TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 5, Pages 2023–2047
S 0002-9947(99)02270-9
Article electronically published on April 13, 1999

THE DISTRIBUTIVITY NUMBERS OF $\mathcal{P}(\omega)$/FIN AND ITS SQUARE

SAHARON SHELAH AND OTMAR SPINAS

Abstract. We show that in a model obtained by forcing with a countable support iteration of Mathias forcing of length ω_2, the distributivity number of $\mathcal{P}(\omega)/\text{fin}$ is ω_2, whereas the distributivity number of r.o.$(\mathcal{P}(\omega)/\text{fin})^2$ is ω_1. This answers a problem of Balcar, Pelant and Simon, and others.

Introduction

A complete Boolean algebra $(B, \leq)$ is called κ-distributive, where κ is a cardinal, if and only if for every family $\langle u_{\alpha i} : i \in I_\alpha, \alpha < \kappa \rangle$ of members of B

$$\prod_{\alpha<\kappa} \sum_{i \in I_\alpha} u_{\alpha i} = \sum_{f \in \prod_{\alpha<\kappa} I_\alpha} \prod_{\alpha<\kappa} u_{\alpha f(\alpha)}$$

holds. It is well-known (see [J, p.152]) that every partially ordered set $(P, \leq)$ which is separative can be densely embedded in a unique complete Boolean algebra, which is usually denoted with r.o.(P). The distributivity number of $(P, \leq)$ is defined as the least κ such that r.o.(P) is not κ-distributive. It is well-known (see [J, p.158]) that the following four statements are equivalent:

1. (1) r.o.(P) is κ-distributive.
2. (2) The intersection of κ open dense sets in P is dense.
3. (3) Every family of κ maximal antichains of P has a refinement.
4. (4) Forcing with P does not add a new subset of κ.

The distributivity number of the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ is denoted with $\mathfrak{h}$. This cardinal was introduced in [BPS], where it has been shown that $\omega_1 \leq \mathfrak{h} \leq 2^\omega$ and the axioms of ZFC do not decide where exactly $\mathfrak{h}$ sits in this interval.

For λ a cardinal let $\mathfrak{h}(\lambda)$ be the distributivity number of $(\mathcal{P}(\omega)/\text{fin})^\lambda$, where by $(\mathcal{P}(\omega)/\text{fin})^\lambda$ we mean the full λ-product of $\mathcal{P}(\omega)/\text{fin}$ in the forcing sense. That is, $p \in (\mathcal{P}(\omega)/\text{fin})^\lambda$ if and only if $p : \lambda \to \mathcal{P}(\omega)/\text{fin} \setminus \{0\}$. The ordering is coordinatewise.

Trivially, $\mathfrak{h}(\lambda) \geq \mathfrak{h}(\gamma)$ holds whenever $\lambda < \gamma$. In fact, if $\langle D_\alpha : \alpha < \mathfrak{h}(\lambda) \rangle$ is a family of dense open subsets of $(\mathcal{P}(\omega)/\text{fin})^\lambda$ whose intersection is not dense, then, letting $D'_\alpha = \{p \in (\mathcal{P}(\omega)/\text{fin})^\gamma : p{\restriction}\lambda \in D_\alpha\}$, clearly the D'_α are dense open in $(\mathcal{P}(\omega)/\text{fin})^\gamma$ and their intersection is not dense.

Since $\mathfrak{h} \leq 2^\omega$, this implies that under CH the sequence $\langle \mathfrak{h}(\lambda) : \lambda \in \textbf{Card} \rangle$ is constant with value $\aleph_1$. In [BPS, 4.14(2)] we read: "We do not know of any further

Received by the editors February 12, 1997 and, in revised form, November 5, 1997.
1991 *Mathematics Subject Classification.* Primary 03E05, 06E05.
The first author is supported by the Basic Research Foundation of the Israel Academy of Sciences; publication 494. The second author is supported by the Swiss National Science Foundation.

properties of this sequence." The most elementary question which arises, and which was explicitly asked by several people, is whether consistently this sequence is not constant. In this paper we give a positive answer by proving the consistency of $\mathfrak{h}(2) < \mathfrak{h}$ with ZFC. In a sequel paper (see [ShSp]), for every $n < \omega$ we will construct a model for $\mathfrak{h}(n+1) < \mathfrak{h}(n)$. In all these models the continuum will be $\aleph_2$, and hence the above sequence will be two-valued. The question of whether more values are possible is tied up with the well-known problem of how to make the continuum bigger than $\aleph_2$, not using finite-support forcing iterations.

The natural forcing to increase $\mathfrak{h}$ is Mathias forcing. We will show that in a model obtained by forcing with a countable support iteration of length ω_2 of Mathias forcing over a model for CH, $\mathfrak{h}(2)$ remains ω_1.

There exists an equivalent game-theoretic definition of $\mathfrak{h}(\lambda)$, which we will use in the sequel. For any ordinal α and any partial ordering P let us consider the following game $G(P, \alpha)$ of length α: Player I and II alternately choose elements $p^I_\beta, p^{II}_\beta \in P$, $\beta < \alpha$, such that for $\beta < \beta' < \alpha$: $p^I_\beta \geq p^{II}_\beta \geq p^I_{\beta'} \geq p^{II}_{\beta'}$. In the end, player II wins if and only if the sequence of moves has no lower bound (this might happen if at some step $\beta < \alpha$, player I does not have a legal move).

We claim that $\mathfrak{h}(\lambda)$ is the minimal cardinal κ such that player II has a winning strategy in the game $G((\mathcal{P}(\omega)/\mathrm{fin})^\lambda, \kappa)$. For one direction, suppose we are given dense open sets $\langle D_\alpha : \alpha < \kappa\rangle$ in $(\mathcal{P}(\omega)/\mathrm{fin})^\lambda$ such that $D = \bigcap\{D_\alpha : \alpha < \kappa\}$ is not dense. By the homogeneity of $(\mathcal{P}(\omega)/\mathrm{fin})^\lambda$ we may assume that D is empty. In fact, if D contains no extension of p, choose $\langle f_\alpha : \alpha < \lambda\rangle$ such that $f_\alpha : p(\alpha) \to \omega$ is one-to-one and onto. Replace D_α by $D'_\alpha = \{\langle f_\alpha[q(\alpha)] : \alpha < \lambda\rangle : q \in D_\alpha \text{ and } q \leq p\}$. Then the D'_α are open dense and their intersection is empty. Now define a strategy for II in $G((\mathcal{P}(\omega)/\mathrm{fin})^\lambda, \kappa)$ as follows: In his αth move let II play $p^{II}_\alpha \in D_\alpha$ such that $p^{II}_\alpha \leq p^I_\alpha$. This is clearly a winning strategy.

Conversely, let σ be a winning strategy for II in $G((\mathcal{P}(\omega)/\mathrm{fin})^\lambda, \kappa)$. We shall make use of (3) above. We define maximal antichains $\langle \mathcal{A}_\alpha : \alpha < \gamma \leq \kappa\rangle$ in $(\mathcal{P}(\omega)/\mathrm{fin})^\lambda$ such that if $\alpha < \beta < \gamma$, then $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$, and for every $p_\beta \in \mathcal{A}_\beta$, if $p_\alpha \in \mathcal{A}_\alpha$ is the unique member with $p_\alpha \geq p_\beta$, then $\langle p_\alpha : \alpha \leq \beta\rangle$ are responses by σ in an initial segment of a play consistent with σ. Suppose $\langle \mathcal{A}_\alpha : \alpha < \delta\rangle$ has been constructed and $\delta < \kappa$ is a limit. If this sequence has no refinement, we are done, otherwise let $\mathcal{B}$ be one. Now it is easy to construct $\mathcal{A}_\delta$ as desired, namely consisting of responses by σ to plays of length $\delta + 1$ with the last coordinate an extension of a member of $\mathcal{B}$. If δ is a successor, construct $\mathcal{A}_\delta$ similarly, where now $\mathcal{B} = \mathcal{A}_{\delta-1}$. It is clear that this construction stops at some $\gamma \leq \kappa$, since otherwise, we could find a play consistent with σ in which II loses.

1. Mathias forcing and Ramsey ultrafilters

Conditions of Mathias forcing are pairs $(u, a) \in [\omega]^{<\omega} \times [\omega]^\omega$ such that $\max(u) < \min(a)$. The ordering is defined as follows: $(u, a) \leq (v, b)$ if and only if $v \subseteq u \subseteq v \cup b$ and $a \subseteq b$. Mathias forcing will be denoted by Q in this paper. Given $p \in Q$ we will write $p = (u^p, a^p)$.

If D is a filter on ω containing no finite sets, then $Q(D)$ denotes Mathias forcing relativized to D; that is, $(u, a) \in Q(D)$ iff $(u, a) \in Q$ and $a \in D$, and the order is as for Q. Note that any two conditions in $Q(D)$ with the same first coordinate are compatible. Therefore, $Q(D)$ is σ-centered; that is, a countable union of centered subsets. It is well-known that Mathias forcing can be decomposed as $Q = Q' * \underset{\sim}{Q}''$, such that Q' is $\mathcal{P}(\omega)$/fin and $\underset{\sim}{Q}'' = Q(\underset{\sim}{G}')$, where $\underset{\sim}{G}'$ is a name for the generic filter added by $\mathcal{P}(\omega)$/fin. In fact, since Q' is σ-closed and hence does not add reals, the map sending (u, a) to $(a, (u, a))$ is a dense embedding of Q in $Q' * \underset{\sim}{Q}''$. The generic filter for $\underset{\sim}{Q}''$, which determines the Mathias real, will be denoted by $\underset{\sim}{G}''$. Here and in the sequel we do not distinguish between a member of $\mathcal{P}(\omega)$/fin and its representatives in $\mathcal{P}(\omega)$. **The above notation will be used throughout the paper.**

The Rudin-Keisler order $\leq_{\mathrm{RK}}$ for ultrafilters on ω is defined by: $D \leq_{\mathrm{RK}} U$ iff there exists a function $f : \omega \to \omega$ such that $D = \{X \subseteq \omega : f^{-1}[X] \in U\}$. In this case D is called a projection of U and it is denoted by $f_*(U)$. If $D \leq_{\mathrm{RK}} U$ and $U \leq_{\mathrm{RK}} D$, we call U and D RK-equivalent. By a result of M. E. Rudin (see [R] or [J, Ex. 38.2., p.480]), in this case there exists a bijection $f : \omega \to \omega$ such that $D = f_*(U)$. Then we say that D and U are RK-equivalent by f.

A nonprincipal ultrafilter D on ω is called a Ramsey ultrafilter iff for every $n, k < \omega$ and every partition $F : [\omega]^n \to k$ there exists $H \in D$ homogeneous for F; that is, $F{\restriction}[H]^n$ is constant. An equivalent definition is as follows (see [J, p.478]): D as above is Ramsey iff for every partition of ω into pieces not in the filter, there exists a filter set which meets each piece at most once. Clearly such a filter is a p-point; that is, for every countable subset of the filter there exists a filter set which is almost contained in every member of it.

We shall use yet another equivalent definition of Ramsey ultrafilter. Let D be a nonprincipal ultrafilter. A function $f \in {}^\omega\omega$ is called unbounded modulo D if $\{n : f(n) > k\} \in D$ for every $k < \omega$; moreover f is called one-to-one modulo D if its restriction to some member of D is one-to-one. Then D is a Ramsey ultrafilter iff every function unbounded modulo D is one-to-one modulo D (see [J, Ex. 38.1., p.479]).

In the following lemma, a forcing P is called ${}^\omega\omega$-bounding iff every function in ${}^\omega\omega$ in the extension V^P is bounded by some function in V. Moreover, an ultrafilter D in V is said to generate an ultrafilter in V^P iff the collection of subsets of ω which belong to V^P and contain an element of D is an ultrafilter in V^P.

Lemma 1.1. *Suppose D_1, D_2 are Ramsey ultrafilters which are not RK-equivalent. Let P be a proper, ${}^\omega\omega$-bounding forcing such that for every filter $G \subseteq P$ which is P-generic over V, D_1 and D_2 generate ultrafilters in $V[G]$. Then in $V[G]$, D_1 and D_2 generate Ramsey ultrafilters which are not RK-equivalent.*

Proof. First, we show that D_1, D_2 are Ramsey ultrafilters in $V[G]$. Here and in the sequel, we denote the ultrafilters generated by D_1, D_2 in $V[G]$ by D_1, D_2 as well. By properness, every $X \in [V]^\omega \cap V[G]$ is covered by a countable set in V. Hence D_1, D_2 generate p-points in $V[G]$. In $V[G]$, let $\langle a_n : n < \omega\rangle$ be a partition of ω such that $a_n \notin D_1$, for all $n < \omega$. As D_1 is a p-point, there exists $X \in D_1 \cap V$ such that $|X \cap a_n| < \omega$, for all $n < \omega$. Let $f \in {}^\omega\omega$ be defined by $f(n+1) > f(n)$ is minimal such that every a_k with $a_k \cap f(n) \neq \emptyset$ satisfies $a_k \cap (X \setminus f(n+1)) = \emptyset$. As P is ${}^\omega\omega$-bounding, we may find a strictly increasing $g \in {}^\omega\omega \cap V$ such that for every $n < \omega$, $[g(n), g(n+1)) \cap \operatorname{range}(f)$ has at least one element. D_1 contains exactly one of the three sets $\bigcup\{[g(3n+i), g(3n+i+1)) : n < \omega\}$, where $i \in \{0, 1, 2\}$. We denote this set by Y. Since D_1 is Ramsey in V, there exists $Z \in D_1 \cap V$ such that $Z \subseteq X \cap Y$ and $|[g(n), g(n+1)) \cap Z| \leq 1$, for all $n < \omega$. We have to verify that $|Z \cap a_n| \leq 1$, for every n. Let $k, l \in Z \cap a_n$. Then $k, l \in X \cap a_n$. By construction of f, there is n_1 such that $X \cap a_n \subseteq [f(n_1), f(n_1+2))$. By construction of g and since f is increasing, there is n_2 such that $f(n_1), f(n_1+1), f(n_1+2) \in [g(n_2), g(n_2+3))$. By construction of Z, there is $n_3 \in \{n_2, n_2+1, n_2+2\}$ such that $k, l \in [g(n_3), g(n_3+1))$. Since $|[g(n_3), g(n_3+1)) \cap Z| \leq 1$, we have that $k = l$.

