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THE NEAR COHERENCE OF FILTERS PRINCIPLE DOES NOT IMPLY THE FILTER DICHOTOMY PRINCIPLE

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Dedicated to Andreas Blass on the occasion of his 60th birthday.

Abstract. We show that there is a forcing extension in which any two ultrafilters on ω are nearly coherent and there is a non-meagre filter that is not nearly ultra. This answers Blass' longstanding question (1989) of whether the principle of near coherence of filters is strictly weaker than the filter dichotomy principle.

1. Introduction

By a *filter* we mean a proper filter on ω. We call a filter non-principal if it contains all cofinite sets. Let $\mathscr{F}$ be a non-principal filter on ω and let $f\colon \omega \to \omega$ be finite-to-one (that means that the preimage of each natural number is finite). Then also $f(\mathscr{F}) = \{X : f^{-1}(X) \in \mathscr{F}\}$ is a non-principal filter. Two filters $\mathscr{F}$ and $\mathscr{G}$ are *nearly coherent* if there is some finite-to-one $f\colon \omega \to \omega$ such that $f(\mathscr{F}) \cup f(\mathscr{G})$ generates a filter. We also say to this situation that $f(\mathscr{F})$ and $f(\mathscr{G})$ are coherent.

The *filter dichotomy principle*, FD, says that for every filter there is a finite-to-one function g such that $g(\mathscr{F})$ is either the filter of cofinite sets (also called the *Fréchet filter*) or an ultrafilter. In the latter case we call $\mathscr{F}$ *nearly ultra*. Talagrand [19] showed that there is a finite-to-one function such that $f(\mathscr{F})$ is Fréchet iff $\mathscr{F}$ is meagre, that is, the set of the characteristic functions of the members of $\mathscr{F}$ is a meagre subset of the space 2^ω.

The *principle of near coherence of filters*, NCF, says that any two filters (equivalently: ultrafilters) are nearly coherent. Blass and Laflamme [5] showed that $\mathfrak{u} < \mathfrak{g}$ implies FD, and that FD implies NCF. The purpose of this paper is to show that NCF does not imply FD.

Main Theorem. *"NCF and not FD" is consistent relative to ZFC.*

Since NCF is equivalent to "$\beta\mathbb{R}_+ \setminus \mathbb{R}_+$ has only one composant" and to "the ideal of compact operators on a Hilbert space is not the sum of two proper subideals" [3] and since FD is weaker than "there are just four slenderness classes of abelian groups" (by [5]; actually by [14] they are equivalent), our result says that it is possible that $\beta\mathbb{R}_+ \setminus \mathbb{R}_+$ has only one composant and the ideal of compact operators

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on a Hilbert space is not the sum of two proper subideals, and yet there are more than four slenderness classes of abelian groups.

We also looked at the splitting number $\mathfrak{s}$ (see [1, 1.3.5]) in our models of NCF and not FD, because by [14], FD together with $\mathfrak{s} > \mathfrak{u}$ implies $\mathfrak{u} < \mathfrak{g}$, and we wanted to know whether $\mathfrak{s}$ also has an influence in our non-implication. We give two types of models of NCF and not FD, one with $\mathfrak{s} = \aleph_2$ and one where we do not know the splitting number. The hope is that the elbow room in our construction will at some time help to solve the open problem of whether $\mathfrak{u} < \mathfrak{g}$ is strictly stronger than FD.

Before we give an outline of the forcing construction, let us first review the known models of FD. P-points and two cardinal characteristics play an important rôle in our topic: We write $A \subseteq^* B$ iff $A \setminus B$ is finite. An ultrafilter $\mathscr{U}$ is called a *P-point* if for every $\gamma < \aleph_1$, for every $A_i \in \mathscr{U}$, $i < \gamma$, there is some $A \in \mathscr{U}$ such that for all $i < \gamma$, $A \subseteq^* A_i$; such an A is called a *pseudo-intersection* or a *diagonalisation* of the A_i, $i < \gamma$. A notion of forcing $\mathbb{P}$ preserves an ultrafilter $\mathscr{U}$ iff $\Vdash_{\mathbb{P}}$ "$(\forall X \in [\omega]^{\aleph_0})(\exists Y \in \mathscr{U})(Y \subseteq X \vee Y \subseteq \omega \setminus X)$", and in the contrary case we say "$\mathbb{P}$ destroys $\mathscr{U}$". If $\mathbb{P}$ preserves $\mathscr{U}$ and $\mathscr{U}$ is a P-point, then $\mathscr{U}$ stays a P-point [6, Lemma 3.2].

$\mathscr{B} \subseteq \mathscr{U}$ is a *base* for $\mathscr{U}$ if for every $X \in \mathscr{U}$ there is some $Y \in \mathscr{B}$ such that $Y \subseteq X$. The cardinal $\mathfrak{u}$ is the smallest cardinal of a base for a non-principal ultrafilter.

A subset $\mathscr{G}$ of $[\omega]^\omega$ is called *groupwise dense* if $(\forall X \in \mathscr{G})(\forall Y \subseteq^* X)(Y \in \mathscr{G})$, and for every partition of ω into finite intervals $\{[\pi_i, \pi_{i+1}) \, : \, i \in \omega\}$ there is an infinite set A such that $\bigcup\{[\pi_i, \pi_{i+1}) \, : \, i \in A\} \in \mathscr{G}$. The *groupwise density number*, $\mathfrak{g}$, is the smallest number of groupwise dense families with empty intersection.

The only models of NCF that have been known so far are also models of FD and $\mathfrak{u} < \mathfrak{g}$, which is possibly strictly stronger than FD. A ground model with CH is extended by an iterated forcing $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\alpha \, : \, \beta \leq \gamma, \alpha < \gamma \rangle$ that is built in the usual way: The iterand $\underset{\sim}{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name and $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \underset{\sim}{\mathbb{Q}}_\alpha$, and at limits we build $\mathbb{P}_\alpha$ with countable supports. The iterands are proper forcings that preserve at least one, indeed all P-points, and thus keep $\mathfrak{u}$ small. (If $\mathfrak{u} < \mathfrak{d}$, which follows from NCF, then by Ketonen [11] every filter witnessing $\mathfrak{u}$ is a P-point, so we do not have to worry whether there is some non-P ultrafilter with a base of a smaller size. Also, since $\mathfrak{u} = \aleph_1$ is the minimum possible, these worries are unnecessary.) Let us write $\mathbf{V}_\alpha$ for $\mathbf{V}^{\mathbb{P}_\alpha}$, an arbitrary extension by a $\mathbb{P}_\alpha$-generic filter G_α. Although $\mathfrak{u}$ is kept small, at least at stationarily many limit steps $\alpha < \aleph_2$ of cofinality $\aleph_1$ the next iterand adds a real that has supersets in all groupwise dense sets in $\mathbf{V}_\alpha$ and thus $\mathfrak{g} = \aleph_2$ [5].

Some types of such models of $\mathfrak{u} < \mathfrak{g}$ are known: An iteration of length $\aleph_2$ with countable support of Blass-Shelah forcing over a ground model of CH [6] gives $\aleph_1 = \mathfrak{u} < \mathfrak{s} = \mathfrak{g} = 2^{\aleph_0} = \aleph_2$, and an iteration of length $\aleph_2$ with countable support of Miller forcing over a ground model of CH [7] gives $\aleph_1 = \mathfrak{u} = \mathfrak{s} < \mathfrak{g} = 2^{\aleph_0} = \aleph_2$. A third type of model of $\mathfrak{u} < \mathfrak{g}$ is given by a countable support iteration of Matet forcing [3]. Other proper tree forcings that preserve P-points can be interwoven into the iteration and, as long as at stationarily many steps of cofinality $\aleph_1$ a real is added that has a superset in each groupwise dense family in the intermediate model, the outcome is $\mathfrak{u} < \mathfrak{g}$.

Now our proof of the main theorem modifies these constructions. First: We preserve only one arbitrary P-point $\mathscr{E} \in \mathbf{V}_0$ that will be fixed forever, and we destroy many others.

