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Homogeneous almost disjoint families

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Abstract. An almost disjoint family is constructed which is isomorphic to any almost disjoint family which can be constructed from it by taking subsets and finite unions. This is applied to the construction of a Boolean algebra with related properties.

Introduction

The motivation for this paper stems from the still unsolved *Toronto Problem* which asks if there is a non-discrete Hausdorff topological space which is homeomorphic to every subspace of the same cardinality as the whole space. Clearly such a space must be of cardinality $\aleph_1$ and it can also be shown that any such space must be scattered of height ω_1. One can ask similar questions for different structures. In particular, R. Bonnet and M. Rubin, and independently A. Dow, have recently [1] shown that it is consistent that there is a *Toronto Boolean algebra* (that is, a Boolean algebra which is isomorphic to every one of its subalgebras of the same cardinality). This algebra, like a Toronto space, is necessarily superatomic. However, unlike the Toronto space problem, it is known that a Toronto Boolean algebra must be of height ω_1. The results of this paper will be shown to be relevant to this problem.

The general problem which is being considered can be thought of as an extension of the study of Jonsson algebras. To see this let E be an equivalence relation on structures. A structure A will be said to be E-Jonsson if and only if every substructure of A of the same cardinality as A is E equivalent to A. If E is equality then E-Jonsson algebras are nothing more than the familiar Jonsson algebras but if E is isomorphism, or homeomorphism in the case of topological spaces, then a broader definition is obtained. In this paper, the equivalence relation,

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$\cong$, to be considered can be defined as follows. Define two Boolean algebras A and B to be equivalent with respect to $\cong$ if and only if A is isomorphic to B and, moreover, the A has the same set of atoms as B.

While it is easy to see that there are no Jonsson Boolean algebras, it is equally easy to see that there is a $\cong$-Jonsson Boolean algebra – consider the algebra of finite and co-finite subsets of the integers. As in the Toronto space problem, the question arises of whether this is the only such algebra. It will be shown in this paper that the answer is consistently negative by constructing an uncountable $\simeq$-Jonsson Boolean algebra with only countably many atoms consistently with ZFC. This is the best one can hope for since $2^{\omega_1} > 2^{\omega}$ implies that there are no such algebras.

The method of proof will be to introduce iteratively isomorphisms for a generically constructed algebra. The difficulty will be in showing that the partial order is ccc. This will be done, not by showing that each factor is ccc, but rather, by showing that the eventual iteration is ccc.

Throughout this paper the term *almost disjoint family* will always refer to families of subsets of ω. Two almost disjoint families $\mathscr{A}$ and $\mathscr{B}$ will be called *isomorphic* if there is a bijection $\Phi : \omega \to \omega$ such that there is a bijection $\Phi^* : \mathscr{A} \to \mathscr{B}$ such that for every $A \in \mathscr{A}$ $\Phi''A \equiv^* \Phi^*(A)$ where $\equiv^*$ refers to equality modulo a finite set. For any almost disjoint family $\mathscr{A}$ let DISJ ($\mathscr{A}$) be a collection of almost disjoint families defined as follows: $\mathscr{B} \in$ DISJ ($\mathscr{A}$) if and only if there is a collection $\{x_\alpha : \alpha \in \omega_1\}$ of disjoint non-empty finite subsets of $\mathscr{A}$ such that $\mathscr{B} = \{\bigcup x_\alpha ; \alpha \in \omega_1\}$. An almost disjoint family $\mathscr{A}$ will be said to be *homogeneous* if and only if $\mathscr{A}$ is isomorphic to $\mathscr{B}$ for every $\mathscr{B} \in$ DISJ ($\mathscr{A}$).

