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CARDINAL PRESERVING IDEALS

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Abstract. We give some general criteria, when κ-complete forcing preserves largeness properties—like κ-presaturation of normal ideals on λ (even when they concentrate on small cofinalities). Then we quite accurately obtain the consistency strength "NS_λ is $\aleph_1$-preserving", for $\lambda > \aleph_2$.

We consider the notion of a κ-presaturated ideal which was basically introduced by Baumgartner and Taylor [3]. It is a weakening of presaturation. It turns out that this notion can be preserved under forcing like the Levy collapse. So in order to obtain such an ideal over a small cardinal it is enough to construct it over an inaccessible and then just to use the Levy collapse.

The paper is organized as follows. In Section 1 the notions are introduced and various conditions on forcing notions for the preservation of κ-presaturation are presented. Models with NS_λ cardinal preserving are constructed in Section 2. The reading of this section requires some knowledge of [6, 7, 9].

The results of the first section are due to the second author and second section to the first.

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Notation. NS_κ denotes the nonstationary ideal over a regular cardinal $\kappa > \aleph_0$, NS^μ_κ denotes the NS_κ restricted to cofinality μ, i.e.

$$\{ X \subseteq \kappa \mid X \cap \{ \alpha < \kappa \mid \mathrm{cf}\,\alpha = \mu \} \in NS_\kappa \},$$

D denotes a normal filter over a regular cardinal $\lambda = \lambda(D) > \aleph_0$,

$$D^+ = \{ A \subseteq \lambda \mid A \neq \emptyset \text{ modulo } D, \text{ i.e. } \lambda - A \notin D \}.$$

$\underset{\sim}{D}^Q$ for a forcing notion Q, such that $\Vdash_Q$ "$\lambda(D)$ is regular" is the Q-name of the normal filter on $\lambda(D)$ generated by D. By forcing with D^+ we mean the forcing with D-positive sets ordered by inclusion.

§1. κ-presaturation and preservation conditions.

The definitions and the facts 1.1–1.5 below are basically due to Baumgartner-Taylor [3].

DEFINITION 1.1. A normal filter (or ideal) D over λ is *κ-presaturated* if

$\Vdash_{D^+}$ "every set of ordinals of cardinality $< \kappa$ can be covered by a set of V of cardinality λ".

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Let us formulate an equivalent definition which is much easier to use.

DEFINITION 1.2. A normal filter D on λ is *κ-presaturated*, if for every $\kappa_0 < \kappa$, and maximal antichains $I_\alpha = \{ A_i^\alpha : i < i_\alpha \}$, i.e.

$$A_i^\alpha \in D^+, \quad [i \neq j \Longrightarrow A_i^\alpha \cap A_j^\alpha \notin D^+], \qquad (\forall A \in D^+)(\exists i < i_\alpha) A \cap A_i^\alpha \in D^+)$$

(for $\alpha < \kappa_0$) for every $B \in D^+$ there is $A^* \in D^+$ such that $A^* \subseteq B$ and

$$(\forall \alpha < \kappa_0)\ |\{ i < i_\alpha : A^* \cap A_i^\alpha \in D^+ \}| \leq \lambda.$$

FACT 1.3. *If $\kappa < \lambda$,*

$$(\forall \Theta < \lambda)\ [\Theta^{<\kappa} < \lambda],\ \{ i < \lambda : \operatorname{cf} i \geq \kappa \} \in D,$$

D is κ-presaturated, then in 1.2 *we can find A^{**} such that*

$$\forall \alpha < \kappa\ \exists! i < i_\alpha\ A^{**} \cap A_i^\alpha \in D^+.$$

PROOF. First pick A^* as in the definition. For every $\alpha < \kappa_0$ let $\langle B_\tau^\alpha \mid \tau < \lambda_\alpha \leq \lambda \rangle$ be an enumeration of

$$\{ A_i^\alpha \cap A^* \mid i < i_\alpha,\ A_i^\alpha \cap A^* \in D^+ \}.$$

Without loss of generality $\min B_\tau^\alpha > \tau$. For $\nu < \lambda$, $\alpha < \kappa_0$ let $\tau_\alpha(\nu)$ be the least τ such that $\nu \in B_\tau^\alpha$ if such τ exists and -1 otherwise. Define $f : \lambda \to \lambda$ by

$$f(\nu) = \bigcup_{\alpha < \kappa_0} \tau_\alpha(\nu).$$

Then there are $\Theta < \lambda$ and a D-positive subset A' of A^* such that $f''(A') = \{\Theta\}$. Since $\Theta^{<\kappa} < \lambda$, using once more the Fodor Lemma, we can find $A^{**} \subseteq A'$ as required. ⊣

DEFINITION 1.4. A normal ideal or filter on λ is *κ-preserving* if and only if

(a) I is precipitous
(b) $\Vdash_{I^+}$ "$\check{\kappa}$ is a cardinal".

REMARKS.

(1) If $\kappa = \lambda^+$, then such an ideal is called presaturated. This notion was introduced by Baumgartner-Taylor [3].
(2) It is unknown if for $\kappa \geq \omega_1$ (b) may hold without (a). But if $2^\lambda = \lambda^+$, then (b) $\Longrightarrow$ (a).
(3) Every precipitous ideal is $|I^+|^+$-preserving.

PROPOSITION 1.5. *Suppose that $\kappa < \lambda$ are regular, $2^\lambda = \lambda^+$ and D is a normal ideal over λ. Then the following conditions are equivalent*:

(a) *D is not κ-presaturated*;
(b) $\Vdash_{D^+} \operatorname{cf} \lambda^+ < \kappa$;
(c) *the forcing with D^+ collapses all the cardinals between λ^+ and some cardinal $< \kappa$*;
(d) *D is not κ-preserving*;
(e) *a generic ultrapower of D is not well founded or it is well founded but it is not closed in V^{D^+} under less than κ-sequences of its elements*;

If in addition $2^{<\kappa} < \lambda$ then also

(f) *the forcing with D^+ adds new subsets to some ordinal $< \kappa$.*

PROOF. (b) $\Longrightarrow$ (c) since $|D^+| = \lambda^+$. The direction (c) $\Longrightarrow$ (a), (b), (d), (e), (f) are trivial. If $\neg$(b) holds, then, as in [3], also $\neg$(a) holds. $\neg$(a) implies $\neg$(d), $\neg$(e) and $\neg$(f). $\dashv$

We are now going to formulate various preservation conditions.

LEMMA 1.6. *Suppose that $\kappa < \lambda$ are regular cardinals, D a normal filter on $\lambda = \lambda(D)$, Q a forcing notion $\Vdash_Q$ "λ regular" and $\underset{\sim}{D}^Q$ denotes the normal filter on λ in V^Q which D generates.*

Then $\underset{\sim}{D}^Q$ is κ-presaturated in V^Q provided that for some K

$(*)^1_{D,K,Q,\kappa}$ *K is a structure with universe $|K|$, unitary function $T = T^K$, and partial unitary functions $p_i = p_i^K$ such that*

(a) *for $t \in K$, $T(t) \in D^+$;*
(b) *$p_i(t)$ is defined if and only if $i \in T(t)$;*
(c) *$\underset{\sim}{\tau}_t = \underset{\sim}{\tau}_t^K = \{ i \in T(t) : p_i(t) \in \underset{\sim}{G}_Q \} \subseteq \lambda$ is not forced to be $\emptyset$ modulo $\underset{\sim}{D}^Q$;*
(d) *if $q \Vdash_Q$ "$\underset{\sim}{T} \in (\underset{\sim}{D}^Q)^+$" for some $q \in Q$ then there is s, $s \in K$, $p_i(s) \geq q$ and $\Vdash_Q$ "$\underset{\sim}{\tau}_s \subseteq \underset{\sim}{T}$;*
(e) *if $\kappa(*) < \kappa$, for $\alpha < \kappa(*)$ $Y_\alpha \subseteq K$ is such that $\{ T(t) : t \in Y_\alpha \}$ is a maximal antichain of D^+ such that*

$$[t \in Y_\alpha,\ s \in Y_\beta,\ \alpha > \beta \Longrightarrow (\exists C \in D)(\forall i \in C \cap T(t) \cap T(s))\ p_i(s) \leq p_i(t)],$$

and $T \in D^+$ then there is s, $s \in K$, $T(s) \subseteq T$ and there are $y_\alpha \subseteq Y_\alpha$, $|y_\alpha| \leq \lambda$ for $\alpha < \kappa(*)$ such that

$$[\alpha < \kappa(*),\ x \in Y_\alpha \setminus y_\alpha \Longrightarrow T(x) \cap T(s) = \emptyset \text{ modulo } D]$$

and

$$\forall \alpha < \kappa(*)\ (\forall x \in y_\alpha)(\exists C \in D)(\forall i \in C \cap T(x) \cap T(s))\ [p_i(x) \leq p_i(s)].$$

PROOF. Let $\kappa(*) < \kappa$, and $\underset{\sim}{I}_\alpha = \{ \underset{\sim}{A}_i^\alpha : i < \underset{\sim}{i}_\alpha \}$ (for $\alpha < \kappa(*)$) be Q-names of maximal antichains of $(\underset{\sim}{D}^Q)^+$ (as in Definition 1.2) (i.e., $\Vdash_Q$ "$\underset{\sim}{I}_\alpha$ is a maximal antichain".

We define by induction on $\alpha < \kappa(*)$ Y_α, h_α such that

(i) $Y_\alpha \subseteq K$, $\{ T(t) : t \in Y_\alpha \}$ a maximal antichain of D^+;
(ii) for every $\beta < \alpha$, $t \in Y_\beta$, $s \in Y_\alpha$ for some $C \in D$

$$i \in C \cap T(s) \cap T(t) \to p_i(t) \leq p_i(s));$$

(iii) $h_\alpha \colon Y_\alpha \to$ ordinals, and $\Vdash$ "$\tau_t \subseteq \underset{\sim}{A}^\alpha_{h_\alpha(t)}$" for $t \in Y_\alpha$.

Using (g) pick s and $\langle y_\alpha \mid \alpha < \kappa(*) \rangle$. Then $\underset{\sim}{\tau}_s$ will be as required in Definition 1.2. $\dashv$

DEFINITION 1.7.

(1) $\mathrm{Gm}(D, \gamma, a)$ (where $a \subseteq \gamma$) is a game which lasts γ moves, in the ith move if $i \in a$ player I, and if $i \notin a$ player II, choose a set $A_i \in D^+$, $A_i \subseteq A_j$ modulo D for $j < i$. If at some stage there is no legal move, player I wins, otherwise player II wins.

(2) If we omit a, it means $a = \{2i + 1 : 2i + 1 < \gamma\}$.
(3) $\mathrm{Gm}^+(D, \gamma, a)$ is defined similarly, but for player II to win, $\bigcap A_i \neq \emptyset$ has to hold as well.

By Galvin-Jech-Magidor [4], the following holds.

PROPOSITION 1.8. *D is precipitous if player I has no winning strategy in* $\mathrm{Gm}^+(D, \omega)$.

PROPOSITION 1.9. *Suppose $\kappa < \lambda$ are regular cardinals, $\forall \Theta < \lambda\ [\Theta^{<\kappa} < \lambda]$, D a normal filter on λ, Q a forcing notion, $\Vdash_Q$ "λ is a regular" and $\underset{\sim}{D}^Q$ denotes the normal filter on λ in V^Q which D generates. Let $(*)^2_{D,K,Q}$ denote the following:*

(a) *K is a partial order, elements of K are of the form $\bar{p} = \langle p_i : i \in T\rangle$, where $T = T_{\bar{p}}$ is D-positive and $p_i \in Q$ for $i \in T$. For $\bar{p}, \bar{q} \in K$, $\bar{p} \geq \bar{q}$ if and only if $T_{\bar{p}} \subseteq T_{\bar{q}}$ and for every $i \in T_{\bar{p}}$, $p_i \geq q_i$ in the ordering of Q;*
(b) *if $\bar{p} \in K$, then $\tau_{\bar{p}} = \{i \in T : p_i \in \underset{\sim}{G}_Q\}$ is not forced to be empty modulo D;*
(c) *if $\underset{\sim}{T}$ is a Q-name, $\underset{\sim}{T} \cap \tau_{\bar{p}}$ not forced to be empty modulo D, then for some $\bar{q} \in K$, $\bar{q} \geq \bar{p}$ and $\Vdash_Q$ "$\tau_{\bar{q}} \subseteq \underset{\sim}{T}$";*
(d) (I) *if $\bar{p}^\alpha = \langle p^\alpha_i \mid i \in T^\alpha\rangle$ $(\alpha < \alpha(*) < \kappa)$ is an increasing sequence in K and $\bigcap_{\alpha<\alpha(*)} T^\alpha \in D^+$ then there is an upper bound in K or* (II) *if $\bar{p}^\alpha$ $(\alpha < \alpha(*) < \kappa)$ is a sequence from K, $T \in D^+$, $T = \bigcap_{\alpha<\alpha(*)} T^\alpha$,*

$$[\alpha < \beta,\ i \in T^\alpha \cap T^\beta \Longrightarrow p^\alpha_i \leq p^\beta_i]$$

then there is $\bar{p}^* \in K$, $T_{\bar{p}^*} \subseteq T$,

$$[i \in T^\alpha \cap T_{\bar{p}^*} \Longrightarrow p^\alpha_i \leq p^*_i];$$

(e) *for every $q \in Q$, $\underset{\sim}{\tau}$ such that $q \Vdash$ "$\underset{\sim}{\tau} \in (D^Q)^+$" there is $\bar{p} \in K$, $p_i \geq q$ and $q \Vdash \tau_{\bar{p}} \subseteq \underset{\sim}{\tau}$;*
(f) *$\langle \emptyset_Q \mid i < \lambda\rangle \in K$, where $\emptyset_Q$ is the minimal element of Q.*