We try to specify what we mean when we use one of the ambiguous expressions "stage α" or "step α"; When we say "at stage α we have" or something similar it means "in $\mathbf{V}^{\mathbb{P}_\alpha}$ holds". When we say "at stage α we do" it refers to the choice of $\underset{\sim}{\mathbb{Q}}_\alpha$, which is a part of $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \underset{\sim}{\mathbb{Q}}_\alpha$. Second: We let $S^2_1 = \{\alpha \in \aleph_2 : \mathrm{cf}(\alpha) = \aleph_1\}$. We build up one non-meagre non-nearly ultra filter $\mathscr{A}$ generated by $\{A_\alpha : \alpha \notin S^2_1\}$. In a stage $\alpha \in \aleph_2 \setminus S^2_1$ we let $\mathbb{Q}_\alpha$ diagonalise $\mathscr{A}$'s initial segment $\mathscr{A}_\alpha :=$ the filter generated in $\mathbf{V}_\alpha$ by $\{A_\beta : \beta < \alpha, \beta \notin S^2_1\}$ and let A_α be a subset of the complement of a diagonalisation built from certain blocks. Looking at sufficiently many combinations of infinitely many blocks guarantees $\mathscr{A} = \mathscr{A}_{\omega_2}$ will not be meagre in the end. Also $\mathscr{A}$ will be very far from being ultra, because at any time it contains a tree of $2^{\aleph_1}$ mutually non-nearly coherent core filters $\Phi(\mathscr{U})$ (see Definition 3.4) among its supersets, and at stages $\alpha \in \aleph_2 \setminus S^2_1$ the filter $\mathscr{A}_\alpha$ even has a pseudo-intersection (see Definition 4.1) in $\mathbf{V}_\alpha$ (if $\mathrm{cf}(\alpha) = \omega$ and α is the limit of a sequence in S^2_1, then we take a cofinal sequence β_n converging to α and take a pseudo-intersection of the centred set $\{X_{\beta_n} : n \in \omega\}$; note that by the definition of iterated forcing the sequence $\langle X_{\beta_n} : n \in \omega\rangle \in \mathbf{V}^{\mathbb{P}_\alpha}$; in the successor case $\alpha = \beta + 1$, $\beta \notin S^2_1$ we use the induction hypothesis and the fact that $\{A_\beta\} \cup \mathscr{A}_\beta$ is centred; if $\alpha = \beta + 1$, $\beta \in S^2_1$, then X_α is a pseudo-intersection of $\mathscr{A}_\beta$; the case of $\mathrm{cf}(\alpha) = \omega$ and α has a maximal predecessor in S^2_1 is similar) and at stages $\alpha \in S^2_1$ the filter $\mathscr{A}_\alpha$ has a pseudo-intersection in $\mathbf{V}_{\alpha+1}$. We strengthen the latter properties of $\mathscr{A}_\alpha$ to a property of every two stages $\beta < \alpha$, $\beta, \alpha \in \aleph_2 \setminus S^2_1$ that is preserved in the iteration and that will allow us to work with stable ordered-union ultrafilters $\mathscr{U}$ on the set $\mathbb{F}$ of all finite, non-empty subsets of ω. The strengthening will be the technical property (P5) of the iteration in Section 5. It prevents the misfortune that for some $\alpha \in S^2_1$, $\mathscr{A}_\alpha$ is all of a sudden nearly ultra.

Third: We get NCF with the aid of a diamond and special iterands. A diamond sequence on S^2_1 is a sequence $\langle S_\alpha : \alpha \in S^2_1\rangle$ such that for all $X \subseteq \aleph_2$ the set $\{\alpha \in S^2_1 : X \cap \alpha = S_\alpha\}$ is stationary. $\diamondsuit(S^2_1)$ says that there is a diamond sequence on S^2_1.

The tricky part is to find suitable iterands $\mathbb{Q}_\alpha$ for $\alpha \in S^2_1$: $\mathbb{Q}_\alpha$ shall preserve $\mathscr{E}$, shall make the ultrafilter handed down by the diamond to be nearly coherent to $\mathscr{E}$ and shall diagonalise $\mathscr{A}_\alpha$ by adding a pseudo-intersection $X_\alpha \in \mathbf{V}_{\alpha+1}$. Thus in the whole extension $\mathscr{A}$ is not mapped by any finite-to-one function to an ultrafilter.

We divide the proof of the Main Theorem into four sections: First we deal with the iterands for the stages $\alpha \notin S^2_1$. In Section 3 we work on the iterands for the stages $\alpha \in S^2_1$. In Section 4 we introduce pseudo-intersections and witnesses. In Section 5 we put the iteration together. In Section 6 we consider the values of the entries of Cichoń's diagramme and of $\mathfrak{s}$, $\mathfrak{r}$, $\mathfrak{u}$, $\mathfrak{g}$, $\mathfrak{h}$ and other cardinal invariants in the two types of models of "NCF and not FD" from Section 5.

2. The iterands $\mathbb{Q}_\alpha$ for $\alpha \notin S^2_1$

In the stages $\alpha \notin S^2_1$ of the iteration $\mathbb{Q}_\alpha$ shall add some set A_α to $\mathscr{A}_\alpha$, the filter generated by $\{A_\beta : \beta < \alpha\}$, so that these additions will guarantee that in the end $\mathscr{A} = \mathscr{A}_{\aleph_2}$ is not meagre. Any forcing that diagonalises all groupwise dense sets in the ground model would accomplish this task. However, we consider here only two candidates: One is Blass-Shelah forcing $\mathbb{Q}$ (see [6] or [1, pages 370 ff.]) and the second is Matet forcing.

Why just these two? Using the first in unboundedly many steps makes $\mathfrak{s} = \aleph_2$ in the final model, whereas taking coboundedly often the latter possibly keeps $\mathfrak{s}$ small.

Now we review Matet forcing, since we shall also use a suborder of it later in the choice of the iterands $\mathbb{Q}_\alpha$ for $\alpha \in S^2_1$.

We let $\mathbb{F}$ be the collection of all non-empty finite subsets of ω. For $a, b \in \mathbb{F}$ we write $a < b$ if $(\forall n \in a)(\forall m \in b)(n < m)$. We will work with proper filters on $\mathbb{F}$, i.e. subsets of $\mathscr{P}(\mathbb{F})$ that are closed under binary intersections and supersets and do not contain the empty set. A sequence $\bar{c}$ of members of $\mathbb{F}$ is called *unmeshed* if for all n, $c_n < c_{n+1}$. The set $(\mathbb{F})^\omega$ denotes the collection of all infinite unmeshed sequences in $\mathbb{F}$. If $\bar{c}$ is a sequence in $(\mathbb{F})^\omega$, we write $(\mathrm{FU})^\omega(\bar{c})$ for the set of all unmeshed sequences whose members are finite unions of some of the c_n's and we write $\mathrm{FU}(\bar{c})$ for the set of all finite unions of members of $\bar{c}$. The symbol $=^*$ denotes equality up to finitely many exceptions.

Definition 2.1. Given $\bar{c}$ and $\bar{d}$ in $(\mathbb{F})^\omega$, we say that $\bar{d}$ is a *condensation of* $\bar{c}$ and we write $\bar{d} \sqsubseteq \bar{c}$ if $\bar{d} \in (\mathrm{FU})^\omega(\bar{c})$. We say $\bar{d}$ is *almost a condensation of* $\bar{c}$ and we write $\bar{d} \sqsubseteq^* \bar{c}$ iff there is an n such that $\langle d_t : t \geq n\rangle$ is a condensation of $\bar{c}$. If there is some $A \in [\omega]^{\aleph_0}$ such that $X =^* \bigcup\{c_n : n \in A\}$, then we let $\bar{c} \cap X$ be defined as $\langle c_n \cap X : n \in \omega, c_n \cap X \neq \emptyset\rangle$.

Note that $\bar{c} \cap X$, when defined, is almost a condensation of $\bar{c}$.

Definition 2.2. 1. In the Matet forcing, $\mathbb{M}$, the conditions are pairs $(w, \bar{c})$ such that $w \in \mathbb{F}$ and $\bar{c} \in (\mathbb{F})^\omega$ and $w < c_0$. The forcing order is $(w', \bar{c}') \geq (w, \bar{c})$ iff $w \subseteq w'$ and $w' \smallsetminus w$ is a union of finitely many of the c_n and $\bar{c}'$ is a condensation of $\bar{c}$.

2. The set $\mathbb{M}^{\mathrm{pr}}$ of *pure conditions of* $\mathbb{M}$ is the set of conditions with $w = \emptyset$. In this case we write $\bar{c}$ instead of $(\emptyset, \bar{c})$. For two pure conditions we let $\bar{c} \leq^* \bar{d}$ iff $\bar{c} \sqsupseteq^* \bar{d}$. We let $\mathrm{set}(\bar{c}) = \bigcup\{c_n : n \in \omega\}$.

In [3] it is shown that $\mathbb{M}$ is proper. In unpublished work, Blass and Laflamme independently have shown that $\mathbb{M}$ preserves P-points. Eisworth's work ([9, Theorem 4] or Theorem 3.5 below) implies this result, as we shall explain below.

We explain why we prefer the Matet forcing: Given a stable ordered union ultrafilter (see Definition 3.1) $\mathscr{U}$ on $\mathbb{F}$, we can thin out the Matet partial order to a σ-centred subforcing. In the thinning process Hindman's theorem ([10, 3.3], below Theorem 3.2) is used, and we do not know how to apply it unless we have forgetful [17, 1.2.5] forcings. "Forgetful" [17, Definition 1.2.5 (3)] means that the possible strengthenings of the part $c_i \subseteq [f(i), f(i+1))$ of a condition $(w, \bar{c})$ do not depend on the part of the condition outside $[f(i), f(i+1))$; in particular not on the part of the condition below $f(i)$, as in the tree forcings, like Laver, Sacks, Miller or the more general examples from Chapter 1.3 of [17]. Moreover in some technical parts of our proof, again for the mentioned property (P5), we shall use the following property: For every condition $(w', \bar{c}')$ in the Matet forcing, the generic real $X = \bigcup\{w : (\exists \bar{c})((w, \bar{c}) \in G)\}$ coincides up to finitely many exceptions with the union over a suitable infinite set of blocks c'_n or there is some $(w, \bar{c}) \in G$ such that $\bar{c}$ and $\bar{c}'$ do not have a common almost condensation. In the next section we consider subforcings in which the second part of this disjunction is excluded.