Section 1

This section is devoted to constructing the parital order which will force, for a certain almost disjoint family $\mathscr{A}$, an isomorphism between $\mathscr{A}$ and $\mathscr{B}$ for any $\mathscr{B} \in$ DISJ ($\mathscr{A}$). This partial order will then be iterated for all instances of $\mathscr{B}$. Given almost disjoint families $\mathscr{A}$ and $\mathscr{B}$ and a bijection $\Psi : \mathscr{A} \to \mathscr{B}$ define the partial order $\mathbb{Q}(\mathscr{B}, \Psi)$ as follows: $p \in \mathbb{Q}(\mathscr{B}, \Psi)$ if and only if $p = (\psi, \Gamma)$ where

1. ψ is a finite partial function from ω to ω
2. Γ is a finite subset of $\mathscr{A}$
3. if x and y are distinct members of Γ then $x \cap y \subseteq \text{domain}(\psi)$ and $\Psi(x) \cap \Psi(x) \subseteq \text{range}(\psi)$
4. $|\text{A} \cap \text{domain}(\psi)| = \Sigma\{|\text{B} \cap \text{range}(\psi); \text{B} \in \Psi(\text{A})\}$ for $\text{A} \in \Gamma$.

Define $(\psi, \Gamma) \le (\phi, \Lambda)$ if and only if

5. $\psi \subseteq \phi$ and $\Gamma \subseteq \Lambda$
6. $(\forall n \in \text{domain}(\psi \setminus \phi))(\forall A \in \Gamma)(n \in A \leftrightarrow \psi(n) \in \Psi(A))$.

LEMMA 1. *The following sets are dense in* $\mathbb{Q}(\mathscr{B}, \Psi)$ *for any* $\mathscr{B}$ *and* Ψ:
(a) $D_n = \{(\psi, \Gamma) \in \mathbb{Q}(\mathscr{B}, \Psi);\ n \in \text{domain}(\psi)\}$
(b) $R_n = \{(\psi, \Gamma) \in \mathbb{Q}(\mathscr{B}, \Psi);\ n \in \text{range}(\psi))\}$
(c) $E_A = \{(\psi, \Gamma) \in \mathbb{Q}(\mathscr{B}, \Psi);\ A \in \Gamma\}$

Proof. Given $n \in \omega$ the only difficulty in extending ψ to include n its domain is that (6) must be satisfied. However note that by (3) there is at most one $A \in \Gamma$ such that $n \in A$. But it is easy to find $m \in (\Psi(A) \setminus \bigcup\{\Psi(B);\ B \in \Gamma \setminus \{A\}\}) \setminus \text{range}(\psi)$ and hence $(\psi \cup \{(n, m)\}, \Gamma) \ge (\psi, \Gamma)$ and $(\psi \cup \{(n, m)\}, \Gamma) \in D_n$. This proves that D_n is dense. A similar proof shows that R_n is dense.

For part (c) it suffices to show that:

7. for each $(\psi, \Gamma) \in \mathbb{Q}(\mathscr{B}, \Psi)$ and $A \notin \Gamma$ it is possible to find $(\psi_d, \Gamma) \ge (\psi, \Gamma)$ such that $|(A \cup \Psi(A)) \cap \text{domain}(\psi_d \setminus \psi)| = 1$ and $(\psi_r, \Gamma) \ge (\psi, \Gamma)$ such that $|(A \cup \Psi(A)) \cap \text{range}(\psi_r \setminus \Psi)| = 1$.

It now follows that E_A is dense for each $A \in \mathscr{A}$ because if $(\psi, \Gamma) \in \mathbb{Q}(\mathscr{B}, \Psi)$ is such that $A \notin \Gamma$ then it is possible to use (7) to find $\psi' \supseteq \psi$ so that $\text{domain}(\psi') \supseteq A \cap A'$ for every $A' \in \Gamma$. It is then possible to apply (7) once more to find $\psi'' \supseteq \psi'$ such that $|A \cap \text{domain}|(\psi'') = \Sigma\{|B \cap \text{range}(\psi'')|;\ B \in \Psi(A)\}$ and $(\psi'', \Gamma) \in \mathbb{Q}(\mathscr{B}, \Psi)$. It is then immediate that $(\psi'', \Gamma \cup \{A\}) \in \mathbb{Q}(\mathscr{B}, \Psi)$ and $(\psi'', \Gamma \cup \{A\}) \ge (\psi, \Gamma)$.