If $(*)^2_{D,K,Q}$ holds then

(1) *if D is κ-presaturated in V, $\{\delta : \mathrm{cf}\,\delta \geq \kappa\ (\text{in } V)\} \in D$, then $\underset{\sim}{D}^Q$ is κ-presaturated in V^Q.*
(2) *If player II wins in* $\mathrm{Gm}(\lambda, D, \gamma)$ *in V $(\gamma \leq \kappa)$, Q κ-complete then he wins in* $\mathrm{Gm}(\lambda, D, \gamma)$ *in V^Q provided that*
$(*)^3_{D,K,Q}$:
(a) *$\bar{p} \in K \Longrightarrow p_i \Vdash_Q$ "$\underset{\sim}{\tau}_{\bar{p}} \neq \emptyset$ modulo $\underset{\sim}{D}^Q$" for $i \in T_{\bar{p}}$*
(b) *for every $S \in D^+$ and $\bar{p} \in K$ if $S \subseteq T_{\bar{p}}$ then*

$$p_i \Vdash S \cap \underset{\sim}{\tau}_{\bar{p}} \neq \emptyset \text{ modulo } \underset{\sim}{D}^Q.$$

In particular Q preserves D^+-ness.
(c) $(*)^2_{D,K,Q}$.
(3) *The same holds if we replace "The player II wins" by "Player I does not have winning strategy" or the game* $\mathrm{Gm}(\lambda, D, \gamma)$ *is replaced by* $\mathrm{Gm}^+(\lambda, D, \gamma)$.

PROOF.

(1) Let $\kappa(*) < \kappa$, $\underset{\sim}{I}_\alpha = \{\underset{\sim}{A}^\alpha_i : i < \underset{\sim}{i}_\alpha\}$ $(\alpha < \kappa(*))$ be $\kappa(*)$ Q-names of maximal antichains of D^+ as mentioned in the definition of κ-presaturitivity (i.e., it is forced that they are like that).

We define by induction on $\alpha \leq \kappa(*)$, Y_α, $j(\nu)$ and $\bar{p}^\nu$ $(\nu \in Y_\alpha)$ such that

(i) Y_α is a set of sequence of ordinals of length α;

(ii) $\bar{p}^\nu \in K$;
(iii) $\beta < \alpha$, $\nu \in Y_\alpha$ implies $\nu|\beta \in Y_\beta$, $\bar{p}^{\nu|\beta} \leq \bar{p}^\nu$ (i.e., $T^\nu \subseteq T^{\nu|\beta}$), $[i \in T^\nu \Longrightarrow p_i^{\nu|\beta} \leq p_i^\nu]$;
(iv) $Y_0 = \{\langle\rangle\}$, $\bar{p}^{\langle\rangle} = \langle p_i^{\langle\rangle} : i < \lambda \rangle$, $p_i^{\langle\rangle} = \emptyset_Q$ (the minimal element);
(v) for δ limit,

$$Y_\delta = \{ \eta : \eta \text{ a sequence of ordinals } \ell g(\eta) = \delta, (\forall i < \delta)\ \eta|i \in Y_i \};$$

(vi) $\{ T^\nu : \nu \in Y_\alpha \}$ is a maximal antichain;
(vii) for α limit, $\nu \in Y_\alpha$,

$$T^\nu = \bigcap_{\alpha' < \alpha} T^{\nu|\alpha'};$$

(viii) for $i \in T^\nu$, $\nu \in Y_{\alpha+1}$, $p_i^\nu \Vdash_Q$ "$i \in \underset{\sim}{A}^\alpha_{j(\nu)}$".

There is no problem to do this for α limit, (vi) is preserved as

$$\forall\Theta < \lambda\, [\Theta^\kappa < \lambda]$$

and $\{ \delta : \mathrm{cf}\,\delta \geq \kappa \} \in D$ by an assumption and Fact 1.3.

Now as D is κ-presaturated there are $B \in D^+$, $y_\alpha \subseteq Y_\alpha$, $|y_\alpha| \leq \lambda$, such that

$$[\nu \in Y_\alpha - y_\alpha \Longrightarrow B \cap T^\nu \notin D^+].$$

So there is $C \in D$ such that

$$(\forall\alpha < \kappa(*))(\forall\nu_1 \neq \nu_2 \in y_\alpha)\, [T^{\nu_1} \cap T^{\nu_2} \cap C = \emptyset].$$

Let

$$B' = \bigcap_{\alpha < \kappa(*)} \bigcup_{\nu \in y_\alpha} (T^\nu \cap C),$$

then $B' \supseteq B$ modulo D, hence $B' \in D^+$.

Apply (d) and get $\bar{p}^*$.

(2) Let us provide a winning strategy for player II. Let $G \subseteq Q$ be generic over V without loss of generality the players choose Q-names for their moves. We will now describe the strategy of player II in $V[G]$.

Player II also chooses T_i, $\bar{t}^j$ $(j \in a, i < \gamma)$, according to the moves. Player II preserves the following (for a fixed winning strategy $\bar{F}$ of player II in $\mathrm{Gm}(D, \gamma, a)$ in V).

(*) the plays $\langle \underset{\sim}{A}_i : i < \gamma^* \leq \gamma \rangle$ and $\langle T_i : i < \gamma^* \rangle$ so far satisfy
  (a) $\langle T_i : i < \gamma^* \rangle$ is a beginning of a play of $\mathrm{Gm}(D, \gamma, a)$ (in V) in which player II uses the strategy $\bar{F}$;
  (b) for $i \in a$, $\underset{\sim}{A}_i \supseteq \underset{\sim}{\tau}_{\bar{t}^i}$, $T_{\bar{t}^i} = T_i$, $\bar{t}^i \in K$;
  (c) for $j < i < \gamma^*$, $\bar{t}^j \leq \bar{t}^i$;
  (d) $V[G] \models$ "$\underset{\sim}{A}_i[G] \in D^+$".

(3) Similar to (2). ⊣

The following proposition shows that it is possible to remove the assumptions on cofinality used in Proposition 1.9 (1).

PROPOSITION 1.10. *Suppose $\kappa < \lambda$ are regular cardinals, Q a κ-complete forcing notion, $\Vdash_Q$ "λ is regular cardinal". Let $\underset{\sim}{D}^Q$ be the Q-name of the normal filter on λ which D generates in V^Q. Assume that the following principle holds*:

$(*)^4_{D,K,Q}$:

(a) *K is a set, its elements are of the form $\bar{p} = \langle p_i : i \in T \rangle$, $p_i \in Q$, $T = T_{\bar{p}} \in D^+$*;
(b) *if $q \in Q$, $T \in D^+$ then there is $\bar{p} \in K$, $\bigwedge_i q \leq p_i$, $T \supseteq T_{\bar{p}}$*;
(c) *let $\underset{\sim}{\tau}_{\bar{p}} = \{ i : p_i \in \underset{\sim}{G} \}$ we assume $p_i \Vdash \underset{\sim}{\tau}_{\bar{p}} \in (D^Q)^+$ (or just $\nVdash \underset{\sim}{\tau}_{\bar{p}} \notin (\underset{\sim}{D}^Q)^+$)*;
(d) *if $\bar{p} \in K$, $T' \subseteq T_{\bar{p}}$, $T' \in D^+$ and $\bigwedge_{i \in T'} p_i \leq q_i \in Q$ then for some $T'' \subseteq T'$ and $\bar{r} = \langle r_i : i \in T'' \rangle \in K$ $\bigwedge_{i \in T''} q_i \leq r_i$*;
(e) *for every $q \in Q$, $\underset{\sim}{\tau}$ such that $q \Vdash \underset{\sim}{\tau} \in (D^Q)^+$, there is $\bar{p} \in K$, $p_i \geq q$, $q \Vdash \tau_{\bar{p}} \subseteq \underset{\sim}{\tau}$*;
(f) *$\langle \emptyset_Q : i < \lambda \rangle \in K$, where $\emptyset_Q$ is the minimal element of Q.*

If D is κ-presaturated then $\underset{\sim}{D}^Q$ is κ-presaturated.

REMARK 1.11. We can omit (b) and get only

$$\nVdash \text{“}\underset{\sim}{D}^Q|\underset{\sim}{\tau}_{\bar{p}} \text{ is not } \kappa\text{-presaturated”}$$

if $\bar{p} \in K$.

PROOF. Let $\kappa(*) < \kappa$, $\underset{\sim}{I}_\alpha = \{ \underset{\sim}{A}^\alpha_i : i < \underset{\sim}{i}_\alpha \}$ for $\alpha < \kappa(*)$ are $\kappa(*)$ Q-names of maximal antichains of $(\underset{\sim}{D}^Q)^+$ which form a counterexample to κ-presaturativity (i.e., at least some $q_0 \in P$ forces this).

We define by induction on $\alpha \leq \kappa(*)$ Y_α, the function $f \restriction Y_\alpha \colon Y_\alpha \to$ ordinals and $\bar{p}^v$ $(v \in Y_\alpha)$ such that

(i) $\bar{p}^v \in K$ let $\bar{p}^v = \langle p^v_i : i \in T^v \rangle$, $T^v \in D^+$;
(ii) for every $v \in Y_\alpha$ and $i \in T^v$ $q_0 \leq p^v_i$;
(iii) $\langle T^v : v \in Y_\alpha \rangle$ is a maximal antichain of D^+;
(iv) for $v \in Y_\alpha$, $i \in T^v$, $p^v_i \Vdash$ "$i \in \underset{\sim}{A}^\alpha_{f(v)}$";
(v) if $\eta \in Y_\beta$, $v \in Y_\alpha$, $\beta < \alpha$, and $T^v \cap T^\eta \in D^+$ then for some $C_{v,\eta} \in D$,

$$(\forall i \in T_v \cap T_\eta \cap C_{v,\eta}) [p^\eta_i \leq p^v_i];$$

(vi) $\beta < \alpha \Longrightarrow Y_\beta \cap Y_\alpha = \emptyset$;
(vii) for $\beta < \alpha$, $v \in Y_\alpha$

$$|\{ \eta \in Y_\beta : T^\eta \cap T^v \in D^+ \}| \leq \lambda.$$

If η, $v \in \bigcup_{\beta < \alpha} Y_\beta$ and $T^\eta \cap T^v \notin D^+$, then we choose $C_{\eta,v} \in D$ disjoint to $T^\eta \cap T^v$. (It occurs when $\eta \neq v \in Y_\beta$.)

Arriving at α, let $\{ \bar{p}^v : v \in Y_\alpha \}$ be maximal such that (i), (ii), (iv), (v), (vi), (vii), holds and $\{ T^v : v \in Y_\alpha \}$ is an antichain on $\underset{\sim}{D}^+$ (not necessarily maximal).

It suffices to prove that it is a maximal antichain, so let $T \in D^+$, $T \cap T^v = \emptyset$ modulo D for every $v \in Y_\alpha$. As D is κ-presaturated, there is $T' \subseteq T$, $T' \in D^+$ such that for every $\beta < \alpha$, $\{ \eta \in Y_\beta : T' \cap T^\eta \in D^+ \}$ has cardinality $\leq \lambda$, and let it be $\{ \eta_{\beta,j} : j < j_\beta \leq \lambda \}$. Let

$$C = \{ \delta < \lambda : \text{if } \beta_1, \beta_2 < \alpha,\ j_1 < \delta,\ j_2 < \delta, \text{ then } \delta \in C_{\eta_{\beta_1,j_1},\eta_{\beta_2,j_2}} \}.$$

Now for each $\beta < \alpha$, let

$$T'_\beta = \{\delta \in T' : (\exists j < j_\beta \cap \delta)\,[\delta \in T^{\eta_{\beta,j}}]\}.$$

Then $T'_\beta \subseteq T'$ and $T' - T'_\beta \notin D^+$ (as $\{T^v : v \in Y_\beta\}$ is a maximal antichain). Let

$$T^* = C \cap T' \cap \bigcap_{\beta<\alpha} T'_\beta.$$

So $T' - T^* \notin D^+$. Hence $T^* \in D^+$, $T^* \subseteq T$. For every $\delta \in T^*$ and $\beta < \alpha$, there is a unique $j^\delta_\beta < \delta$, $\delta \in T^{\eta_{\beta,j^\delta_\beta}}$ (if j_1, j_2 are candidates use the definition of $C_{\eta_{\beta,j_1},\eta_{\beta,j_2}}$ (there is one as $\delta \in T'_\beta$). Now for $\beta_1 < \beta_2 < \alpha$ (for our δ) by the choice of $C_{\eta_{\beta_1 j_1},\eta_{\beta_2 j_2}}$

$$p_\delta^{\eta_{\beta_1,j^\delta_{\beta_1}}} \le p_\delta^{\eta_{\beta_2,j^\delta_{\beta_2}}}.$$