3. σ-CENTRED SUBFORCINGS OF $\mathbb{M}$

In this section we look for the iterands $\mathbb{Q}_\alpha$ for $\alpha \in S^2_1$. To define these (necessarily non-complete, as they will destroy some P-points) subforcings of $\mathbb{M}$, we first introduce some properties of filters on the set $\mathbb{F}$ of all non-empty finite subsets for ω. Our nomenclature follows Blass [2] and Eisworth [9].

Note that if we used the Blass-Shelah forcing or the Matet forcing as iterand $\mathbb{Q}_\alpha$ at stationarily many stages $\alpha \in S^2_1$, then we would get $\mathfrak{u} < \mathfrak{g}$, which implies FD.

Definition 3.1. A non-principal filter $\mathscr{F}$ on $\mathbb{F}$ is said to be an *ordered-union* filter if it has a basis of sets of the form $\mathrm{FU}(\bar{d})$ for $\bar{d} \in (\mathbb{F})^\omega$. Let μ be an uncountable cardinal. An ordered-union filter is said to be $< \mu$*-stable* if, whenever it contains $\mathrm{FU}(\bar{d}_\alpha)$ for $\bar{d}_\alpha \in (\mathbb{F})^\omega$, $\alpha < \kappa$, for some $\kappa < \mu$, then it also contains some $\mathrm{FU}(\bar{e})$ for some $\bar{e}$ that is almost a condensation of $\bar{d}_\alpha$ for $\alpha < \kappa$. For "$< \aleph_1$-stable" we say "stable".

Ordered-union ultrafilters need not exist, as their existence implies the existence of Q-points [2] and there are models without Q-points [16]. With the help of Hindman's theorem one shows that under MA(σ-centred) stable (even $< 2^\omega$-stable) ordered-union ultrafilters exist [2]. We will construct suitable stable ordered-union ultrafilters for the choice of $\mathbb{Q}_\alpha$, $\alpha \in S^2_1$, by induction on $\aleph_1$ using CH and Hindman's theorem:

Theorem 3.2 (Hindman, [10, Corollary 3.3])**.** *If the set $\mathbb{F}$ is partitioned into finitely many pieces, then there is a set $\bar{d} \in (\mathbb{F})^\omega$ such that $\mathrm{FU}(\bar{d})$ is included in one piece.*

The theorem also holds if instead of $\mathbb{F}$ we partition only $\mathrm{FU}(\bar{c})$ for some $\bar{c} \in (\mathbb{F})^\omega$; the homogeneous sequence $\bar{d}$ given by the theorem is then a condensation of $\bar{c}$.

Definition 3.3. Given an ordered-union ultrafilter $\mathscr{U}$ on $\mathbb{F}$ we let $\mathbb{M}(\mathscr{U})$ consist of all pairs $(s, \bar{c}) \in \mathbb{M}$, such that $s \in \mathbb{F}$ and $\mathrm{FU}(\bar{c}) \in \mathscr{U}$. The forcing order is the same as in the Matet forcing.

It is well known [13, 3] that Matet forcing $\mathbb{M}$ can be decomposed into two steps $\mathbb{P} * \mathbb{M}(\mathscr{U})$, such that $\mathbb{P}$ is $\aleph_1$-closed (that is, every descending sequence of conditions of countable length has a lower bound) and adds a stable ordered-union ultrafilter $\mathscr{U}$ on the set $\mathbb{F}$.

In order to state a preservation property of $\mathbb{M}(\mathscr{U})$, we need the following definition.

Definition 3.4. Let $\mathscr{U}$ be a filter on $\mathbb{F}$. The *core of* $\mathscr{U}$ is the filter $\Phi(\mathscr{U})$ such that

$$X \in \Phi(\mathscr{U}) \text{ iff } (\exists\, \mathrm{FU}(\bar{c}) \in \mathscr{U})(\bigcup_{n \in \omega} c_n \subseteq X).$$

If $\mathscr{U}$ is ultra on $\mathbb{F}$, then $\Phi(\mathscr{U})$ is not diagonalised (see [9, Prop. 2.3]) and also all finite-to-one images of $\Phi(\mathscr{U})$ are not diagonalised (the same proof). So $\Phi(\mathscr{U})$ is not meagre.

The *Rudin-Blass ordering* on filters on ω is defined as follows: Let $\mathscr{F} \leq_{RB} \mathscr{G}$ iff there is a finite-to-one f such that $f(\mathscr{F}) \subseteq f(\mathscr{G})$. The following property of stable ordered-union ultrafilters $\mathscr{U}$ will be important for our proof:

Theorem 3.5 (Eisworth [9, "$\rightarrow$" Theorem 4, "$\leftarrow$" Cor. 2.5, this direction works also with non-P ultrafilters])**.** *Let $\mathscr{U}$ be a stable ordered-union ultrafilter on $\mathbb{F}$ and* *let $\mathscr{V}$ be a P-point. Iff $\mathscr{V} \not\geq_{RB} \Phi(\mathscr{U})$, then $\mathscr{V}$ continues to generate an ultrafilter after we force with $\mathbb{M}(\mathscr{U})$.*

In the decomposition $\mathbb{M} = \mathbb{P} * \mathbb{M}(\underset{\sim}{\mathscr{U}})$, the stable ordered-union ultrafilter $\mathscr{U}$ in the intermediate model fulfills $\Phi(\mathscr{U}) \not\leq_{\mathrm{RB}} \mathscr{V}$ for any P-point $\mathscr{V}$ in the ground model, and hence by Theorem 3.5, $\mathbb{M}$ preserves P-points.

We shall not add $\mathscr{U}$ by forcing, but working with a $\sqsubseteq^*$-descending sequence $\bar{c}^\alpha$, $\alpha < \aleph_1$, with the property that $\mathrm{FU}(\bar{c}^\alpha)$, $\alpha < \aleph_1$, generates an ultrafilter $\mathscr{U}$ on $\mathbb{F}$. Then this is a stable ordered-union ultrafilter.

4. Filters with pseudo-intersections

In this section we work with some properties of filters on ω, in the direction of building a non-meagre non-nearly-ultra filter $\mathscr{A}$.

Definition 4.1. Let $\mathscr{A} \subseteq [\omega]^{\aleph_0}$ be such that for all n for all $X_0, \dots, X_n \in \mathscr{A}$, $\bigcap_{i \leq n} X_i$ is infinite. This is called "$\mathscr{A}$ is centred" or "$\mathscr{A}$ has the finite intersection property".

(a) By $\mathrm{fil}(\mathscr{A})$ we denote the filter on ω generated by $\mathscr{A} \cup \{\omega \smallsetminus n : n < \omega\}$. Later, when we are working in the context of an intermediate forcing extension $\mathbf{V}_\alpha$, the non-absolute definition $\mathrm{fil}(\mathscr{A})$ is to be interpreted in that $\mathbf{V}_\alpha$.

(b) $X \in [\omega]^{\aleph_0}$ is a *pseudo-intersection of $\mathscr{A}$* (or *diagonalises $\mathscr{A}$*) if for all $A \in \mathscr{A}$, $X \subseteq^* A$.

Note that $X \in [\omega]^{\aleph_0}$ is a pseudo-intersection of $\mathscr{A}$ if it is a pseudo-intersection of $\mathrm{fil}(\mathscr{A})$. Only centred sets can have pseudo-intersections.

Definition 4.2.

(1) $\mathscr{R}^* = \{R \subseteq \omega \times \omega \ : \ (\forall m)(\exists^{<\aleph_0} n)(mRn) \wedge (\forall n)(\exists^{<\aleph_0} m)(mRn)\}$. The quantifier $\exists^{<\aleph_0}$ means that there are finitely many and at least one.

(2) For $R, S \in \mathscr{R}^*$ we let $R^{-1} = \{(n,m) \ : \ (m,n) \in R\}$ and we let $R \circ S = \{(m,r) \ : \ (\exists n)((m,n) \in R \wedge (n,r) \in S)\}$. Note that the order is different from the one known in the composition of functions: We first "map" with R, then with S. For $\mathscr{R} \subseteq \mathscr{R}^*$ and $S \in \mathscr{R}^*$ we let $S \circ \mathscr{R} = \{S \circ R \ : \ R \in \mathscr{R}\}$.

(3) For $A \subseteq \omega$, $R \in \mathscr{R}^*$ we let $R(A) = \{n \ : \ mRn, m \in A\}$.

(4) For $\bar{c} = \langle c_n \ : \ n \in \omega \rangle \in (\mathbb{F})^\omega$, $R \in \mathscr{R}^*$ we let $R(\bar{c}) = \langle R(c_n) \ : \ n \in \omega \rangle$. This can be meshed or even be not pairwise disjoint (that is, not unmeshed), but it does not matter.