To prove (7) choose $m \in A \setminus \bigcup\{B \in \mathscr{A};\ B \in \Gamma\}$ and $k \in \omega \setminus \bigcup\{B \in \mathscr{B};\ B \in \Psi''\Gamma\}$. Let $\psi' = \psi \cup \{(m, k)\}$ and note that $(\psi', \Gamma) \in \mathbb{Q}(\mathscr{B}, \Psi)$ since (4) and (6) have not been violated by the choice of m and k. A similar proof establishes the second half of (7). □

It now follows that if G is $\mathbb{Q}(\mathscr{B}, \Psi)$ generic and $\psi_G = \bigcup\{\psi;\ (\psi, \Gamma) \in G\}$ then ψ_G is a permutation of ω which witnesses that $\mathscr{A}$ is isomorphic to $\mathscr{B}$. Moreover $\psi_G^* = \Psi$. The largest part of this paper will be devoted to proving that $\mathbb{Q}(\mathscr{B}, \Psi)$ satisfies the countable chain condition. The following lemma will be useful in doing this.

LEMMA 2. *Suppose that* (ψ, Γ) *and* (ψ, Λ) *are in* $\mathbb{Q}(\mathscr{B}, \Psi)$. *Then* (ψ, Γ) *and* (ψ, Λ) *are compatible if there is* $k \in \omega$ *and a bijection* $\varphi : \Gamma \to \Lambda$ *such that the following conditions are satisfied*:
(a) φ *is the identity when restricted to* $\Gamma \cap \Lambda$
(b) $A \cap \varphi(B) \subseteq k$ *for every* $\{A, B\} \in [\Gamma]^2$
(c) $\Psi(A) \cap \Psi(\varphi(B)) \subseteq k$ *for every* $\{A, B\} \in [\Gamma]^2$
(d) $A \cap k = \varphi(A) \cap k$ and $\Psi(A) \cap k = \Psi(\varphi(A)) \cap k$ *for all* $A \in \Gamma$
(e) $|A \cap \varphi(A) \setminus \text{domain}(\psi)| = |\Psi(A) \cap \Psi(|\varphi(A)) \setminus \text{range}(\psi)|$ *for* $A \in \Gamma$.

Proof. First note that $(\psi, \Gamma \cup \Lambda)$ satisfies conditions (1) and (2) in the definition of $\mathbb{Q}(\mathscr{B}, \Psi)$ but that ψ must be extended in order to satisfy (3). In particular ψ must be extended so that domain $(\psi) \supseteq \bigcup\{\bigcap a; a \in [\Gamma \cup \Lambda]^2\}$ and range $(\psi) \supseteq \bigcup\{\bigcap a; a \in [\Psi''(\Gamma \cup \Lambda)]^2\}$. From the facts that (ψ, Γ) and (ψ, Λ) are both in $\mathbb{Q}(\mathscr{B}, \Psi)$, and hence satisfy (3), and that (d) is satisfied it follows that it suffices to extend ψ so that domain $(\psi) \supseteq \bigcup\{A \cap \varphi(A); A \in \Gamma\}$ and range $(\psi) \supseteq \bigcup\{\Psi(A) \cap \Psi(\varphi(A)); A \in \Gamma\}$. Note that $\{A \cap \varphi(A)\backslash\text{domain}(\psi); A \in \Gamma\}$ and $\{\Psi(A) \cap \Psi(\varphi(A))\backslash\text{range}(\psi); A \in \Gamma\}$ are both disjoint families because, according to (3) of the definition of $\mathbb{Q}(\mathscr{B}, \Psi)$, $\{A\backslash\text{domain}(\psi); A \in \Gamma\}$ and $\{\Phi(A)\backslash\text{range}(\psi); A \in \Gamma\}$ are. But it now follows from (e) that it is easy to extend ψ to ψ' so that $(\psi', \Gamma \cup \Lambda)$ is in $\mathbb{Q}(\mathscr{B}, \Psi)$ and $(\psi', \Gamma \cup \Lambda) \geq (\psi, \Gamma)$ and $(\psi', \Gamma \cup \Lambda) \geq (\psi, \Lambda)$. □