Hence using κ-completeness of Q, we can find p_δ,

$$\bigwedge_{\beta<\alpha} p_\delta^{\eta_{\beta,j^\delta_\beta}} \le p_\delta.$$

Now $\langle p_\delta : \delta \in T^*\rangle$ satisfies much, almost contradicting the maximality of $\{\bar{p}^v : v \in Y_\alpha\}$ and non-maximality of $\{T^v : v \in Y_\alpha\}$ (and the choice of a f). But repairing this is easy. Without loss of generality $T^* \subseteq T^{\eta_{0,0}}$. Using (d) of $(*)^4_{D,K,Q}$ we obtain $\bar{p}' = \langle p'_\delta \mid \delta \in T^a\rangle \in K$ such that $T^a \subseteq T^*$ and $\bigwedge_{i\in T^a} p_i \le p'_i$. We have to take care of (iv). So we have $\bar{p}' = \langle p'_\delta : \delta \in T^a\rangle \in K$, $\bigwedge_{v\in Y_\alpha} T^a \cap T^v = \emptyset$ (modulo D), $\bar{p}'$ satisfies everything except (v). Now $\nVdash \underset{\sim}{\tau}_{\bar{p}'} \notin (\underset{\sim}{D}^+)^Q$ so for some $r \in Q$ $r \Vdash \underset{\sim}{\tau}_{\bar{p}'} \in (\underset{\sim}{D}^+)^Q$. Hence for some r_1, $r \le r_1 \in Q$, and j the following holds:

$$(*)\qquad r_1 \Vdash \underset{\sim}{\tau}_{\bar{p}'} \cap \underset{\sim}{A}^\alpha_j \in D^+.$$

For each $i \in T^*$ choose if possible p''_i, $p'_i \le p''_i \in Q$ $p''_i \Vdash$ "$i \in \underset{\sim}{A}^\alpha_j$". Set $T^b = \{i : p''_i \text{ is defined}\}$. It is necessarily in D^+ (otherwise this contradicts $(*)$). Applying (d) of $(*)^4_{K,Q,D}$ we get a final $\bar{p}$ contradicting "Y_α maximal but $\{T^v : v \in Y_\alpha\}$ is not".

For $\alpha = 0$, use (b) with our q_0. ⊣

PROPOSITION 1.12. *Let D, D_1 be normal filters over a regular cardinal λ. Suppose that the following holds*

C_{D,D_1}: *for every $T \in D^+$, $F(T) = \{\delta < \lambda : T \cap \delta \text{ is stationary}\} \in D_1^+$.*

Let Q be a κ-closed forcing for a regular $\kappa < \lambda$. Then in V^Q C_{D,D_1} is not forced to fail provided that

$(*)^5_{D,K,D_1,Q}$:

(α) $(*)^4_{D,K,Q}$ *or* $(*)^2_{D,K,Q}$ *and*

(β_1) *for* $\bar{p} \in K$

$$\{\delta < \lambda : \emptyset \Vdash \underset{\sim}{\tau}_{\bar{p}} \cap \delta \text{ stationary}\} \in D_1^+$$

and

$$\emptyset \Vdash (D_1^+)^V \subseteq (D_1^Q)^+$$

or

(β_2) *for* $\bar{p} \in K$

$$\{\delta < \lambda : \emptyset \nVdash \underset{\sim}{\tau}_{\bar{p}} \cap \delta \text{ nonstationary}\} \in D_1^+$$

and Q *satisfies* λ-*c.c.*

PROOF. Suppose otherwise. Let

$$\Vdash \text{“}\underset{\sim}{S} \in (\underset{\sim}{D}^Q)^+ \text{ and } F(\underset{\sim}{S}) \notin (\underset{\sim}{D}_1^Q)^+\text{”}.$$

Set

$$T = \{ i < \lambda \mid \text{some } p_i \text{ forces “} i \in \underset{\sim}{S}\text{”}\}.$$

Using (d) for $\bar{p} = \langle \emptyset : i < \lambda \rangle$ (see (f)) find $\bar{r} \in K$, $\bar{r} = \langle r_i \mid i \in T' \rangle$ such that $T' \subseteq T$ and $r_i \geq p_i$. Then $\Vdash \underset{\sim}{\tau}_{\bar{r}} \subseteq \underset{\sim}{S}$. If ($\beta_1$) holds, then we are done. If (β_2) is true, then by λ-c.c. of Q and normality of D_1 find $C \in D_1$ so that $\Vdash$ “$\check{C} \cap F(\underset{\sim}{S}) = \emptyset$”. Then also $\Vdash$ “$\check{C} \cap F(\underset{\sim}{\tau}_{\bar{r}}) = \emptyset$”. The set

$$\{\delta \in C \mid \text{for some } q_\delta,\ q_\delta \Vdash \underset{\sim}{\tau}_{\bar{r}} \cap \check{\delta} \text{ is stationary}\}$$

is in D_1^+. Hence it is nonempty. So there is $\delta \in C$ such that $q_\delta \Vdash \underset{\sim}{\tau}_{\bar{r}} \cap \check{\delta}$ is stationary. But then

$$q_\delta \Vdash \text{“}\check{\delta} \in F(\underset{\sim}{\tau}_{\bar{r}}) \text{ and } \check{\delta} \in \check{C}\text{”}$$

which contradicts the choice of C. ⊣

LEMMA 1.13. *Suppose that* μ *is a regular cardinal,* $\lambda > \mu$ *is an inaccessible* $Q = \mathrm{Col}(\mu, {<}\lambda)$, *and* D *is a normal filter on* λ. *Let* $T \in D^+$ *and* $\bar{p} = \langle p_i \mid i \in T \rangle$ *be a sequence of conditions in* Q. *Then there is* $C \in D$ *such that for every* $i \in T \cap C$ $p_i \Vdash \underset{\sim}{\tau}_{\bar{p}} \in (\underset{\sim}{D}^Q)^+$, *where* $\underset{\sim}{\tau}_{\bar{p}} = \{ i : p_i \in \underset{\sim}{G}\}$.

PROOF. Set $S = \{ i \in T : p_i \nVdash \tau_{\bar{p}} \in (\underset{\sim}{D}^Q)^+\}$. If $\lambda - S \in D$, then let $C = \lambda - S$. It will be as required since

$$\Vdash (\underset{\sim}{\tau} \subseteq \tau_{\bar{p}} \text{ and } \tau_{\bar{p}} - \underset{\sim}{\tau} \subseteq \lambda - C)$$

i.e., $\underset{\sim}{\tau}$ and $\tau_{\bar{p}}$ are the same modulo $\underset{\sim}{D}^Q$ where $\tau = \{ i \in T \cap C \mid p_i \in \underset{\sim}{G}\}$.

Let us now assume that $S \in D^+$. For every $i \in S$ there is $q_i \geq p_i$ $q_i \Vdash \underset{\sim}{\tau}_{\bar{p}} \notin (\underset{\sim}{D}^Q)^+$. Set $\underset{\sim}{\tau}^* = \{ i \in S : q_i \in \underset{\sim}{G}\}$. Since Q satisfies λ-c.c. and D is λ-complete, there exists q forcing “$\underset{\sim}{\tau}^* \in (\underset{\sim}{D}^Q)^+$”. Then for some $q' \geq q$, $i_0 \in S$ $q' \geq q_{i_0}$. So

$$q' \Vdash \text{“}\{ i \in S : q_i \in \underset{\sim}{G}\} \in (D^Q)^+\text{”}.$$

Hence $q' \Vdash$ “$\underset{\sim}{\tau}_{\bar{p}} \in (D^Q)^+$”. Which is impossible since $q_{i_0} \Vdash$ “$\tau_{\bar{p}} \notin (D^Q)^+$”. Contradiction. So $S \notin D^+$. ⊣

REMARK 1.14. It is possible to replace the Levy collapse by any λ-c.c. forcing.

PROPOSITION 1.15. *Suppose that* μ *is a regular cardinal* $\lambda > \mu$ *is an inaccessible,* Q *is the Levy collapse* $\mathrm{Col}(\mu, {<}\lambda)$ *and* D, D_1 *are normal filters over* κ. *Then*

(1) $(*)^2_{D.K.Q}$ *holds if* $\{\delta < \lambda \mid \mathrm{cf}\,\delta \geq \mu\} \in D$ *and we let*

$$K = \{ \langle p_i \mid i \in T \rangle \mid T \in D^+,\ p_i \in Q\};$$

(2) $(*)^4_{D,K,Q}$ *holds if we let*

$$K = \{ \langle p_i \mid i \in T \cap C \rangle \mid T \in D^+,\ p_i \in Q,\ C \text{ is as in Lemma 1.13}\};$$

(3) $(*)^3_{D.K.Q}$ *holds if*

$$\{\delta < \lambda \mid \operatorname{cf}\delta \geq \mu\} \in D$$

and K is as in (2);

(4) $(*)^5_{D.K.D_1.Q}$ *holds if*

$$\{\delta < \lambda \mid \operatorname{cf}\delta < \mu\} \in D, \qquad \{\delta < \lambda \mid \delta \text{ is an inaccessible}\} \in D_1$$

and K is as in (2).

PROOF. (1), (2), (3) Easy checking.

(4) Suppose that in V C_{D,D_1} holds. Let us show the condition (β_2) of $(*)^5_{D.K.D_1.Q}$. Let $\bar{p} = \langle p_i \mid i \in T \rangle \in K$ and $C \in D_1$. We should find some $\delta \in C$ and $q \in Q$ $q \Vdash \underset{\sim}{\tau}_{\bar{p}} \cap \delta$ is stationary.

Using C_{D,D_1} find an inaccessible $\delta \in C$ such that $T \cap \delta$ is stationary and for every $i < \delta$ $p_i \in \operatorname{Col}(\mu, {<}\delta)$. Now, as in Lemma 1.13, there is a stationary $S \subseteq T \cap \delta$ such that every p_i $(i \in S)$ forces in $\operatorname{Col}(\mu, {<}\delta)$ that "$\underset{\sim}{\tau}_{\bar{p}} \cap \delta$ is stationary". If D concentrates on ordinals of cofinality $<\mu$, then without loss of generality all elements of T are of some fixed cofinality $<\mu$. The forcing $\operatorname{Col}(\mu, {<}\lambda)$ is μ-closed, so it would preserve the stationarity of $(\tau_{\bar{p}} \cap \delta)^{\operatorname{Col}(\mu,<\delta)}$. ⊣

COROLLARY 1.16. *Let λ be an inaccessible, $\mu < \lambda$ be a regular cardinal and $Q = \operatorname{Col}(\mu, {<}\lambda)$. Then the following properties are preserved in the generic extension (i.e., for any normal filter D on λ)*

(1) *κ-presaturatedness, for $\kappa \leq \mu$;*
(2) *the existence of winning strategy for II in games defined in Definition* 1.7 *for $\gamma \leq \mu$;*
(3) *the nonexistence of winning strategy for I for $\gamma \leq \mu$;*
(4) *reflection of stationary subsets of ordinals of cofinality $<\mu$ provided that λ is Mahlo and in V the reflecting ordinals are inaccessibles.*

It is possible to use (4) in order to give an alternative to the Harrington-Shelah [12] proof of "every stationary subset of $\aleph_2$ consisting of ordinals of cofinality ω reflects" from a Mahlo cardinal.

Let λ be a Mahlo cardinal. Use the Backward Easton iteration in order to add α^+ Cohen subsets to every regular $\alpha < \lambda$. Now iterate the forcing for shooting clubs through compliments of nonreflecting subsets of

$$\{\alpha < \lambda \mid \operatorname{cf}\alpha = \omega\}$$

as it is done in [12]. Such forcing will preserve all the cardinals. Since it is possible to pick a submodel N such that $\delta = N \cap \lambda$ is an inaccessible cardinal and use Cohen subsets of δ for the definition of N-generic clubs. We refer to [17] for similar construction with $\diamond$.

Let us now give an example of a forcing notion satisfying the preservation conditions but not λ-c.c. The forcing notion we are going to consider was introduced by J. Baumgartner [2] and in a slightly different form by U. Avraham [1].

PROPOSITION 1.17. *Suppose $\mu < \lambda$ are regular cardinals, $\forall\Theta < \lambda$ $(\Theta^{<\mu} < \lambda)$, D is the closed unbounded filter on λ restricted to some cofinalities $\geq\mu$ (or to a stationary* *set consisting of ordinals of such cofinalities) and $S \in D$ is a stationary subset of λ containing all ordinals of cofinality $<\mu$. Let*

$$Q = \{ A \mid A \text{ is a set of pairwise disjoint closed intervals} \subseteq \lambda \text{ of cardinality } <\mu, \text{ and } [\alpha, \beta] \in A \text{ implies } \alpha \in S \}$$

ordered by inclusion (so $\Vdash_Q$ "S contains a club $\underset{\sim}{C}_Q = \{ \alpha \mid \text{for some } \beta\ [\alpha, \beta] \in \underset{\sim}{G}_Q \}$"). Define

$$K = \{ \langle p_i \mid i \in T \rangle \mid T \in D^+,\ T \subseteq \{ \delta \in S \mid \operatorname{cf} \delta \geq \mu \}, p_i \in Q \text{ for some } j \geq i,\ [i, j] \in p_i \text{ and } p_i \Vdash \tau_{\bar{p}} \in (D^Q)^+ \text{ for every } i \in T \}.$$

Then $(*)^4_{D,K,Q}$, $(*)^2_{D,K,Q}$, $(*)^3_{D,K,Q}$ hold.