Note that R^{-1} and $R \circ S$ are also in $\mathscr{R}^*$. The next definition will play a crucial rôle in the iteration.

Definition 4.3. We say $(\bar{c}, \mathscr{R})$ is a *witness over $\mathscr{A}$* when:

(a) $\mathscr{A} \subseteq [\omega]^{\aleph_0}$ is centred,

(b) $\bar{c} \in (\mathbb{F})^\omega$,

(c) $\mathscr{R}$ is a countable subset of $\mathscr{R}^*$,

(d) $\mathscr{R} \neq \emptyset$,

(e) for every $R \in \mathscr{R}$, the set $R(\mathrm{set}(\bar{c}))$ is a pseudo-intersection of $\mathscr{A}$.

The purpose of $R \in \mathscr{R}^*$ is to increase infinite sets in a gentle manner, as with finite-to-one functions: $f(\mathscr{F}) = \{X \ : \ f^{-1}X \in \mathscr{F}\} = \{X \ : \ R(X) \in \mathscr{F}\}$, where xRy iff $f(y) = x$. Since f is a finite-to-one function, we have $R \in \mathscr{R}$. Iff for every

$R \in \mathscr{R}^*$ there is some $X \in \mathscr{F}$ such that $R(X) \notin \mathscr{V}$, then $\mathscr{F}$ is not Rudin-Blass below $\mathscr{V}$. We shall use the "if"-direction of this criterion for $\mathscr{F} = \Phi(\mathscr{U})$ and $\mathscr{V} = \mathscr{E}$ in the final section.

However, our main use of $\mathscr{R}^*$ is the following: We use countable subsets $\mathscr{R}$ of $\mathscr{R}^*$ to map pseudo-intersections of $\mathscr{A}_\alpha$ to other pseudo-intersections of $\mathscr{A}_\alpha$, as in the definition of "witness". The union of all used countable parts (that is, $\bigcup_{\varepsilon < \aleph_1} \mathscr{R}_\varepsilon$ from Lemma 5.4; note we require $\zeta < \varepsilon \to \mathscr{R}_\zeta \subseteq \mathscr{R}_\varepsilon$) at a limit step $\alpha \in S^2_1$ does not exhaust $(\mathscr{R}^*)^{\mathbf{V}_\alpha}$, since $\mathscr{A}_\alpha$ is centred and there are $\{(n, 2n) : n \in \omega\}$ and $\{(n, 2n+1) : n \in \omega\}$, which can never be both elements of $\mathscr{R}$, such that there is a $\bar{c} \in (\mathbb{F})^\omega$ with $(\bar{c}, \mathscr{R})$ witnessing over $\mathscr{A}_\beta$ for some $\beta < \alpha$. However, it does so modulo composition with "shifts" $R_{\bar{c},\bar{d}} = \{(m,n) : (\exists i)(m \in c_i \wedge n \in d_i)\}$ for $\bar{c}, \bar{d} \in (\mathbb{F})^\omega$. This will follow from the technical requirement (γ) in the proof of Lemma 5.4.

Lemma 4.4. (1) *Assume that* $\mathbf{V}$ *is a transitive class in* $\mathbf{V}^+$. *If in* $\mathbf{V}$, $(\bar{c}, \mathscr{R})$ *is a witness over* $\mathscr{A}$, *then this also holds in* $\mathbf{V}^+$.

(2) $(\bar{c}, \mathscr{R})$ *witnesses over* $\mathscr{A}$ *iff for every* $A \in \mathscr{A}$, $(\bar{c}, \mathscr{R})$ *witnesses over* $\{A\}$.

(3) $(\bar{c}, \mathscr{R})$ *witnesses over* $\mathscr{A}$ *iff* $(\bar{c}, \mathscr{R})$ *witnesses over* $\mathrm{fil}(\mathscr{A})$.

(4) *If* $(\bar{c}, \mathscr{R})$ *witnesses over* $\mathscr{A}$ *and* $\bar{d} \sqsubseteq^* \bar{c}$, *then* $(\bar{d}, \mathscr{R})$ *witnesses over* $\mathscr{A}$.

(5) *If* $\mathscr{R} \subseteq \mathscr{R}^*$ *and* $\bar{c}$, $\bar{d} \in (\mathbb{F})^\omega$ *and* $R = R_{\bar{c},\bar{d}} = \{(m,n) : (\exists i)(m \in c_i \wedge n \in d_i)\}$, *then* $R(\bar{c}) = \bar{d}$ *and* $R^{-1} = R_{\bar{d},\bar{c}}$ *and* $(\bar{d}, R_{\bar{d},\bar{c}} \circ \mathscr{R})$ *witnesses over* $\mathscr{A}$ *iff* $(\bar{c}, \mathscr{R})$ *witnesses over* $\mathscr{A}$.

Proof. (1) The definition of witnessing contains only bounded quantifiers and an existential quantifier. Items (2), (3), and (4) are obvious. Proof of (5): We compute $R^{-1}(d_k) = \{(n,m) : \exists i (m \in d_i \wedge n \in c_i)\}(d_k) = c_k$ so $R^{-1} = R_{\bar{d},\bar{c}}$. Although $R \circ R^{-1} = \{(n,m) : (\exists i)(n, m \in c_i)\}$ is in general not a subset of the diagonal, we have $(R \circ R^{-1})(c_n) = c_n$. Now $(R^{-1} \circ S)(d_k) = \{(m,r) : \exists n((m,n) \in R^{-1} \wedge (n,r) \in S)\}(d_k) = S(R^{-1}(d_k)) = S(c_k)$. So we have $(\forall A \in \mathscr{A})(\forall S \in \mathscr{R})(S(\mathrm{set}(\bar{c})) \subseteq^* A)$ iff $(\forall A \in \mathscr{A})(\forall S \in \mathscr{R})(R^{-1} \circ S(\mathrm{set}(\bar{d})) \subseteq^* A)$. □

5. The iteration

We start with a ground model $\mathbf{V}$ that fulfills CH and $\diamondsuit(S^2_1)$ (and hence $2^{\aleph_1} = \aleph_2$).

In a countable support iteration of proper forcings of iterands of size $\leq \aleph_1$ each real appears in a $\mathbf{V}_\alpha$ for some α with countable cofinality, and a reflection property ensures that each ultrafilter $\mathscr{U}$ in the final model has $\aleph_1$-club many $\alpha \in \aleph_2$ such that $\mathscr{U} \cap V_\alpha$ has a $\mathbb{P}_\alpha$-name and is an ultrafilter in $\mathbf{V}_\alpha$ (see [6, Item 5.6 and Lemma 5.10]). A subset of $\aleph_2$ is called $\aleph_1$-club if it is unbounded in $\aleph_2$ and closed under suprema of strictly ascending sequences of lengths $\aleph_1$. By well-known techniques based on coding $\mathbb{P}_\alpha$-names for ultrafilters as subsets of $\aleph_2$ (e.g., such a coding is carried out in [15, Claim 2.8]) and based on the maximal principle (see, e.g., [12, Theorem 8.2]) it is safe to assume that an enumeration $\langle \underset{\sim}{f}_\alpha : \alpha < \aleph_2 \rangle$ exists and that the $\diamondsuit(S^2_1)$-sequence $\langle \underset{\sim}{S}_\alpha : \alpha \in S^2_1 \rangle$ gives an $\aleph_1$-club set of $\mathbb{P}_\alpha$-names $\underset{\sim}{S}_\alpha$ for an ultrafilter in $\mathbf{V}^{\mathbb{P}_\alpha}$ such that for any ultrafilter $\mathscr{U} \in V^{\mathbb{P}_{\aleph_2}}$ there are stationarily many $\alpha \in S^2_1$ with $\mathscr{U} \cap V^{\mathbb{P}_\alpha} = S_\alpha$. For names $\underset{\sim}{x}$ and objects x we use the rule $\underset{\sim}{x}[G] = x$.

We fix a diamond sequence $\langle \underset{\sim}{S}_\alpha : \alpha \in S^2_1 \rangle$. Only S_α gets a second letter: $S_\alpha = \mathscr{D}$ to make clearer that it is an ultrafilter. We also fix a P-point $\mathscr{E} \in \mathbf{V}$ that will be preserved throughout our iteration. Let $\underset{\sim}{f}_\alpha$, $\alpha \in \aleph_2 \setminus S^2_1$, be an enumeration of all $\mathbb{P}_{\aleph_2}$-names for finite-to-one, monotone, surjective functions from ω to ω, each appearing cofinally often, such that $\underset{\sim}{f}_\alpha$ is a $\mathbb{P}_\alpha$-name. We assume that $\underset{\sim}{\pi}_\alpha$ is a $\mathbb{P}_\alpha$-name such that for all evaluations f_α, π_α of $\underset{\sim}{f}_\alpha$, $\underset{\sim}{\pi}_\alpha$ respectively, $\pi_\alpha(0) = 0$, $\pi_\alpha(n+1) = \max(f_\alpha^{-1}(n)) + 1$.