Second, we show that D_1, D_2 do not become RK-equivalent in $V[G]$. Otherwise, in $V[G]$ we had a bijection $f : \omega \to \omega$ such that $f_*(D_1) = D_2$. Let $f_1 \in {}^\omega\omega$ be defined such that $f_1(n+1) > f_1(n)$ is minimal with

$$f_1(n+1) \geq \max[\{f(k) : k < f_1(n)\} \cup \{f^{-1}(k) : k < f_1(n)\}].$$

As P is ${}^\omega\omega$-bounding, we may find a strictly increasing $g \in {}^\omega\omega \cap V$ such that for every $n < \omega$, $[g(n), g(n+1)) \cap$ range(f_1) has at least two elements. Each of D_1 and D_2 contains one of the three sets

$$C_i = \bigcup\{[g(3n+i), g(3n+i+1)) : n < \omega\},$$

where $i \in \{0,1,2\}$. Suppose $C_i \in D_1$ and $C_j \in D_2$. By Ramseyness in V, there exist $X \in D_1 \cap V$, $Y \in D_2 \cap V$ such that $X \subseteq C_i$, $Y \subseteq C_j$ and $|X \cap [g(3n+i), g(3n+i+1))| \leq 1$, $|Y \cap [g(3n+j), g(3n+j+1))| \leq 1$, for all $n < \omega$. Let x_n be the unique element of $X \cap [g(3n+i), g(3n+i+1))$ in the case that this set is not empty, and let y_n be the unique element of $Y \cap [g(3n+i-1), g(3n+i+1))$ if this set is not empty. Note that by construction, $f(x_n) \in [g(3n+i-1), g(3n+i+1))$. Hence $\{x_n : f(x_n) = y_n\} \in D_1$, since otherwise, f would map a set in D_1 to a set disjoint to a member of D_2. Consequently, $\{y_n : f(x_n) = y_n\} \in D_2$. Choose $X_1 \in D_1 \cap V$ and $Y_1 \in D_2 \cap V$ such that $X_1 \subseteq \{x_n : f(x_n) = y_n\}$ and $Y_1 \subseteq \{y_n : f(x_n) = y_n\}$. Define

$$f' = \{(x,y) : \exists n (x \in [g(3n+i), g(3n+i+1)) \cap X_1 \wedge y \in [g(3n+i-1), g(3n+i+2)) \cap Y_1)\}.$$

Then $f' \in V$ and f' is a map with $\mathrm{dom}(f') = f^{-1}[f[X_1] \cap Y_1] \in D_1$ and $f'(x) = f(x)$ for all $x \in \mathrm{dom}(f')$. Therefore, f' witnesses in V that D_1, D_2 are RK-equivalent, a contradiction. □

In the sequel we shall have the following situation: Given are two models of ZFC, $V_0 \subseteq V_1$, and in V_1 we have D which is an ultrafilter on $([\omega]^\omega)^{V_0}$. That is, $D \subseteq ([\omega]^\omega)^{V_0}$ is a filter and for every $a \in ([\omega]^\omega)^{V_0}$, either $a \in D$ or $\omega \setminus a \in D$. Then we call D Ramsey if every function in V_0 which is unbounded modulo D is one-to-one modulo D. We will say that some real $r \in ([\omega]^\omega)^{V_1}$ induces D if $D = \{a \in ([\omega]^\omega)^{V_0} : r \subseteq^* a\}$.

An easy genericity argument together with the σ-closedness of $\mathcal{P}(\omega)/\mathrm{fin}$ shows that $\Vdash_{\mathcal{P}(\omega)/\mathrm{fin}} \underset{\sim}{G'}$ is a Ramsey ultrafilter.

In [M], Mathias has shown that $r \in [\omega]^\omega$ is Mathias generic over V if and only if r is an almost intersection of a $\mathcal{P}(\omega)/\mathrm{fin}$-generic filter G', that is, $r \subseteq^* a$ for all $a \in G'$. It follows that every infinite subset of a Mathias generic real is Mathias generic as well. This will be used in the proof of the following well-known fact.

Lemma 1.2. *Let $(N, \in)$ be a countable model of ZF^- (in particular, N must be able to prove the above mentioned result of Mathias). If $p \in Q \cap N$ there exists $q \in Q$ such that $q \leq p$, $u^p = u^q$, and for every $a \in [\omega]^\omega$ with $u^q \subseteq a \subseteq u^q \cup a^q$, a is Mathias generic over N. In particular, q is (N,Q)-generic below p.*

Proof. Since N is countable, in V we may find $b \in [\omega]^\omega$ which is Mathias generic over N and contains p in its induced generic filter; that is, $u^p \subseteq b \subseteq u^p \cup a^p$. Let $q = (u^p, b \setminus u^p)$. Then every a as in the lemma is an infinite subset of b, and hence Mathias generic over N. □

2. Outline of the proof

Let V be a model of CH and let $\langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration of Mathias forcing, that is $\forall \alpha < \omega_2$, $\Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha$ is Mathias forcing". **This notation will be kept throughout the paper.**

The following theorem is folklore. In the proof, a set $C \subseteq \omega_2$ will be called ω_1-club if C is unbounded in ω_2 and closed under increasing sequences of length ω_1.

Theorem 2.1. *If G is P_{ω_2}-generic over V, where $V \models CH$, then $V[G] \models \mathfrak{h} = \omega_2$.*

Proof. In $V[G]$ let $\langle D_\nu : \nu < \omega_1 \rangle$ be a family of open dense subsets of $\mathcal{P}(\omega)/\text{fin}$ $\setminus\{0\}$. By a standard Löwenheim-Skolem argument, for every α belonging to some ω_1-club $C \subseteq \omega_2$, for every $\nu < \omega_1$ it is true that $D_\nu \cap V[G_\alpha]$ belongs to $V[G_\alpha]$ and is open dense in $(\mathcal{P}(\omega)/\text{fin})^{V[G_\alpha]} \setminus \{0\}$. Now for a given $A \in (\mathcal{P}(\omega)/\text{fin})^{V[G]} \setminus \{0\}$, by properness and genericity there exists $\alpha \in C$ such that $A \in G(\alpha)'$, where $G(\alpha)$ is the $\underset{\sim}{Q}_\alpha[G_\alpha]$-generic filter determined by G and $G(\alpha)'$ is its first component according to the decomposition of Mathias forcing defined in §1. As $\alpha \in C$, $G(\alpha)'$ clearly meets every D_ν, $\nu < \omega_1$. But now r_α, the $\underset{\sim}{Q}_\alpha$-generic real (determined by $G(\alpha)''$) is below each member of $G(\alpha)'$, hence below A and in $\bigcap_{\nu<\omega_1} D_\nu$. This proves that $\bigcap_{\nu<\omega_1} D_\nu$ is dense. □

The rest of this paper proves:

Theorem 2.2. *In the notation of Theorem 2.1, $V[G] \models \mathfrak{h}(2) = \omega_1$.*

The proof consists of the following two propositions. By S^2_1 we will denote the ordinals in ω_2 of cofinality ω_1. We will tacitly use the well-known results from [B, §5], where it has been shown that for $\alpha < \omega_2$ we can define a quotient forcing $P_{\omega_2}/\underset{\sim}{G}_\alpha$, also denoted by $P_{\alpha\omega_2}$, where $\underset{\sim}{G}_\alpha$ is a P_α-name for the P_α-generic filter.

Proposition 2.3. *There exists an ω_1-club $C \subseteq S^2_1$ such that for every $\alpha \in C$ the following holds: If $\underset{\sim}{r}$ is a $P_{\omega_2}/\underset{\sim}{G}_\alpha$-name such that $\Vdash_{P_{\omega_2}/\underset{\sim}{G}_\alpha}$ "$\underset{\sim}{r}$ induces a Ramsey ultrafilter on $([\omega]^\omega)^{V[\underset{\sim}{G}_\alpha]}$"[1], then there exists a $P_{\omega_2}/\underset{\sim}{G}_\alpha$-name $\underset{\sim}{r}^1$ such that $\Vdash_{P_{\omega_2}/\underset{\sim}{G}_\alpha}$ "$\underset{\sim}{r}^1 \in V[\underset{\sim}{G}_{\alpha+1}]$ and $\underset{\sim}{r}^1$ and $\underset{\sim}{r}$ generate the same ultrafilters on $([\omega]^\omega)^{V[\underset{\sim}{G}_\alpha]}$".*

Proposition 2.4. *Suppose that $V \models CH$ and $\underset{\sim}{r}$ is a Q-name such that $\Vdash_Q$ "$\underset{\sim}{r}$ induces a Ramsey ultrafilter $\mathcal{D}$ on $([\omega]^\omega)^V$". Then $\Vdash_Q$ "$\underset{\sim}{\underset{\sim}{D}}$ and $\underset{\sim}{G}'$ are RK-equivalent by some function $f \in ({}^\omega\omega) \cap V$".*

[1] Added in proof: In addition we have to assume that the filter induced by $\underset{\sim}{r}$ is forced to be a P-filter in $V[\underset{\sim}{G}_{\omega_2}]$, i.e. every countable subset of the filter in $V[\underset{\sim}{G}_{\omega_2}]$ has an almost intersection in the filter.

It is easy to see that Theorem 2.2 follows from Propositions 2.3 and 2.4; fix C as in Proposition 2.3. In $V[G]$ define a winning strategy for player II in the game $G((\mathcal{P}(\omega)/\mathrm{fin})^2, \omega_1)$ as follows:

> Play in such a way that whenever $\langle (p^I_\nu, p^{II}_\nu) : \nu < \omega_1 \rangle$ is a play, there exists $\alpha \in C$ such that $\langle p^{II}_\nu(0) : \nu < \omega_1 \rangle$ and $\langle p^{II}_\nu(1) : \nu < \omega_1 \rangle$ generate Ramsey ultrafilters on $([\omega]^\omega)^{V[G_\alpha]}$ which are not RK-equivalent by any $f \in ({}^\omega\omega)^{V[G_\alpha]}$.

First we show that such a strategy exists in $V[G]$. Then we show that it is winning. We work in $V[G]$. For $x \in V[G]$, let $o(x) = \min\{\alpha < \omega_2 : x \in V[G_\alpha]\}$. Let $\Gamma : \omega_1 \to (\omega_1)^2$ be a bijection such that $\Gamma(\alpha) = (\beta, \delta)$ implies $\beta \leq \alpha$. For each $\alpha < \omega_2$, $V[G_\alpha] \models \mathrm{CH}$. Hence we can choose $g_\alpha : \omega_1 \to V[G_\alpha]$ which enumerates all triples $(a, \pi, f) \in V[G_\alpha]$ such that $a \in [\omega]^\omega$, $\pi : [\omega]^n \to k$ for some $n, k < \omega$, and $f \in {}^\omega\omega$. In his αth move, II plays $(p^{II}_\alpha(0), p^{II}_\alpha(1)) \leq (p^I_\alpha(0), p^I_\alpha(1))$ such that, *if* $\Gamma(\alpha) = (\beta, \delta)$, $\xi \in C$ is minimal with $\xi \geq \sup\{o((p^I_\nu(0), p^I_\nu(1))) : \nu < \beta\}$, and $(a, \pi, f) = g_\xi(\delta)$, *then* for $i \in \{0, 1\}$ we have

(1) $p^{II}_\alpha(i) \subseteq a$ or $p^{II}_\alpha(i) \cap a = \emptyset$,
(2) $p^{II}_\alpha(i)$ is homogeneous for π,
(3) $f[p^{II}_\alpha(0)] \cap p^{II}_\alpha(1) = \emptyset$.

Since C is ω_1-club, it is easy to verify that this strategy is as desired.

Suppose that $\langle p_\nu : \nu < \omega_1 \rangle$ are moves of player II that are consistent with this strategy. Suppose this play is won by I. Hence there exists $(r_0, r_1) \in ([\omega]^\omega)^2 \cap V[G]$ with $(r_0, r_1) \leq p_\nu$, for all $\nu < \omega_1$. So we get $\alpha \in C$, and Ramsey ultrafilters G_i on $([\omega]^\omega)^{V[G_\alpha]}$, for $i < 2$, such that G_i is generated by $\langle p_\nu(i) : \nu < \omega_1 \rangle$, and G_0 is not RK-equivalent to G_1 by any $f \in {}^\omega\omega \cap V[G_\alpha]$. Then G_i is generated by r_i. By Proposition 2.3 we obtain that r_i belongs to $V[G_{\alpha+1}]$, and hence by Proposition 2.4, G_0 and G_1 are both RK-equivalent to $G(\alpha)'$ by some $f \in {}^\omega\omega \cap V[G_\alpha]$. By construction, this is impossible. By the game-theoretic characterization of $\mathfrak{h}(2)$ (see Introduction), this implies $V[G] \models \mathfrak{h}(2) = \omega_1$.

3. Iteration of Mathias forcing

Throughout this section $\langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$ denotes a countable support iteration of Mathias forcing of length γ. By [Sh*b*, p.96ff.], we may assume that elements of P_γ are hereditarily countable. **We shall always assume this in the sequel.** For $p \in P_\gamma$, the collection of $\beta \in \gamma$ such that in the transitive closure of p there exists a P_β-name for a condition in $\underset{\sim}{Q}_\beta$, is denoted by $\mathrm{cl}(p)$. By our assumption, $\mathrm{cl}(p)$ is a countable subset of γ. Note that if $\langle r_\alpha : \alpha < \gamma \rangle$ is a sequence of P_γ-generic Mathias reals, then only $\langle r_\alpha : \alpha \in \mathrm{cl}(p) \rangle$ are needed in order to evaluate p. Letting $a^* = \mathrm{cl}(p)$, we can define P_{a^*} as the countable support iteration of Mathias forcing with domain a^*. So P_{a^*} is isomorphic to P_δ, where $\delta = \mathrm{o.t.}(a^*)$. The question arises whether we can view p as a condition in P_{a^*}. It should be clear that this is not obvious.

In this section we prove that P_γ has a dense subset P'_γ which can be equipped with an order $\leq'$, such that forcing with $(P_\gamma, \leq)$ is equivalent to forcing with $(P'_\gamma, \leq')$, and the definition of $(P'_\gamma, \leq')$ is absolute for Π^1_1-correct models of ZF^- (up to some trivial restrictions). This will be used in the following sections to show that potential counterexamples to Propositions 2.3 and 2.4 must be added by an iteration of countable length (see Lemma 4.2). In particular, it will be obvious that if $p \in P'_\gamma$, then $p \in P'_{a^*}$, where $a^* = \mathrm{cl}(p)$.

We shall present these results for Mathias forcing only, although they can be generalized to include many more forcing notions. They are true for the class of Suslin proper forcings in the sense of [JSh2]. But this is not the optimal level of generalization, since our results are also true for all standard tree forcings such as Sacks, Laver, Miller forcing. But these are not Suslin proper in the sense of [JSh2] since their incompatibility relation is not analytic. In [Sh630] the first author gives a framework which includes all of these forcings.

Lemma 3.1. *Let $\langle P_\alpha, \dot{Q}_\beta : \alpha \le \gamma, \beta < \gamma\rangle$ be a countable support iteration of Mathias forcing. Let $(N, \in)$ be a countable model of ZF$^-$. Let $a^* \subseteq \gamma$ be closed such that $a^* \in N$ and $a^* \subseteq N$ (so a^* is countable in V). Let $\langle P_{a^*\cap\alpha}, \dot{Q}_\alpha : \alpha \in a^*\rangle$ be a countable support iteration with domain a^* of Mathias forcing.*

If $N \models p \in P_{a^*}$, there exists $q \in P_\gamma$ with cl$(q) = a^*$ such that q is (N, P_{a^*}, p)-generic; that is, if $\langle r_\alpha : \alpha < \gamma\rangle$ is a sequence of P_γ-generic Mathias reals over V with q belonging to its induced generic filter, then $\langle r_\alpha : \alpha \in a^*\rangle$ is $(P_{a^*})^N$-generic over N, with p belonging to its induced filter.

Proof. The proof follows closely Shelah's proof ([Sh*b*, p.90]) of preservation of properness by countable support iterations. By induction on $j \le \max a^*$, $j \in a^*$, we prove the following:

(∗) For every $i < j$, $i \in a^*$, for every $\mathbf{p}$ a P_i-name for an element of $(P_{a^*\cap j})^N \cap N$, and for every $q \in P_i$, *if* q is $(N, P_{a^*\cap i}, \mathbf{p}{\restriction}a^* \cap i)$-generic with $\mathrm{cl}(q) = a^* \cap i$, *then* there exists $r \in P_j$ with $\mathrm{cl}(r) = a^* \cap j$ such that r is $(N, P_{a^*\cap j}, \mathbf{p})$-generic, and $r{\restriction}i = q$.