Remark. Notice that for every $\delta < \lambda$ and $p \in Q$ there are $q \geq p$ and $[i, j] \in q$ so that $i \leq \delta \leq j$. But if $\delta \notin S$ then also $i < \delta$. This implies that $\underset{\sim}{C}_Q$ is a club contained in S.

Proof. Let $\bar{p} = \langle p_i \mid i \in T \rangle$ be so that $T \in D^+$,

$$T \subseteq \{ \delta \in S \mid \operatorname{cf} \delta \geq \mu \},$$

$p_i \in Q$ and for some $j \geq i$, $[i, j] \in p_i$. Let us show that $\nVdash$ "$\underset{\sim}{\tau}_{\bar{p}}$ is not stationary". Suppose otherwise. Let $\underset{\sim}{E}$ be a name of a club such that $\Vdash \underset{\sim}{E} \cap \underset{\sim}{\tau}_{\bar{p}} = \emptyset$. For every $i \in T$ pick $q_i \geq p_i$ forcing "$i \notin \underset{\sim}{E}$". Pick an increasing continuous sequence $\langle M_i \mid i < \lambda \rangle$ of elementary submodels of some $H(\tau)$ for τ big enough so that $|M_i| < \lambda$ and $M_i \cap \lambda \in \lambda$. Set $C^* = \{ \delta \mid M_\delta \cap \lambda = \delta \}$. Clearly, C^* is a club. Pick $\delta^* \in C^* \cap T$. Then $q_{\delta^*} \Vdash \delta^* \in \underset{\sim}{E}$ since otherwise for some $\alpha < \delta^*$, $r \geq q_{\delta^*}$ $r \Vdash \underset{\sim}{E} \cap \delta^* \subseteq \alpha$. But since $\operatorname{cf} \delta^* \geq \mu$ and $|r| < \mu$, for some β, $\alpha < \beta < \delta^*$ there exists $r' \in Q \cap M_\beta$, $r' \geq r | \delta^*$ forcing "$\underset{\sim}{E} \cap \delta^* \not\subseteq \alpha$". But $r' \cup r \in Q$, since $r \geq q_{\delta^*} \geq p_{\delta^*}$ and $[\delta^*, j] \in p_{\delta^*}$ for some $j \geq \delta^*$. This leads to the contradiction.

So for every $\bar{p}$ as above some $q \in Q$ forces "$\underset{\sim}{\tau}_{\bar{p}}$ is stationary". Now the arguments of Lemma 1.13 apply. So for every $\bar{p} = \langle p_i \mid i \in T \rangle$ as above there exists $C \in D$ such that for every $i \in T \cap C$ $p_i \Vdash \underset{\sim}{\tau}_{\bar{p}} \in (\underset{\sim}{D}^Q)^+$.

Let us show that for any $q \in Q$, $\underset{\sim}{T}$ such that $q \Vdash \underset{\sim}{T} \in (\underset{\sim}{D}^Q)^+$ there exists $\bar{p} \in K$, $p_i \geq q$ and $q \Vdash$ "$\tau_{\bar{p}} \subseteq \underset{\sim}{T}$ modulo $(\underset{\sim}{D}^Q)^+$". The set

$$T = \{ \delta \mid \text{for some } p_\delta \geq q,\ p_\delta \Vdash \delta \in \underset{\sim}{T} \cap \underset{\sim}{C}_Q \}$$

is in D^+. But if $p_\delta \Vdash \delta \in \underset{\sim}{C}_Q$, then for some δ' $[\delta, \delta'] \in p_\delta$. Now fix $\langle p_\delta \mid \delta \in T \rangle$ as above in order to obtain $\bar{p} \in K$ as required.

The checking of the rest of the conditions is routine. ⊣

What happens if D concentrates on small cofinality? Does forcing with Q of Proposition 1.17 preserve $<\mu$-presaturedness? Strengthening the assumptions it is possible to obtain a positive answer. Namely the following holds.

Proposition 1.18. *Let μ, λ, S, Q be as in Proposition 1.17, assume that D is a club filter restricted to some cofinality $<\mu$ (or to a stationary set of such cofinality) and there exists a set $S^- \in D$ consisting of ordinals of cofinality $<\mu$ and for every*

$\delta \in S^-$ there is a set $A_\delta \subseteq \delta$, $|A_\delta| < \mu$ consisting of ordinals of cofinality $\geq \mu$, and the following holds:

(∗) *"for every club $C \subseteq \lambda$ $\{\delta \in S^- \mid \sup(A_\delta \cap C) = \delta\} \in D$".*

Let

$$K = \{\langle p_i \mid i \in T\rangle \mid T \in D^+,\ p_i \in Q,\ T \subseteq S^-,\ \text{for every } i \in T\ (\exists \alpha < i)(\forall \xi \in A_i - \alpha)(\exists \eta)([\xi, \eta] \in p_i)\}.$$

Then $(*)^4_{D|\underset{\sim}{S}^*, K, Q}$ holds, where

$$\underset{\sim}{S}^* = \{\delta \in S^- \mid \sup(A_\delta - \underset{\sim}{C}_Q) < \delta\}.$$

Remarks.

(1) The property $(*)$ is true in L and also it can be easily forced.

(2) If we are interested only in a condition forcing $<\mu$-presaturation then there is no need in $\underset{\sim}{S}^*$.

Proof. Let us check that for $\bar{p} = \langle p_i \mid i \in T\rangle \in K \not\Vdash$ "$\tau_{\bar{p}}$ is not stationary". Suppose otherwise, let $\underset{\sim}{C}$ be a name of club disjoint to $\underset{\sim}{\tau}_{\bar{p}}$. Pick $q_i \geq p_i$ forcing "$i \notin \underset{\sim}{C}$". Let $\langle M_\alpha \mid \alpha < \lambda\rangle$ and C^* be as in the previous proposition. Pick $\delta^* \in C^* \cap T$ such that $\text{otp}(C^* \cap \delta^*) = \delta^*$ and $\sup(A_{\delta^*} \cap C^*) = \delta^*$. By the assumption $q_{\delta^*} \Vdash$ "$\delta^* \notin \underset{\sim}{C}$". So for some $r \geq q_{\delta^*}$ and $\alpha < \delta^*$ $r \Vdash$ "$\underset{\sim}{C} \cap \delta^* \subseteq \alpha$". But it is possible to find $\delta \in (A_{\delta^*} \cap C^*) - \alpha$ such that $r|\delta \in M_\delta$. So some $r' \geq r|\delta$, $r' \in M_\delta$ forces "$\underset{\sim}{C} \cap \delta^* \nsubseteq \alpha$". But $r \cup r' \in Q$. Contradiction. ⊣

§2. Constructions of cardinal preserving ideals.

By Jech-Magidor-Mitchell-Prikry [13] it is possible to construct a model with a normal μ-preserving ideal over μ^+ for any regular μ from one measurable. Actually $NS^\mu_{\mu^+}$ can be such ideal. We shall examine here $NS^\tau_{\mu^+}$ for $\tau < \mu$ and NS_{μ^+}.

In order to formulate the results we need the following definition of [9]:

Definition 2.1. Let $\vec{\mathscr{F}} = \langle \mathscr{F}(\alpha, \beta) \mid \beta < \gamma\rangle$ be a sequence of ultrafilters over α. Let $\delta < \gamma$, $\rho > 0$ be ordinals and $\lambda \leq \alpha$ be a regular cardinal. Then

(a) δ is an up-repeat point for $\vec{\mathscr{F}}$ if for every $A \in \mathscr{F}(\alpha, \delta)$ there is δ', $\delta < \delta' < \gamma$ such that $A \in \mathscr{F}(\alpha, \delta')$;

(b) δ is a (λ, ρ)-repeat point for $\vec{\mathscr{F}}$ if (a1) $\text{cf}\,\delta = \lambda$ and (a2) for every

$$A \in \bigcap\{\mathscr{F}(\alpha, \beta) \mid \delta \leq \beta < \delta + \rho\}$$

there are unboundedly many ξs in δ such that

$$A \in \bigcap\{\mathscr{F}(\alpha, \xi') \mid \xi \leq \xi' < \xi + \rho\}.$$

If $\vec{\mathscr{F}}$ is the maximal sequence of measures of the core model we will simply omit it.

Theorem 2.2. *The exact strength of*

(1) *"NS_{μ^+} is μ-preserving for a regular $\mu > \aleph_2$ + GCH" or;*

(2) *"$NS^{\aleph_0}_{\mu^+}$ is μ-preserving for a regular $\mu > \aleph_2$ + GCH" or;*

(3) *"NS_{μ^+} is $\aleph_2$-preserving for a regular $\mu > \aleph_2$ + GCH" or;*

(4) "$NS^{\aleph_0}_{\mu^+}$ *is* $\aleph_2$*-preserving for a regular* $\mu > \aleph_2$ + GCH"

is the existence of an up-repeat point.

PROOF. Use the model of [9] with a presaturated NS_κ over an inaccessible κ and the Levy collapse $\mathrm{Col}(\mu, <\kappa)$. By the results of Section 1, NS_κ will be μ-preserving in the generic extension. ⊣

The rest of the section will be devoted to the construction of NS_κ ω_1-preserving from an $(\omega, \kappa^+ + 1)$-repeat point. The following theorem will then follow by results of [9], [10].

THEOREM 2.3.

(1) *The existence of an* $(\omega, \kappa^+ + 1)$*-repeat point is sufficient for* "NS_κ *is* ω_1*-preserving* $+\kappa$ *is an inaccessible* +GCH".
(2) *The strength of* "$NS^{\aleph_0}_{\mu^+}$ *is* ω_1*-preserving* +GCH" *for a regular* $\mu > \aleph_2$ *is* (ω, μ)*-repeat point.*
(3) *The strength of* "NS_{μ^+} *is* ω_1*-preserving* +GCH", *where* μ *is a regular cardinal* $>\aleph_2$ *is* $(\omega, \mu + 1)$*-repeat point.*

Suppose that $\vec{U}$ is a coherent sequence of ultrafilters with an $(\omega, \kappa^+ + 1)$-repeat point α at κ. Define the iteration $\mathscr{P}_\tau$ for τ in the closure of

$$\{ \beta \mid (\beta = \kappa) \text{ or } (\beta < \kappa \text{ and } \beta \text{ is an inaccessible or } \beta = \gamma + 1 \text{ and } \gamma \text{ is an inaccessible}) \}.$$

On the limit stages use the limit of [6]. For the benefit of the reader let us give a precise definition. See also a more recent presentation [11].

Let A be a set consisting of αs such that $\alpha < \kappa$ and $\alpha > o(\alpha) > 0$. Denote by A^ℓ the closure of the set $\{ \alpha + 1 \mid \alpha \in A \} \cup A$. For every $\alpha \in A^\ell$ define by induction $\mathscr{P}_\alpha$ to be the set of all elements p of the form $\langle \underset{\sim}{p}_\gamma \mid \gamma \in g \rangle$ where

(1) g is a subset of $\alpha \cap A$.
(2) g has an Easton support, i.e., for every inaccessible $\beta \le \alpha$, $\beta > |\mathrm{dom}\, g \cap \beta|$;
(3) for every $\gamma \in \mathrm{dom}\, g$ $p{\restriction}\gamma = \langle \underset{\sim}{p}_\beta \mid \beta \in \gamma \cap g \rangle \in \mathscr{P}_\gamma$ and $p{\restriction}\gamma \Vdash_{\mathscr{P}_\gamma}$ "$\underset{\sim}{p}_\gamma \in \underset{\sim}{Q}_\gamma$".

Let $p = \langle \underset{\sim}{p}_\gamma \mid \gamma \in g \rangle$, $q = \langle \underset{\sim}{q}_\gamma \mid \gamma \in f \rangle$ be elements of $\mathscr{P}_\alpha$. Then $p \ge q$ (p is stronger than q) if the following holds:

(1) $g \supseteq f$
(2) for every $\gamma \in f$ $p{\restriction}\gamma \Vdash_{\mathscr{P}_\gamma}$ "$\underset{\sim}{p}_\gamma \ge \underset{\sim}{q}_\gamma$ in the forcing $\underset{\sim}{Q}_\gamma$"
(3) there exists a finite subset b of f so that for every $\gamma \in f \setminus b$, $p{\restriction}\gamma \Vdash$ "$\underset{\sim}{p}_\gamma \ge^* \underset{\sim}{q}_\gamma$ in $\underset{\sim}{Q}_\gamma$", where $^*\ge$ is the direct extension ordering of $\underset{\sim}{Q}_\gamma$.

If $b = \emptyset$, then let $p \ge^* q$.

Suppose that τ is an inaccessible and $\mathscr{P}_\tau$ is defined. Define $\mathscr{P}_{\tau+1}$. Let $C(\tau^+)$ be the forcing for adding τ^+-Cohen subsets to τ, i.e.,

$$\{ f \in V^{\mathscr{P}_\tau} \mid f \text{ is a partial function from } \tau^+ \times \tau \text{ into } \tau,\ |f|^{V^{\mathscr{P}_\alpha}} < \tau \text{ and for every } \beta < \tau^+ \ \{ \beta' \mid (\beta, \beta') \in \mathrm{dom}\, f \} \text{ is an ordinal} \}.$$

$\mathscr{P}_{\tau+1}$ will be

$$\mathscr{P}_\tau * C(\tau^+) * \mathscr{P}(\tau, o^{\vec{U}}(\tau)),$$

where $\mathscr{P}(\tau, o^{\vec{U}}(\tau))$ is the forcing of [6, 7] with the slight change described below.