We construct (carefully) by induction on $\alpha \leq \aleph_2$ a countable support iteration of proper forcings $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \aleph_2, \alpha \leq \aleph_2 \rangle$ and two sequences of names $\langle \underset{\sim}{A}_\alpha : \alpha \in \aleph_2 \smallsetminus S^2_1 \rangle$ and $\langle \underset{\sim}{X}_\alpha : \alpha \in S^2_1 \rangle$ such that

(P1) For all $\alpha < \aleph_2$, $\Vdash_{\mathbb{P}_\alpha}$ "$\underset{\sim}{\mathbb{Q}}_\alpha$ is proper".

(P2) For all $\alpha \leq \aleph_2$, $\Vdash_{\mathbb{P}_\alpha}$ "fil($\mathscr{E}$) is ultra".

(P3) For $\alpha \in \aleph_2 \setminus S^2_1$, $\underset{\sim}{A}_\alpha$ is a $\mathbb{P}_{\alpha+1}$ name. We write $A_\alpha = \underset{\sim}{A}_\alpha[G_{\alpha+1}]$. For each $\alpha \in \aleph_2 \smallsetminus S^2_1$, $\{A_\beta : \beta \in \alpha \smallsetminus S^2_1\}$ is centred and $f_\alpha(A_\alpha) \neq^* \omega$. We let $\mathscr{A}_\alpha = \mathrm{fil}(\{A_\beta : \beta \in \alpha \smallsetminus S^2_1\})$. So A_α shows that $f_\alpha(\mathscr{A}_{\alpha+1})$ is not the Fréchet filter.

(P4) For $\alpha \in S^2_1$, $\underset{\sim}{X}_\alpha$ is a $\mathbb{P}_{\alpha+1}$-name. Let $\underset{\sim}{\mathscr{A}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathscr{A}_\alpha$. If $\alpha \in S^2_1$ and if $\underset{\sim}{S}_\alpha$ is a $\mathbb{P}_\alpha$-name $\mathscr{D}$ for an ultrafilter in $V^{\mathbb{P}_\alpha}$, then $\Vdash_{\mathbb{P}_{\alpha+1}}$ "$\mathscr{D}$ and fil($\mathscr{E}$) are nearly coherent, and $\underset{\sim}{X}_\alpha$ is a pseudo-intersection of $\underset{\sim}{\mathscr{A}}_\alpha$".

(P5) For $\beta < \gamma$, $\beta, \gamma \in \aleph_2 \smallsetminus S^2_1$, if $G_\gamma \subseteq \mathbb{P}_\gamma$ is generic over $\mathbf{V}$ and $G_\beta = \mathbb{P}_\beta \cap G_\gamma$, then

$$\text{if } \mathbf{V}_\beta \models \text{“}(\bar{c}, \mathscr{R}) \text{ is a witness over } \mathscr{A}_\beta\text{”},$$

$$\text{then } \mathbf{V}_\gamma \models \text{“}(\exists \bar{d})(\bar{d} \sqsubseteq^* \bar{c} \wedge (\bar{d}, \mathscr{R}) \text{ is a witness over } \mathscr{A}_\gamma)\text{”}.$$

Now we prove that such an iteration exists. We start with the Fréchet filter $\mathscr{A}_0 \in \mathbf{V}_0$. We say "$\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\delta : \gamma \leq \alpha', \delta < \alpha' \rangle$ and $\langle A_\gamma : \gamma \in \alpha' \smallsetminus S^2_1 \rangle$ and $\langle X_\gamma : \gamma \in \alpha' \cap S^2_1 \rangle$ have properties (P1) to (P5)" if all requirements (P1), (P3), and (P4) hold for $\alpha < \alpha'$, (P2) holds for $\alpha \leq \alpha'$ and (P5) holds for $\beta < \gamma \leq \alpha'$.

The following lemma is for the successor steps $\alpha \mapsto \alpha + 1$ for $\alpha \notin S^2_1$.

Lemma 5.1. *Assume that $\alpha \in \aleph_2 \smallsetminus S^2_1$ and that $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\delta : \gamma \leq \alpha, \delta < \alpha \rangle$, $\langle \underset{\sim}{A}_\gamma : \gamma \in \alpha \smallsetminus S^2_1 \rangle$ and $\langle \underset{\sim}{X}_\gamma : \gamma \in \alpha \cap S^2_1 \rangle$ are defined with the properties (P1) to (P5). Then there are some $\underset{\sim}{\mathbb{Q}}_\alpha$, $\underset{\sim}{A}_\alpha$ such that*

(a) $\Vdash_{\mathbb{P}_\alpha}$ *"$\underset{\sim}{\mathbb{Q}}_\alpha$ is proper and preserves P-points, so in particular* fil($\mathscr{E}$)*",*

(b) $\Vdash_{\mathbb{P}_\alpha}$ *"$\{\underset{\sim}{A}_\alpha\} \cup \underset{\sim}{\mathscr{A}}_\alpha$ is centred",*

(c) $\Vdash_{\mathbb{P}_\alpha}$ *"$\underset{\sim}{A}_\alpha$ is disjoint from infinitely many intervals $[\underset{\sim}{\pi}_\alpha(n), \underset{\sim}{\pi}_\alpha(n+1))$",*

(d) *property (P5) still holds.*

Proof. We let $\underset{\sim}{\mathbb{Q}}_\alpha$ be the $\mathbb{P}_\alpha$-name for Blass-Shelah forcing [6] or $\mathbb{M}$ or Miller forcing or any proper forcing that preserves that fil($\mathscr{E}$) is a P-point and adds $\langle m_i : i < \omega \rangle \in {}^\omega\omega$, such that

$$(\forall f \in ({}^\omega\omega)^{\mathbf{V}_\alpha})(\exists^\infty i)(f(m_i) < m_{i+1}). \tag{5.1}$$

Then we let $\underset{\sim}{A}_\alpha$ be a name for

$$A_\alpha = \bigcup_{i \in \omega} [\pi_\alpha(m_i + 1), \pi_\alpha(m_{i+1})). \tag{5.2}$$

Then claim (c) is true, because $A_\alpha \cap \bigcup_{i \in \omega} [\pi_\alpha(m_i), \pi_\alpha(m_i + 1)) = \emptyset$.

For claim (b) we let $B \in \mathscr{A}_\alpha$; actually $B \in [\omega]^{\aleph_0} \cap \mathbf{V}_\alpha$ is sufficient. We thin out B to $C \in [B]^\omega$ that contains at most one point in each interval $[\pi_\alpha(n), \pi_\alpha(n+1))$ and no point in $[0, \pi_\alpha(2))$. We let $f_C \colon \omega \to C$ be its increasing enumeration and let $\tilde{f}_C$ be the iteration of f_C, that is, $\tilde{f}_C(0) = 0$, $\tilde{f}_C(n+1) = f_C(\tilde{f}_C(n))$. Then by

(5.1) applied to $\tilde{f}_C \circ \pi_\alpha$, we get

$$(\exists^\infty i)(\tilde{f}_C(\pi_\alpha(m_i)) \in C \cap [\pi_\alpha(m_i+1), m_{i+1}) \subseteq [\pi_\alpha(m_i+1), \pi_\alpha(m_{i+1}))),$$

and hence $A_\alpha \cap C$ is infinite.

Now we check property (P5). The only new cases are $\beta \leq \alpha$ and $\gamma = \alpha + 1$. By induction hypothesis and by transitivity of $\sqsubseteq^*$ we need to consider only $\beta = \alpha$. If $\mathbf{V}_\alpha \models$ "$(\bar{c}, \mathscr{R})$ witnesses over $\mathscr{A}_\alpha[G_\alpha]$", then $\mathbf{V}_{\alpha+1} \models$ "there is some $\bar{d} \sqsubseteq^* \bar{c}$ such that $(\bar{d}, \mathscr{R})$ witnesses over $\mathscr{A}_{\alpha+1}[G_{\alpha+1}]$": We work in $\mathbf{V}_\alpha$. Let $\mathscr{R}$ be enumerated as R_n, $n \in \omega$. First we thin out $\bar{c}$ to $\langle c'_k : k < \omega \rangle$ such that

$$(\forall k \in \omega)(\forall i \leq k)(R_i(c'_k) \cap \pi_\alpha(k+1) = \emptyset). \tag{5.3}$$

We let $\bar{c}' = \langle c'_k : k < \omega \rangle$. In $\mathbf{V}_{\alpha+1}$ let

$$w = \{n < \omega : (\forall k \leq n)(R_k(c'_n) \subseteq A_\alpha)\}.$$

From $\mathscr{R} \subseteq \mathbf{V}_\alpha$ and (5.1), applied to $f(k) = \min\{\ell > k+1 : (\forall i \leq k)(R_i(c'_k) \subseteq \pi_\alpha(\ell))\}$, together with equation (5.3), it follows that there are infinitely many i such that $(\forall k \leq m_i)(R_k(c'_{m_i}) \subseteq [\pi_\alpha(m_i+1), \pi_\alpha(m_{i+1})))$. Since $m_i + 1 < m_{i+1}$, we have that for these i, by equation (5.2), $\emptyset \neq [\pi_\alpha(m_i+1), \pi_\alpha(m_{i+1})) \subseteq A_\alpha$, and hence w is infinite.