Case 1. $j = \min a^*$. Then $P_{a^*\cap j} = \{\emptyset\}$. We let $r = \emptyset$.

Case 2. $a^* \cap j = (a^* \cap \beta) \cup \{\beta\}$ for some $\beta < j$. By induction hypothesis we may assume $\beta = i$. Choose $\langle r_\alpha : \alpha < i\rangle$ P_i-generic over V such that q belongs to the induced generic filter. Then $\langle r_\alpha : \alpha \in a^* \cap i\rangle$ is $(P_{a^*\cap i})^N$-generic over N with $\mathbf{p}[r_\alpha : \alpha < i]{\restriction}a^* \cap i$ belonging to the induced filter. Hence $x := (\mathbf{p}[r_\alpha : \alpha < i](i))[r_\alpha : \alpha \in a^* \cap i]$ is well-defined and $N[r_\alpha : \alpha \in a^* \cap i] \models$ "x is a Mathias condition". By Lemma 1.2, choose a Mathias condition $y \le x$ which is $(N[r_\alpha : \alpha \in a^* \cap i], \underset{\sim}{Q}_i[r_\alpha : \alpha \in a^* \cap i])$-generic. In V we may choose a P_i-name $q(i)$ for y such that q forces the above to hold for $q(i)$. Then $r = q^\frown\langle q(i)\rangle$ is as desired.

Case 3. $\bigcup a^* \cap j = j$. Let $\langle i_n : n < \omega\rangle$ be increasing and cofinal in $a^* \cap j$ with $i_0 = i$. Let $\langle D_n : n \in \omega\rangle$ list all subsets of $(P_{a^*\cap j})^N$ which belong to N and are dense in the sense of N. We define sequences $\langle q_n : n < \omega\rangle$ and $\langle \mathbf{p}_n : n < \omega\rangle$ such that $q_0 = q$, $\mathbf{p}_0 = \mathbf{p}$, and for all $n < \omega$ the following hold:

(1) $\mathbf{p}_{n+1}$ is a P_{i_n}-name for an element of $(P_{a^*\cap j})^N$.
(2) $q_n \in P_{i_n}$ and q_n is $(N, P_{a^*\cap i_n}, \mathbf{p}_n{\restriction}a^* \cap i_n)$-generic.
(3) $q_{n+1}{\restriction}i_n = q_n$.
(4) $q_n \Vdash_{P_{i_n}}$ "$\mathbf{p}_{n+1} \in D_n \cap N$ and $\mathbf{p}_{n+1} \le \mathbf{p}_n$".

Suppose that we have already gotten q_n and $\mathbf{p}_n$. Choose $\langle r_\alpha : \alpha < i_n\rangle$ P_{i_n}-generic over V with q_n belonging to its induced generic filter. Let $s = \mathbf{p}_n[r_\alpha : \alpha < i_n]$.

Hence $s \in (P_{a^* \cap j})^N \cap N$ by (4) in case $n > 0$, and by assumption on $\mathbf{p}_0$ otherwise. In N we can define

$$D'_n = \{t_0 \in P_{a^* \cap i_n} : \exists t_1 (t_0{}^\wedge t_1 \in D_n \text{ and } t_0{}^\wedge t_1 \leq s)\}.$$

Then N thinks that D'_n is dense below $s \restriction i_n$ in $P_{a^* \cap i_n}$. By (2), $s \restriction i_n$ belongs to the $(P_{a^* \cap i_n})^N$-generic filter induced by $\langle r_\alpha : \alpha \in a^* \cap i_n \rangle$. By genericity this filter meets $D'_n \cap N$, and hence there is $t \in D_n \cap N$ with $t \leq s$ and $t \restriction i_n$ belonging to the filter. In V we find a P_{i_n}-name $\mathbf{p}_{n+1}$ for t such that q_n forces the above properties of t to hold for $\mathbf{p}_{n+1}$.

By induction hypothesis, $(*)$ is true for $i = i_n$, $j = i_{n+1}$. Therefore there exists $q_{n+1} \in P_{i_{n+1}}$, such that (3) holds and (2) holds for $n+1$ instead of n.

This finishes the construction. Now let $r = \bigcup_{n<\omega} q_n$. Then r is as desired, as is easily seen.

Since a^* is closed, the three cases are exhaustive. □

We start defining $(P'_\gamma, \leq')$. For α an ordinal, define P'_α as follows:

> $p \in P'_\alpha$ iff p is a function, $\mathrm{dom}(p) \in [\alpha]^{\leq\omega}$, and for all $i \in \mathrm{dom}(p)$ there exists $u_i^p \in [i]^{\leq\omega}$ such that $p(i)$ is the code of a Borel function with domain the set of all functions $r : u_i^p \to {}^\omega\omega$ and target the set of Mathias conditions. For $i \notin \mathrm{dom}(p)$, we let $u_i^p = \emptyset$.

For any well-ordered set a^*, we can similarly define P'_{a^*}. If $p \in P'_\alpha$, we let $\mathrm{cl}(p) = \bigcup\{u_i^p : i \in \mathrm{dom}(p)\} \cup \mathrm{dom}(p)$.

Remark 3.2. We can view P'_γ as a subset of P_γ. Given $p \in P'_\gamma$ and $i \in \mathrm{dom}(p)$, and $\langle r_j : j < i \rangle$ P_i-generic over V, by absoluteness we have that $p(i)\langle r_j : j < u_i^p \rangle$ is a Mathias condition in the extension. By the existential completeness of forcing, there exists a P_i-name τ_i such that $\Vdash_{P_i} p(i)\langle \mathbf{r}_j : j \in u_i^p \rangle = \tau_i$. Now we can identify p with $\langle \tau_i : i < \gamma \rangle \in P_\gamma$. **In the sequel we shall tacitly make use of this identification**.

We want to define a partial order $\leq'$ on P'_γ such that forcing with $(P'_\gamma, \leq')$ will be equivalent to forcing with $(P_\gamma, \leq)$. First, for $p \in P'_\alpha$ we define by induction on $\alpha \leq \gamma$ when some family of reals $\langle r_j : j \in u \rangle$ with $\mathrm{cl}(p) \subseteq u$ satisfies p:

- $\alpha = 0$: The only member of P_0 is $\emptyset$, and we stipulate that every sequence of reals satisfies $\emptyset$;
- $\alpha = \beta + 1$: $\langle r_j : j \in u \rangle$ satisfies p if $\langle r_j : j \in u \rangle$ satisfies $p \restriction \beta$ and the filter of Mathias conditions induced by r_β contains $p(\beta)\langle r_j : j \in u_\beta^p \rangle$;
- $\alpha = \bigcup \alpha$: $\langle r_j : j \in u \rangle$ satisfies p if $\langle r_j : j \in u \rangle$ satisfies $p \restriction \beta$ for all $\alpha < \beta$.

Now let $p, q \in P'_\gamma$. We define

> $p \leq' q$ iff $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$, $u_i^q \subseteq u_i^p$ for all $i \in \mathrm{dom}(p)$, and for every family of reals $\langle r_j : j \in u \rangle$ such that $\mathrm{cl}(p) \subseteq u$ and $\langle r_j : j \in u \rangle$ satisfies p; for every $i \in \mathrm{dom}(q)$ we have
>
> $$p(i)\langle r_j : j \in u_i^p \rangle \leq q(i)\langle r_j : j \in u_i^q \rangle,$$
>
> where $\leq$ denotes the Mathias order.

Being a Borel code is a Π^1_1 property (see [J, p. 538]). Therefore, by the definitions and absoluteness of Π^1_1 statements we obtain that the definition of $(P'_\gamma, \leq')$ is very much absolute.

Fact 3.3. *Let $(N,\in)$ be a countable transitive model of ZF^- with $\gamma \in N$. Then $N \models p \in P'_\gamma$ iff $p \in P'_\gamma \cap N$ and $N \models$ cl(p) is countable. Moreover, for every $p,q \in (P'_\gamma)^N$ we have that $N \models p \le' q$ iff $p \le' q$.*

Later we will use variants of this fact without proof. In particular, we will have that γ is countable in N. Then "$N \models$ cl(p) is countable" follows, and we do not have to assume that N is transitive.

We want to prove equivalence of the forcings $(P_\gamma, \le)$ and $(P'_\gamma, \le')$. We start with the following easy observation:

Lemma 3.4. *If $p,q \in P'_\gamma$, then $p \le' q$ implies $p \le q$.*

Proof. By induction on $\alpha \le \gamma$ we prove that this is true for P'_α.

$\alpha = 0$: clear.

$\alpha = \beta + 1$: $p \le' q$ clearly implies $p{\restriction}\beta \le' q{\restriction}\beta$. By induction hypothesis we conclude $p{\restriction}\beta \le q{\restriction}\beta$. Let G_β be P_β-generic over V with $p{\restriction}\beta \in G_\beta$. Let $\langle r_j : j < \beta\rangle$ be the sequence of Mathias reals determined by G_β. It is clear that $\langle r_j : j < \beta\rangle$ satisfies $p{\restriction}\beta$. By assumption we have $p(\beta)\langle r_j : j \in u^p_\beta\rangle \le q(\beta)\langle r_j : j \in u^q_\beta\rangle$. By our identification (see Remark 3.2) we have $p(\beta)\langle r_j : j \in u^p_\beta\rangle = p(\beta)[G_\beta]$ and $q(\beta)\langle r_j : j \in u^q_\beta\rangle = q(\beta)[G_\beta]$. Consequently, $p{\restriction}\beta \Vdash_{P_\beta} p(\beta) \le q(\beta)$, and hence $p \le q$.

$\alpha = \bigcup \alpha$: clear by induction hypothesis and definition of the partial orders. □

The next lemma shows that P'_γ is a dense subset of P_γ. In the proof we will use the following coding of Mathias conditions by reals $x \in {}^\omega\omega$ with the property $\forall i,j(0 < i < j \Rightarrow x(i) < x(j))$: such x codes the Mathias condition $(\operatorname{ran} x{\restriction}[1,x(0)),\ \operatorname{ran} x{\restriction}[x(0),\infty))$. Hence we may assume that a P_i-name for a Mathias condition is a sequence $\langle f_n : n < \omega\rangle$ such that $f_n : A_n \to \omega$, where A_n is a countable antichain of P_i.

For $p \in P_\gamma$ and a sequence of reals $\bar r = \langle r_j : j \in u\rangle$ with cl$(p) \subseteq u$, we define by induction on $i \le \gamma$, $i \in$ dom(p),

(a) $\bar r$ *evaluates* $p(i)$;
(b) $p(i)[\bar r]$, if $\bar r$ evaluates p.

Case 1. $i = 0$. $\bar r$ evaluates $p(i)$, $p(i)[\bar r] = p(i)$.

Case 2. $i > 0$. Then $p(i) = \langle f_n : n < \omega\rangle$, where $f_n : A_n \to \omega$ and $A_n \subseteq P_i$ is a countable antichain. We define that $\bar r$ evaluates γ if:

(1) for every $n < \omega$, every $q \in A_n$, and every $\beta \in$ dom(q), $\bar r$ evaluates $q(\beta)$;
(2) for every $n < \omega$ there exists a unique $q \in A_n$ such that for all $\beta \in$ dom(q), $q(\beta)[\bar r]$ belongs to the filter on Q induced by r_β;
(3) the real x defined by $x(n) = f_n(q)$, where $q \in A_n$ is the unique member as in (2), codes a Mathias condition (i.e. $\forall i,j(0 < i < j < \omega \Rightarrow x(i) < x(j))$).

If (1)–(3) hold, $p(i)[\bar r]$ is defined as the Mathias condition coded by x.

The set of sequences $\bar r = \langle r_j : j \in$ cl$(p(i))\rangle$ which evaluate $p(i)$ is a Borel set with code $p(i)$; it is not difficult, though tedious, to show that it has a $\Delta^1_1(p(i))$-definition (see [JSp], where the details are worked out). First, $\bar r$ evaluates $p(i)$ iff there exists a sequence of reals which are the evaluations by $\bar r$ of all the names that belong to the transitive closure of $p(i)$, such that $p(i)$ can be evaluated from these using $\bar r$.

Since $p(i)$ is hereditarily countable, there is only one existential real quantifier, and the others are number quantifiers. Second, if such a sequence of reals exists, then it is unique, hence we can turn this statement into a universal statement. Now by Suslin's theorem (see [J, p.502]) we are done.

By a similar argument, the map sending $\bar{r}$, which evaluates $p(i)$, to $p(i)[\bar{r}]$ has a Borel definition.

Lemma 3.5. *For every $p \in P_\gamma$ there exists $p' \in P'_\gamma$ such that $p' \leq p$.*

Proof. For each $i \in \mathrm{dom}(p)$ let $u^i_p = \mathrm{cl}(p(i))$. Then u^i_p is countable. We define $p'(i) : \{\bar{r} : \bar{r} : u^p_i \to {}^\omega\omega\} \to Q$ (Q is Mathias forcing) by cases as follows: If $\bar{r}$ evaluates $p(i)$, we let $p'(i)(\bar{r}) = p(i)[\bar{r}]$, otherwise we let $p(i)(\bar{r})$ be the maximum element of Q. By the remarks above, $p'(i)$ is a total Borel function as desired. Now let $p' = \langle p'(i) : i \in \mathrm{dom}(p)\rangle$. Then clearly $p' \in P'_\gamma$. By induction on $i \in \mathrm{dom}(p')$ it is easy to prove that if $\bar{r} = \langle r_j : j < i\rangle$ is P_i-generic over V and contains $p' \restriction i$ in its generic filter, then $\bar{r}$ evaluates $p(i)$ and $p'(i)(\bar{r}) = p(i)[\bar{r}]$; hence $p' \restriction i \Vdash_{P_i} p'(i) = p(i)$. □

In order to conclude that forcings $(P_\gamma, \leq)$ and $(P'_\gamma, \leq')$ are equivalent, it is enough to prove the following:

Lemma 3.6. *For all $p, q \in P'_\gamma$ with $p \leq q$ there exists $r \in P'_\gamma$ with $r \leq' p$ and $r \leq' q$.*

Corollary 3.7. *Forcings $(P_\gamma, \leq)$ and $(P'_\gamma, \leq')$ are equivalent.*

Proof of 3.7. By Lemma 3.5 it is enough to show that $(P'_\gamma, \leq)$ and $(P'_\gamma, \leq')$ are equivalent. Let D be dense open in $(P'_\gamma, \leq)$, and let $p \in P'_\gamma$. Let $q \in D$, $q \leq p$. By Lemma 3.6 there is $r \in P'_\gamma$ with $r \leq' p$ and $r \leq' q$. By 3.4 we have $r \leq q$, and hence $r \in D$. Therefore D is dense in $(P'_\gamma, \leq')$. Conversely, if D is dense in $(P'_\gamma, \leq')$, then D is dense in $(P'_\gamma, \leq)$ by Lemma 3.4.

From Lemma 3.6 it follows that for all $p, q \in P'_\gamma$, p, q are incompatible with respect to $\leq$ iff they are incompatible with respect to $\leq'$. Therefore every $(P'_\gamma, \leq)$-name is a $(P'_\gamma, \leq')$-name and vice versa.