The change is in the definition of $U(\tau, \gamma, t)$ the ultrafilter extending $U(\tau, \gamma)$ for $\gamma < o^{\vec{U}}(\tau)$ and a coherent sequence t or more precisely in the definition of the master conditions sequence. Let

$$j^\tau_\beta \colon V \to N^\tau_\beta \cong V^\tau / U(\tau, \beta)$$

for $\beta < o^{\vec{U}}(\tau)$. Pick some well ordering W of V_λ, for a big enough λ so that for every inaccessible $\delta < \lambda$, $W | V_\delta \colon V_\delta \leftrightarrow \delta$. Let γ be some fixed ordinal below $o^{\vec{U}}(\tau)$. Let us for a while drop the indices τ, γ in $j^\tau_\gamma, N^\tau_\gamma$.

Let $\langle \underset{\sim}{D}_{\gamma'} \mid \gamma' < \tau^+ \rangle$ be the $j(W)$-least enumeration of all E-dense open subsets of $j(\mathscr{P}_\tau * C(\tau^+))$ which are in N. Where a subset D of a forcing notion $\mathscr{P}$ E-dense is if for every $p \in \mathscr{P}$ there exists $q \in D$ which is an Easton extension of p. Define an E-increasing sequence $\langle \underset{\sim}{p}_{\gamma'} \mid \gamma' < \tau^+ \rangle$ of elements of $\mathscr{P}_{j(\tau)} * C(j(\tau)^+)/\mathscr{P}_{\tau+1}$ so that for every $\gamma' < \tau^+ \underset{\sim}{p}_{\gamma'}$ there will be a $j(W)$-least E-extension of $\langle \underset{\sim}{p}_{\gamma''} \mid \gamma'' < \gamma' \rangle$ in $\underset{\sim}{D}_{\gamma'}$ compatible with $j''(G \cap C(\tau^+))$.

Now as in [6, 7] set $A \in U(\tau, \gamma, t)$ if for some r in the generic subset of $\mathscr{P}_\tau * C(\tau^+)$, some $\gamma' < \tau^+$, a name $\underset{\sim}{A}$ of A and a $\mathscr{P}_\tau * C(\tau)$-name $\underset{\sim}{T}$, in N

$$r \cap \{\langle \check{t}, \underset{\sim}{T} \rangle\} \cup \underset{\sim}{p}_{\gamma'} \Vdash \check{\tau} \in j(\underset{\sim}{A}).$$

Note that the set $D = \{ q \mid (\check{\tau} \in j(\underset{\sim}{A}) \}$ is E-dense and it belongs to N. So it appears in the list $\langle \underset{\sim}{D}_{\gamma'} \mid \gamma' < \tau^+ \rangle$. Hence some $\underset{\sim}{p}_{\gamma'}$ is in D.

Let G be a generic subset of $\mathscr{P}_\kappa * C(\kappa^+)$. Recall that by [7] the sequence

$$\langle U(\gamma, \gamma', \emptyset) \mid \gamma' < o^{\vec{U}}(\gamma) \rangle$$

is commutative Rudin-Keisler increasing sequence of ultrafilters, for every γ.

Now let

$$j^* \colon V[G] \to M^* \cong V[G]^\kappa / U(\kappa, \gamma, \emptyset).$$

Then $M^* = M[j^*(G)]$ for a model M of ZFC which is contained in M^*. We would like to have the exact description M.

The next definition is based on the Mitchel notion of complete iteration see [15, 14, 16].

DEFINITION 2.4. Suppose that N is a model of set theory, $\vec{V}$ a coherent sequence in N, κ is a cardinal. The complete iteration $j \colon N \to M$ of $\vec{V}$ at κ will be the direct limit of $j_\nu \colon N \to M_\nu$, $\nu < \ell_\kappa$ for some ordinal ℓ_κ, where ℓ_κ, j_ν, M_ν are defined as follows. Set $M_0 = N$, $j_0 = \text{id}$, $\vec{V_0} = \vec{V}$, $C_0(\alpha, \beta) = \emptyset$ for all α and β. If $o^{\vec{V}}(\kappa) = 0$, then $\ell_\kappa = 1$. Suppose otherwise:

CASE 1. *$o^{\vec{V}}(\kappa)$ is a limit ordinal.*

If j_ν, M_ν, $\vec{V}_\nu$, $\vec{C}_\nu$ are defined then set

$$j_{\nu\nu+1} \colon M_\nu \to M_{\nu+1} \cong M^{\alpha_\nu} / V_\nu(\alpha_\nu, \beta_\nu),$$

where α_ν is the minimal ordinal α so that

(i) $\alpha \geq \kappa$;

(ii) α is less than the first $\kappa' > \kappa$ with $o^{\vec{V}}(\kappa') > 0$;

(iii) for some $\beta < o^{\vec{V}_\nu}(\alpha)$ $\vec{C}_\nu(\alpha, \beta)$ is bounded in α and β_ν is the minimal β satisfying (iii) for α_ν.

If there is not such an α_ν then set $\ell_\kappa = \nu$, $j = j_\nu$, $M = M_\nu$, $\vec{C} = \vec{C}_\nu$. Define $\vec{C}_{\nu+1}|(\alpha_\nu, \beta_\nu) = \vec{C}_\nu|(\alpha_\nu, \beta_\nu)$, $\alpha_{\nu+1} = j_{\nu\nu+1}(\alpha_\nu)$

$$\vec{C}_{\nu+1}(\alpha_{\nu+1}, j_{\nu\nu+1}(\beta)) = \begin{cases} \vec{C}_\nu(\alpha_\nu, \beta_\nu), & \text{if } \beta \neq \beta_\nu \\ \vec{C}_\nu(\alpha_\nu, \beta_\nu) \cup \{\alpha_\nu\}, & \text{if } \beta = \beta_\nu. \end{cases}$$

CASE 2. $o^{\vec{V}}(\kappa) = \tau + 1$.

Define j_ν, M_ν, $\vec{V}_\nu$ and $\vec{C}_\nu$ as in Case 1, only in (ii) let α be less than the image of κ under the embedding of N by $V(\kappa, \tau)$.

THEOREM 2.5. *Let $j^*: V[G] \to M^*$ be*

(a) *the ultrapower of $V[G]$ by the ultrafilter $U(\tau, \gamma, \emptyset)$*

or

(b) *the direct limit of the ultrapowers with the Rudin-Kiesler increasing sequence of the ultrafilters $\langle U(\tau, \gamma', \emptyset) \mid \gamma' < \gamma \rangle$.*

Then M^* is a generic extension of M, where M is the complete iteration of $\vec{U}|(\tau, \gamma+1)$ at τ, if (a) *holds, and of $\vec{U}|(\tau, \gamma)$ at τ, if* (b) *holds.*

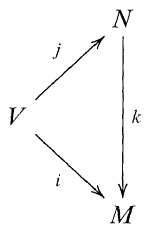
MAIN LEMMA. *Let M be as in the theorem and let i be the canonical embedding of V into M. Then there is $G^* \subseteq i(\mathscr{P}_\kappa * C(\kappa^+))$ so that $G^* \in V[G]$ and G^* is $M[G]$ generic.*

PROOF. Let us prove the lemma by induction on the pairs (τ, γ) ordered lexicographically. Suppose that it holds for all $(\tau, \gamma') < (\kappa, \alpha)$. Let us prove the lemma for (κ, γ).

CASE 1. $\alpha = \alpha' + 1$.

Let N be the ultrapower of V by $U(\kappa, \alpha)$ and $j: V \to N$ the canonical embedding. We just simplify the previous notations, where $N = N^\alpha_\kappa$, $j = j^\alpha_\kappa$. Then M is the complete iteration of $j(\vec{U})|\kappa+1$ at κ in N if α' is a limit ordinal, or for successor α', the above iteration should be performed ω-times in order to obtain M. Let us concentrate on the first case; similar and slightly simpler arguments work for the second one.

Denote by $k: N \to M$ the above iteration. Then the following diagram is commutative



By the inductive assumption there exists $G' \in N[G]$ M-generic subset of

$$\mathscr{P}_{k(\kappa)} * C((k(\kappa))^+).$$

If $\alpha' = 0$ then $k(\alpha') = o^{i(\vec{U})}(k(\kappa)) = 0$ and $C(k(\kappa)^+)$ is only the forcing used over $k(\kappa)$. Suppose now that $\alpha' > 0$. Then $k(\alpha') > 0$ and the forcing over $k(\kappa)$ is $C(k(\kappa)^+)$ followed by $\mathscr{P}(k(\kappa), k(\alpha'))$. Let us define in $N[G]$ a $M[G']$-generic subset of $\mathscr{P}(k(\kappa), k(\alpha'))$. Recall that a generic subset of $\mathscr{P}(k(\kappa), k(\alpha'))$ can be reconstructed from a generic sequence $b_{k(\kappa)}$ to $k(\kappa)$, where $b_{k(\kappa)}$ is a combination of a cofinal in $k(\kappa)$ sequences so that $b^{-1}_{k(\kappa)}(\{n\})$ is a sequence appropriate for the ultrafilters $i(U)(k(\kappa), \delta)$ with δ on the depth n. We refer to [7] for detailed definitions.

Let us use the indiscernibles of the complete iteration $\vec{C}$ in order to define $b_{k(\kappa)}$. Namely, only $\langle\, \vec{C}(k(\kappa), \delta) \mid \delta < k(\alpha')\,\rangle$ will be used.

Set

$$b_{k(\kappa)} = \{\, \langle \tau, n, \xi \rangle \mid n < \omega,\ \xi < k(\kappa)^+,\ \tau < k(\kappa),$$
$$\text{for some } \delta < k(\alpha') \text{ coded by } \langle n, \xi \rangle \text{ in sense of [7] } \tau \in \vec{C}(k(\kappa), \delta))\,\}.$$

Let G'' be the subset of $\mathscr{P}(k(\kappa), k(\alpha'))$ generated by $b_{k(\kappa)}$. Let us show that G'' is $M[G']$-generic subset. Let $D \in M[G']$ be a dense open subset of $\mathscr{P}(k(\kappa), k(\alpha'))$ and $\underset{\sim}{D}$ its canonical name. Since M is the direct limit for some ν less than the length of the complete iteration, for some $\underset{\sim}{D}_\nu \in M_\nu$, $\underset{\sim}{D}$ is the image of $\underset{\sim}{D}_\nu$. Let us work in M_ν. Denote by G_ν the appropriate part of G' and by t a coherent sequence which generates $b_{k(\kappa)}|\alpha_\nu$. Let $k_\nu \colon M \to M_\nu$ be the part of the complete iteration k on the step ν.

Case 2. $\mathrm{cf}^{M_\nu}(k_\nu(\alpha')) < k_\nu(\alpha')$.

Then $k_\nu(\alpha')$ changes its cofinality to $\mathrm{cf}^{M_\nu}(k_\nu(\alpha'))$ after the forcing with

$$\mathscr{P}(k_\nu(\kappa), k_\nu(\alpha')).$$

For every T so that $\langle t, T \rangle \in \mathscr{P}(k_\nu(\kappa), k_\nu(\alpha'))$, there exists $i < \mathrm{cf}^{M_\nu}(k_\nu(\alpha'))$ and T^* so that $\langle t, T^* \rangle$ is a condition stronger than $\langle t, T \rangle$ and $\langle t, T^* \rangle$ forces

"some $\langle t', T' \rangle \in \check{D}_\nu$ with t' on the level i is in the generic set".

In order to find such T^* just use the $k_\nu(\alpha')$-completeness of the ultrafilters involved in the forcing $\mathscr{P}(k_\nu(\kappa), k_\nu(\alpha'))$ and the Prikry property. Now for every $t' \in T^*$ which is appropriate for i or some $j \geq i$ there exists T' such that $\langle t', T' \rangle \in D_\nu$. The same property remains true for $k_{\nu'}(T^*)$ for every $\nu \leq \nu' <$ length of the iteration k. Pick $\nu' \geq \nu$ to be large enough in order to contain elements of $b_{k(\kappa)}$ appropriate for i. Let t' be a coherent sequence generating $b_{k(\kappa)}|\alpha_{\nu'}$. It is possible to pick such t' in $k_{\nu'}(T^*)$. But then for some T' $\langle t', T' \cap k_{\nu'}(T^*) \rangle \in k_{\nu\nu'}(D_\nu)$. The image of $\langle t', T' \cap k_{\nu'}(T^*) \rangle$ under the rest of the iteration will be in G''.

Case 3. $\mathrm{cf}^{M_\nu}(k_\nu(\alpha')) = k_\nu(\alpha')$.

Then $k_\nu(\alpha')$ changes its cofinality to ω.

For every T so that $\langle t, T \rangle \in \mathscr{P}(k_\nu(\kappa), k_\nu(\alpha'))$ there exist $n < \omega$ and T^* so that $\langle t, T^* \rangle$ is a condition stronger than $\langle t, T \rangle$ and $\langle t, T^* \rangle$ forces "some $\langle t', T' \rangle \in \check{D}_\nu$, with t' containing the first $\check{n}$ elements of the canonical ω-sequence to $k_\nu(\kappa)$, is in the generic set".