We let $\bar{d} = \langle \bar{c}'_n : n \in w \rangle$. Then $\bar{d} \sqsubseteq^* \bar{c}$, and if $C \in \mathscr{A}_{\alpha+1}[G_{\alpha+1}]$, then: In the first case, if $C \in \mathscr{A}_\alpha[G_\alpha]$, then $(\forall R \in \mathscr{R})(R(\mathrm{set}(\bar{d}) \subseteq R(\mathrm{set}(\bar{c}')) \subseteq^* C)$. In the second case, if $C = A_\alpha$, then $(\forall R \in \mathscr{R})(R(\mathrm{set}(\bar{d})) \subseteq^* C)$ by the choice of w. Hence we have for all $C \in \mathrm{fil}(\{A_\alpha\} \cup \mathscr{A}_\alpha)$, $(\forall R \in \mathscr{R})(R(\mathrm{set}(\bar{d})) \subseteq^* C)$. Obviously $\mathbb{P}_{\alpha+1}$ preserves P-points. □

Now we consider two kinds of limit steps, those with countable cofinality and those with cofinalities $\aleph_1$ or $\aleph_2$. For (P2) we use a well-known preservation theorem: The countable support limit of forcings preserves each P-point that is preserved by all approximations [6, Theorem 4.1]. We also use the fact that the countable support limit of proper forcings is proper [18, III, 3.2]. So our iteration preserves $\aleph_1$. It preserves $\aleph_2$, because any collapse would appear at some intermediate step $\mathbb{P}_\alpha$, but $\mathbb{P}_\alpha$ has size $\aleph_1$ and the $\aleph_2$-c.c.

Lemma 5.2. *Let $\alpha = \lim_n \alpha_n$ be the limit of a strictly increasing sequence of ordinals in $\aleph_2$. If for each n, $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha_n, \gamma \leq \alpha_n \rangle$ and the two sequences of names $\langle \underset{\sim}{A}_\gamma : \gamma \in \alpha_n \setminus S^2_1 \rangle$ and $\langle \underset{\sim}{X}_\gamma : \gamma \in \alpha_n \cap S^2_1 \rangle$ fulfill* (P1) *to* (P5)*, then also $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha, \gamma \leq \alpha \rangle$ and the sequence of names $\langle \underset{\sim}{A}_\gamma : \gamma \in \alpha \setminus S^2_1 \rangle$ and $\langle \underset{\sim}{X}_\gamma : \gamma \in \alpha \cap S^2_1 \rangle$ fulfill* (P1) *to* (P5).

Proof. Again we have to check property (P5) for the new instance $\beta \in \alpha \setminus S^2_1$ and α itself, given $\beta < \alpha$ and a witness $(\bar{c}, \mathscr{R})$ over $\mathscr{A}_\beta$ in $\mathbf{V}_\beta$. By induction hypothesis we may possibly increase β and assume that $\alpha_{n_0} = \beta$ for some n_0. Then we choose $\bar{c}^n \in \mathbf{V}_{\alpha_n}$, $n \in [n_0, \omega)$, in a $\sqsubseteq^*$-descending manner using (P5) between $\mathbf{V}_{\alpha_n}$ and $\mathbf{V}_{\alpha_{n+1}}$, all the time for the same $\mathscr{R}$, and in the end we find some $\bar{d}$ such that for all n, $\bar{d} \sqsubseteq^* \bar{c}^n$. □

Lemma 5.3. *Let $\alpha = \lim_{\varepsilon < \kappa} \alpha_\varepsilon$ be the limit of a strictly increasing sequence of ordinals in ω_2 and let κ be $\aleph_1$ or $\aleph_2$. If for all ε, $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha_\varepsilon, \gamma \leq \alpha_\varepsilon \rangle$ and two sequences of names $\langle \underset{\sim}{A}_\gamma : \gamma \in \alpha_\varepsilon \setminus S^2_1 \rangle$ and $\langle \underset{\sim}{X}_\gamma : \gamma \in \alpha_\varepsilon \cap S^2_1 \rangle$ fulfill (P1) to (P5), then also $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha, \gamma \leq \alpha \rangle$ and the sequences of names $\langle \underset{\sim}{A}_\gamma : \gamma \in \alpha \setminus S^2_1 \rangle$ and $\langle \underset{\sim}{X}_\gamma : \gamma \in \alpha \cap S^2_1 \rangle$ fulfill (P1) to (P5).*

Proof. Property (P5) is vacuously true in limit steps. Properties (P3) and (P4) are obviously true. □

Finally we carry out the successor step $\alpha \mapsto \alpha+1$ for $\alpha \in S^2_1$:

Lemma 5.4. *Let $\alpha \in S^2_1$. Assume that $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\delta : \gamma \le \alpha, \delta < \alpha\rangle$, $\langle \underset{\sim}{A}_\gamma : \gamma \in \alpha \setminus S^2_1\rangle$ and $\langle \underset{\sim}{X}_\gamma : \gamma \in \alpha \cap S^2_1\rangle$ fulfill (P1) to (P5) and that the member of the diamond sequence $\underset{\sim}{S}_\alpha$ is a $\mathbb{P}_\alpha$-name for a non-principal ultrafilter $\mathscr{D}$ on ω.*

Then there are some $\underset{\sim}{\mathbb{Q}}_\alpha$ and $\underset{\sim}{X}_\alpha$, such that $\langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\delta : \gamma \le \alpha+1, \delta < \alpha+1\rangle$, $\langle \underset{\sim}{A}_\gamma : \gamma \in (\alpha+1) \setminus S^2_1\rangle$ and $\langle \underset{\sim}{X}_\gamma : \gamma \in (\alpha+1) \cap S^2_1\rangle$ have properties (P1) to (P5).

Proof. Let $G_\alpha \subseteq \mathbb{P}_\alpha$ be generic over $\mathbf{V}$ and let $G_\beta = \mathbb{P}_\beta \cap G_\alpha$ for $\beta < \alpha$. We write $\mathscr{A}_\alpha$ for $\underset{\sim}{\mathscr{A}}_\alpha[G_\alpha]$. Let $\langle \alpha_\varepsilon : \varepsilon < \omega_1\rangle \in \mathbf{V}$ be increasing continuous with limit α, and each α_ε has cofinality $\aleph_0$ for $1 \le \varepsilon < \omega_1$ and let $\alpha_0 = 0$, $\alpha_1 = \omega+\omega$.

Using Lemma 4.4(1), we can find a sequence $\langle (\xi_\varepsilon, \bar{d}_\varepsilon, \mathscr{R}'_\varepsilon, R'_\varepsilon, B_\varepsilon) : \varepsilon < \omega_1\rangle$, such that

- (a) for every $\zeta < \omega_1$, the sequence $\langle (\xi_\varepsilon, \bar{d}_\varepsilon, \mathscr{R}'_\varepsilon, R'_\varepsilon, B_\varepsilon) : \varepsilon < \zeta\rangle$ belongs to $\mathbf{V}_{\alpha_\zeta}$, $\xi_\varepsilon < \alpha_{\varepsilon+1}$, and $\mathrm{cf}(\xi_\varepsilon) = \aleph_0$,
- (b) $R'_\varepsilon \in (\mathscr{R}^*)^{\mathbf{V}_{\xi_\varepsilon}}$,
- (c) in $\mathbf{V}_{\xi_\varepsilon}$ we have $\bar{d}_\varepsilon \in (\mathbb{F})^\omega$ and $\mathscr{R}'_\varepsilon \in ([(\mathscr{R}^*)]^{\aleph_0})^{\mathbf{V}_{\xi_\varepsilon}}$ and $(\bar{d}_\varepsilon, \mathscr{R}'_\varepsilon)$ is a witness over $\underset{\sim}{\mathscr{A}}_{\xi_\varepsilon}[G_{\xi_\varepsilon}]$,
- (d) $B_\varepsilon \subseteq \mathbb{F}$,
- (e) every such tuple $(\xi_\varepsilon, \bar{d}_\varepsilon, \mathscr{R}'_\varepsilon, R'_\varepsilon, B_\varepsilon)$ appears in the sequence $\aleph_1$ times.