It follows that if G is a $(P'_\gamma, \leq)$-generic filter, then G is also $(P'_\gamma, \leq')$-generic, and if G' is $(P'_\gamma, \leq')$-generic, then $G = \{p \in P'_\gamma : \exists q \in G'(q \leq p)\}$ is $(P'_\gamma, \leq')$-generic, and then $V[G] = V[G']$. □

The following will be crucial for proving Lemma 3.6:

Lemma 3.8. *Let a^* be a countable closed set of ordinals, and let $p \in P'_{a^*}$. Let $(N, \in)$ be a countable elementary substructure of $(H(\chi), \in)$ for some large enough regular χ, such that $p, a^* \in N$. There exists $q \in P'_{a^*}$, $q \leq' p$, such that for every sequence of reals $\bar{r} = \langle r_l : l \in a^*\rangle$ which satisfies q, $\bar{r}$ is $(P_{a^*}, \leq)$-generic over N.*

Proof. By induction on $j \in a^*$ we prove the following:

(∗) For every $i < j$, $i \in a^*$, for every $P_{a^* \cap i}$-name $\mathbf{p}$ for a member of $N \cap P_{a^* \cap j}$, and for every $q \in P'_{a^* \cap i}$, *if* every sequence of reals $\bar{r} = \langle r_l : l \in a^* \cap i\rangle$ which satisfies q is $P_{a^* \cap i}$-generic over N, and $q \Vdash_{P_{a^* \cap i}} \mathbf{p} \restriction i \in G_{a^* \cap i}$, *then* there exists $r \in P'_{a^* \cap j}$ such that $r \restriction a^* \cap i = q$, every $\langle r_l : l \in a^* \cap i\rangle$ which satisfies r is $P_{a^* \cap j}$-generic over N, and $r \Vdash_{P_{a^* \cap j}} \mathbf{p} \in G_{a^* \cap j}$.

Case 1. $j = \min a^*$. Let $r = \emptyset$.

Case 2. $a^* \cap j = (a^* \cap \beta) \cup \{\beta\}$ for some $\beta < j$. By induction hypothesis we may assume $\beta = i$. Let $\bar{r} = \langle r_l : l \in a^* \cap i \rangle$ satisfy q. By assumption, $\bar{r}$ is $P_{a^* \cap i}$-generic over N and $\mathbf{p}[\bar{r}] \restriction a^* \cap i$ belongs to the generic filter induced by $\bar{r}$. By absoluteness, $x := (\mathbf{p}[\bar{r}](i))[\bar{r}]$ is a Mathias condition in V, say $x = (u^x, a^x)$. Using $N[\bar{r}]$ as a code, we may effectively construct $u \in [\omega]^\omega$ which is Mathias-generic over $N[\bar{r}]$ with x belonging to the generic filter induced by u. Let $y = (u^x, a^x \cap u)$. Then every real r_i which satisfies y is Mathias-generic over $N[\bar{r}]$ (see Lemma 1.2). Moreover, the function sending $\bar{r}$ to y is Borel. Denote it by $r(i)$. Then we may let $r = q^\wedge \langle r(i) \rangle$.

Case 3. $a^* \cap j$ is unbounded in $N \cap j$. We choose $\langle i_n : n < \omega \rangle$ increasing and cofinal in $N \cap j$ with $i_0 = i$. Let $\langle D_n : n < \omega \rangle$ list all dense subsets of $P_{a^* \cap j}$ in N. We define two sequences $\langle q_n : n < \omega \rangle$ and $\langle \mathbf{p}_n : n < \omega \rangle$ such that $q_0 = q$, $\mathbf{p} = \mathbf{p}_0$, and for all $n < \omega$ the following hold:

(1) $\mathbf{p}_{n+1}$ is a P_{i_n}-name for a member of $P_{a^* \cap j} \cap N$;
(2) $q_n \in P'_{a^* \cap i_n}$, and for every $\bar{r} = \langle r_l : l \in a^* \cap i_n \rangle$ which satisfies q_n, $\bar{r}$ is $P_{a^* \cap i_n}$-generic over N, and $q_n \Vdash_{P_{a^* \cap i_n}} \mathbf{p}_n \restriction a^* \cap i_n \in G_{a^* \cap i_n}$;
(3) $q_{n+1} \restriction i_n = q_n$;
(4) $q_n \Vdash_{P_{a^* \cap i_n}} \mathbf{p}_{n+1} \in D_n \cap N$ and $\mathbf{p}_{n+1} \leq \mathbf{p}_n$.

The construction is analogous to the proof of Lemma 3.1.

Now let $r = \bigcup_{n<\omega} q_n$, and let $\bar{r} = \langle r_l : l \in a^* \cap j \rangle$ satisfy r. We have to show that $\bar{r}$ is $P_{a^* \cap j}$-generic over N. Let $G \subseteq P_{a^* \cap j}$ be the filter induced by $\bar{r}$. Then $r \in G$. We have to show that $D_n \cap G \neq \emptyset$ for all $n < \omega$. Let $n < \omega$. We claim that $p_{n+1} := \mathbf{p}_{n+1}[\bar{r} \restriction i_n] \in G \cap D_n$. By (2) and (3), $\bar{r} \restriction i_n$ is $P_{a^* \cap i_n}$-generic over N, and hence $p_{n+1} \in D_n$ by (4). To prove $p_{n+1} \in G$ it is enough to show that $p_{n+1} \restriction i_m \in G_{a^* \cap i_m}$ for all $n < m < \omega$. For this, by induction on m show (using (4)) that $p_m \leq p_{n+1}$. This suffices, since by (2), $p_m \restriction a^* \cap i_m \in G_{a^* \cap i_m}$. This finishes the proof of $(*)$.

Applying $(*)$ for $i = \min(a^*)$ and $j = \max(a^*)$, we get $q \in P'_{a^*}$ such that every $\bar{r} = \langle r_l : l \in a^* \rangle$ which satisfies q is $(P_{a^*}, \leq)$-generic over N and contains p in its induced filter. We have to show that $q \leq' p$. By contradiction, suppose that $\bar{r} = \langle r_l : l \in a^* \rangle$ satisfies q and there is $i \in \mathrm{dom}(q)$ such that $q(i)\langle r_l : l \in a^* \cap i \rangle \not\leq p(i)\langle r_l : l \in u_i^p \rangle$. We can choose r'_i which satisfies $q(i)\langle r_l : l \in a^* \cap i \rangle$, but not $p(i)\langle r_l : l \in u_i^p \rangle$. Choose $\langle r'_l : l \in a^* \setminus (i+1) \rangle$ arbitrary such that $\bar{r}' := \langle r_l : l \in a^* \cap i \rangle^\wedge \langle r'_l : l \in a^* \setminus i \rangle$ satisfies q. By the above, $\bar{r}'$ is P_{a^*}-generic over N, containing p in its generic filter. But this is impossible by the choice of r'_i.

We are now able to give the proof of Lemma 3.6.

Proof of 3.6. Let $p, q \in P'_\gamma$ with $P_\gamma \models p \leq q$. Let $a^* = \mathrm{cl}(p)$. Hence we have $p, q \in P'_{a^*} \subseteq P_{a^*}$. We need the following claim:

Claim. $P_{a^*} \models p \leq q$.

Proof of Claim. Otherwise, let $i \in \mathrm{dom}(p)$ be minimal s.t. $\neg(p \restriction i \Vdash_{P_{a^* \cap i}} p(i) \leq q(i))$. Choose $r \in P_{a^* \cap i}$ such that $P_{a^* \cap i} \models r \leq p \restriction i$ and $r \Vdash_{P_{a^* \cap i}} p(i) \not\leq q(i)$.

Let $(N, \in)$ be a countable elementary substructure of $(H(\chi), \in)$, χ large enough and regular, containing everything relevant. By Lemma 3.1 there exists $q_1 \in P_i$ which is $(N, P_{a^* \cap i}, r)$-generic. Let $\bar{r} = \langle r_j : j < i \rangle$ be P_i-generic over V with q_1 belonging to the induced filter. Then $\langle r_j : j \in a^* \cap i \rangle$ is $P_{a^* \cap i}$-generic over N, with r belonging to the induced filter. We conclude that on the one hand,

$V[r_j : j < i] \models p(i)[r_j : j < i] \leq q(i)[r_j : j < i]$, but on the other hand,

$$N[r_j : j \in a^* \cap i] \models p(i)[r_j : j \in a^* \cap i] \not\leq q(i)[r_j : j \in a^* \cap i].$$

But $p(i)[r_j : j < i] = p(i)[r_j : j \in a^* \cap i]$, and similarly for $q(i)$. Since the Mathias order is absolute, we have a contradiction. □

Let $(N, \in)$ be as in the proof of the Claim. By Lemma 3.8, there exists $r \in P'_{a^*}$ with $r \leq' p$ such that every sequence of reals $\bar{r} = \langle r_j : j \in a^* \rangle$ which satisfies r is P_{a^*}-generic over N. Given such $\bar{r}$ and $i \in \mathrm{dom}(p)$, $p \restriction i$ belongs to the generic filter on $P_{a^*} \cap N$ induced by $\bar{r} \restriction a^* \cap i$, and hence by the Claim, $N[\bar{r} \restriction a^* \cap i] \models p(i)[\bar{r} \restriction a^* \cap i] \leq q(i)[\bar{r} \restriction a^* \cap i]$. But $p(i)[\bar{r} \restriction a^* \cap i] = p(i)(r_j : j \in u_i^p)$, and similarly for q. By absoluteness of the Mathias order and by $r \leq' p$ we obtain

$$r(i)(r_j : j \in u_i^r) \leq p(i)(r_j : j \in u_i^p) \leq q(i)(r_j : j \in u_i^q).$$

Since $\bar{r}$ and i were arbitrary we conclude that $r \leq' q$.

The proof of Corollary 3.7 now being complete, **throughout the rest of this paper we identify** $(P_\gamma, \leq)$ **with** $(P'_\gamma, \leq')$**.**

Definition 3.9. If $u \subseteq \gamma$ is finite and $p, q \in P_\gamma$, then $q \leq_u p$ is defined by $q \leq p$ and for all $\alpha \in u$, $q \restriction \alpha \Vdash_{P_\alpha}$ "$q(\alpha)$ and $p(\alpha)$ have the same first coordinate".

By arguments that are standard by now, we obtain the following lemma. Note that it makes sense only in the light of Corollary 3.7. For the proof, make a similar inductive construction as we did several times. At successor steps, use Lemma 1.2 to get generic conditions which are pure extensions, if required by u.

Lemma 3.10. *Let $(N, \in)$ be a countable model of ZF$^-$ such that γ is countable in N. If $p \in P_\gamma \cap N$, and $u \in [\gamma]^{<\omega}$, there exists $q \in P_\gamma$ such that $q \leq_u p$ and q is (N, P_γ)-generic.*

For the proof that potential counterexamples to Propositions 2.3 and 2.4 are added by an iteration of countable length, we will also need the following lemma.

Lemma 3.11. *Suppose $a^* \subseteq \gamma$ is a countable closed set of ordinals, P_{a^*} is a countable support iteration of Mathias forcing with domain a^*, and $p \in P_{a^*}$. Let $(N, \in)$ be a countable model of ZF$^-$ with $\gamma \in N$, and suppose that $a^* \subseteq N$, $a^* \in N$, $p \in N$, and $N \models p \in P_{a^*}$.*

There exists $q \in P_{a^*}$ and a P_{a^*}-name $\bar{\mathbf{r}}'_\gamma = \langle \mathbf{r}'_l : l < \gamma \rangle$ such that $q \leq p$ and, letting $\bar{\mathbf{r}}_{a^*} = \langle \mathbf{r}_l : l \in a^* \rangle$ be a name for the P_{a^*}-generic sequence of Mathias reals, we have

$$q \Vdash_{P_{a^*}} \text{"}\bar{\mathbf{r}}'_\gamma \text{ is } P_\gamma\text{-generic over } N, \text{ and } \forall l \in a^* (\mathbf{r}'_l = \mathbf{r}_l)\text{"}.$$

Proof. By induction on $j \leq \gamma$, $j \in N$, we prove the following:

(∗) Suppose $i \in j$, $i \in N$, $q \in P_{a^* \cap i}$, and $\bar{\mathbf{r}}'_i = \langle \mathbf{r}'_l : l < i \rangle$ is a $P_{a^* \cap i}$-name such that $q \leq p \restriction a^* \cap i$ and

$$q \Vdash_{P_{a^* \cap i}} \bar{\mathbf{r}}'_i \text{ is } P_i\text{-generic over } N \text{ and } \forall l \in a^* \cap i (\mathbf{r}'_l = \mathbf{r}_l).$$

Then there exists $r \in P_{a^* \cap j}$ and $\bar{\mathbf{r}}'_j = \langle \mathbf{r}'_l : l < j \rangle$ such that $r \restriction a^* \cap j = q$, $r \leq p \restriction a^* \cap j$, $\bar{\mathbf{r}}'_j \restriction i = \bar{\mathbf{r}}'_i$, and

$$r \Vdash_{P_{a^* \cap j}} \text{"}\bar{\mathbf{r}}'_j \text{ is } P_j\text{-generic over } N \text{ and } \forall l \in a^* \cap j (\mathbf{r}'_l = \mathbf{r}_l).$$

Case A. $N \cap j = (N \cap \beta) \cup \{\beta\}$, for some $\beta < j$. Then $j = \beta + 1$, since $N \models \mathrm{ZF}^-$, and so $\beta + 1 \in N$. Hence we may assume $\beta = i$.

Case A1. $i \in a^*$. Let $\bar{r}_{a^*\cap i} = \langle r_l : l \in a^* \cap i\rangle$ be $P_{a^*\cap i}$-generic over V with q in its generic filter. Let $\bar{r}'_i = \bar{\mathbf{r}}'_i[\bar{r}_{a^*\cap i}]$. Then $N[\bar{r}'_i] \in V[\bar{r}_{a^*\cap i}]$ and $N[\bar{r}'_i] \models$ ZF$^-$. By assumption we have $p(i)[\bar{r}_{a^*\cap i}] = p(i)[\bar{r}'_i]$. Let x be this common value. Then x is a Mathias condition. By Lemma 1.2, in $V[\bar{r}_{a^*\cap i}]$ we may choose a Mathias condition $y \le x$ such that every $z \in [\omega]^\omega$ with $u^y \subseteq z \subseteq u^y \cup a^y$ is Mathias generic over $N[\bar{r}'_i]$. In V we have a $P_{a^*\cap i}$-name q_i such that q forces that all of the above holds for q_i instead of y. Now let $r = q^\wedge\langle q_i\rangle$ and $\mathbf{r}'_i = \mathbf{r}_i$.

Case A2. $i \notin a^*$. Then $P_{a^*\cap j} = P_{a^*\cap i}$. Since N is countable, in V there exists a $P_{a^*\cap i}$-name $\mathbf{r}'_i$ such that q forces that $\mathbf{r}'_i$ is Mathias generic over $N[\bar{\mathbf{r}}'_i]$. We let $r = q$ and $\bar{\mathbf{r}}'_j = \bar{\mathbf{r}}'^\wedge_i\langle \mathbf{r}'_i\rangle$.

Case B. $N \cap j$ is unbounded in $N \cap j$.

