that $s_l\leqslant^* f_{\alpha_l}$. Since $otp(\aleph_{\omega+1}\cap N_0)=\omega_1$, we can find an $\alpha^*\in\aleph_{\omega+1}\cap N_0$ such that $\sup\{\alpha_l:l\in\omega\}\leqslant\alpha^*$.

Now, by the choice of $\dot{h}_l$, $l\in\omega$, the following claim together with Claim 6.1.1 contradicts to ($\otimes$) and hence proves the theorem:

Claim 6.1.2. $V_1\models\Vdash_Q$ "$\dot{g}_{\alpha^*}\not\leqslant^*\dot{h}_l$ *for all* $l\in\omega$".

Proof. Assume to the contrary that, in V_1, we have

$$q\Vdash_Q \text{“}\dot{g}_{\alpha^*}\restriction(\omega\setminus k)\leqslant\dot{h}_l\restriction(\omega\setminus k)\text{”}$$

for some $q\in Q$ and k, $l\in\omega$. We may assume that

$$s_l\restriction(\omega\setminus k)<f_{\alpha^*}\restriction(\omega\setminus k)$$

and $\sup(\mathrm{dom}(q))<f_{\alpha^*}(m)$ for all $m\in\omega\setminus k$ as well. Working further in V_1, let $m^*=k+1$ and let $q'\leqslant q$ be such that

$$q'\Vdash_Q \text{“}\dot{h}_l(m^*)=j^*\text{”}$$

for some $j^*\in\omega$. Then $q''=q\cup(q'\restriction S_l)$ also forces the same statement. Since $f_{\alpha^*}(m^*)\in\aleph_{m^*}\setminus(s_l(m^*)\cup\mathrm{dom}(q))$, we have

$$f_{\alpha^*}(m^*)\notin\mathrm{dom}(q'').$$

Hence

$$q^*=q''\cup\{\langle f_{\alpha^*}(m^*),j^*+1\rangle\}$$

is an element of Q and $q^*\leqslant q''\leqslant q$. But

$$q^*\Vdash_Q \text{“}\dot{g}_{\alpha^*}(m^*)=\dot{\varphi}\circ f_{\alpha^*}(m^*)=j^*+1>j^*=\dot{h}_l(m^*)\text{”}.$$

This is a contradiction. □

A similar but slightly simpler proof yields the following:

Theorem 6.2. *Suppose that* $V_0,P,\eta,\dot{\eta},V_1$ *are as in the previous theorem. Then*

$V_1\models\mathbb{C}(\aleph_\omega)$ *does not have the weak Freese-Nation property.*

Proof. Suppose to the contrary that $F\in V_1$ is a weak Freese-Nation mapping on $\mathbb{C}(\aleph_\omega)$. Let $X=\{x_\xi:\xi<\aleph_\omega\}$ be the canonical free subset of $\mathbb{C}(\aleph_\omega)$ completely generating the whole $\mathbb{C}(\aleph_\omega)$. We have $X\in V_0$.

Let $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ be as in the proof of the previous theorem. For $n \in \omega$, let $c_n = x_n \cdot -\sum_{m<n} x_m$. Note that $\{c_n : n \in \omega\}$ is a partition of $\mathbb{C}(\aleph_\omega)$. For each $\alpha < \aleph_{\omega+1}$ and $n \in \omega$, let

$$b_{\alpha,n} = \sum_{m>n} (x_{f_\alpha(m)} \cdot c_m).$$

Let χ be sufficiently large and let $N \prec \langle \mathscr{H}(\chi), \in \rangle$ be such that N contains everything we need in the course of the proof, $|\aleph_\omega \cap N| = \aleph_0$, and $otp(\aleph_{\omega+1} \cap N) = \omega_1$. The last two conditions are possible because of $V_0 \models (\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$. In V_0, let $\{\xi_{n,k} : k \in \omega\}$ be an enumeration of $(\aleph_n \setminus \aleph_{n-1}) \cap N$ for each $n \in \omega$. In $V_1 = V_0[\eta]$ let

$$b^* = \sum_{n \in \omega} \left(c_n \cdot \sum_{l \leqslant \eta(n)} x_{\xi_{n,l}} \right).$$