Section 2

The almost disjoint family mentioned in Section 1 will be constructed generically. In particular let $\mathbb{A}$ be the partial order consisting of finite partial functions from $\omega \times \omega_1$ to 2. For $f \in \mathbb{A}$ define $\Delta(f) = \{\alpha \in \omega_1; (i, \alpha) \in \text{domain}(f)$ for some $i \in \omega\}$. If f and g are in $\mathbb{A}$ then define $f \geq g$ if and only if $f \supseteq g$ and for every $i \in \omega$ there is at most one $\alpha \in \omega_1$ such that $(i, \alpha) \in \text{domain}(f\backslash g)$ and $f(i, \alpha) = 1$. For $f \in \mathbb{A}$ define $A_f(\alpha) = \{i \in \omega; f(i, \alpha) = 1\}$ and if G is $\mathbb{A}$ generic define $A_G(\alpha) = \{i \in \omega; i \in A_f(\alpha)$ for some $f \in G\}$.

If G is $\mathbb{A}$ generic over V then it is possible to define a finite support iteration of partial orders $\mathbb{P}(\alpha)$ for $\alpha \in \omega_2$. Let $\mathscr{A} = \{A_G(\alpha); \alpha \in \omega_1\}$. If $\mathbb{P}(\alpha)$ has been defined, $\mathscr{B}_\alpha$ is a $\mathbb{P}(\alpha)$ name for a member of DISJ $(\mathscr{A})$ and Ψ_α is a $\mathbb{P}(\alpha)$ name for a bijection from $\mathscr{A}$ to $\mathscr{B}$ which have been chosen according to some scheme then $\mathbb{P}(\alpha + 1)$ is defined to be $\mathbb{P}(\alpha)*\mathbb{Q}(\mathscr{B}_\alpha, \Psi_\alpha)$. The choice of $\mathscr{B}_\alpha$ will be according to some plan which will ensure that all candidates will have been encountered by the end of the iteration. The choice of Ψ_α is a bit more complicated.

Let $F_\alpha : \omega_1 \to \alpha$ be a bijection. Let $\{M_\alpha^\tau : \tau \in \omega_1\}$ be a continuous increasing sequence of elementary submodels of

$$(H(\omega_2), \mathscr{A}, \mathscr{B}_\alpha, \{\mathbb{Q}(\mathscr{B}_\beta, \Psi_\beta); \beta \in \alpha\}, F_\alpha, \in).$$

Define $S_\alpha(x)$ to be the least τ such that $x \in M_\alpha^\tau$. For any bijection $\Phi : \mathscr{A} \to \mathscr{B}$, where $\mathscr{B} \in \text{DISJ}(\mathscr{A})$, define a relation $Z(\Phi)$ on ω_1 according to the rule: $Z(\Phi)(\xi, \zeta)$ holds if and only if $S_\alpha(A) = \xi$ and $S_\alpha(\Phi(A)) = \zeta$ for some $A \in \mathscr{A}$. The bijection $\Psi_\alpha : \mathscr{A} \to \mathscr{B}_\alpha$ is defined so that:

$$|\{\gamma \in \eta; Z(\Psi_\alpha)(\gamma, \eta)\}| \leq 1 \qquad \text{for every } \eta \in \omega_1 \tag{1}$$

$$0 < |\zeta \Delta \xi| < \aleph_0 \quad \text{if } Z(\Psi_\alpha)(\zeta, \xi). \tag{2}$$

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This is easily done by induction on ω_1. It will occasionally be more convenient to deal with the associated function $\Psi_\alpha^+ : \omega_1 \to [\omega_1]^{<\omega}$ defined by $\Psi_\alpha^+(\eta) = a$ if and only if $\Psi_\alpha(A_\eta) = \bigcup\{A_\nu ; \nu \in a\}$.