Pick $\nu' \geq \nu$ in order to first reach n elements of the canonical ω-sequence to $k(\kappa)$. Then proceed as in Case 2.

CASE 4. $\mathrm{cf}^{M_\nu}(k_\nu(\alpha')) = (k_\nu(\alpha'))^+$ *in* M_ν.

Let D be a dense open subset of $\mathscr{P}(k_\nu(\kappa), k_\nu(\alpha'))$. Set

$$D' = \{ \langle p, T\rangle \in \mathscr{P}(k_\nu(\kappa), k_\nu(\alpha')) \mid \text{for some level } \delta < (k_\nu(\kappa))^+, \\ \text{for every } \langle \xi_1, \dots, \xi_n\rangle \text{ such that } \xi_n \in \mathrm{Suc}_{T,\mu}(p^\frown\langle \xi_1, \dots, \xi_{n-1}\rangle) \\ \text{for some } \mu \geq \delta,\ \langle p^\frown\langle \xi_1, \dots, \xi_n\rangle, T_{p^\frown\langle \xi_1,\dots,\xi_n\rangle}\rangle \in D\}.$$

CLAIM. *D' is a dense open.*

The proof is similar to Lemma 3.11 of [6]. Define $D_1 = D'$, for every $n < \omega$ set $D_{n+1} = D_n'$ and finally $D_\omega = \bigcap_{n<\omega} D_n$.

CLAIM. *For every condition $\langle p, T\rangle$ there exists a stronger condition $\langle p, T^*\rangle$ in D_ω.*

We refer to Lemma 1.4 [6] for the proof. Just replace σ there by "$\in D_\omega$".

Let $\langle t, T^*\rangle \in D_\omega$. Then for some n $\langle t, T^*\rangle \in D_n$. Now by simple induction it is possible to show that there is $\delta < (k_\nu(\kappa))^+$ so that for every $\xi \in \mathrm{Suc}_{T^*,\delta}(t)$ $\langle t^\frown\langle\xi\rangle, T^*_{t^\frown\langle\xi\rangle}\rangle \in D$.

Now pick a large enough part of the iteration k to reach the level δ. Continue as in Case 2.

It completes the definition of a generic subset of $\mathscr{P}_{k(\kappa)+1}$. Let us refer to it as $G_{k(\kappa)+1}$. We now turn to the construction of the generic object for the forcing between $k(\kappa)+1$ and $i(\kappa)+1$. Let us define it by induction on δ, $k(\kappa)+1 \leq \delta \leq i(\kappa)+1$. Suppose that for every $\delta' < \delta$ a M-generic subset $G_{\delta'}$ of $\mathscr{P}_{\delta'}$ is defined in $V[G]$. Define G_δ.

If there is some $\tau > \delta$ and an indiscernible τ' for it $\tau' \leq \delta$, then use the inductive assumptions to produce G_δ. Suppose now that there is no τ, τ' as above. Notice, that then $\delta = k(\delta^*)$ for some $\delta^* \leq \delta$. Let us split the proof according to the following two cases.

CASE A. $\delta = \delta' + 1$.

Then

$$\mathscr{P}_\delta / G_{\delta'} = C(\delta^+) * \mathscr{P}(\delta, o^{k(\vec{U})}(\delta))$$

if $o^{k(\vec{U})}(\delta) > 0$ and $\mathscr{P}/G_{\delta'} = C(\delta^+)$ otherwise. So G_δ will be $G_{\delta'} * (G' * G'')$, where G'' may be empty.

Let us define first $G' \subseteq C(\delta^+)$. We use as inductive assumption that $k(p_\nu)|\delta \in G_{\delta'}$ for every $\nu < \kappa^+$ where $\langle p_\nu \mid \nu < \kappa^+\rangle$ is the master condition sequence for $U(\kappa, \alpha, \emptyset)$. Set

$$G' = \bigcup\{ k(p_\nu)(\delta) \mid C(\delta^+) \mid \nu < \kappa^+ \}.$$

Let us check that G' is $M[G_{\delta'}]$-generic subset of $C(\delta^+)$. Suppose that $D \in M[G_{\delta'}]$ is a dense open subset of $C(\delta^+)$. Let $\underset{\sim}{D}$ be a canonical $P_{\delta'}$-name of D. By the assumption on δ there are indiscernibles $\kappa < \alpha_{\nu_1} < \cdots < \alpha_{\nu_n} < \delta$ such that the support of $\underset{\sim}{D}$ is a subset of $\{\kappa, \alpha_{\nu_1}, \dots, \alpha_{\nu_n}\}$. Since the forcing $C(\delta^+)$ is δ-closed $\underset{\sim}{D}$

can be replaced by it dense open subset with support κ alone. Let us assume that $\underset{\sim}{D}$ is already such a subset. Then $\underset{\sim}{D} = k(\underset{\sim}{D}^*)$ for dense open $\underset{\sim}{D}^* \in N$ subset of $C((\delta^*)^+)$. By the choice of the master condition sequence for some $\nu < \kappa^+$

$$p_\nu|\delta^* \Vdash p_\nu(\delta^*)|C((\delta^*)^+) \in \underset{\sim}{D}^*.$$

But this implies

$$k(p_\nu)(\delta)|C(\delta^+) \in D.$$

So $G' \cap D \neq \emptyset$.

Suppose now that $o^{k(\vec{U})}(\delta) > 0$. We need to define G'' a $M[G_{\delta'} * G']$-generic subset of $\mathscr{P}(\delta, o^{k(\vec{U})}(\delta))$. In this case δ is a limit of indiscernibles, i.e., $\vec{C}(\delta, \tau)$ is unbounded in δ for every $\tau < o^{k(\vec{U})}(\delta))$. So we are in the situation considered above. The only difference is that some $p_\nu' s$ may contain information about the generic sequence b_δ to δ. In order to preserve $k(p_\nu)|\delta + 1$ in the generic set, we need to start b_δ according to $k(p_\nu)(\delta)$. Notice, that further elements of the master condition sequence do not increase the coherent sequence given by p_ν.

CASE B. *δ is a limit ordinal.*

Set

$$G_\delta = \{\, p \in \mathscr{P}_\delta \mid \text{for every } \delta' < \delta \;\; p|\delta' \in G_{\delta'} \,\}.$$

Let us show that G_δ is M-generic subset of $\mathscr{P}_\delta$. Consider two cases.

CASE B.1. *There are unboundedly many in δ indiscernibles for ordinals $\geq \delta$.*

Since $\delta = k(\delta^*)$, δ is a limit of indiscernibles for δ. Then δ is measurable in M and the direct limit is used on the stage δ. Let $\langle \tau_i \mid i < \lambda \rangle$ be a cofinal sequence of indiscernibles.

Let $D \in M$ be a dense open subset of $\mathscr{P}_\delta$. Then

$$\{\, \tau < \delta \mid D \cap H(\tau) \text{ is a dense open subset of } \textstyle\bigcup_{\tau' < \tau} \mathscr{P}_{\tau'} \,\}$$

contains a club in M. The sequence $\langle \tau_i \mid i < \lambda \rangle$ is almost contained in every club of M so there is i_0 such that $D \cap \mathscr{P}_{\tau_0}$ is dense. But then $G_{\tau_0} \cap D \neq \emptyset$. Hence $G_\delta \cap D \neq \emptyset$.

CASE B.2. *There are only boundedly many in δ indiscernibles for ordinals $\geq \delta$.*

Then, since $\delta = k(\delta^*)$, there is no indiscernibles for ordinals $\geq \delta$ below δ. So there are unboundedly many in δ ordinals τ which are in the range of k.

Let D be a dense open subset of $\mathscr{P}_\delta$ in M. Define

$$D' = \{\, p \in \mathscr{P}_\delta \mid \text{for some } \beta < \delta \;\; p|\beta \Vdash \text{“} p \setminus \beta \in D/\underset{\sim}{G}_\beta \text{”} \,\}.$$

Set $D_0 = D$, $D_{n+1} = D_n'$ for every $n < \omega$ and let $D_\omega = \bigcup_{n<\omega} D_n$.

CLAIM B.2.1. *D_ω is E-dense subset of $\mathscr{P}_\delta$.*

PROOF. Let $p \in \mathscr{P}_\delta$ define $p^*_E \geq p$ as in Lemma 1.4 [6] where σ is replaced by belonging to D_ω. Suppose that $p^* \notin D_\omega$. Then there is $p' \in D_\omega$, $p' \geq p^*$. Let $\beta \in \operatorname{dom} p^* \cap \operatorname{dom} p'$ be the last on which an information about cofinal sequences is added. But then for some $p'' \in \mathscr{P}_\beta$

$$p''{}^\frown p^*(\beta) \Vdash p^* \setminus \beta \in D_1/G_\beta.$$

Then, as in Lemma 1.4 [6], it is possible to go down until finally for some n and some $p'' \in \mathscr{P}_{\min(\operatorname{dom} p^*)}$

$$p'' \Vdash p^* \in D_n/G_{\min(\operatorname{dom} p^*)}$$

which contradicts the definition of p^*. ⊣

It is enough to show that $G_\delta \cap D_\omega \neq \emptyset$. Since then, for some n, $G_\delta \cap D_n \neq \emptyset$. Now use the fact that all initial segments of G_δ are M-generic. So let us prove for every E-dense set D that $G_\delta \cap D \neq \emptyset$. Without loss of generality we can assume that D is in the range of k, since it is always possible to find some $\tau < \delta$ in the range of k above the support of D and the intersection of τ E-dense open subsets of $\mathscr{P}_\delta/G_\tau$ is E-dense open.

Let $D^* \in N$ be an E-dense open subset of $\mathscr{P}_{\delta^*}$ so that $k(D^*) = D$. By the definition of the master condition sequence $\langle p_\nu \mid \nu < \kappa^+ \rangle$, for some $\tau_0 < \kappa^+$ $p_{\nu_0}|\delta^* \in D^*$. Then $k(p_{\nu_0})|\delta \in D$. But, by the choice of $\langle G_{\delta'} \mid \delta' < \delta \rangle$ $k(p_{\nu_0})|\delta' \in G_{\delta'}$ for every $\delta' < \delta$. So $k(p_{\nu_0})|\delta \in G_\delta$.

So G_δ is an M-generic subset of $\mathscr{P}_\delta$. It completes Case B and hence also Case 1 of the lemma.

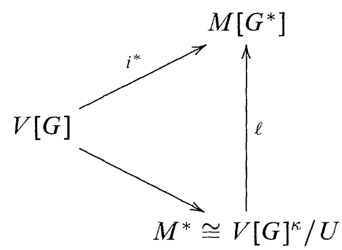
CASE 5. *α is a limit ordinal.*

Use the inductive assumption and the definitions of the generic sets of Case 1. Define a generic subset of $\mathscr{P}_{k(\kappa)}$ as in Case B.1. ⊣

Let $G^* \in V[G]$ be the M-generic subset of $\mathscr{P}_{i(\kappa)} * C(i(\kappa)^+)$ defined in the Main Lemma. Then $G^* \cap \mathscr{P}_\kappa * C(\kappa^+) = G$ and for every $\nu < \kappa^+$ $i(p_\nu) \in G^*$. Define the elementary embedding $i^*\colon V[G] \to M[G^*]$ by $i^*(\underset{\sim}{a}[G]) = (i(\underset{\sim}{a}))[G^*]$. Then $i^*|V = i$, $i^*(G)$ and, if

$$U^* = \{ A \subset \kappa \mid \kappa \in i^*(A) \}$$

then $U^* = U(\kappa, \alpha, \emptyset)$. So the following diagram is commutative



where $\ell([f]_{U^*}) = i^*(f)(\kappa)$.

It remains to show that $\ell = \mathrm{id}$. Notice, that it is enough to prove that every indiscernible in $\vec{C}$ is of the form $i^*(f)(\kappa)$ for some $f \in V[G]$. Examining the

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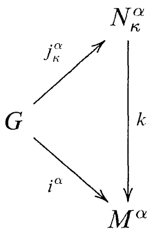
construction of G^*, it is not hard to see that every indiscernible is an element of the generic sequence b_δ for some $\delta \in k''(N)$.

So indiscernibles are interpretations of forcing terms with parameters in $k''(N)$. But such elements can easily be represented by functions on κ in $V[G]$. Let $\mu \in \vec{C}$ and $\mu = (t(\delta))[G^*]$ for $\delta = k(\delta^*)$, where t is a term. Let $\delta^* = [f]_{U(\kappa,\alpha)}$ for some $f: \kappa \to \kappa$, $f \in V$. Define a function $g \in V[G]$ on κ as follows: $g(\tau) = t(f(\tau))[G]$. Then

$$i^*(g)(\kappa) = t(t(f)(\kappa))[G^*] = (t(\delta))[G^*] = \mu. \quad \dashv$$

Let us now turn to the construction of ω_1-preserving ideal. Suppose that α^* is an $(\omega, \kappa^+ + 1)$-repeat point for $\vec{U}$ in V. Let $\alpha^* \leq \alpha < \alpha^* + \kappa^+$ be an ordinal. Denote by M^* the complete iteration of $j_\kappa^\alpha(\vec{U})|((j_\kappa^\alpha(\kappa), (j_\kappa^\alpha(\alpha^* + \kappa^+))$ in N_κ^α. Let i^α, k^α be the canonical elementary embeddings making the diagram



commutative.

