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Homogeneity of Infinite Permutation Groups

Saharon Shelah[1,2] and Simon Thomas[2]

[1] Institute of Mathematics, The Hebrew University, Jerusalem, Israel
[2] Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

1. Introduction

Let Ω be an infinite set and λ be a (possibly finite) cardinal such that $\lambda < |\Omega|$. A group G of permutations of Ω is said to be λ-homogeneous if for each pair A, B of subsets of Ω with $|A| = |B| = \lambda$, there exists an element $g \in G$ such that $g[A] = B$. It has been known for some time that if the permutation group G is m-homogeneous for some $m \in \omega$, then G is also n-homogeneous for all $n < m$ (see [1–3, 6]). More recently, Neumann [4] has shown that if G is λ-homogeneous for some $\lambda \geqq \omega$, then G is n-homogeneous for all $n \in \omega$, (Neumann actually obtained the stronger result that G is n-transitive for all $n \in \omega$.) In [4], Neumann asked whether λ-homogeneity implies θ-homogeneity for all cardinals such that $\lambda > \theta$. In this paper, we shall show that this does not hold, assuming Martin's Axiom (MA) and $2^{\omega} > \omega_2$.

Theorem 1.1 (MA and $2^{\omega} > \omega_2$). *Let $\omega_1 < \kappa < 2^{\omega}$. Then there exists a group G of permutations of κ with the following properties.*

(a) *G is λ-homogeneous for all $\omega < \lambda < \kappa$.*

(b) *If $g \in G \setminus 1$, then $|g[\omega] \cap \omega| < \omega$. In particular, since G preserves the almost disjoint family $A = \{g[\omega] \mid g \in G\}$, G is not ω-homogeneous.*

Nikos [5] has independently given another example of an ω_1-homogeneous but not ω-homogeneous permutation group. However, the following problem is still open.

Question 1.2. It is consistent with **ZFC** that every ω_1-homogeneous permutation group is ω-homogeneous?

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Offprint requests to: S. Thomas

2. The Proof of the Theorem

Throughout we assume MA and $\omega_1 < \kappa < 2^\omega$. We begin with two combinatorial lemmas, the first of which is an easy consequence of MA.

Lemma 2.1 (MA). *Let $\mathcal{I}$ be an almost disjoint family of infinite subsets of ω such that $\omega \leqq |\mathcal{I}| < 2^\omega$. Then there exists a partition, $\omega = \bigcup_{n\in\omega} D_n$, such that*

(a) *$|D_n| = \omega$ for all $n \in \omega$; and*
(b) *for each $n \in \omega$ and $Y \in \mathcal{I}$, $|Y \cap D_n| < \omega$.* □

Lemma 2.2 (MA). *Let $|X| = \theta > \omega$ and let $\mathcal{I}$ be an almost disjoint family of countable subsets of X such that $|\mathcal{I}| < 2^\omega$. Then there exists a partition, $X = \bigcup_{\alpha<\theta} D_\alpha$, such that*

(a) *$|D_\alpha| = \omega$ for all $\alpha < \theta$; and*
(b) *for each $\alpha < \theta$ and $Y \in \mathcal{I}$, $|Y \cap D_\alpha| < \omega$.*

Proof. It is easily seen that there is a partition, $X = \bigcup_{\alpha<\theta} S_\alpha$, such that for each $\alpha < \theta$

(i) $|S_\alpha| = \omega$; and
(ii) the set $\mathcal{I}_\alpha = \{Y \cap S_\alpha \mid Y \in \mathcal{I}, |Y \cap S_\alpha| = \omega\}$

is either empty or infinite.

By applying Lemma 2.1 to those S_α such that $\mathcal{I}_\alpha \neq \phi$, we can obtain a partition with the desired properties. □

Now we start the construction of the permutation group $G \leqq \mathrm{Sym}(\kappa)$. [Here $\mathrm{Sym}(\kappa)$ denotes the group of all permutations of κ.] By MA, $\kappa^\lambda = 2^\omega$ for all $\omega < \lambda < \kappa$. Let $\{X_\alpha \mid \alpha < 2^\omega\}$ be an enumeration of the subsets $X \subsetneqq \kappa$ such that $\omega < |X| < \kappa$. We shall define inductively an increasing chain of groups $G_\alpha \leqq \mathrm{Sym}(\kappa)$, $\alpha < 2^\omega$, so that the following conditions are satisfied.

2.3. $|G_\alpha| \leqq |\alpha| + \omega$ for all $\alpha < 2^\omega$.

2.4. If $\beta < \alpha$ and $|X_\beta| = \lambda$, then there exists $\pi \in G_\alpha$ such that $\pi[\lambda] = X_\beta$.