The remainder of this section is devoted to showing that $\mathbb{A}*\mathbb{P}(\omega_2)$ satisfies the countable chain condition. Suppose that $\{p_\alpha \in \mathbb{A}*\mathbb{P}(\omega_2); \alpha \in \omega_1\}$ are given. It may be assumed that all the conditions are determined in the sense that $(p_\alpha \mid \xi) \Vdash "p_\alpha(\xi) = (\psi_\alpha(\xi), \Gamma_\alpha(\xi))"$. Let $p_\alpha = (f_\alpha, q_\alpha)$ and let $\Sigma(\alpha) = \text{support}(q_\alpha)$. It may be assumed that $\{\Sigma(\alpha); \alpha \in \omega_1\}$ form a Δ-system with root Σ and that $\{\Delta(f_\alpha); \alpha \in \omega_1\}$ form a Δ-system with root Δ. By extending f_α it is also possible to assure that

$$\{\mu \in \omega_1; \mu \in (\Gamma_\alpha(\xi) \cup \Psi_\xi'' \Gamma_\alpha(\xi)) \text{ for some } \xi \in \Sigma(a)\} \supseteq \Delta(f_\alpha)$$

and that there is a fixed $k \in \omega$ such that domain $(f_\alpha) = k \times \Delta(f_\alpha)$ for each α. Finally it may be assumed that there is a bijection $I_{\nu,\mu} : \Delta(f_\nu) \times k \times \Sigma \to \Delta(f_\mu) \times k \times \Sigma$ which is an isomorphism of the structures $(\Delta(f_\nu), k, \Sigma, \{(\psi_\nu(\xi), \Gamma_\nu(\xi)); \xi \in \Sigma\})$ and $(\Delta(f_\mu), k, \Sigma, \{(\psi_\mu(\xi), \Gamma_\mu(\xi)); \xi \in \Sigma\})$ and which is the identity on $\Delta \times k \times \Sigma$.

Now choose $\gamma \in \omega_1$ such that if $\xi \in \zeta$ and $\{\zeta, \xi\} \subseteq \Sigma$ then $S_\zeta^{-1}\xi \subseteq \gamma$. Next choose α and β in ω_1 such that $(\Delta(f_\alpha)\setminus\Delta) \cup (\Delta(f_\beta)\setminus\Delta) \cap \gamma = 0$. Notice that this implies that

$$\text{If } \{\xi, \zeta\} \subseteq \Sigma \text{ and } \xi \in \zeta \text{ and } \gamma \in M_\zeta^\xi \cap \omega_1 \text{ then } M_\zeta^\xi \cap \omega_1 \supseteq M_\xi^\vartheta \cap \omega_1 \text{ for } \vartheta \in \tau. \tag{3}$$

Note that if $f' = f_\alpha \cup f_\beta$ then $f' \in \mathbb{A}$ and $f' \geq f_\alpha$ and $f' \geq f_\beta$. To show that p_α and p_β are compatible it suffices to show that there is $f \supseteq f'$ (not $f \geq f'$) such that:

$$f \Vdash "|A_\zeta \cap A_{I_{\alpha,\beta}(\zeta)} \setminus \text{domain}(\psi_\xi)| = |\bigcup\{A_\vartheta ; \vartheta \in \Psi_\xi^+(\zeta)\} \cap (\bigcup\{A_\vartheta ; \vartheta \in \Psi_\xi^+(I_{\alpha,\beta}(\zeta))\}\setminus \text{range}(\psi_\xi)| \quad \text{for } \xi \in \Sigma \text{ and } \zeta \in \Delta(f_\alpha) \tag{4}$$

$$f \geq f_\alpha \quad \text{and} \quad f \geq f_\beta. \tag{5}$$

The reason that this suffices is that the isomorphism $I_{\alpha,\beta}$ ensures that (a) and (d) of Lemma 2 are satisfied while f' forces that (b) and (c) will be satisfied because of (5).

To find f suffices to find positive integers z_ζ satisfying the following system of equations:

$$\{z_\zeta - d_\zeta(\xi) = \Sigma\{z_\nu - r_\zeta(\xi); \nu \in \Psi_\xi^+(\zeta)\}; \zeta \in \Delta(f_\alpha) \text{ and } \xi \in \Sigma\} \tag{6}$$

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where $d_\zeta(\xi) = |A_\zeta \backslash \text{domain}\,(\psi_\alpha(\xi))|$ and $r_\zeta(\xi) = |A_\zeta \backslash \text{range}\,(\psi_\alpha(\xi))|$. To see this let $\{J(\zeta); \zeta \in \Delta(f_\alpha)\}$ be disjoint subsets of $\omega \backslash k$ such that $|J(\zeta)| = z_\zeta$. Let $J = \bigcup\{J(\zeta); \zeta \in \Delta(f_\alpha\}$ and define f so that domain $(f) = \text{domain}\,(f') \cup (J \times (\Delta(f_\alpha) \cup \Delta(f_\beta)))$ and if $i \geq k$ then $f(i, \zeta) = 1$ if and only if $i \in J(\zeta)$ or $i \in J(I_{\alpha,\beta}(\zeta))$. It is easily checked that f is as required.