We now choose $(\bar{c}_\varepsilon, \mathscr{R}_\varepsilon)$ by induction on $\varepsilon < \omega_1$ such that

- (α) $(\bar{c}_\varepsilon, \mathscr{R}_\varepsilon) \in \mathbf{V}_{\alpha_\varepsilon}$ is a witness over $\mathscr{A}_{\alpha_\varepsilon}$ and $\mathrm{id}_\omega \in \mathscr{R}_0$,
- (β) if $\zeta < \varepsilon$ then $\bar{c}_\zeta \sqsupseteq^* \bar{c}_\varepsilon$ and $\mathscr{R}_\zeta \subseteq \mathscr{R}_\varepsilon$,
- (γ) for all $\zeta < \varepsilon$, if $(\bar{d}_\zeta, \mathscr{R}'_\zeta)$ witnesses over $\mathscr{A}_{\xi_\zeta}$, then there are some $\zeta \le \zeta' \le \varepsilon$ and some $\bar{c}' \sqsupseteq^* \bar{c}_{\zeta'}$ (the direction is not a mistake) and some $\bar{d}' \sqsubseteq^* \bar{d}_\zeta$ such that $R_{\bar{d}',\bar{c}'} = \{(m,n) : (\exists i)(m \in d'_i \wedge n \in c'_i)\}$ and $R_{\bar{d}',\bar{c}'} \circ \mathscr{R}'_\zeta \subseteq \mathscr{R}_{\zeta'}$,
- (δ) $\omega \setminus R'_\varepsilon(\mathrm{set}(\bar{c}_{\varepsilon+1})) \in \mathrm{fil}^{\mathbf{V}_{\alpha_{\varepsilon+1}}}(\mathscr{E})$,
- (ε) $\omega \setminus R'_\varepsilon(\mathrm{set}(\bar{c}_{\varepsilon+1})) \in \mathscr{D}$,
- (ζ) $\mathrm{FU}(\bar{c}_{\varepsilon+1})$ is included in B_ε or disjoint from B_ε.

We start the induction with $\alpha_0 = 0$, $\mathscr{R}_0 = \{\mathrm{id}_\omega\}$, and we take an arbitrary $\bar{c}_0 \in (\mathbb{F})^\omega$. $\mathscr{A}_0$ is the Fréchet filter.

At limit steps ε we take the $\bar{c}_\varepsilon \sqsubseteq^* \bar{c}_\zeta$ for all $\zeta < \varepsilon$ and we take $\mathscr{R}_\varepsilon = \bigcup_{\zeta<\varepsilon} \mathscr{R}_\zeta$. (($\gamma$) is automatically fulfilled at limit steps.)

We carry out the successor step. Suppose $(\bar{c}_\delta, \mathscr{R}_\delta)$, $\delta < \varepsilon$, are given and that ε is countable. We show how to fulfill (γ) in successor steps $\varepsilon = \varepsilon'+1$: We enumerate all tasks for item (γ) as $(\bar{e}^n, \hat{\mathscr{R}}^n, \zeta_n)$, $n \in \omega$, and we build $(\bar{c}_{\varepsilon'}, \mathscr{R}_\varepsilon)$ by induction in ω steps as the limit of $(\bar{c}_{\varepsilon'}, \mathscr{R}^n)$ that is increasing, actually constant in the first component, in $(\le^*, \subseteq)$ and witnessing over $\mathscr{A}_{\alpha_\varepsilon}$. By the induction hypothesis (P5) below α, we may strengthen the $\bar{e}_\zeta$ and increase the ξ_ζ, and hence we may assume that the ζ_n fulfill $\xi_{\zeta_n} = \alpha_\varepsilon$ and that $(\bar{e}^n, \hat{\mathscr{R}}^n)$ witnesses over $\mathscr{A}_{\alpha_\varepsilon}$.

We start with $\mathscr{R}^{-1} = \mathscr{R}_{\varepsilon'}$.

We assume that

- $(\bar{e}^n, \hat{\mathscr{R}}^n)$ is a witness over $\mathscr{A}_{\alpha_\varepsilon}$ (this is the current task for (γ)),
- $(\bar{c}_{\varepsilon'}, \mathscr{R}^{n-1})$ are already constructed witnessing over $\mathscr{A}_{\alpha_\varepsilon}$, and

- (γ) holds for $(\bar{d}_\zeta, \mathscr{R}'_\zeta)$ that are enumerated among the tasks $(\bar{e}^m, \hat{\mathscr{R}}^m)$, $m < n$, with $\hat{\mathscr{R}}^{n-1}$ in the place of $\mathscr{R}_{\zeta'}$ and with $\bar{c}' = \bar{c}_{\varepsilon'}$ and with $\bar{d}' = \bar{e}^m$.

Then $R^{-1}_{\bar{e}^n, \bar{c}_{\varepsilon'}}$ maps $\bar{e}^n$ into $\bar{c}_{\varepsilon'}$. So $(\bar{c}_{\varepsilon'}, \{R_{\bar{e}^n, \bar{c}_{\varepsilon'}} \circ S \,:\, S \in \hat{\mathscr{R}}^n\})$ witnesses over $\mathscr{A}_{\alpha_\varepsilon}$. Hence we may let $\{R_{\bar{e}^n, \bar{c}_{\varepsilon'}} \circ S \,:\, S \in \hat{\mathscr{R}}^n\} \cup \mathscr{R}^{n-1} = \mathscr{R}^n$.

In the end we let $\mathscr{R}_\varepsilon = \bigcup_{n \in \omega} \mathscr{R}^n$. Then $(\bar{c}_{\varepsilon'}, \mathscr{R}_\varepsilon)$ witnesses over $\mathscr{A}_{\alpha_\varepsilon}$ and the property (γ) is carried on. Now we strengthen $\bar{c}_{\varepsilon'}$ three times in order to fulfill items (δ), (ε), and (ζ), and we call the outcome $\bar{c}_\varepsilon$. For (ζ) we use the mentioned stronger form of Hindman's Theorem.

Now we let $\mathscr{U} = \mathrm{fil}(\{\mathrm{FU}(\bar{c}_\varepsilon) \,:\, \varepsilon < \omega_1\})$. It is a stable ordered-union ultrafilter by (ζ) and (β). Then we take $\mathbb{Q}_\alpha = \mathbb{M}(\mathscr{U})$. It is σ-centred and hence proper. So (P1) holds.

In $\mathbf{V}_\alpha$, the P-point $\mathscr{E}$ and the ultrafilter $\mathscr{D}$ are both not Rudin-Blass above $\Phi(\mathscr{U})$, as is secured by (δ) and (ε). All potential Rudin-Blass finite-to-one maps are covered by the enumeration $\{R'_\varepsilon \,:\, \varepsilon \in \aleph_1\} = (\mathscr{R}^*)^{\mathbf{V}_\alpha}$. By Eisworth's Theorem, Theorem 3.5, the successor $\mathbb{Q}_\alpha$ preserves "fil($\mathscr{E}$) is an ultrafilter". So (P2) holds also for $\mathbb{P}_{\alpha+1}$. Item (P3) is vacuous for $\alpha \in S^2_1$.

Now we prove (P4), that $\Vdash_{\mathbb{Q}_\alpha}$ "$\underset{\sim}{\mathscr{D}}$ and fil($\mathscr{E}$) are nearly coherent and $\underset{\sim}{\mathscr{A}}_\alpha$ has a diagonalisation". First: The near coherence comes from density arguments for $\mathbb{M}(\mathscr{U})$: Let the generic real $X_\alpha = \bigcup\{w \,:\, (\exists w)((w, \bar{c}) \in G_{\alpha+1})\}$ be enumerated increasingly by e_α. Then the generalised inverse of this enumeration $g_\alpha(k) = \min\{n \,:\, e_\alpha(n) \geq k\}$ is a finite-to-one function that makes $\mathscr{E}$ and $\mathscr{D}$ nearly coherent: Given $(w, \bar{c}) \in \mathbb{M}(\mathscr{U})$ and $E \in \mathscr{E}$ and $D \in \mathscr{D}$ we get by (δ) some $E' \subseteq E$, $E' \in \mathscr{E}$ and some $\bar{d} \geq^* \bar{c}$ such that E' avoids $\mathrm{set}(\bar{d})$, and by (ε) we find some $D' \in \mathscr{D}$, $D' \subseteq D$ that avoids $\mathrm{set}(\bar{d})$. Now, for two suitable $k < k'$, we have $[\max(d_k), \min(d_{k'})) \cap D' \neq \emptyset$ and $[\max(d_k), \min(d_{k'})) \cap E' \neq \emptyset$. So $(w, \langle d_k, d_{k'}\rangle{}^\frown\bar{d} | [k'+1, \infty))$ is stronger than $(w, \bar{c})$ and it forces that $g_\alpha(E) \cap g_\alpha(D) \neq \emptyset$. Since this works for any two sets, $\mathscr{E}$ and $\mathscr{D}$ are nearly coherent by g_α.

Second: X_α diagonalises $\mathscr{A}_\alpha$, since by property (α), $(\bar{c}_\varepsilon, \mathrm{id}_\omega)$ is a witness over $\mathscr{A}_{\alpha_\varepsilon}$, and this is a complicated way to say that $\mathrm{set}(\bar{c}_\varepsilon)$ is a pseudo-intersection of $\mathscr{A}_{\alpha_\varepsilon}$. Since this holds for all $\varepsilon < \omega_1$, by genericity X_α diagonalises $\mathscr{A}_\alpha$.