Claim 6.2.1. *For every $\alpha \in \aleph_{\omega+1} \cap N$, there is $n_\alpha \in \omega$ such that $b_{\alpha,n_\alpha} \leqslant b^*$.*

Proof. In V_0, let $t_\alpha \in {}^\omega\omega$ be such that $f_\alpha(n) = \xi_{n,t_\alpha(n)}$ for all $n \in \omega$. Note that t_α is well-defined by $f_\alpha \in N$. Since η is dominating, there is an $n_\alpha \in \omega$ such that $t_\alpha \restriction (\omega \setminus n_\alpha) \leqslant \eta \restriction (\omega \setminus n_\alpha)$. By definition of $b_{\alpha,n}$ and b^*, it follows that $b_{\alpha,n_\alpha} \leqslant b^*$. $\square$

Now, let $N_1 = N[\eta]$. Note that we have $N_1 \prec \mathscr{H}(\chi)^{V_1}$. Let $\langle d_l : l \in \omega \rangle$ be an enumeration of $F(b^*) \cap \mathbb{C}(\aleph_\omega) \restriction b^* \cap N_1$ and $\dot{d}_l$, $l \in \omega$ be corresponding P-names. Since P is proper, we can choose these names so that $\{\dot{d}_l : l \in \omega\} \subseteq N$.

By c.c.c. of P and elementarity of N, we can find $S_l \in [\aleph_\omega]^{\aleph_0} \cap N$ for each $l \in \omega$ such that

$$\Vdash_P \text{“}\dot{d}_l \in \langle \{x_\xi : \xi \in S_l\} \rangle^{\mathrm{cm}}_{\mathbb{C}(\aleph_\omega)}\text{”}.$$

For $l \in \omega$, let $s_l \in \prod_{n\in\omega} \aleph_n \cap N$ be defined by $s_l(0) = 0$ and

$$s_l(m) = \sup \aleph_m \cap S_l$$

for $m \in \omega \setminus \{0\}$. Since $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ was taken to be a scale on $\langle \prod_{n\in\omega} \aleph_n, \leqslant^* \rangle$, there is $\alpha_l^* \in \aleph_{\omega+1} \cap N$ for each $l \in \omega$ such that $s_l <^* f_{\alpha_l^*}$. By $otp(\aleph_{\omega+1} \cap N) = \omega_1$, we can find an $\alpha^* \in \aleph_{\omega+1} \cap N$ such that $\alpha_l^* < \alpha^*$ for all $l \in \omega$.

Since $b_{\alpha^*,n_{\alpha^*}} \in N_1$ and $b_{\alpha^*,n_{\alpha^*}} \leqslant b^*$ by Claim 6.2.1, the following claim gives the desired contradiction:

Claim 6.2.2. *In $V_1, b_{\alpha^*,n_{\alpha^*}} \nleqslant d_l$ for all $l \in \omega$.*

Proof. For $l \in \omega$, let $m \in \omega$ be such that
(1) $n_{\alpha^*} < m$, and
(2) $s_l(m) < f_{\alpha^*}(m)$.

Note that we have $c_m \nleqslant d_l$, since $d_l \leqslant b^*$ and $c_m \nleqslant b^*$ by definition of b^*. By (2), $f_{\alpha^*}(m) \notin S_l$ (note that we also have $\omega < f_{\alpha^*}(m)$). Hence $x_{f_{\alpha^*}(m)} \cdot c_m \nleqslant d_l$.

But $x_{f_{\alpha^*}(m)} \cdot c_m \leqslant b_{\alpha^*,n_{\alpha^*}}$ by (1) and by the definition of $b_{\alpha^*,n_{\alpha^*}}$. It follows that $b_{\alpha^*,n_{\alpha^*}} \not\leqslant d_l$. □

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