2.5. If $g \in G_\alpha \setminus 1$, then $|g[\omega] \cap \omega| < \omega$.

Let $G_0 = 1$. If α is a limit ordinal, let $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. Now suppose that $\alpha = \beta + 1$. We shall find a suitable permutation $\pi \in \mathrm{Sym}(\kappa)$ so that $G_\alpha = \langle G_\beta, \pi\rangle$ satisfies our requirements.

Let $\lambda = \bigcup_{\xi<\lambda} C_\xi$ be a partition into countably infinite subsets such that for each $\xi < \lambda$ and $g \in G_\beta$, $|g[\omega] \cap C_\xi| < \omega$. (By 2.5, $\{g[\omega] \mid g \in G_\beta\}$ is an almost disjoint family, and so the existence of such a partition follows from Lemma 2.2.) Let $X_\beta = \bigcup_{\xi<\lambda} D_\xi$, $\kappa \setminus \lambda = \bigcup_{\eta<\kappa} E_\eta$ and $\kappa \setminus X_\beta = \bigcup_{\eta<\kappa} F_\eta$ be partitions with the same property. For each $\xi < \lambda$ and $\eta < \kappa$, let $\mathbb{P}^0_\xi$, $\mathbb{P}^1_\eta$ be the sets of finite injective functions from C_ξ to D_ξ, E_η to F_η respectively. Let $\mathbb{P} = \left(\prod_{\xi<\lambda} \mathbb{P}^0_\xi\right) \times \left(\prod_{\eta<\kappa} \mathbb{P}^1_\eta\right)$ be the finite support product. We regard each element $p \in \mathbb{P}$ as a finite function from κ to κ. With this convention,

$q \leqq p$ if and only if $q \supseteqq p$. Let W be the set of reduced words from the free product $G^*_\beta\langle x\rangle$ which involve x. If $w(\bar{g}, x) \in W$ and $p \in \mathbb{P}$, then we regard $w(\bar{g}, p)$ as a partial function from κ to κ in the obvious way. Note that there are only finitely many $\gamma \in \kappa$ such that $w(\bar{g}, p)(\gamma)$ is defined. If $A \subseteq \kappa$, then $w(\bar{g}, p)[A] = \{w(\bar{g}, p)(\gamma) \mid \gamma \in A, w(\bar{g}, p)(\gamma)$ is defined$\}$.

Let $\mathbb{Q}$ be the poset consisting of elements of the form $\langle p, \mathscr{F}\rangle$, where $p \in \mathbb{P}$ and $\mathscr{F}$ is a finite subset of W. We define $\langle q, \mathscr{G}\rangle \leqq \langle p, \mathscr{F}\rangle$ if and only if

2.6. $q \supseteqq p$ and $\mathscr{G} \supseteqq \mathscr{F}$, and

2.7. for each $w(\bar{g}, x) \in \mathscr{F}$, $w(\bar{g}, q)[\omega] \cap \omega = w(\bar{g}, p)[\omega] \cap \omega$.

Lemma 2.8. *For each $w(\bar{g}, x) \in W$, the set $\{\langle q, \mathscr{F}\rangle \mid w(\bar{g}, x) \in \mathscr{F}\}$ is dense in $\mathbb{Q}$.*

Proof. Clearly $\langle q, \mathscr{F} \cup \{w(\bar{g}, x)\}\rangle \leqq \langle q, \mathscr{F}\rangle$. □

Lemma 2.9. *For each $\gamma \in \kappa$, the set $\{\langle q, \mathscr{F}\rangle \mid \gamma \in \operatorname{dom} q\}$ is dense in $\mathbb{Q}$.*

Proof. Let $\langle p, \mathscr{F}\rangle \in \mathbb{Q}$ with $\gamma \notin \operatorname{dom} p$. Suppose, for example, that $\gamma \in C_\xi$. Let $w(\bar{g}, x) = g_1 x^{n_1} g_2 \dots g_r x^{n_r} g_{r+1} \in \mathscr{F}$, where $g_i \in G_\beta$ for $1 \leqq i \leqq r+1$ and $n_j \in \mathbb{Z} \setminus 0$ for $1 \leqq j \leqq r$. For each $\delta \in D_\xi \setminus \operatorname{ran} p$, let $q_\delta = p \cup \{\langle \gamma, \delta\rangle\}$. Suppose that for infinitely many $\delta \in D_\xi \setminus \operatorname{ran} p$, there exists $a_\delta \in \omega$ such that $w(\bar{g}, q_\delta)(a_\delta) \in \omega \setminus w(\bar{g}, p)[\omega]$.