Next we prove (P5) in the new cases, that is for soe $\beta \in \alpha \smallsetminus S^2_1$ and for $\alpha + 1$. So assume that $\beta < \alpha$, $(\bar{d}, \mathscr{R})$ is a witness over $\mathscr{A}_\beta$ in $\mathbf{V}_\beta$ and all later models. For some ε_0, we have $\beta < \alpha_{\varepsilon_0}$. By (γ)[1] we have some $\varepsilon \geq \varepsilon_0$ and some $\bar{d}' \sqsubseteq^* \bar{d}$ and some $\bar{c}' \sqsupseteq^* \bar{c}_\varepsilon$ such that $R_{\bar{d}', \bar{c}'} \circ \mathscr{R} \subseteq \mathscr{R}_\varepsilon$. Then, by the choice of $\mathscr{U}$ = the filter generated by $\mathrm{FU}(\bar{c}_\delta)$, $\delta < \aleph_1$, and of the forcing $\mathbb{M}(\mathscr{U})$, a density argument shows that in $\mathbf{V}_{\alpha+1}$, for all $R \in \mathscr{R}_\varepsilon$, the set $R(\mathrm{set}(\bar{c}_\varepsilon) \cap X_\alpha)$ is a pseudo-intersection of $\mathscr{A}_\alpha$.

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[1] Suppose we wanted to simplify (P5) to (P5)$'$: For $\beta < \gamma$, $\beta, \gamma \in \aleph_2 \smallsetminus S^2_1$, if $G_\gamma \subseteq \mathbb{P}_\gamma$ is generic over $\mathbf{V}$ and $G_\beta = \mathbb{P}_\beta \cap G_\gamma$, then

$$\text{if } \mathbf{V}_\beta \models \text{“}\mathrm{set}(\bar{c}) \text{ is a pseudo-intersection of } \mathscr{A}_\beta\text{”},$$
$$\text{then } \mathbf{V}_\gamma \models \text{“}(\exists \bar{d})(\bar{c} \leq^* \bar{d} \wedge \mathrm{set}(\bar{d}) \text{ is a pseudo-intersection of } \mathscr{A}_\gamma)\text{”},$$

and dispense with the complex requirement (γ). Then we would get stuck just at this point here, not knowing how to continue upwards for (P5)$'$ from an arbitrary $\bar{d}$ with $\mathrm{set}(\bar{d})$ being a pseudo-intersection of $\mathscr{A}_\beta$. Also, requirements (δ) to (ζ) lead to arbitrary $\bar{d}$. So some complexity is necessary, even if at first sight the various items of the list with the Greek letters are not so intertwined and it seems that we carry the original (P5) and (γ) with us only for having more technique. Something like (P5)$'$ is needed, because otherwise $\mathscr{A}_\alpha$ could become nearly ultra for some $\alpha \in S^2_1$ and by bad luck be in addition nearly coherent to S_α. Then we could not diagonalise $\mathscr{A}_\alpha$ and make S_α and fil($\mathscr{E}$) nearly coherent without destroying $\mathscr{E}$.

So $(\bar{c}_\varepsilon \cap X_\alpha, \mathscr{R}_\varepsilon)$ witnesses over $\mathscr{A}_\alpha$ and since by the definition of the forcing order of $\mathbb{M}(\mathscr{U})$ the set X_α splits only finitely many of the $c_{\varepsilon,n}$, that is, there is an infinite set Y such that X_α is almost the union over the $c_{\varepsilon,n}$, $n \in Y$, we get $\bar{c}_\varepsilon \cap X_\alpha \sqsubseteq^* \bar{c}'$. Since $R_{\bar{d}',\bar{c}'} \circ \mathscr{R} \subseteq \mathscr{R}_\varepsilon$, we have $(\bar{c}_\varepsilon \cap X_\alpha, R_{\bar{d}',\bar{c}'} \circ \mathscr{R})$ witnesses over $\mathscr{A}_\alpha$. Now see: For all X, $R_{\bar{d}',\bar{c}'}(X) \supseteq^* R_{\bar{d}',\bar{c}_\varepsilon}(X)$. So we have that

$$(\bar{c}_\varepsilon \cap X_\alpha, R_{\bar{d}',\bar{c}_\varepsilon} \circ \mathscr{R}) \text{ witnesses over } \mathscr{A}_\alpha. \tag{5.4}$$

We have $R_{\bar{d}',\bar{c}_\varepsilon}(\bar{d}' \cap R_{\bar{c}_\varepsilon,\bar{d}'}(X_\alpha)) =^* \bar{c}_\varepsilon \cap X_\alpha$, since X_α splits only finitely many $c_{\varepsilon,n}$. We may write $R_{\bar{d}' \cap R_{\bar{c}_\varepsilon,\bar{d}'}(X_\alpha), \bar{c}_\varepsilon \cap X_\alpha}$ for $R_{\bar{d}',\bar{c}_\varepsilon}$ in (5.4), as this is equivalent to it. Hence Lemma 4.4(5) fits literally and we get that $(\bar{d}' \cap R_{\bar{c}_\varepsilon,\bar{d}'}(X_\alpha), \mathscr{R})$ is a witness over $\mathscr{A}_\alpha = \mathscr{A}_{\alpha+1}$. Of course, $\bar{d}' \cap R_{\bar{c}_\varepsilon,\bar{d}'}(X_\alpha) \sqsubseteq^* \bar{d}$. □

Now we show that forcing with $\mathbb{P}_{\aleph_2}$ gives with the filter $\mathscr{A}$ generated by $\{A_\alpha : \alpha \in \aleph_1 \setminus S^2_1\}$ a counterexample to the filter dichotomy principle: If $f(\mathscr{A})$ were ultra, then f would appear in some intermediate step, say in $\mathbb{P}_{\alpha_0}$. By known reflection properties of countable support iterations of proper forcings, at an $\aleph_1$-club of later steps α we would have that $f(\mathscr{A}_\alpha)$ is ultra in $\mathbf{V}_\alpha$. Hence there is some $\alpha \in S^2_1$ such that the member $\underset{\sim}{S}_\alpha$ diamond sequence would guess this ultrafilter $S_\alpha = \mathscr{D} = f(\mathscr{A}_\alpha)$. But X_α diagonalises $\mathscr{A}_\alpha$ and hence $f(X_\alpha)$ diagonalises $f(\mathscr{A}_\alpha)$, and this contradicts the fact that as shown in the proof of Lemma 5.4 there is a finite-to-one $g_\alpha \in \mathbf{V}_{\alpha+1}$ coming from the inverse function of the enumeration of X_α, with $g_\alpha(f(\mathscr{A}_\alpha)) = g_\alpha(\mathscr{D}) = g_\alpha(\mathscr{E})$ being a P-point.

The filter $\mathscr{A}$ is not meagre, as $f_\alpha(A_\alpha) \neq^* \omega$ by (P3), and f_α, $\alpha < \aleph_2$, enumerates all finite-to-one functions in $\mathbf{V}_{\aleph_2}$.

By (P4) and the guessing strength of the diamond, all ultrafilters are nearly coherent to $\mathscr{E}$. So NCF holds in $\mathbf{V}_{\aleph_2}$.

So we have proved the Main Theorem.

6. The values of some cardinals in our models

For the definitions of the cardinal characteristics we refer the reader to [1] or [4]. The generic real r added by the Blass-Shelah forcing is not split by any real in the ground model [1, Lemma 7.4.25].

So in the type of models we get when increasing $\mathfrak{s}$ we have $\mathfrak{b} = \mathfrak{u} = \mathfrak{r} = \mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{g} = \aleph_1$ and $\mathfrak{s} = \mathrm{unif}(\mathcal{M}) = \mathrm{unif}(\mathcal{N}) = \mathfrak{d} = \aleph_2$. This follows from the well-known inequalities in Cichoń's diagramme and from Vojtăś' inequalities that $\mathfrak{r}$ is greater than or equal to both covering numbers [20] and its dual $\mathfrak{s}$ is less than or equal to both uniformities.

However, in the type of models built from Matet iterands $\mathbb{M}$ or Miller iterands in stages $\alpha \notin S^2_1$, we do not know the splitting number nor the uniformities.

We have in the iteration of $\mathbb{M}$ and $\mathbb{M}(\mathscr{U})$ $\mathfrak{b} = \mathfrak{u} = \mathfrak{r} = \mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{g} = \aleph_1$ and $\mathfrak{d} = \aleph_2$. From NCF and not FD it follows that in both kinds of models $\mathfrak{u} = \mathfrak{g} = \mathfrak{g}_f$ and $\mathfrak{mcf} = \aleph_2$. $\mathfrak{g}_f$ is the smallest number of groupwise dense ideals whose intersection is empty and $\mathfrak{mcf}$ is the minimal cofinality of the ultrapower $(\omega^\omega, \leq_{\mathscr{U}})$ for a non-principal ultrafilter $\mathscr{U}$. Note that Brendle constructed a model of $\kappa = \mathfrak{g} < \mathfrak{g}_f = \mathfrak{b} = \kappa^+$ by a c.c.c. forcing [8].

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