Case B1. $j \in a^*$. Since a^* is closed and $a^* \subseteq N$, we conclude that either $a^* \cap j$ is bounded in $a^* \cap j$, or else $a^* \cap j$ is unbounded in j. In the first case we may assume $i > \max(a^* \cap j)$, and proceed as in Case A2. In the latter case, a similar diagonalization as in 3.1 and 3.8 works.

Case B2. $j \notin a^*$. Since a^* is closed, $a^* \cap j$ is bounded below j. Hence we may assume $i > \max(a^* \cap j)$. Then $P_{a^*\cap j} = P_{a^*\cap i}$, and as in Case A2, in V there exists a $P_{a^*\cap i}$-name $\langle \mathbf{r}'_l : i \le l < j\rangle$ such that q forces that $\langle \mathbf{r}'_l : i \le l < j\rangle$ is $P_j/\bar{\mathbf{r}}'_i$-generic over N. We let $r = q$ and $\bar{\mathbf{r}}'_j = \underset{\sim}{\bar{\mathbf{r}}}'^\wedge_i\langle \mathbf{r}'_l : i \le l < j\rangle$.

4. Proof of Proposition 2.3

The following lemma will give us the ω_1-club for Proposition 2.3.

Lemma 4.1. *Suppose $V \models CH$. Let $\langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \le \omega_2, \beta < \omega_2\rangle$ be a countable support iteration of Mathias forcing. Let G_{ω_2} be P_{ω_2}-generic over V and, for $\delta < \omega_2$, r_δ the $\underset{\sim}{Q}_\delta[G_\delta]$−generic real determined by G_{ω_2}. Then the set S of $\delta \in S^2_1$ such that for some $\alpha_\delta < \delta$*

$$(*)\qquad \mathcal{P}(\omega)^{V[\{G_{\alpha_\delta}, r_\delta\}]} = \mathcal{P}(\omega)^{V[G_{\delta+1}]}$$

is nonstationary.

Proof. Suppose that S is stationary. We will derive a contradiction. For $\delta \in S$ choose $p_\delta \in P_{\delta+1}$ forcing $(*)$. Since $\delta \in S^2_1$ and p_δ is hereditarily countable, without loss of generality we may assume that $p_\delta(\delta)$ is a P_{α_δ}-name and $\sup(\mathrm{dom}(p_\delta \restriction \delta)) < \alpha_\delta$. Otherwise, increase α_δ, and then $(*)$ still holds, of course. By Fodor's Theorem and $V[G_\alpha] \models CH$ for $\alpha < \omega_2$, there exist $\alpha^* < \omega_2$, $p \in P_{\alpha^*}$ and a stationary $S_1 \subseteq S$ such that $\forall\delta \in S_1(\alpha_\delta = \alpha^* \wedge p_\delta\restriction\delta = p)$. Hence in $V[G_{\alpha^*}]$ we can compute $p_\delta(\delta)[G_\delta]$ for $\delta \in S_1$. Again by the CH in $V[G_{\alpha^*}]$ and the $\aleph_2$-completeness of the nonstationary ideal on ω_2, there exist a stationary $S_2 \subseteq S_1$ and $q \in \underset{\sim}{Q}_{\alpha^*}[G_{\alpha^*}]$ such that $\forall\delta \in S_2(p_\delta(\delta)[G_\delta] = q)$.

Let $G(\omega_2)$ be $Q^{V[G_{\omega_2}]}$-generic over $V[G_{\omega_2}]$, where Q is Mathias forcing, such that $q \in G(\omega_2)$. Let r_{ω_2} be the corresponding Mathias real, and let $G_{\omega_2+1} = G * G(\omega_2)$. By Theorem 2.1, $\mathcal{P}(\omega)$/fin is $\aleph_1$-distributive in $V[G_{\omega_2}]$. Since Mathias forcing is the composition of $\mathcal{P}(\omega)$/fin and some σ-centered forcing, it follows that $V[G_{\omega_2+1}] \models \mathfrak{c} = \omega_2$. By properness and $V \models$ CH we have $V[G_{\alpha^*}, r_{\omega_2}] \models$ CH. (If you do not see this, let $V = L$ and use [J, 15.3., p.130].) Hence there exists $\alpha^* < \alpha < \omega_2$ such that $r_\alpha \notin V[G_{\alpha^*}, r_{\omega_2}]$. Hence in $V[G_{\omega_2}]$ there exists $q_1 \in Q^{V[G_{\omega_2}]} \cap G(\omega_2)$, $q_1 \leq q$, forcing this. Let $\alpha < \gamma < \omega_2$ such that $q_1 \in V[G_\gamma]$. By genericity there exists $\delta \in S_2 \cap [\gamma, \omega_2)$ such that, if $q_1 = (u, a)$, then $u \subseteq r_\delta \subseteq u \cup a$, that is, q_1 belongs to the generic filter generated by r_δ. Let $q_2 = (u, a \cap r_\delta)$. Then $q_2 \in Q^{V[G_{\omega_2}]}$ and $q_2 \leq q_1$.

Let r be $Q^{V[G_{\omega_2}]}$-generic over $V[G_{\omega_2}]$ such that $u \subseteq r \subseteq u \cup (a \cap r_\delta)$. Then r is an infinite subset of r_δ. By the remark preceding Lemma 1.2, we have that r is $Q^{V[G_\delta]}$-generic over $V[G_\delta]$. From $(*)$ and the choice of q we conclude that $r_\alpha \in V[G_{\alpha^*}, r]$. On the other hand, q_1 belongs to the generic filter induced by r, and we conclude $r_\alpha \notin V[G_{\alpha^*}, r]$, a contradiction. $\square$

Let $C \subseteq S^2_1 \setminus S$ be ω_1-club, where S is as in Lemma 4.1. We claim that C serves for Proposition 2.3. By contradiction, suppose that this is false. Hence there exist $\alpha \in C$, $p^* \in P_{\omega_2}/\underset{\sim}{G}_\alpha$, and $\underset{\sim}{r}$ such that

(+) $p^* \Vdash_{P_{\omega_2}/\underset{\sim}{G}_\alpha} \underset{\sim}{r}$ induces a Ramsey ultrafilter on $([\omega]^\omega)^{V[\underset{\sim}{G}_\alpha]}$ which is not induced by any real in $V[\underset{\sim}{G}_{\alpha+1}]$.

We may assume that $\mathrm{cl}(\underset{\sim}{r}) \subseteq \mathrm{cl}(p^*)$.

Since forcing $P_{\omega_2}/\underset{\sim}{G}_\alpha$ is equivalent to a countable support iteration of length ω_2 of Mathias forcing in $V[G_\alpha]$ (see [B, §5]), for notational simplicity we assume $V[G_\alpha] = V$ for the moment, and later we shall remember that really $V = V[G_\alpha]$ for some $\alpha \in C$ and derive a final contradiction.

First, we show that by the absoluteness results from §3, we may assume that $\underset{\sim}{r}$ is added by an iteration of countable length. Let $a^* = \mathrm{cl}(p^*)$. So $a^* \subseteq \omega_2$ is countable. We may assume that $0 \in a^*$ and a^* is closed.

Lemma 4.2. *Assuming (+), it is true that $p^* \Vdash_{P_{a^*}}$ "$\underset{\sim}{r}$ induces a Ramsey ultrafilter on $([\omega]^\omega)^V$ which is not induced by any real in $V[\underset{\sim}{G}_0]$".*

Proof. (a) $p^* \Vdash_{P_{a^*}} \underset{\sim}{r}$ induces an ultrafilter on $([\omega]^\omega)^V$: Otherwise, there exists $a \in ([\omega]^\omega)^V$ and $p \in P_{a^*}$ such that $p \leq p^*$ and $p \Vdash_{P_{a^*}}$ "$\underset{\sim}{r} \cap a$ and $\underset{\sim}{r} \cap (\omega \setminus a)$ are both infinite." Let χ be large enough and regular, and let $(N, \in) \prec (H(\chi), \in)$ be countable, containing everything relevant. By Lemma 3.1 choose $q \in P_{\omega_2}$ such that q is (N, P_{a^*}, p)-generic, and let $\langle r_\alpha : \alpha \in \omega_2\rangle$ be P_{ω_2}-generic over V, with induced filter G, such that $q \in G$. Then $\langle r_\alpha : \alpha \in a^*\rangle$ is P_{a^*}-generic over N with p, and hence also p^*, in its generic filter, denoted by G_{a^*}. Then clearly $p^* \in G$. We obtain that $V[G] \models$ "$\underset{\sim}{r}[G] \subseteq^* a$ or $\underset{\sim}{r}[G] \subseteq^* \omega \setminus a$", and $N[G_{a^*}] \models |\underset{\sim}{r}[G_{a^*}] \cap a| = |\underset{\sim}{r}[G_{a^*}] \cap (\omega \setminus a)| = \omega$. But clearly $\underset{\sim}{r}[G] = \underset{\sim}{r}[G_{a^*}]$, a contradiction.

(b) $p^* \Vdash_{P_{a^*}}$ "No real in $V[\underset{\sim}{G}_0]$ induces the same ultrafilter (on $([\omega]^\omega)^V$) as $\underset{\sim}{r}$": Otherwise, there is $p \in P_{a^*}$, $p \leq p^*$, and a P_1-name $\underset{\sim}{r}^1$ such that $p \Vdash_{P_{a^*}} \underset{\sim}{r}$ and $\underset{\sim}{r}^1$ induce the same ultrafilter. Choose $(N, \in)$ as in (a), containing everything relevant. We can get $q \in P_{a^*}$, $q \leq p$, as in Lemma 3.11. Let $\bar{r}_{a^*} = \langle r_l : l \in a^*\rangle$ be P_{a^*}-generic over V containing q in its generic filter. By Lemma 3.11, in $V[\bar{r}_{a^*}]$ there exists $\bar{r}'_{\omega_2} = \langle r'_l : l < \omega_2\rangle$ such that $\bar{r}'_{\omega_2}$ is P_{ω_2}-generic over N and $r_l = r'_l$, for all $l \in a^*$. We obtain that $\underset{\sim}{r}[\bar{r}'_{\omega_2}] = \underset{\sim}{r}[\bar{r}_{a^*}]$ and $\underset{\sim}{r}^1[\bar{r}'_{\omega_2}] = \underset{\sim}{r}^1[\bar{r}_{a^*}]$. Let the common value be

r, r^1, respectively. In $N[\bar{r}'_{\omega_2}]$ we have some $x \in ([\omega]^\omega)^V$ such that $r \subseteq^* x$ but $r^1 \not\subseteq^* x$, or conversely. Since $N[\bar{r}'_{\omega_2}] \in V[\bar{r}_{a^*}]$, we have that in $V[\bar{r}_{a^*}]$, r and r^1 do not generate the same ultrafilters on $([\omega]^\omega)^V$, a contradiction.

(c) $p^* \Vdash_{P_{a^*}} \underset{\sim}{r}$ induces a *Ramsey* ultrafilter on $([\omega]^\omega)^V$. Otherwise, there exist $p \in P_{a^*}$ and $f \in ({}^\omega\omega)^V$ such that if $\underset{\sim}{D}$ is a P_{a^*}-name for the filter induced by $\underset{\sim}{r}$ we have that $p \Vdash_{P_{a^*}} f$ is unbounded but not one-to-one modulo $\underset{\sim}{D}$. Let $(N, \in)$ be as above containing everything relevant. We can get $q \in P_{a^*}$, $q \leq p$, as in Lemma 3.11. Let $\bar{r}_{a^*} = \langle r_l : l \in a^* \rangle$ be P_{a^*}-generic over V containing q in its generic filter. By Lemma 3.11, in $V[\bar{r}_{a^*}]$ there exists $\bar{r}'_{\omega_2} = \langle r'_l : l < \omega_2 \rangle$ such that $\bar{r}'_{\omega_2}$ is P_{ω_2}-generic over N and $r_l = r'_l$, for all $l \in a^*$. We obtain that $\underset{\sim}{r}[\bar{r}'_{\omega_2}] = \underset{\sim}{r}[\bar{r}_{a^*}]$. Let r be the common value. Then r induces the same filter, say D, in $V[\bar{r}_{a^*}]$ and in $N[\bar{r}'_{\omega_2}]$, and also f is unbounded modulo D in both models. Hence by construction, on the one hand we have that $V[\bar{r}_{a^*}] \models f$ is not one-to-one modulo D, but on the other hand, $N[\bar{r}'_{\omega_2}] \models f$ is one-to-one modulo D. Since $N[\bar{r}'_{\omega_2}] \in V[\bar{r}_{a^*}]$ we have a contradiction. □

Continuing the proof of Proposition 1, let $\delta = \text{o.t.}(a^*)$. Then $\delta < \omega_1$, and clearly P_{a^*} and P_δ are isomorphic. Then our assumption (+) becomes:

(++) $p^* \Vdash_{P_\delta} \underset{\sim}{r}$ induces a Ramsey ultrafilter on $([\omega]^\omega)^V$ which is not induced by any real in $V[\underset{\sim}{G_0}]$.

Let $\underset{\sim}{D}$ be a P_δ-name for the filter on $([\omega]^\omega)^V$ induced by $\underset{\sim}{r}$. In V, let $(N, \in)$ be a countable elementary substructure of $(H(\chi), \in)$, where χ is a large enough regular cardinal, such that $\delta, p^*, \underset{\sim}{D}, \underset{\sim}{r} \in N$. **This N will be fixed for the rest of this section.** Let G_0 be Q_0-generic, containing a (N, Q_0)-generic condition below $p^*(0)$. In $V[G_0]$ we define

$$\mathcal{Y} = \{Y : \exists (N[G_0], P_\delta/\underset{\sim}{G_0})\text{-generic } q(q \leq p^*\restriction[1,\delta) \wedge q \Vdash_{P_\delta/\underset{\sim}{G_0}} \text{“}\underset{\sim}{D} \cap N = Y\text{”})\}.$$

Since every Ramsey ultrafilter is a p-point (see §1), and every $Y \in \mathcal{Y}$ is a countable subset of the denotation of $\underset{\sim}{D}$ in a $P_\delta/\underset{\sim}{G_0}$-generic extension of $V[G_0]$, and $\underset{\sim}{D}$ is forced to be a Ramsey ultrafilter on $([\omega]^\omega)^V$, we conclude that such Y is definable from $([\omega]^\omega)^N$ and a member of $([\omega]^\omega)^V$, and hence $\mathcal{Y} \subseteq V$.[2]

Lemma 4.3. *$\mathcal{Y}$ is a Σ^1_2 set in $V[G_0]$.*

Proof. We show that $Y \in \mathcal{Y}$ is equivalent to saying:

> There exists a countable model $(M, \in)$ such that $N[G_0] \cup \{N[G_0], Y\} \subseteq M$, $(M, \in) \models ZF^-$, and $(M, \in) \models \exists q \in P_\delta/\underset{\sim}{G_0}(q$ is $(N[G_0], P_\delta/\underset{\sim}{G_0})$-generic and $q \Vdash_{P_\delta/\underset{\sim}{G_0}}$ “$\underset{\sim}{D} \cap ([\omega]^\omega)^N = Y$”$)$.

It is well-known (see [J, the proof of 41.1., pp.527f.]) that the quantification over countable models as above is equivalent to quantifying over structures (ω, R), where R is a well-founded binary relation, which makes the formula no worse (and no better) than Σ^1_2, and that the rest is arithmetical.

[2]Added in proof: This argument works only if $Y \in V$, which is false in general. It is here that we need that $\underset{\sim}{D}$ is forced to be a P-filter in $V[\underset{\sim}{G_\delta}]$ (see footnote 1).