We can suppose that the new information $\langle \gamma, \delta\rangle$ is first used at the same place in the word $w(\bar{g}, x)$ during each computation of $w(\bar{g}, q_\delta)(a_\delta)$.

First suppose that infinitely many distinct a_δ are used. Then we must have that $g_{r+1}(a_\delta) = \delta$ and $n_r < 0$. But this means that $g_{r+1}[\omega] \cap D_\xi$ is infinite, which is a contradiction.

Hence we can suppose that we always use the same element $a \in \omega$ as a_δ. Suppose that the new information $\langle \gamma, \delta\rangle$ is first used after computing

$$\sigma = g_{i+1} p^{n_{i+1}} \dots g_r p^{n_r} g_{r+1}(a).$$

Then we must have that $\sigma = \gamma$ and $n_i > 0$. A moment's thought shows that actually $n_i = 1$. In order that each computation terminate, we must have that $i = 1$. But this means that $g_1[D_\xi] \cap \omega$, and hence also $D_\xi \cap g_1^{-1}[\omega]$, is infinite. This is a contradiction.

Hence there is a cofinite subset $D \subseteq D_\xi$ such that $\langle p \cup \{\langle \gamma, \delta\rangle\}, \mathscr{F}\rangle \leqq \langle p, \mathscr{F}\rangle$ for all $\delta \in D$. □

A similar argument yields

Lemma 2.10. *For each $\gamma \in \kappa$, the set $\{\langle q, \mathscr{F}\rangle \mid \gamma \in \operatorname{ran} q\}$ is dense in $\mathbb{Q}$.* □

Lemma 2.11. $\mathbb{Q}$ *is c.c.c.*

Proof. Suppose not, and let $\{p_i, \mathscr{F}_i\rangle \mid i < \omega_1\}$ be pairwise incompatible. We can suppose that the following conditions hold.

2.12. For all $i < j < \omega_1$, $|\mathscr{F}_i| = |\mathscr{F}_j| = n$ and $|\operatorname{dom} p_i| = |\operatorname{dom} p_j| = m$.

2.13. The partial functions $\{p_i \mid i < \omega_1\}$ are pairwise compatible in $\mathbb{P}$. There exists a fixed element $f \in \mathbb{P}$ such that $p_i \cap p_j = f$ for all $i < j < \omega_1$.

Let $i<\omega_1$ and let $w(\bar{g}, x) \in \mathscr{F}_i$. Let H be the subgroup of G_β generated by the parameters $\bar{g}$. Then there exist countable subsets $A \subseteq \lambda$ and $B \subseteq \kappa$ such that

(i) $\omega \subseteq S = \bigcup_{\xi \in A} C_\xi \cup \bigcup_{\eta \in B} E_\eta = \bigcup_{\xi \in A} D_\xi \cup \bigcup_{\eta \in B} F_\eta$; and

(ii) $H[S] = S$.

Suppose that $w(\bar{g}, p_i \cup p_j)[\omega] \cap \omega \supsetneqq w(\bar{g}, p_i)[\omega] \cap \omega$. Then clearly $(\operatorname{dom} p_j \backslash \operatorname{dom} f) \cap S \neq \phi$. Since S is countable, 2.13 implies that there are only countably many $j \neq i$ for which this occurs. So we can suppose that if $i<j<\omega_1$ and $w(\bar{g}, x) \in \mathscr{F}_i$, then $w(\bar{g}, p_i \cup p_j)[\omega] = w(\bar{g}, p_j)[\omega] \cap \omega$.

We then obtain the following condition.

2.14. If $i<j<\omega_1$, then there exist $w(\bar{g}, x) \in \mathscr{F}_j$ and $a_i \in \omega$ such that

$$w(\bar{g}, p_i \cup p_j)(a_i) \in \omega \backslash w(\bar{g}, p_j)[\omega].$$

Fix a particular $j \geqq \omega$, and suppose that there are infinitely many $i<j$ for which 2.14 holds with respect to

$$w(\bar{g}, x) = g_1 x^{n_1} g_2 \ldots g_r x^{n_r} g_{r+1} \in \mathscr{F}_j.$$

Suppose that for two such i, $k<j$, we have that $a_i = a_k = a$. Clearly the new information in $p_i \cap p_j$, $p_k \cup p_j$ is first used at the same place in the word $w(\bar{g}, x)$ during the computations of $w(\bar{g}, p_i \cup p_j)(a)$, $w(\bar{g}, p_k \cup p_j)(a)$ respectively. But this means that p_i, p_k are incompatible, a contradiction. Hence for all but finitely many of these i, p_i is used to evaluate $x^\varepsilon g_{r+1}(a_i)$, where $\varepsilon = \pm 1$. If $\varepsilon = 1$, then

$$(\operatorname{dom} p_i \backslash \operatorname{dom} f) \cap g_{r+1}[\omega] \neq \phi;$$

while if $\varepsilon = -1$, then

$$(\operatorname{ran} p_i \backslash \operatorname{ran} f) \cap g_{r+1}[\omega] \neq \phi.$$

Hence we obtain the following condition.