To prove that it is possible to solve the system of equations note that the equation $z_\zeta - d_\zeta(\xi) = \Sigma\{z_v - r_\zeta(\xi); v \in \Psi_\xi^+(\zeta)\}$ is equivalent to the equation

$$\begin{aligned} z_\zeta &- (|A_\zeta| - |A_\zeta \cap \text{domain}\,(\psi_\alpha(\xi))|) \\ &= \Sigma\{z_v - (|A_v| - |A_v \cap \text{range}\,(\psi_\alpha(\xi))|); v \in \Psi_\xi^+(\zeta)\} \end{aligned}$$

which, in turn, by (1.4) is equivalent to $z_\zeta - |A_\zeta| = \Sigma\{z_v - |A_v|; v \in \Psi_\xi^+(\zeta)\}$. Consequently, the system of equations (6) is equivalent to

$$\{z_\zeta - |A_\zeta| = \Sigma\{z_v - |A_v|; v \in \Psi_\xi^+(\zeta)\}; \zeta \in \Delta(f_\alpha) \text{ and } \xi \in \Sigma\}. \tag{7}$$

Clearly solving (7) with positive integers is equivalent to solving the homogeneous equations

$$\{z_\zeta = \Sigma\{z_v; v \in \Psi_\xi^+(\zeta)\}; \zeta \in \Delta(f_\alpha) \text{ and } \xi \in \Sigma\} \tag{8}$$

with nonzero rational solutions.

To find non-zero rational solutions for (8) proceed by induction on $|\Sigma|$. Assume that the assertion is true when $|\Sigma| \leq k$ and that $|\Sigma| = k + 1$. Let μ be that largest member of Σ and let $\{z_\zeta; \zeta \in \Delta(f_\alpha)\}$ be non-zero rational solutions for $\{z_\zeta = \Sigma\{z_v; v \in \Psi_\xi^+(\zeta)\}; \zeta \in \Delta(f_\alpha) \text{ and } \xi \in \Sigma \backslash \{\mu\}\}$. Now proceed by induction on τ to solve the family of equations

$$\begin{aligned} &\{z_\zeta = \Sigma\{z_v; v \in \Psi_\xi^+(\zeta)\}; \zeta \in \Delta(f_\alpha) \text{ and } \xi \in \Sigma \backslash \{\mu\}\} \\ &\quad \cup \{z_\zeta = \Sigma\{z_v; v \in \Psi_\mu^+(\zeta)\}; \zeta \in \Delta(f_\alpha) \cap M_\mu^\tau \text{ and } \Psi_\mu^+(\zeta) \subseteq M_\mu^\tau\}. \end{aligned} \tag{9}$$

Suppose that the family of equations (9) has been solved and that ϑ is the first ordinal bigger than τ such that the family of equations

$$\begin{aligned} &\{z_\zeta = \Sigma\{z_v; v \in \Psi_\xi^+(\zeta)\}; \zeta \in \Delta(f_\alpha) \text{ and } \xi \in \Sigma \backslash \{\mu\}\} \\ &\quad \cup \{z_\zeta = \Sigma\{z_v; v \in \Psi_\mu^+(\zeta)\}; \zeta \in \Delta(f_\alpha) \cap M_\mu^\vartheta \text{ and } \Psi_\mu^+(\zeta) \subseteq M_\mu^\vartheta\} \end{aligned} \tag{10}$$

contains a new equation. Property (2.1) of $Z(\Psi_\mu)$ ensures that (10) contains precisely one more equation than (9). Moreover (3) implies that