If $Y \in \mathcal{Y}$, then choosing a countable $(M, \in)$ which is elementarily embeddable into $(H(\chi)^{V[G_0]}, \in)$ and contains $N[G_0] \cup \{N[G_0], Y\}$, we easily see that one implication holds.

Conversely, if $(M, \in), Y, q$ are given, as above, then by Lemma 3.10, in $V[G_0]$ choose $q_1 \le q$ which is $(M, P_\delta/\underset{\sim}{G}_0)$-generic. Here we use again the fact that $P_\delta/\underset{\sim}{G}_0$ is equivalent to a countable support iteration of Mathias forcing. Then clearly q_1 is also $(N[G_0], P_\delta/\underset{\sim}{G}_0)$-generic, and $q_1 \Vdash_{P_\delta/G_0}$ "$\underset{\sim}{D} \cap ([\omega]^\omega)^N = Y$" holds in $V[G_0]$. In fact, let G_1 be $P_\delta/\underset{\sim}{G}_0$-generic over $V[G_0]$, containing q_1. Then G_1 is $P_\delta/\underset{\sim}{G}_0$-generic over M and contains q. By assumption on M, G_1 is $P_\delta/\underset{\sim}{G}_0$-generic over N. Moreover, $\underset{\sim}{r}[G_0 * G_1]$ is the same real in $V[G_0 * G_1]$ and $N[G_0 * G_1]$. Hence we are done. □

The crucial fact, whose proof will require considerable space, is that $\mathcal{Y}$ is uncountable. Then we obtain that in $V[G_0]$, $\mathcal{Y}$ is an uncountable Σ^1_2 set which is a subset of V. By a well-known result of descriptive set theory (see the remark after Corollary 4.10, below), either $\mathcal{Y}$ has a perfect subset, or else $\mathcal{Y}$ is the union of $\aleph_1$ countable Borel sets. The first case will be ruled out by a theorem which says that Mathias forcing does not add a perfect set of old reals. In the second case we shall remember that really $V = V[G_\alpha]$ for some $\alpha \in C$, and by the definition of C we will obtain a contradiction.

In order to prove that $\mathcal{Y}$ is uncountable, by fusion we shall build a perfect tree of $(N[G_0], P_\delta/\underset{\sim}{G}_0)$-generic conditions which all decide $\underset{\sim}{D} \cap N$ in different ways. This is much harder than it might seem at first glance. The crucial lemma will be Lemma 4.7 below.

Definition 4.4. (1) For $u \in [\delta]^{<\omega}$ and $p \in P_\delta$, let

$$E(p, u) = \{a \in ([\omega]^\omega)^V : \exists q \le_u p(q \Vdash_{P_\delta} a \in \mathcal{D})\}.$$

(2) Suppose $\bar{x} = \langle \underset{\sim}{x}_\alpha : \alpha \in u \rangle$ is such that every $\underset{\sim}{x}_\alpha$ is a P_α-name for a finite subset of ω with elements larger than the members of the first coordinate of $p(\alpha)$. Then by $p \cup \bar{x}$ we denote the condition $\bar{p} \in P_\delta$ by $\bar{p}(\alpha) = p(\alpha)$ for $\alpha \notin u$, and first coordinate of $\bar{p}(\alpha) =$ first coordinate of $p(\alpha)$, and second coordinate of $\bar{p}(\alpha) =$ (second coordinate of $p(\alpha)) \cup \underset{\sim}{x}_\alpha$, for $\alpha \in u$. Moreover, by $\bar{x} \cup p$ we denote the condition $\bar{q} \in P_\delta$ by $\bar{q}(\alpha) = p(\alpha)$ for $\alpha \notin u$, first coordinate of $\bar{q}(\alpha) =$ (first coordinate of $p(\alpha)) \cup \dot{x}_\alpha$ and second coordinate of $\bar{q}(\alpha) =$ (second coordinate of $p(\alpha)) \setminus (\max(\underset{\sim}{x}_\alpha) + 1)$ for $\alpha \in u$.

Lemma 4.5. *The ordering $\le_u$ has the pure decision property; that is, for τ a P_δ-name for a member of $\{0, 1\}$ and $p \in P_\delta$ there exists $q \le_u p$ such that q decides τ.*

Proof. We prove it by induction on $\max(u)$. Let $\alpha_0 = \max(u)$ and $u_0 = u \setminus \{\alpha_0\}$. We may regard τ as a P_{α_0}-name for a $P_\delta/\underset{\sim}{G}_{\alpha_0}$-name. First, if $\alpha_0 = 0$, then by the pure decision property of Mathias forcing (proved in [B, 9.3.]) there exists $q(0) \in Q$, $q(0) \le_{\{0\}} p(0)$, deciding the disjunction "$\exists q_1 \in P_\delta/\underset{\sim}{G}_0 (q_1 \le p \restriction [1, \delta) \wedge q_1 \Vdash_{1\delta} \tau = 0) \vee \exists q_1 \in P_\delta/\underset{\sim}{G}_0 (q_1 \le p \restriction [1, \delta) \wedge q_1 \Vdash_{1\delta} \tau = 1)$". By the maximum principle of forcing we may find q_1 such that $q(0)^\wedge q_1 \le_{\{0\}} p$ and $q(0)^\wedge q_1$ decides τ.

For the inductive step, as in the case $\alpha_0 = 0$, we know that for some $q_1 \in P_\delta/\underset{\sim}{G}_{\alpha_0}$, $q_1 \leq_{\{\alpha_0\}} p \restriction [\alpha_0, \delta)$, $p \restriction \alpha_0 \Vdash_{P_{\alpha_0}}$ "q_1 decides τ"; moreover, by induction hypothesis there exists $q_0 \leq_{u_0} p$, $q_0 \in P_{\alpha_0}$, which decides whether for such q_1, $q_1 \Vdash \tau = 0$ or $q_1 \Vdash \tau = 1$. Then $q_0{}^\wedge q_1$ is as desired. □

Lemma 4.6. *Let $p \in P_\delta$, $u \in [\mathrm{dom}(p)]^{<\omega}$, $n \in \omega$ and $\bar{x} = \langle \underset{\sim}{x}_\alpha : \alpha \in u\rangle$ such that $\underset{\sim}{x}_\alpha$ is a P_α-name for the first n members of the infinite part of $p(\alpha)$. Suppose also that for no $q \leq p$, $E(q, u)$ is a filter.*

Then for $i \in \{0, 1\}$ there exist $q_i \leq_u p$ and disjoint $a_i \in [\omega]^\omega$ such that $q_i \cup \bar{x} \Vdash$ "$a_i \in \underset{\sim}{D}$".

Proof. First note that if $q \leq p$, for every $k \in \omega$ we may find a disjoint sequence $\langle a_i : i < k\rangle$ of members of $[\omega]^\omega$ and $\langle q_i : i < k\rangle$ such that $q_i \leq_u q$ and $q_i \Vdash$ "$a_i \in \underset{\sim}{D}$". In fact, since $E(q, u)$ is not a filter, there exist $a'_0, a'_1 \in E(q, u)$ such that $a'_0 \cap a'_1 \notin E(q, u)$. Let $q'_i \leq_u q$ force "$a'_i \in \underset{\sim}{D}$". By the pure decision property of $\leq_u$, as proved in Lemma 4.5., there exists $q_0 \leq_u q'_0$ deciding whether $a_0 := a'_0 \setminus a'_1$ or $a'_0 \cap a'_1$ belongs to $\underset{\sim}{D}$. Then clearly $q_0 \Vdash$ "$a_0 \in \underset{\sim}{D}$". Hence we may let $q_1 = q'_1$, $a_1 = a'_1$. Now proceeding by induction we easily construct $\langle a_i : i < k\rangle$ and $\langle q_i : i < k\rangle$ as desired.

For $\alpha \in u$ let $\langle \underset{\sim}{y}^i_\alpha : i < 2^n\rangle$ be an enumeration (of names) of all the subsets of (the denotation) of $\underset{\sim}{x}_\alpha$, and let $\langle \bar{y}_i : i < n^*\rangle$ enumerate all $\bar{y}_\sigma = \langle \underset{\sim}{y}^{\sigma(\alpha)}_\alpha : \alpha \in u\rangle$, where $\sigma \in {}^u(2^n)$. Now using the observation above we easily construct q_τ and $a_\tau \in [\omega]^\omega$, for every $\tau \in {}^{\leq n^*}(n^* + 1)$, such that the following requirements hold:

(1) $q_\emptyset = p$, $a_\emptyset = \omega$,
(2) $\langle a_{\tau^\frown\langle i\rangle} : i < n^* + 1\rangle$ is a partition of ω,
(3) $\tau \subseteq \sigma \Rightarrow q_\tau \geq_u q_\sigma$,
(4) $|\tau| > 0 \Rightarrow \bar{y}_{|\tau|-1} \cup q_\tau \Vdash$ "$a_\tau \in \underset{\sim}{D}$".

Now choose $q_0 \leq_u p$ such that for every $i < n^*$ and $\tau \in {}^{<n^*}(n^* + 1)$, $\bar{y}_i \cup q_0$ decides for which j, $a_{\tau^\frown\langle j\rangle}$ belongs to $\underset{\sim}{D}$. For this we use again the pure decision property of $\leq_u$. Then clearly we may find $\tau_1 \in {}^{n^*}(n^* + 1)$ such that, letting $a_1 := \bigcup\{A_{\tau_1|j} : 1 \leq j \leq n^*\}$, $a_0 := \omega \setminus a_1$ and $q_1 := q_{\tau_1}$, the conclusion of the lemma holds. □

The following lemma shows that the assumption of Lemma 4.6 holds. As always, we implicitly regard $P_\delta/\underset{\sim}{G}_0$ as a countable support iteration of Mathias forcing.

Lemma 4.7. *In $V[G_0]$, for no $q \in P_\delta/\underset{\sim}{G}_0$ with $q \leq p^*\restriction[1, \delta)$, and for no $u \in [\mathrm{dom}(q)]^{<\omega}$ is it true that $E(q, u)$ is a filter.*

Proof. Suppose by way of contradiction that for some $q \leq p^*\restriction[1, \delta)$ and $u \in [\mathrm{dom}(q)]^\omega$, $E(q, u)$ is a filter. By the pure decision property of $\leq_u$, then $E(q, u)$ is an ultrafilter. By the transitivity of the ordering $\leq_u$ we have that for every $q' \leq_u q$, $E(q', u) \subseteq E(q, u)$ and hence $E(q', u)$ is a filter. By the pure decision property again, we obtain $E(q', u) = E(q, u)$. This fact will be used several times in the sequel.

In V let $\underset{\sim}{E}, \underset{\sim}{q}$ be Q_0-names for $E(q,u), q$. Without loss of generality we may assume that the above properties of $E(q,u), q$ are forced by $p^*(0)$ to hold for $\underset{\sim}{E}, \underset{\sim}{q}$. Moreover we may certainly assume $\underset{\sim}{E}, \underset{\sim}{q} \in N$.

Let $G_0 = G_0' * G_0''$ be the decomposition of G_0 according to the decomposition of Mathias forcing $Q_0 = Q_0' * \underset{\sim}{Q}_0''$. Let $p^*(0) = (u^{p^*}, a^{p^*})$. In $V[G_0']$ we can define

$$D_1 = \{a \in [\omega]^\omega : \exists a' \in G_0' (u^{p^*}, a') \Vdash \text{“} a \in \underset{\sim}{E} \text{”}\}.$$

By hypothesis and as $Q(G_0')$ has the pure decision property (see [JSh]), we conclude that D_1 is an ultrafilter. Working in $V[G_0']$, we distinguish two cases according to whether G_0' is a projection of D_1 or not. In both cases we derive a contradiction.

Case 1. $G_0' \leq_{RK} D_1$.

Let $f \in {}^\omega\omega$ witness this. As Q_0' is σ-closed and hence does not add new reals, $f \in V$. As $N' := N[G_0'] \prec (H(\chi)^{V[G_0']}, \in)$ (see [Sh*b*, 2.11., p.88]) and $D_1 \in N'$, we may assume $f \in N'$, and hence $f \in N$ by properness. Since $G_0' \cap N$ is countable, there exists $a \in G_0'$ such that $G_0' \cap N = \{b \in N : a \subseteq^* b\}$.

We work in $V[G_0']$. By Case 1 there exists $b \in D_1$ such that $f[b] \subseteq a$. Let $x \in Q(G_0')$ with u^{p^*} as its first coordinate be such that

$$x \Vdash_{Q(G_0')} b \in \underset{\sim}{E}. \tag{1}$$

Note that x is trivially $(N[G_0], Q(G_0'))$-generic, since $Q(G_0')$ is *ccc*. By Lemma 3.10 there exists a $Q(G_0')$-name $\underset{\sim}{q_1}$ for a $(N[\underset{\sim}{G_0}], P_\delta/\underset{\sim}{G_0})$-generic condition, such that $p^*(0) \Vdash_{Q(G_0')} \underset{\sim}{q_1} \leq_u \underset{\sim}{q}$. By the remark at the beginning of this proof, we have

$$p^*(0) \Vdash_{Q(G_0')} E(\underset{\sim}{q}, u) = E(\underset{\sim}{q_1}, u). \tag{2}$$

We conclude that $x * \dot{q}_1$ is a $(N[G_0'], Q(G_0') * (P_\delta/\underset{\sim}{G_0}))$-generic condition below p^*. By (1) and (2), there is a $Q(G_0')$-name $\underset{\sim}{q_2}$ such that $p^*(0) \Vdash \underset{\sim}{q_2} \leq_u \underset{\sim}{q_1}$ and $x * \underset{\sim}{q_2} \Vdash b \in \underset{\sim}{D}$. Clearly, $x * \underset{\sim}{q_2}$ is $(N[G_0'], Q(G_0') * (P_\delta/\underset{\sim}{G_0}))$-generic. Let G_1 be $Q(G_0') * (P_\delta/\underset{\sim}{G_0})$-generic over $V[G_0']$ such that $x * \underset{\sim}{q_2} \in G_1$. We conclude that $b \in \underset{\sim}{D}[G_0' * G_1]$.

Note that we must have that $f_*(\underset{\sim}{D}[G_0' * G_1]) \neq G_0'$. Otherwise, $\underset{\sim}{D}[G_0' * G_1]$ could be computed from f and G_0' in $V[G_0']$. For this we use that $\underset{\sim}{D}[G_0' * G_1]$ is Ramsey. Moreover, since any two RK-comparable Ramsey ultrafilters are actually RK-equivalent (see [J, Ex. 38.4, p.480]), and G_0' is induced by some real in $V[G_0]$ (namely the Mathias real determined by G_0), the same is true for $\underset{\sim}{D}[G_0' * G_1]$ (the function witnessing equivalence moves the Mathias real to a real inducing $\underset{\sim}{D}[G_0' * G_1]$). Here, G_0 is the Q_0-generic filter determined by G_1. This contradicts our basic assumption (++).

Hence this inequality holds in $N'[G_1]$. Therefore there exists $a_1 \in N \cap G_0'$ such that $f^{-1}[a_1] \notin \underset{\sim}{D}[G_0' * G_1]$. Let $b_1 = b \setminus f^{-1}[a_1]$. So $b_1 \in \underset{\sim}{D}[G_0' * G_1]$. We obtain that

$f[b_1] \cap a_1 = \emptyset$, $f[b_1] \subseteq a$, and $a \subseteq^* a_1$. Hence $f[b_1]$ is finite. But then $f[b_1] \notin G'_0$, a contradiction.