2.15. If $\omega \leqq j < \omega_1$, then there exist $h_1, \ldots, h_s \in G_\beta$ and $\varepsilon_1, \ldots, \varepsilon_s \in \{1, -1\}$ for some $s \leqq n$, and a partition, $j = F \cup B^j_{\varepsilon_1}(h_1) \cup \ldots \cup B^j_{\varepsilon_s}(h_s)$, such that

(i) F is finite, and $B^j_{\varepsilon_k}(h_k)$ is infinite for $1 \leqq k \leqq s$;

(ii) if $\varepsilon_k = 1$ and $i \in B^j_{\varepsilon_k}(h_k)$, then $(\operatorname{dom} p_i \backslash \operatorname{dom} f) \cap h_k[\omega] \neq \phi$;

(iii) if $\varepsilon_k = -1$ and $i \in B^j_{\varepsilon_k}(h_k)$, then $(\operatorname{ran} p_i \backslash \operatorname{ran} f) \cap h_k[\omega] \neq \phi$.

Claim 2.16. *Suppose that $C \subseteq \omega_1$ is countably infinite. Then there exists an infinite subset $S \subseteq C$ and an ordinal $\sup(S) < \sigma_s < \omega_1$ such that if $\sigma_s < j < \omega_1$, $g \in G_\beta$, $\varepsilon = \pm 1$ and $|B^j_\varepsilon(g) \cap S| = \omega$, then $B^j_\varepsilon(g) \subseteq \sigma_s$.*

Proof of claim. Let $C \subseteq \omega_1$ be countable, and suppose that there exist distinct g_ℓ, $1 \leqq \ell \leqq t$, such that for some $j_\ell > \sup(C)$, $1 \leqq \ell \leqq t$,

$$|C \cap B^{j_1}_1(g_1) \cap \ldots \cap B^{j_t}_1(g_t)| = \omega.$$

Since $\{g[\omega] \mid g \in G_\beta\}$ is an almost disjoint family, this implies that $|\operatorname{dom} p_i \backslash \operatorname{dom} f| \geqq t$ for $i < \omega_1$. Hence there exists an infinite subset $S \subseteq C$ and a finite subset $\{g_\ell | 1 \leqq \ell \leqq m_1\} \subseteq G_\beta$ such that if $j > \sup(C)$, $g \in G_\beta$ and $|B_1^j(g) \cap S| = \omega$, then $g = g_\ell$ for some $1 \leqq \ell \leqq m_1$. By shrinking S if necessary, we may also suppose that there are elements $\{h_\ell | 1 \leqq \ell \leqq m_2\}$ such that if $j > \sup(C)$, $h \in G_\beta$ and $|B_{-1}^j(h) \cap S| = \omega$, then $h = h_\ell$ for some $1 \leqq \ell \leqq m_2$. Hence there exists an ordinal $\sup(S) < \sigma_s < \omega_1$ such that if $\sigma_s < j < \omega_1$, $g \in G_\beta$, $\varepsilon = \pm 1$ and $|B_\varepsilon^j(g) \cap S| = \omega$, then $B_\varepsilon^j(g) \subseteq \sigma_s$. This completes the proof of Claim 2.16.

Set $\sigma_0 = 0$. Using Claim 2.16, we can find ordinals $\sigma_0 < \sigma_1 < \ldots < \sigma_{n+1} < \omega_1$ and countably infinite subsets $S_i \subseteq \sigma_i \setminus \sigma_{i-1}$, $1 \leqq i \leqq n+1$, such that if $j > \sigma_{n+1}$, $g \in G_\beta$, $\varepsilon = \pm 1$ and $|B_\varepsilon^j(g) \cap S_i| = \omega$, then $B_\varepsilon^j(g) \subseteq \sigma_i$. But this clearly contradicts 2.15. Hence $\mathbb{Q}$ is c.c.c. $\square$

By MA, there exists an element $\pi \in \mathrm{Sym}(\kappa)$ such that $G_\alpha = \langle G_\alpha, \pi \rangle$ satisfies 2.3–2.5. Hence $G = \bigcup_{\alpha < 2^\omega} G_\alpha$ satisfies the requirements of Theorem 1.1.

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