$$\{z_\zeta = \Sigma\{z_\nu ; \nu \in \Psi^+_\xi(\zeta)\}; \zeta \in \Delta(f_\alpha) \text{ and } \xi \in \Sigma\setminus\{\mu\}\}$$
$$= \{z_\zeta = \Sigma\{z_\nu ; \nu \in \Psi^+_\mu(\zeta)\}; \zeta \in \Delta(f_\alpha) \cap \mathrm{M}^9_\mu \text{ and } \Psi^+_\mu(\zeta) \subseteq \mathrm{M}^9_\mu\}$$
$$\cup \{z_\zeta = \Sigma\{z_\nu ; \nu \in \Psi^+_\mu(\zeta)\}; \zeta \notin \Delta(f_\alpha) \cap \mathrm{M}^9_\mu \text{ and } \Psi^+_\mu(\zeta) \cap \mathrm{M}^9_\mu = 0\}. \qquad (11)$$

Hence the equations in (9) can be divided into two disjoint sets with no variable appearing in both sets. Hence multiplying these (non-zero) solutions by appropriate constants will yield a solution to (10). This completes the proof of the countable chain condition and proves the following.

THEOREM 1. *It is consistent with ZFC that there is a homogeneous almost disjoint family of size* $\aleph_1$.

Section 3

In this section it will be shown how to construct a Boolean algebra which is somewhat like a Toronto Boolean algebra but has small height. The following definition will clarify what is meant by this. If $\mathfrak{B}$ is a Boolean algebra then define at $(\mathfrak{B})$ to be the ideal generated by the atoms of $\mathfrak{B}$. It is now possible to define a sequence of quotient algebras in a fashion similar to the construction of derived subspaces of a topological space. Let $\mathfrak{B}^0 = \mathfrak{B}$. Let $\mathfrak{J}^\alpha = \bigcup\{\mathfrak{J}^\beta ; \beta \in \alpha\}$ and define $\mathfrak{B}^{\alpha+1} = \mathfrak{B}/(\mathfrak{J}^\alpha \cup \mathrm{at}\,(\mathfrak{B}^\alpha))$. If α is a limit ordinal then let $\mathfrak{B}^\alpha = \mathfrak{B}/\mathfrak{J}^\alpha$. The *height of* $\mathfrak{B}$ is defined to be the least ordinal α such that $\mathfrak{B}^\alpha = \mathfrak{B}^{\alpha+1}$.

Let $\mathscr{A}$ be the homogeneous almost disjoint family of subsets of ω of size $\aleph_1$ such that for each $n \in \omega$ there is a finite subset of $\mathscr{A}$ whose intersection is $\{n\}$ – trivial modifications to any almost disjoint family always make it possible to assume that this is the case. Let $\mathfrak{B}(\mathscr{A})$ be the subalgebra of $P(\omega)$ generated by $\mathscr{A}$. Clearly $\mathfrak{B}$ is a Boolean algebra of height 2 and cardinality $\aleph_0 + |\mathscr{A}|$. Note that at $(\mathfrak{B}(\mathscr{A})) = \omega$. Now let $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}(\mathscr{A})$. For each $A \in \mathfrak{A}$ there is a finite subset $a(\mathrm{A}, \mathscr{A})$ of $\mathscr{A}$ such that $\mathrm{A} \equiv^* \bigcup a(\mathrm{A}, \mathscr{A})$ or $\mathrm{A}\setminus\omega \equiv^* \bigcup a(\mathrm{A}, \mathscr{A})$. Let

$$\mathrm{A}^+(\mathfrak{A}) = \begin{cases} \mathrm{A} & \text{if } \mathrm{A} \equiv^* \bigcup a(\mathrm{A}, \mathscr{A}) \\ \omega\setminus\mathrm{A} & \text{if } \omega\setminus\mathrm{A} \equiv^* \bigcup a(\mathrm{A}, \mathscr{A}). \end{cases}$$

Call $\mathrm{A} \in \mathfrak{A}$ minimal for $\mathfrak{A}$ if A is infinite and for every $\mathrm{B} \in \mathfrak{A}$, if $\mathrm{B} \subseteq \mathrm{A}$ then either B is finite or $\mathrm{A}\Delta\mathrm{B}$ is finite.