As in the Main Lemma find in $V[G]$ a $\mathscr{P}(\kappa, \alpha)$-name of a M^α-generic subset $\underset{\sim}{G}'$ of

$$\mathscr{P}_{i^*(\kappa)} * C(i^\alpha(\kappa^+))/\mathscr{P}_{\kappa+1}$$

so that each $k^\alpha(p_\nu^\alpha) \in \underset{\sim}{G}'$ for $\nu < \kappa^+$, where $\langle p_\nu^\alpha \mid \nu < \kappa^+ \rangle$ is the master condition sequence for $U(\kappa, \alpha, \emptyset)$. Let $\underset{\sim}{G}''$ be obtained from $\underset{\sim}{G}'$ by removing all the information on the generic subset of $C(i^\alpha(\kappa^+))$ except $i^{\alpha''}(G \cap C(\kappa^+))$.

We shall define a presaturated filter $U^*(\kappa, \alpha)$ on κ in $V[G]$ extending $U(\kappa, \alpha)$ so that

(i) all generic ultrapowers of $U^*(\kappa, \alpha)$ are generic extensions of M^α;
(ii) every set $U(\kappa, \alpha, \emptyset)$ is $U^*(\kappa, \alpha)$-positive.

Property (i) will insure that the forcing for shooting clubs over $i^\alpha(\kappa)$ will be κ-closed forcing. Property (ii) is needed for the iteration of forcings for shooting clubs over κ in $V[G]$.

Denote $G \cap \mathscr{P}_\kappa$ by $\underline{G}$ and $G \cap C(\kappa^+)$ by $\bar{G}$. Let $\bar{G}(\nu)$ denote the ν-th function of $\bar{G}$ for $\nu < \kappa^+$. Let j^* be the embedding of $V[G]$ into the ultrapower of $V[G]$ by $U(\kappa, \alpha, \emptyset)$. Let $\langle \alpha_\nu \mid \nu < \kappa^+ \rangle$ be the list of all the indiscernibles for $i_\kappa^\alpha(\kappa)$ of the complete iteration used to define M^α. For every $\xi < \kappa^+$ extend $\underset{\sim}{G}$ to $\underset{\sim}{G}_\xi^*$ by adding to $\underset{\sim}{G}''$ conditions $\langle \xi + \nu + 1, \kappa, \alpha_\nu \rangle$ for every $\nu < \kappa^+$, i.e., the $\xi + \nu + 1$ generic function moves κ to α_ν.

Define a filter $U^*(\kappa, \alpha)$ on κ in $V[G]$ as follows:

$A \subseteq \kappa$ belongs to $U^*(\kappa, \alpha)$ if and only if for some $r \in G$ some $\underset{\sim}{r}'$ such that $\emptyset \Vdash \underset{\sim}{r}' \in \underset{\sim}{G}''$, for every $\underset{\sim}{p} \in \underset{\sim}{C}(j_\kappa^\alpha(\kappa^+))(0)$ such that in $N_\kappa^\alpha[j^*(G)]$ $p \notin j^*(\bar{G}(0))$, and for some $\underset{\sim}{r}^*$ such that $\emptyset \Vdash \underset{\sim}{r}^* \in \underset{\sim}{G}^*_{\xi(\underset{\sim}{p})}$ in M^α

$$r \cup 1_{\mathscr{P}(\kappa,\alpha)} \cup \underset{\sim}{p} \cup \underset{\sim}{r}^* \Vdash_{\mathscr{P}_{i^\alpha(\kappa)+1}} \check{\kappa} \in i^\alpha(\underset{\sim}{A})$$

where $\xi(p) < \kappa^+$ is the minimal such that $p|\delta \notin j^*(G(0))$, for δ the $\xi(p)$ member of some fixed enumeration of $j^\alpha_\kappa(\kappa)$.

CLAIM 1. $U(\kappa, \alpha, \emptyset) \supseteq U^*(\kappa, \alpha)$.

PROOF. Let $A \in U(\kappa, \alpha, \emptyset)$, then for some $\nu < \kappa^+$ some r, $\langle \emptyset, T \rangle \in \mathscr{P}(\kappa, \alpha)$, in N^α_κ

$$r \cup \{\langle \emptyset, T \rangle\} \cup \underset{\sim}{p}_\nu \Vdash \check{\kappa} \in j^\alpha_\kappa(\underset{\sim}{A}).$$

Let us split $\underset{\sim}{p}_\nu$ into three parts $\underset{\sim}{p}_1 = \underset{\sim}{p}_\nu|\mathscr{P}_{j^\alpha_\kappa}$, $\underset{\sim}{p}_2 = \underset{\sim}{p}_\nu \cap C(j^\alpha_\kappa(\kappa^+))(0)$ and $\underset{\sim}{p}_3 =$ the rest of $\underset{\sim}{p}_\nu$. Then, using k^α, in M^α

$$r \cup \{\langle \emptyset, T \rangle\} \cup k^\alpha(\underset{\sim}{p}_\nu) \Vdash \check{\kappa} \in i^\alpha(\underset{\sim}{A}).$$

Extend $\underset{\sim}{p}_2$ to some p_2', still remaining in the master condition sequence, in order to make its domain above all the indices of the generic sequences listed in $\underset{\sim}{p}_3$. Extend p_2' to some $p_2'' \in C(j^\alpha_\kappa(\kappa^+))(0)$ which is incompatible with a member of the master condition sequence. Then, using k^α, in M^*

$$r \cup \{\langle \emptyset, T \rangle\} \cup k^\alpha(\underset{\sim}{p}_1) \cup k^\alpha(p_2'') \cup k^\alpha(\underset{\sim}{p}_3) \Vdash \check{\kappa} \in i^\alpha(\underset{\sim}{A}).$$

By the definition of $U^*(\kappa, \alpha)$, it implies that A is a $U^*(\kappa, \alpha)$-positive set.

So every member of $U(\kappa, \alpha, \emptyset)$ is $U^*(\kappa, \alpha)$-positive. The fact that $U(\kappa, \alpha, \emptyset)$ is an ultrafilter completes the proof of the claim.

CLAIM 2. *$U^*(\kappa, \alpha)$ is normal precipitous cardinal preserving filter and its generic ultrapower is a generic extension of M^α.*

PROOF. Let $U = U^*(\kappa, \alpha) \cap V[\underline{G}, \bar{G}(0)]$. The arguments of Theorem 1 show that a generic ultrapower by U is isomorphic to a generic extension of the complete iteration of N^α_κ with $j^\alpha_\kappa(\vec{U})|j^\alpha_\kappa(\kappa)$ above κ. Actually the forcing with U is isomorphic to $\mathscr{P}(\kappa, \alpha)$ followed by $C(j^\alpha_\kappa(\kappa^+))(0)$ (in the sense of this iteration) over $V[\underline{G}, \bar{G}(0)]$. Clearly, the generic function form $j^\alpha_\kappa(\kappa^+)$ into $j^\alpha_\kappa(\kappa^+)$ produced by this forcing is incompatible with $j^*(\bar{G}(0))$ which belongs to $V[\underline{G}, \bar{G}(0)]$.

Now the forcing with $U^*(\kappa, \alpha)$ over $V[G]$ does the following: First it picks $\underset{\sim}{p} \in \underset{\sim}{C}(j^\alpha_\kappa(\kappa^+))(0)$ incompatible with $j^*(\bar{G}(0))$ and then $G^*_{\xi(\underset{\sim}{p})}$ is added. $G^*_{\xi(\underset{\sim}{p})}$ insures that a generic ultrapower is a generic extension of M^α. So the forcing with $U^*(\kappa, \alpha)$ is isomorphic to $\mathscr{P}(\kappa, \alpha)$ followed by a portion of $C(i^\alpha(\kappa^+))$, which is κ-closed.

Force over $V[G]$ with the forcing Q_κ which is the Backward-Easton iteration of the forcings adding δ^+-Cohen subsets to every regular $\delta < \kappa$ with $o^{\vec{U}}(\delta) > 0$. Fix a generic subset H of Q_κ. All the filters $U^*(\kappa, \alpha)$ extend in the obvious fashion in $V[G, H]$. Let us use the same notations for the extended filters.

Set

$$F = \bigcap \{ U^*(\kappa, \alpha) \mid \alpha^* \leq \alpha \leq \alpha^* + \kappa^+ \}.$$

Then F is a $\leq \kappa$-preserving filter in $V[G, H]$ and forcing with it is isomorphic to $\mathscr{P}(\kappa, \alpha)$ (for some $\alpha, \alpha^* \leq \alpha \leq \alpha^* + \kappa^+$) followed by κ-closed forcing. Let us shoot clubs through elements of F, then through the sets of generic points and so on, as was done in [5, 9]. Denote this forcing by B. Let R be its generic subset over $V[G, H]$. We shall show that NS_κ is ω_1-preserving ideal in $V[G, H, R]$. The forcing with NS_κ consists of two parts:

(a) embedding of B into $\mathscr{P}(\kappa, \alpha)$;
(b) (κ-closed forcing) $*(i^\alpha(B)/i''^\alpha(R))$.

By the choice of $i^\alpha(\kappa)$, the forcing $i^\alpha(B)$ is a shooting club through sets containing a club (the club of indiscernibles for $i^\alpha(\kappa)$). So it is κ-closed forcing and part (b) does not cause any problem.

Let us examine part (a) and show that this forcing preserves ω_1. Recall that B is the direct limit of $\langle B_\beta \mid \beta < \kappa^+ \rangle$ where each B_β is of cardinality κ and for a limit β, $B_\beta =$ the direct limit of $B_{\beta'} (\beta' < \beta)$, if cf $\beta = \kappa$ and $B_\beta =$ the inverse limit of $B_{\beta'}$ $(\beta' < \beta)$ otherwise. The forcing of (a) is

$$\mathscr{P} = \{ \pi \in V[G, H] \mid \text{for some } \beta < \kappa^+ \ \pi \text{ is an embedding of } B_\beta \text{ into } \mathscr{P}(\kappa, \alpha) \}.$$

For $\pi_1, \pi_2 \in \mathscr{P}$ let $\pi_1 \geq \pi_2$ if $\pi_1 | \operatorname{dom} \pi_2 = \pi_2$.

Claim. *NS_κ is an ω_1-preserving ideal in $V[G, H, R]$.*

Proof. Suppose otherwise. Then for some $\alpha^* \leq \alpha \leq \alpha^* + \kappa^+$ some condition in the forcing $\mathscr{P}(\kappa, \alpha)^*$ (the forcing isomorphic to the adding of κ^+-Cohen subsets of κ^+) $*\mathscr{P}$ over $V[G, H]$ forces ω_1 to collapse. Let us assume that the empty condition already forces this. Consider the case when $\Vdash_{\mathscr{P}(\kappa,\alpha)} \operatorname{cf} \check{\kappa} = \check{\omega}$. The remaining cases are simpler.

Let S be a $V[G, H]$-generic subset of $\mathscr{P}(\kappa, \alpha) * (\kappa^+$-Cohen subsets of $\kappa^+)$. Denote $V[G, H, S]$ by $\bar{V}$. Let $\underset{\sim}{f}$ be a $\mathscr{P}$-name in $\bar{V}$ of a function from ω to $\omega_1^{\bar{V}}$. Pick an elementary submodel $\tilde{N}$ of $\langle H(\lambda), \varepsilon, B, \mathscr{P}, \kappa, \underset{\sim}{f} \rangle$, for λ big enough, satisfying the following three conditions:

(1) $|N| = \kappa$;
(2) $N \supseteq H^{\bar{V}}(\kappa)$;
(3) $N \cap \kappa^+ = \delta$ for some δ such that $\operatorname{cf}^{V[G.H]} \delta = \kappa$.

Then $B \cap N = B_\delta$. Also B_δ is a direct limit of $\langle B_{\delta'} \mid \delta' < \delta \rangle$. Pick in $V[G, H]$ a cofinal sequence $\langle \delta_\beta \mid \beta < \kappa \rangle$ to δ and in $\bar{V}$ a cofinal sequence $\langle \tau_n \mid n < \omega \rangle$ to κ. Consider the subsets of κ, $\langle \underset{\sim}{A}_\beta \mid \beta < \kappa \rangle$ such that $B_{\delta_\beta+1}/B_{\delta_\beta}$ is the forcing for shooting club into $\underset{\sim}{A}_\beta$. Assume for simplicity that all A_βs are in V. Let

$$A = \Delta_{\beta<\kappa} A_\beta = \{ \gamma < \kappa \mid \text{for every } \beta < \gamma, \ \gamma \in A_\beta \}.$$

The A_βs and A are in

$$\bigcap \{ U(\kappa, \gamma) \mid \alpha^* \leq \gamma \leq \alpha^* + \kappa^+ \}.$$

Pick $\pi_0 \in \mathscr{P} \cap N$ deciding the value of $\underset{\sim}{f}(0)$. Without loss of generality $\operatorname{dom} \pi_0 = B_{\delta_{\beta_0}+1}$ for some $\beta_0 < \delta$. Let n_0 be the least $n < \omega$ such that $\delta_{\tau_n} > \beta_0$. Denote $\delta_{\tau_{n_0}}$ by ε_0. Let C_0 be the generic club through A_{β_0} defined by π_0 and S. As in Lemma 3.6 [9], it is possible to find an element of the generic sequence to κ, $\tau(0) \in A - \tau_{n_0}$

such that $C \cap \tau$ is a $V[G|\tau(0),\ H|\tau(0)]$-generic club through $A \cap \tau$. Choosing $\tau(0)$ more carefully, it is possible to also satisfy the following

$$A \cap \tau(0) = \Delta_{\beta<\tau(0)}(A_\beta \cap \tau(0)).$$

Now pick $\pi_0' \in \mathscr{P} \cap N$ to be an extension of π_0 with domain B_{δ_r} such that the clubs of π_0' intersected with $\tau(0)$ are $V[G|\tau(0),\ H|\tau(0)]$-generic for $B_\delta|\tau(0)$. It is possible since N satisfies condition (2).