Case 2. $G'_0 \not\leq_{\mathrm{RK}} D_1$.

In V let $\underset{\sim}{D}_1$ be a Q'_0-name for D_1, and let $\underset{\sim}{G'_0}$ be the canonical name for the Q'_0-generic filter. Then by hypothesis there exists $t_0 \in [a^{p^*}]^\omega$ such that

$$t_0 \Vdash_{Q'_0} \text{“}\underset{\sim}{G'_0} \not\leq_{\mathrm{RK}} \mathcal{D}_1\text{”}.$$

We may certainly assume $\underset{\sim}{D}_1, t_0 \in N$.

In V let g be Q'-generic over N such that $t_0 \in g$, where Q is Mathias forcing and $Q = Q' * \underset{\sim}{Q''}$ its canonical decomposition. In $N[g]$ let $d = \underset{\sim}{D}_1[g]$. By elementarity we conclude

$$N[g] \models d \text{ is an u.f.}, g \text{ a Ramsey u.f. and } g \not\leq_{\mathrm{RK}} d. \tag{3}$$

In [GSh] it was shown that for any ultrafilter D on ω there exists a proper forcing Q_D such that whenever G is a Ramsey ultrafilter with $G \not\leq_{\mathrm{RK}} D$, then after forcing with Q_D, G still generates an ultrafilter but D does not. Moreover Q_D is ${}^\omega\omega$-bounding. Hence by Lemma 1.1, every such G generates a Ramsey ultrafilter in every Q_D-generic extension.

Definition 4.8. Conditions in Q_D are $f = \langle h, E; E_0, E_1, \dots \rangle$ where $h : \omega \to \{-1, 1\}$, and the sets $E, E_0, E_1, \dots$ belong to the ideal dual to D and partition ω.

The ordering is defined as follows: $\langle h, E; E_0, E_1, \dots \rangle \leq \langle h', E'; E'_0, E'_1, \dots \rangle$ if and only if

$E \supseteq E'$,
$E, E_0, E_1, \dots$ is a coarser partition than $E', E'_0, E'_1, \dots$,
$h \restriction E' = h' \restriction E'$,
for all i: $h \restriction E'_i \in \{h' \restriction E'_i, -h' \restriction E'_i\}$.

A Q_D-generic filter G determines a generic real $s = \bigcup\{h_f : f \in G\}$.

By standard arguments one proves that whenever $s \in {}^\omega\{-1, 1\}$ is Q_D-generic, f belongs to the generic filter which s generates, and s_f is defined by

$$s_f(n) = \begin{cases} s(n), & n \in E^f, \\ -s(n), & n \notin E^f, \end{cases} \tag{4}$$

then s_f is Q_D-generic as well and f belongs to its generic filter. Here E^f is the second coordinate of f. Hence especially $-s$, where $(-s)(n) = -s(n)$, is also Q_D-generic.

In $N[g]$ we have the forcing Q_d. In V, choose $s \in {}^\omega\{-1, 1\}$ Q_d-generic over $N[g]$. By the properties of Q_d and (3), g generates a Ramsey ultrafilter in $N[g][s]$.

Finally, in V choose $t_1 \subseteq t_0$ $Q(g)$-generic over $N[g][s]$. Since every infinite subset of t_1 is also $Q(g)$-generic and, as just noticed, $-s$ is also Q_d-generic, without loss of generality, we may assume that

$$V \models t_1 \Vdash_{Q'_0} \text{“}s^{-1}(1) \in \underset{\sim}{D}_1\text{”}.$$

Otherwise, work with some $t_2 \in [t_1]^\omega$ and $-s$. Hence, by the definition of D_1, and since Q'_0 does not add reals, we may assume:

$$V \models (u^{p^*}, t_1) \Vdash_{Q_0} \text{“}s^{-1}(1) \in \underset{\sim}{E}\text{”}. \tag{5}$$

Claim 1. There exists a Q-name $\underset{\sim}{q}' \in N[g, s, t_1]$ such that

$$N[g, s, t_1] \models (u^{p^*}, t_1) \Vdash_Q \text{“}\underset{\sim}{q}' \in P_\delta/\underset{\sim}{G}_0 \wedge \underset{\sim}{q}' \leq_u \underset{\sim}{q} \wedge \underset{\sim}{q}' \Vdash_{P_\delta/\underset{\sim}{G}_0} \text{‘}\underset{\sim}{r} \subseteq^* s^{-1}(1)\text{’”}. \tag{6}$$

Proof. Otherwise, there exist $(u', t') \leq (u^{p^*}, t_1)$ and $\underset{\sim}{q}'$ such that in $N[g, s, t_1]$, $\underset{\sim}{q}'$ is a Q-name for a condition in $P_\delta/\underset{\sim}{G}_0$, and

$$N[g, s, t_1] \models (u', t') \Vdash_Q \text{“}\underset{\sim}{q}' \leq_u \underset{\sim}{q} \wedge \underset{\sim}{q}' \Vdash_{P_\delta/\underset{\sim}{G}_0} \text{‘}\underset{\sim}{r} \setminus s^{-1}(1) \text{ is infinite’”}.$$

Such $\underset{\sim}{q}'$ exists by the existential completeness of forcing and the pure decision property of $\leq_u$.

By Lemma 3.10, in V there exists $\bar{q} \in P_\delta$ such that $\bar{q} \leq_u (u', t') * \underset{\sim}{q}'$ and $\bar{q}$ is $(N[g, s, t_1], P_\delta)$-generic. Since by the observation at the very beginning of the present proof we know that

$$p^*(0) \Vdash_Q \text{“}\underset{\sim}{E} = E(\bar{q}{\restriction}[1, \delta), u)\text{”},$$

by (5) and the definition of $\underset{\sim}{E}$, there exists $\bar{\bar{q}} \in P_\delta$ such that $\bar{\bar{q}} \leq_u \bar{q}$ and $\bar{\bar{q}} \Vdash_{P_\delta}$ “$\underset{\sim}{r} \subseteq^* s^{-1}(1)$”. Now choose G P_δ-generic over V such that $\bar{\bar{q}} \in G$. Then clearly $V[G] \models \underset{\sim}{r}[G] \subseteq^* s^{-1}(1)$ and $N[g, s, t_1][G] \models |\underset{\sim}{r}[G] \setminus s^{-1}(1)| = \omega$. But $\underset{\sim}{r}[G]$ is the same real in both models, a contradiction. □

Let us abbreviate the formula “…” in (6) by $\phi(\underset{\sim}{q}', s)$.

Since t_1 is $Q(g)$-generic and g generates a Ramsey ultrafilter in $N[g][s]$, there exists $(u', t') \in Q(g)$ such that $u' \subseteq t_1 \subseteq t'$ and

$$N[g, s] \models (u', t') \Vdash_{Q(g)} \text{“}(u^{p^*}, \underset{\sim}{t}) \Vdash_Q \text{‘}\phi(\underset{\sim}{q}', s)\text{’ ”}, \tag{7}$$

where $\underset{\sim}{t}$ is the canonical name for the generic real added by $Q(g)$, and in the formula $\phi(s)$, $\underset{\sim}{q}'$ is now a $Q(g)$-name for the above $\underset{\sim}{q}'$.

Since s is Q_d-generic over $N[g]$ and $(u', t') \in N[g]$, there exists $f \in Q_d$ such that f belongs to the Q_d-generic filter induced by s, and in $N[g]$ the following holds:

$$f \Vdash_{Q_d} \text{“}N[g][\underset{\sim}{s}] \models ((u', t') \Vdash_{Q(g)} \text{‘}(u^p, \underset{\sim}{t}) \Vdash_Q \ \phi(\underset{\sim}{q}', \underset{\sim}{s})\text{’)”},$$

where $\underset{\sim}{s}$ is the canonical Q_d-name for the Q_d-generic real and in $\phi(\underset{\sim}{q}', \underset{\sim}{s})$, $\underset{\sim}{q}'$ denotes now a $Q_d * Q(g)$-name for the $\underset{\sim}{q}'$ in (7). By the definition of Q_d we have $\omega \setminus E^f \in d$.

Claim 2. $V \models t_1 \Vdash_{Q'}$ “$\omega \setminus E^f \in \underset{\sim}{D}_1$”.

Proof. As g is Q'-generic over N, $\omega \setminus E^f \in d = \underset{\sim}{D}_1[g]$, and Q' does not add reals, there exists $u \in g$ such that

$$N \models u \Vdash_{Q'} \text{“}\omega \setminus E^f \in \underset{\sim}{D}_1\text{”}.$$

By elementarity we conclude that this is true in V. But clearly we have $t_1 \subseteq^* u$. □

Let s_f be defined as in Definition 4.8. By the remarks after 4.8, s_f is Q_d-generic over $N[g]$, and clearly f belongs to the generic filter determined by s_f. Hence (5) holds if s is replaced by s_f. Clearly $N[g][s] = N[g][s_f]$, and hence t_1 is $Q(g)$-generic over $N[g][s_f]$, and consequently $N[g][s][t_1] = N[g][s_f][t_1] =: N^*$.

Let G^* be Q-generic over V, containing a (N^*, Q)-generic condition below (u^{p^*}, t_1). Then by Claim 2, $\omega \setminus E^f \in \underset{\sim}{E}[G^*]$. But also $s^{-1}(1), s_f^{-1}(1) \in \underset{\sim}{E}[G_0^*]$. In fact, in $N^*[G^*]$ we have $q_1 := \underset{\sim}{q}'[s][t_1][G^*]$ and $q_2 := \underset{\sim}{q}'[s_f][t_1][G^*]$ with the property that $P_\delta/\underset{\sim}{G_0} \models q_1, q_2 \leq_u q$, and $q_1 \Vdash_{P_\delta/\underset{\sim}{G_0}} \underset{\sim}{r} \subseteq^* s^{-1}(1)$, and $q_2 \Vdash_{P_\delta/\underset{\sim}{G_0}} \underset{\sim}{r} \subseteq^* s_f^{-1}(1)$. Otherwise, as in the proof of Claim 1, in $V[G^*]$ we could find $(N[G^*], P_\delta/\underset{\sim}{G_0})$-generic conditions $\bar{q}_1 \leq_u q_1$ and $\bar{q}_2 \leq_u q_2$ forcing the opposite. By choosing filters which are $P_\delta/\underset{\sim}{G_0}$-generic over V and contain $\bar{q}_1, \bar{q}_2$ respectively, we obtain a contradiction. Consequently, $s^{-1}(1)$, $s_f^{-1}(1)$, and $\omega \setminus E^f$ belong to $\underset{\sim}{E}[G^*]$. But $s^{-1}(1), s_f^{-1}(1)$ are complementary on $\omega \setminus E^f$, and hence $\underset{\sim}{E}[G^*]$ is not a filter, a contradiction. □

Using 4.6, 4.7 and [B, Lemma 7.3], by standard arguments on proper forcing we obtain the following corollary.

Corollary 4.9. *In $V[G_0]$, there exist $\langle q_s : s \in {}^{<\omega}2\rangle$, $\langle a_s : s \in {}^{<\omega}2\rangle$ such that the following hold:*

(1) *if $s \subseteq t$, then $a_s \supseteq a_t$ and $a_{s\frown\langle 0\rangle} \cap a_{s\frown\langle 1\rangle} = \emptyset$,*

(2) if $f \in {}^\omega 2$, then $\langle q_{f\restriction n} : n < \omega\rangle$ is a descending chain in $P_\delta/\underset{\sim}{G_0}$ which has a lower bound q_f such that:
 - q_f is $(N[G_0], P_\delta/\underset{\sim}{G_0})$-generic,
 - $q_f \Vdash \forall n(a_{f\restriction n} \in \underset{\sim}{D})$,
 - q_f decides $\underset{\sim}{D} \cap N$.

Corollary 4.10. *$\mathcal{Y}$ is uncountable.*

From Lemma 4.3 and Corollary 4.10 we conclude that $\mathcal{Y}$ is an uncountable Σ^1_2 set in $V[G_0]$ which is a subset of V. By well-known results from descriptive set theory, $\mathcal{Y}$ is the union of ω_1 Borel sets, say $\langle B_\alpha : \alpha < \omega_1\rangle$, and this decomposition is absolute for models computing ω_1 correct (see [J, Theorem 95, p.520, its proof on p.526 using the Shoenfield tree, and Lemma 40.8, p.525, where its absoluteness is proved]). If one of the B_α is uncountable it contains a perfect subset (see [J, Theorem 94, p.507]). This case will be ruled out by Lemma 4.11.

Otherwise, each B_α is countable. Now $\mathcal{Y}$ and hence $\langle B_\alpha : \alpha < \omega_1\rangle$ is coded by a real x. We may assume that x also codes $\langle q_s : s \in {}^{<\omega}2\rangle$ and $\langle a_s : s \in {}^{<\omega}2\rangle$ from 4.9. Now remember that V here is really $V[G_\alpha]$ where $\alpha \in C$ (C coming from 4.1), and hence $V[G_0] = V[G_{\alpha+1}]$. Clearly there exists $\beta < \alpha$ such that $x \in V[G_\beta, r_\alpha]$. Then also $\langle B_\alpha : \alpha < \omega_1\rangle$, $\langle q_s : s \in {}^{<\omega}2\rangle, \langle a_s : s \in {}^{<\omega}2\rangle \in V[G_\beta, r_\alpha]$, and hence, as each B_α is countable, $\mathcal{Y} \subseteq V[G_\beta, r_\alpha]$. But from this we conclude $\mathcal{P}(\omega)^{V[G_\beta, r_\alpha]} = \mathcal{P}(\omega)^{V[G_{\alpha+1}]}$, as a new real in $V[G_{\alpha+1}] \setminus V[G_\beta, r_\alpha]$ would give a new branch through $\langle a_s : s \in {}^{<\omega}2\rangle$ and hence a new member in $\mathcal{Y}$. But $\alpha \in C$, and hence $(*)$ in 4.1 fails for it, a contradiction.

Therefore, in order to finish the proof of Proposition 2.3 it suffices to prove the following lemma:

Lemma 4.11. [3] *Suppose $q \in Q$, where Q is Mathias-forcing, τ is a Q-name, and*

$$q \Vdash_Q \text{“}\tau \subseteq {}^{<\omega}2 \text{ is a perfect tree”}.$$

Then $q \Vdash_Q$ “$[\tau] \nsubseteq V$” .

Proof. By applying the pure decision property of Q repeatedly, without loss of generality, we may assume that if $q = (s, a)$, then for every $t \in [a]^{<\omega}$ and $n \in \omega$ there exists $m \in \omega$ such that $(s \cup t, a \setminus m)$ decides the value of $\tau \cap {}^{<n}2$. Hence if we let

$$T_t = \{\nu \in {}^{<\omega}2 : \exists n((s \cup t, a \setminus n) \Vdash_Q \text{“}\nu \in \tau\text{”})\},$$

then T_t is a tree with no finite branches.