THEOREM 2. *If there is a homogeneous almost disjoint family of subsets of* ω *of size* $\aleph_1$ *then there is a Boolean algebra* $\mathfrak{B}$ *of height* 2 *and cardinality* $\aleph_1$ *such that if* $\mathfrak{U} \subseteq \mathfrak{B}$ *is any subalgebra of* $\mathfrak{B}$ *containing* at $(\mathfrak{B})$ *then* $\mathfrak{A}$ *is isomorphic to* $\mathfrak{B}$.

Proof. Let $\mathscr{A}$ be a homogeneous almost disjoint family of size $\aleph_1$ and let $\mathfrak{B} = \mathfrak{B}(\mathscr{A})$. It suffices to show that if $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$ containing ω then $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$. For each $A \in \mathfrak{A}$ such that $A^+(\mathfrak{A}) = A$ there is $B \subseteq A$ such that B is minimal. Hence it is possible to find a family $\mathscr{D}$ such that if B and B′ are distinct members of $\mathscr{D}$ then $B \not\equiv^* B'$ and, furthermore, for every $A \in \mathfrak{A}$ there is $B \in \mathscr{D}$ such that $B \subseteq^* A^+(\mathfrak{A})$. Then $\{a(B); B \in \mathscr{D}\} \in \mathrm{DISJ}\,(\mathscr{A})$ and consequently $\{a(B); B \in \mathscr{D}\}$ and $\mathscr{A}$ are isomorphic. Since $\mathfrak{A}$ contains ω it follows that if $A \in \mathfrak{A}$ and $A' \equiv^* A$ then $A' \in \mathfrak{A}$. Hence the isomorphism between $\{a(B); B \in \mathscr{D}\}$ and $\mathscr{A}$ induces an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$.

For an almost disjoint family $\mathscr{A}$ and $X \subseteq \omega$ let $\mathscr{A}(X) = \{A \cap X; A \in \mathscr{A}\}$. An almost disjoint family will be called *hereditary* if $\mathscr{A}$ is isomorphic to $\mathscr{A}(X)$ whenever $|\mathscr{A}| = |\mathscr{A}(X)|$ – the isomorphism going from X to ω. This definition is relevant to the present discussion because of the following fact.

THEOREM 3. *If there is a homogeneous hereditary almost disjoint family of subsets of* ω *of size* $\aleph_1$ *then there is a Toronto boolean algebra of height* 2.

Proof. Let $\mathscr{A}$ be a homogeneous hereditary almost disjoint family of size $\aleph_1$ and let $\mathfrak{B} = \mathfrak{B}(\mathscr{A})$. Let $\mathfrak{A}'$ be an uncountable subalgebra of $\mathfrak{B}$. Let $X' = \bigcup \mathrm{at}\,(\mathfrak{A}')$ and then choose $X \subseteq X'$ such that $|a \cap X| = 1$ for each $a \in \mathrm{at}\,(\mathfrak{A}')$. Let $\mathfrak{A}$ be the subalgebra of $\mathfrak{B}$ generated by $X \cup \{A \in \mathfrak{A}'; A \text{ is infinite}\}$ and note that $\mathfrak{A}$ is isomorphic to $\mathfrak{A}'$.

Using the fact that $\mathscr{A}$ is hereditary it follows that $\mathscr{A}(X)$ is isomorphic to $\mathscr{A}$ and hence $\mathfrak{B}$ is isomorphic to $\mathfrak{B}(X)$ = the algebra generated by $X \cup \{B \in \mathfrak{B}; B \cap X \text{ is infinite}\}$. Then use the fact that $\mathscr{A}$, and hence $\mathscr{A}(X)$, is homogeneous together with the fact that $\mathscr{A}(X)$ and $\mathfrak{A}$ satisfy the hypothesis of Theorem 2 to establish that $\mathfrak{B}(X)$ is isomorphic to $\mathfrak{A}$. This completes the proof. □

The question of whether it is consistent that there is a homogeneous hereditary almost disjoint family of size $\aleph_1$ remains open.

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