Now find in N an extension π_1 of π_0' deciding $\underset{\sim}{f}(1)$. Define π_1' and $\tau(1)$ as above. Continue the process for all $n < \omega$. Finally set $\pi = \bigcup_{n<\omega} \pi_n$. It is enough to show that $\pi \in \mathscr{P}$, since then $\pi \Vdash \underset{\sim}{f} \in V_1$. Let us prove that π and S produce a $V[G, H]$-generic subset of B_δ. Suppose that $\mathscr{D} \in V[G, H]$ is a dense subset of B_δ. Then

$$X = \{\, \tau < \kappa \mid \mathscr{D} \cap H(\tau) \text{ is a dense subset of } B_{\delta_r}|\tau \,\}$$

contains a club in $V[G, H]$. Since the generic sequence to κ is almost contained in every club of $V[G, H]$, for some $n < \omega$, $\tau(n) \in X$. But then the generic subset produced by π_n' intersects $\mathscr{D}$. So the same is true for π.

Let us now turn to successor cardinals. We would like to make NS_κ ω_1 preserving for $\kappa = \mu^+$ for a regular $\mu > \aleph_1$. It is possible to use the model constructed above, collapse κ to μ^+ and apply the results of Section 1. But an $(\omega, \kappa^+ + 1)$-repeat point was used in the construction of the model. It turns out that an $(\omega, \mu + 1)$-repeat point suffices for NS_{μ^+} and an (ω, μ)-repeat point for $NS_{\mu^+}|\{\, \alpha < \mu^+ \mid \operatorname{cf}\alpha < \mu \,\}$. On the other hand, precipitousness of $NS_{\mu^+}^{\aleph_0}$ implies an (ω, μ)-repeat point, by [9].

Let us preserve the notations used above. Assume that $\mu < \kappa$ is a regular cardinal and some $\alpha^* < o^{\vec{U}}(\kappa)$ is an $(\omega, \mu + 1)$-repeat point. Let G be a generic subset of $\mathscr{P}_\kappa * C(\kappa^+)$. Over V[G] instead of the forcing Q_κ, in the previous construction, use $\operatorname{Col}(\mu, \kappa)$ the Levy collapse of all the cardinals τ, $\mu < \tau < \kappa$ on μ. Let H be a generic subset of $\operatorname{Col}(\mu, \kappa)$. Denote by $H(\tau)$ the generic function from μ on τ where $\tau \in (\mu, \kappa)$.

Now the forcing for shooting clubs should come. In order to prevent collapsing cardinals by this forcing, $j(\kappa)$ was made a limit of κ^+ indiscernibles for the measures

$$\{\, U(\kappa, \alpha) \mid \alpha^* \le \alpha < \alpha + \kappa^+ \,\}.$$

But now we have only μ measures. So the best we can do is to make $j(\kappa)$ a limit of μ indiscernibles and then its cofinality in V will be $\mu < \kappa$. It looks slightly paradoxical since usually $\operatorname{cf} j(\kappa) = \kappa^+$, but it is possible by [8]. In order to explain the idea of [8] which will be used here let us give an example of a precipitous ideal I on ω_1 so that:

$$(*) \qquad \Vdash_{I^+} \text{“}\operatorname{cf}^V(j(\check{\omega}_1)) = \check{\omega}\text{”}.$$

Example. Suppose that κ is a measurable cardinal. Let U be a normal measure on κ and $j\colon V \to N \cong V^\kappa/U$ the canonical embedding. Let $\mathscr{P}_\kappa$ be the Backward-Easton iteration of the forcings $C(\alpha^+)$ for all regular $\alpha < \kappa$. Let $G_\kappa * H_\kappa$ be a generic subset of V. Collapse κ to ω_1 by the Levy collapse. Let R_κ be $V[G_\kappa * H_\kappa]$-generic subset of $\operatorname{Col}(\omega_1, \kappa)$. Denote $V[G_\kappa * H_\kappa * R_\kappa]$ by V_1. We shall define a precipitous ideal satisfying $(*)$ in V_1.

Let $j_0 = j$, $N_0 = N$, $\kappa_0 = \kappa$, $U_0 = U$. Set $N_1 \cong N_0^{j_0(\kappa)}/j_0(U)$, $\kappa_1 = j_0(\kappa)$, $U_1 = j_0(U)$ and let $j_1 \colon N_0 \to N_1$ be the canonical embedding. Continue the definition for all $n < \omega$. Set j_ω, N_ω to be the direct limit of $\langle j_n, N_n \mid n < \omega \rangle$. Then $j_\omega(\kappa) = \bigcup_{n<\omega} \kappa_n$. Set $\kappa_\omega = j_\omega(\kappa)$. Notice, that

$$\bigcup(j''(\kappa^+)) = (\kappa_1^+)^{N_0} \qquad \text{and} \qquad \bigcup(j_{n+1}((\kappa_n^+)^{N_n})) = (\kappa_{n+1}^+)^{N_n+1}.$$

So $\bigcup(j''_\omega(\kappa^+)) = (\kappa_\omega^+)^{N_\omega}$.

Define a filter U^* in $V[G_\kappa * H_\kappa R]$ as follows:

$A \in U^*$ if and only if for some $r \in G_\kappa * H_\kappa * R$ for some $n < \omega$ in N_ω

$$r \cup p_n \Vdash \check{\kappa} \in j_\omega(\underset{\sim}{A})$$

where p_n is the name of condition in the forcing $C(j_\omega(\kappa^+))$ defined as follows:

let $\underset{\sim}{h}$ be the name of the generic function from ω onto $(\kappa^+)^V$ in

$$\mathrm{Col}(\omega, j_\omega(\kappa))/G_\kappa * H_\kappa * R_\kappa.$$

Set

$$p_n = \{\langle j_\omega(\underset{\sim}{h}(0)), \kappa, \kappa_1\rangle, \langle j_\omega(h(\underset{\sim}{1})), \kappa, \kappa_2\rangle, \dots, \langle j_\omega(\underset{\sim}{h}(n)), \kappa, \kappa_{n+1}\rangle\}.$$

The meaning of the above is that the value on κ of $j_\omega(h(m))$th function from $j_\omega(\kappa)$ to $j_\omega(\kappa)$ is forced to be κ_{m+1}.

It is not difficult to see now that U^* is a normal precipitous ideal on ω_1 and a generic ultrapower with it is isomorphic to a $V[G_\kappa * H_\kappa * R_\kappa]$-generic extension of $N_\omega[G_\kappa, H_\kappa, R_\kappa]$. Also for a generic embedding j^*, $j^*|V = j_\omega$. Hence $j^*(\kappa) = \kappa_\omega$ which is of cofinality ω in V.

As in [13], it is possible to extend U^* to the closed unbounded filter with the same property. Using the Namba forcing, it is possible to construct a precipitous ideal satisfying $(*)$ on $\aleph_2$. Starting from a measurable which is a limit of measurable, it is possible to build such an ideal over an inaccessible or even measurable. Since then it is possible to change the cofinality of the ordinal of cofinality κ^+ to ω in N and that is what was needed to catch all κ_ns in the above construction. We do not know if one measurable is sufficient for a precipitous ideal satisfying $(*)$ over $\kappa > \aleph_2$.

Note also that by Proposition 1.5 if I is λ-preserving, then $\mathrm{cf}^V j(\kappa) \geq \lambda$. In particular, if I is presaturated then $\mathrm{cf}^V j(\kappa) = \kappa^+$.

Let us now return to the construction of NS_{μ^+} ω_1-preserving. Let $\alpha^* \leq \alpha < \alpha^* + \mu$ be an ordinal. Define M^* as in the construction of NS_κ ω_1-preserving for an inaccessible κ. Now $\mathrm{cf}^V i^\alpha(\kappa)$ will be μ. We shall define, in $V[G, H]$, $U^*(\kappa, \alpha)$ extending $U(\kappa, \alpha)$ satisfying conditions (i) and (ii) from page 1545. To do so simply combine the definition there with the definition of the example. We leave the details to the reader. The rest of the construction does not differ from an inaccessible cardinal case.

The above results give equiconsistency for $NS_\kappa|$ (singular) and NS_κ, but for $\kappa > \aleph_2$. For $\kappa = \aleph_2$, we do not know if the assumption of the existence of an $(\omega, \omega_1 + 1)$-repeat point (or of an (ω, ω_1)-repeat point for $NS_{\aleph_2}^{\aleph_0}$) can be weakened. By [5], a measurable is sufficient for the precipitousness of $NS_{\aleph_2}^{\aleph_0}$ and a measurable of order 2 for $NS_{\aleph_2}$. Let us show that ω_1-preservingness requires stronger assumptions.

LEMMA 2.6. *Suppose that $\kappa > \aleph_1$ is a regular cardinal, $2^\kappa = \kappa^+$, $2^{\aleph_0} = \aleph_1$. I is a normal ω_1-preserving ideal over κ so that $\{\alpha < \kappa \mid \operatorname{cf}\alpha = \omega\} \notin I$. Then $\exists\tau\ o(\tau) \geq 2$ in the core model.*

PROOF. Without loss of generality, assume that $\neg\exists\alpha\ o(\alpha) = \alpha^{++}$. Denote by $\mathscr{K}(\vec{\mathscr{F}})$ the core model with the maximal sequence $\vec{F}$. We refer to Mitchell papers [15, 14] for the definitions and properties of $\mathscr{K}(\vec{\mathscr{F}})$ that we are going to use.

The set

$$A = \{\alpha < \kappa \mid \alpha \text{ is regular in } \mathscr{K}(\vec{\mathscr{F}}) \text{ and of cofinality } \omega \text{ in } V\}$$

is I-positive. If the set

$$A^* = \{\alpha \in A \mid \alpha \text{ is not measurable in } \mathscr{K}(\vec{\mathscr{F}})\}$$

is bounded in κ, then $o(\kappa) \geq 2$. Suppose otherwise. Let $j\colon V \to M$ be a generic elementary embedding so that the set $\{\alpha < \kappa \mid \operatorname{cf}\alpha = \omega\}$ belongs to a generic ultrafilter G_I. Then $j(A^*)$ is unbounded in $j(\kappa)$. Pick the mimimal $\alpha \in j(A^*) - \kappa$.

CLAIM. $\operatorname{cf}^{\mathscr{K}(\vec{\mathscr{F}})}(\alpha) = (\kappa^+)^{\mathscr{K}(\vec{\mathscr{F}})}$.

PROOF. Suppose otherwise. Then, since α is regular in $\mathscr{K}(\vec{F})$ and $j|\mathscr{K}(\vec{F})$ is an iterated ultrapower of $\mathscr{K}(\vec{F})$ by $\vec{F}$, α is a limit indiscernible of this iteration. By Proposition 1.5 ${}^{\omega}M \cap V[G_I] \subseteq M$. Since α is not a measurable in $\mathscr{K}(j(\vec{F}))$, it implies that $\operatorname{cf}^{\mathscr{K}(\vec{F})} \alpha > \omega$. But then, using ${}^{\omega}M \cap V[G_I] \subseteq M$ and the arguments of [14] and [9], we obtain some τ with $o^{\vec{F}}(\tau) \geq (\omega_1)^{\mathscr{K}(\vec{F})}$.

By [14], $(\kappa^+)^{\mathscr{K}(\vec{F})} = (\kappa^+)^V$. But in $V[G_I]$, $\operatorname{cf}\alpha = \omega$ and hence $\operatorname{cf}(\kappa^+)^V = \omega$ which contradicts Proposition 1.5.

LEMMA 2.7. *Suppose that $\kappa > \aleph_1$ is a regular cardinal, $2^\kappa = \kappa^+$, $2^{\aleph_0} = \aleph_1$ and I is a normal ω_1-preserving ideal. If there exists an ω-club C so that every $\alpha \in C$ is regular in $\mathscr{K}(\vec{F})$, then $\exists\tau\ o^{\vec{F}}(\tau) \geq \omega_1$ in $\mathscr{K}(\vec{F})$.*

The proof is similar to Lemma 2.6; just consider the ω_1th member of $j(C) - \kappa$.

THEOREM 2.8. *Assume* GCH, *if $NS_{\aleph_2}^{\aleph_0}$ is ω_1-preserving then $\exists\tau\ o^{\vec{F}}(\tau) \geq \omega_1$ in $\mathscr{K}(\vec{F})$.*

The proof follows from Lemma 2.7.

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