We shall define a Q-name η for a real in $[\tau] \setminus V$. To this end, for every $t \in [b]^{<\omega}$, we construct $b \in [a]^\omega$, $\eta_t \in T_t$ and $n(t) \in \omega$ such that the following hold:

(1) $(s \cup t, b \setminus (\max(t) + 1)) \Vdash_Q$ “$\eta_t \in \tau$”;

(2) if $T_t \cap [\eta_t]$ has infinitely many branches, hence by compactness a non-isolated one, and x_t is the lexicographically least such one, then for every $m \in b \setminus (\max(t) + 1)$, $\eta_{t\cup\{m\}}$ is not an initial segment of x_t, but $\eta_{t\cup\{m\}} \restriction n(t \cup \{m\})$ is; moreover, $\lim_{m\to\omega} n(t \cup \{m\}) = \infty$;

(3) if $T_t \cap [\eta_t]$ has finitely many branches, then for every $m \in b \setminus (\max(t)+1)$,
- if $T_{t\cup\{m\}}$ has a member extending η_t which does not belong to T_t, then $\eta_{t\cup\{m\}}$ is like that, say among the shortest the lexicographically least one;
- if $T_{t\cup\{m\}}$ has no such member, then $\eta_{t\cup\{m\}} = \eta_t$.

The construction of b, $\langle \eta_t : t \in [b]^{<\omega}\rangle$ and $\langle n(t) : t \in [b]^{<\omega}\rangle$ is by fusion: Suppose that an initial segment of b, say t, has been fixed and for some $b' \in [a \setminus t]^\omega$, for every $t' \in \mathcal{P}(t)$ and $m \in b'$, $\eta_{t'}, n(t')$ and $\eta_{t'\cup\{m\}}, n(t' \cup \{m\})$ have been defined such that (1), (2), (3) hold for $\eta_{t'}, \eta_{t'\cup\{m\}}, n(t'), n(t' \cup \{m\})$ and b'. Now the least element of b', say k, is put into b. Then successively for each $t' \in \mathcal{P}(t)$, first count how many branches $T_{t'\cup\{k\}} \cap [\eta_{t'\cup\{k\}}]$ has, and then accordingly define $\eta_{t'\cup\{k\}\cup\{m\}}$ and maybe $n(t' \cup \{k\} \cup \{m\})$ (if we are in case (2)) for $m \in b'$, all the time shrinking b' to make sure that in the end, for some $b'' \in [b']^\omega$, for every $t' \in \mathcal{P}(t \cup \{k\})$, (1), (2) and (3) hold for $\eta_{t'}$ and b''. The construction is totally straightforward, so we leave the rest to the reader.

We define a Q-name as follows:

$$\eta = \bigcup\{\eta_t : t \in [b]^{<\omega} \wedge ((s \cup t, b \setminus (\max(t) + 1)) \in \underset{\sim}{G})\}.$$

Here $\underset{\sim}{G}$ is the canonical name for the Q-generic filter. By construction we conclude:

$$(s, b) \Vdash_Q \text{“}\eta \in [\tau] \cup \tau\text{”}.$$

Suppose now that some $(s \cup t, b^*) \leq (s, b)$ forces that η belongs to V, so, without loss of generality, there exists $\eta^* \in V$ such that

$$(s \cup t, b^*) \Vdash_Q \text{“}\eta = \eta^*\text{”}.$$

From this we will derive a contradiction. Then the Lemma will be proved. Clearly we have $\eta^* \in {}^\omega\omega \cup {}^{<\omega}\omega$. We distinguish the following cases:

[3] The referee informed us that this is a corollary of a theorem of Groszek and Slaman which implies that if a proper forcing P adds new reals, then every perfect set in V^P contains a new real.

Case 1. $T_t \cup [\eta_t]$ has infinitely many branches.

Subcase 1a: $\eta^* = x_t$. By construction, if $m \in b^*$, then $(s \cup t \cup \{m\}, b^* \setminus (m+1)) \Vdash_Q$ "$\eta_{t \cup \{m\}} \subseteq \eta$" and $\eta_{t\cup\{m\}} \not\subseteq x_t$, a contradiction.

Subcase 1b: $\eta^* \restriction n \neq x_t \restriction n$ for some n. If $m \in b^*$ with $n(t \cup \{m\}) \geq n$, then by construction $(s \cup t \cup \{m\}, b^* \setminus (m+1)) \Vdash_Q$ "$\eta \restriction n(t \cup \{m\}) = x_t \restriction n(t \cup \{m\})$", a contradiction.

Case 2. $T_t \cap [\eta_t]$ has only finitely many branches.

Subcase 2a: $\eta^* \in [T_t] \cup T_t$. Since τ is forced to be a perfect tree, there exists $u \in [b^*]^{<\omega}$ such that $T_{t \cup u}$ has a member above η_t which is not in T_t. But then by construction $(s \cup t \cup u, b^* \setminus (\max(u)+1)) \Vdash_Q$ "$\eta \notin [T_t] \cup T_t$", a contradiction.

Subcase 2b: $\eta^* \restriction n \notin T_t$ for some n. By construction of T_t, there exists m such that $(s \cup t, b^* \setminus m) \Vdash_Q$ "$\tau \cap {}^{\leq n}2 = T_t \cap {}^{\leq n}2$". But $(s \cup t, b^* \setminus m) \Vdash_Q$ "$\eta \restriction n \in \tau$", a contradiction. $\square\square$

5. Proof of Proposition 2.4

The proof will use several ideas from the proof of Proposition 2.3. Suppose that Proposition 2.4 is false, that is, there exist Q-names $\underset{\sim}{D}$ and $\underset{\sim}{r}$, and $p \in Q$ such that p forces that $\underset{\sim}{r}$ induces a Ramsey ultrafilter $\underset{\sim}{D}$ on $([\omega]^\omega)^V$ which is not RK-equivalent to $\underset{\sim}{G'}$ by any $f \in {}^\omega\omega \cap V$.

First note that a σ-centered forcing P does not add such $\underset{\sim}{D}$. In fact, since $V \models \mathrm{CH}$, such $\mathcal{D}$ is forced to be generated by a $\subseteq^*$-descending chain $\langle \underset{\sim}{a_\alpha} : \alpha < \omega_1 \rangle$ of members of $([\omega]^\omega)^V$. For every $\alpha < \omega_1$, choose $p_\alpha \in P$ and $a_\alpha \in ([\omega]^\omega)^V$ such that $p_\alpha \Vdash_P \underset{\sim}{a_\alpha} = a_\alpha$. Since P is σ-centered, there exists $X \in [\omega_1]^{\omega_1}$ such that p_α, p_β are compatible whenever $\alpha, \beta \in X$. By the *ccc* of P, there exists a P-generic filter G which contains p_α for uncountably many $\alpha \in X$. Then clearly $\underset{\sim}{D}[G] \in V$, as $\underset{\sim}{D}[G]$ is generated by $\langle a_\alpha : \alpha \in X \rangle$. The argument shows that no condition in P forces that $\underset{\sim}{D}$ does not belong to V.

Since $Q(\underset{\sim}{G'})$ is forced to be σ-centered, by what we have just proved, we may assume that $\underset{\sim}{D}$ is a Q'-name. As usual, we write $p = (u^p, a^p)$. For $t \in Q'$ we define

$$D_t = \{a \in ([\omega]^\omega)^V : t \Vdash_{Q'} \text{“}a \in \underset{\sim}{D}\text{”}\}.$$

The following claim follows immediately from the definitions:

Claim 1. For all $t \in Q'$ with $t \leq a^p$, we have that $a \in D_t$ if and only if $(u^p, t) \Vdash_Q$ "$\underset{\sim}{r} \subseteq^* a$".

Claim 2. Suppose that $(N, \in)$ is a countable model of ZF^- such that $\underset{\sim}{r}, p \in N$, and $\underset{\sim}{r}$ is hereditarily countable in N. Then for every $a \in [\omega]^\omega \cap N$ and $t \in Q' \cap N$ with $t \leq a^p$, it is true that $(u^p, t) \Vdash_Q$ "$\underset{\sim}{r} \subseteq^* a$" implies that $N \models (u^p, t) \Vdash_Q$ "$\underset{\sim}{r} \subseteq^* a$".

Proof of Claim 2. Otherwise, there exists $q \in N \cap Q$ such that $q \leq (u^p, t)$ and $N \models q \Vdash_Q$ "$\underset{\sim}{r} \cap (\omega \setminus a)$ is infinite").

By Lemma 1.2, there exists $q' \in Q$ such that $q' \leq q$ and q' is (N, Q)-generic. Let G be Q-generic over V, containing q'. Then by assumption $\underset{\sim}{r}[G] \subseteq^* a$. On the other hand, $N[G] \models |\underset{\sim}{r}[G] \cap (\omega \setminus a)| = \omega$. As $\underset{\sim}{r}[G]$ is the same real in $V[G]$ and $N[G]$ we have a contradiction. $\square$

By assumption, and since Q' does not add reals, we conclude:

$$a^p \Vdash_{Q'} \text{"}\underset{\sim}{D} \text{ and } \underset{\sim}{G'} \text{ are Ramsey ultrafilters which are not RK-equivalent."}$$

Choose a countable elementary substructure $(N, \in) \prec (H(\chi), \in)$ where χ is a large enough regular cardinal, such that $\underset{\sim}{D}, \underset{\sim}{r}, p \in N$.

In V, let g be Q'-generic over N such that $a^p \in g$. In $N[g]$, let $d = \underset{\sim}{D}[g]$. By elementarity we conclude

(1) $N[g] \models$ "g and d are Ramsey ultrafilters which are not RK-equivalent."

In V, choose $s \in {}^{\omega}\{-1, 1\}$ Q_d-generic over $N[g]$, where Q_d is the forcing from 4.8, defined in $N[g]$ from the ultrafilter d. From (1), Lemma 1.1, and [GSh], we conclude that g generates a Ramsey ultrafilter in $N[g][s]$.

Finally, in V choose $t_1 \leq a^p$ $Q(g)$-generic over $N[g][s]$. Since every infinite subset of t_1 is also $Q(g)$-generic and $-s$ is also Q_d-generic, without loss of generality we may assume that $s^{-1}(1) \in D_{t_1}$.

By Claims 1 and 2, we conclude:

(2) $$N[g][s][t_1] \models (u^p, t_1) \Vdash_Q \text{"}\underset{\sim}{r} \subseteq^* s^{-1}(1)\text{"}.$$

Since g generates a Ramsey ultrafilter in $N[g][s]$, by the remark preceding Lemma 1.2, we conclude that t_1 is $Q(g)$-generic over $N[g][s]$. Since $Q(g)^{N[g]}$ is dense in $Q(g)^{N[g][s]}$, there exists $(u', t') \in Q(g)^{N[g]}$ such that $u' \subseteq t_1 \subseteq t'$ and

(3) $$N[g, s] \models (u', t') \Vdash_{Q(g)} \text{"}(u^p, \underset{\sim}{t}) \Vdash_Q \text{'}\underset{\sim}{r} \subseteq^* s^{-1}(1)\text{' "}.$$

Here $\underset{\sim}{t}$ is the canonical name for the generic real added by $Q(g)$.

Since s is Q_d-generic over $N[g]$ and all the parameters in the formula "..." of (3) belong to $N[g]$, there exists $f \in Q_d$ such that f belongs to the Q_d-generic filter induced by s, and in $N[g]$ the following holds:

$$f \Vdash_{Q_d} \text{"}N[g][\underset{\sim}{s}] \models [(u', t') \Vdash_{Q(g)} \text{'}(u^p, \underset{\sim}{t}) \Vdash_Q \text{'}\underset{\sim}{r} \subseteq^* \dot{s}^{-1}(1)\text{"}]\text{"}.$$

Here $\underset{\sim}{s}$ is the canonical Q_d-name for the Q_d-generic real. By definition of Q_d, $\omega \setminus E^f \in d$.

Claim 3. $V \models \omega \setminus E^f \in D_{t_1}$.

Proof of Claim 3. As g is Q'-generic over N, $\omega \setminus E^f \in d = \underset{\sim}{D}_1[g]$, and Q' does not add reals, there exists $w \in g$ such that

$$N \models w \Vdash_{Q'} \text{"}\omega \setminus E^f \in \underset{\sim}{D}\text{"}.$$

By elementarity we conclude that this is true in V, so by definition of D_w, $\omega \setminus E^f \in D_w$. Clearly we have $t_1 \leq w$, so $\omega \setminus E^f \in D_{t_1}$. $\square$

Let s_f be defined as in the remark after 4.8. Then s_f is also Q_d-generic over $N[g]$, and clearly f belongs to the generic filter determined by s_f. Hence (3) holds if s is replaced by s_f.

Clearly $N[g][s] = N[g][s_f]$, and hence t_1 is $Q(g)$-generic over $N[g][s_f]$, and consequently $N[g][s][t_1] = N[g][s_f][t_1]$.

From (3) we conclude:

$$N[g][s_f][t_1] \models (u^p, t_1) \Vdash_Q \text{“}\underset{\sim}{r} \subseteq^* s_f^{-1}(1)\text{”}. \tag{4}$$

From Claim 3 together with Claims 1 and 2 we conclude:

$$N[g][s_f][t_1] \models (u^p, t_1) \Vdash_Q \text{“}\underset{\sim}{r} \subseteq^* \omega \backslash E^f\text{”}. \tag{7}$$

Since $s^{-1}(1), s_f^{-1}(1)$ are complementary on $\omega \setminus E^f$, (2), (4) and (5) imply that $\underset{\sim}{r}$ is forced to be finite, a contradiction. □□□

References

[B] J.E. Baumgartner, *Iterated forcing,* Surveys in Set Theory, A.R.D. Mathias (ed.), London Math. Soc. Lect. Notes Ser. 8, Cambridge Univ. Press, Cambridge (1983), 1–59. MR **87c:**03099

[BPS] B. Balcar, J. Pelant and P. Simon, *The space of ultrafilters on N covered by nowhere dense sets,* Fund. Math. 110 (1980), 11–24. MR **82c:**54003

[GSh] M. Goldstern and S. Shelah, *Ramsey ultrafilters and the reaping number – Con*($\mathfrak{r} < \mathfrak{u}$), Ann. Pure Appl. Logic 49 (1990), 121–142. MR **91m:**03050

[J] T. Jech, *Set theory,* Academic Press, New York (1978). MR **80a:**03062

[JSh] H. Judah and S. Shelah, Δ^1_2*-sets of reals,* Ann. Pure Appl. Logic 42 (1989), 207–233. MR **90f:**03081

[JSh2] H. Judah and S. Shelah, *Souslin forcing,* J. Symb. Logic 53/4 (1988), 1188–1207. MR **90h:**03035

[JSp] H. Judah and O. Spinas, *Large cardinals and projective sets,* Arch. Math. Logic 36 (1997), 137–155. MR **98m:**03102

[M] A.R.D. Mathias, *Happy families,* Ann. Math. Logic 12 (1977), 59–111. MR **58:**10462

[R] M.E. Rudin, *Partial orders on the types of βN,* Trans. Amer. Math. Soc. 155 (1971), 353–362. MR **42:**8459

[Sh*b*] S. Shelah, *Proper forcing,* Lecture Notes in Math., vol. 940, Springer (1982). MR **84h:**03002

[Sh630] S. Shelah, *Paper 630,* to appear.

[ShSp] S. Shelah, O. Spinas, *The distributivity numbers of finite products of $\mathcal{P}(\omega)$/fin,* Fund. Math. 158 (1998), 81–93. CMP 99:01.

Institute of Mathematics, Hebrew University, Givat Ram, 91904 Jerusalem, Israel
E-mail address: shelah@math.huji.ac.il

Mathematik, ETH-Zentrum, 8092 Zürich, Switzerland
E-mail address: spinas@math.ethz.ch