

GROUPS ISOMORPHIC TO ALL THEIR  
NON-TRIVIAL NORMAL SUBGROUPS\*

BY

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ABSTRACT

In answer to a question of P. Hall, we supply another construction of a group which is isomorphic to each of its non-trivial normal subgroups.

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## 1. Introduction

In the early 1970's, the following question was asked by Philip Hall of J.C. Lennox (see [4]) and appeared later in [3] in the form:

Must a non-trivial group, which is isomorphic to each of its non-trivial normal subgroups, be either free of infinite rank, simple or infinite cyclic?

This was answered in the negative by Obraztsov in [6], where he used the technique of graded diagrams developed by Ol'shanskii in [7].

In this note we provide a different construction of a non-trivial group which is isomorphic to each of its non-trivial normal subgroups. This construction makes special use of the notion of homogeneous and universal structures in model theory and well-known facts about HNN extensions and free products with amalgamation. In our view, the following construction yields a transparent proof of the existence of such groups and shows how basic ideas from model theory can be used to resolve questions in other areas of mathematics. While Obraztsov's proof is based on the deep theory of the geometry of defining relations in groups, our approach is accessible to a graduate student with basic knowledge of model theory and group theory. Our main result is the following:

**MAIN THEOREM:** *Let  $\kappa$  be an uncountable cardinal such that  $\kappa = \kappa^{<\kappa}$ . Then there exists a group  $G$  of cardinality  $\kappa$  with a descending principal series*

$$G \supset G_1 \supset \cdots \supset G_n \supset \cdots$$

*such that  $\bigcap_{n < \omega} G_n = 1$ ,  $G_n \cong G$  for each  $n < \omega$ , and any normal subgroup of  $G$  is equal to some  $G_n$ .*

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## 2. Homogeneous and universal structures

In this section we consider a class of models  $\mathcal{M}$  and give a brief summary of its relevant properties. Notation and terminology observed here follow that in [1].

Let  $\mathcal{M}$  be the collection of relational structures in the language  $L$  of groups with constant symbol 1 (whose value in any model will be the group identity), the axioms of groups and an additional sequence  $P_n$  ( $n \in \omega$ ) of unary predicates having the following properties: If  $A$  is a group and  $P_n A$  denotes  $P_n$  applied to

$A$ , then  $P_n A$  is a normal subgroup of  $A$  (denoted  $P_n A \triangleleft A$ ) and  $P_{n+1} A \subseteq P_n A$ . In terms of first order language, this is to say that:

- (1)  $1 \in P_n A$ .
- (2) If  $x, y \in P_n A$ , then  $xy^{-1} \in P_n A$ .
- (3) If  $x \in P_n A$  and  $z \in A$ , then  $z^{-1}xz \in P_n A$ .
- (4) If  $x \in P_{n+1} A$ , then  $x \in P_n A$ .

Thus the class  $\mathcal{M}$  of all such  $L$ -models is defined to be

$$\{(P_n A)_{n < \omega} : P_n A \text{ is a group, } P_n A \triangleleft P_0 A \text{ and } P_{n+1} A \subseteq P_n A \ (n < \omega)\}.$$

To simplify notation, we let  $P_0 A = A$  and abbreviate the sequence  $(P_n A)_{n < \omega}$  of groups as  $(P_n A)$ .

Let  $(P_n A), (P_n B) \in \mathcal{M}$ . The relational structure  $(P_n A)$  is said to be a **sub-structure** of  $(P_n B)$  (equivalently,  $(P_n B)$  is an **extension** of  $(P_n A)$ ) if  $P_n A = P_n B \cap P_0 A$  for all  $n < \omega$ . We denote this by  $(P_n A) \leq (P_n B)$ . Let  $\varphi: P_0 A \rightarrow P_0 B$  be a map. We say that  $\varphi$  is a **homomorphism** of  $(P_n A)$  into  $(P_n B)$  if  $\varphi$  is a group homomorphism and  $\varphi(P_n A) = P_n B \cap \varphi(P_0 A)$  ( $n < \omega$ ). A homomorphism  $\varphi$  is called an **embedding** of  $(P_n A)$  into  $(P_n B)$  if  $\varphi$  is one-to-one. Hence if  $(P_n A) \leq (P_n B)$ , then the identity map  $id: (P_n A) \rightarrow (P_n B)$  is an embedding. If  $\varphi$  is one-to-one and onto, then  $\varphi$  is called an **isomorphism** and  $(P_n A)$  is said to be **isomorphic** to  $(P_n B)$ . This we denote by  $(P_n A) \cong (P_n B)$ . We refer to the cardinality of  $P_0 A$  as the **cardinality** of  $(P_n A)$ .

PROPOSITION 2.1: *The following properties hold in  $\mathcal{M}$ :*

- (I)  $\mathcal{M}$  contains structures of arbitrarily large cardinality.
- (II) If  $(P_n A) \in \mathcal{M}$  and  $(P_n A) \cong (P_n B)$ , then  $(P_n B) \in \mathcal{M}$ .
- (III) If  $(P_n A_1), (P_n A_2) \in \mathcal{M}$ , there exist  $(P_n C) \in \mathcal{M}$  and embeddings

$$\varphi_i: (P_n A_i) \hookrightarrow (P_n C) \quad (i = 1, 2).$$

- (IV) (Amalgamation Property) Let  $(P_n A), (P_n B_1), (P_n B_2) \in \mathcal{M}$  such that

$$\varphi_i: (P_n A) \hookrightarrow (P_n B_i) \quad (i = 1, 2)$$

are embeddings. Then there is some  $(P_n C) \in \mathcal{M}$  and embeddings

$$\gamma_i: (P_n B_i) \hookrightarrow (P_n C) \quad (i = 1, 2) \quad \text{such that } \gamma_1 \circ \varphi_1 = \gamma_2 \circ \varphi_2.$$

- (V) The union of any chain of structures  $(P_n A) \in \mathcal{M}$  is still in  $\mathcal{M}$ .

(VI) $_{\kappa}$  Let  $\kappa$  be an uncountable cardinal. If  $(P_n A) \in \mathcal{M}$  and  $(P_n C) \leq (P_n A)$  such that  $(P_n C)$  is of cardinality  $< \kappa$ , then there exists  $(P_n B) \in \mathcal{M}$  of cardinality  $< \kappa$  such that  $(P_n C) \leq (P_n B) \leq (P_n A)$ .

*Proof:* (II) This is clear by the definition of a homomorphism between elements of  $\mathcal{M}$ . (V) is obvious.

(I) Let  $\alpha$  be an infinite cardinal. Take any group  $H$  of cardinality  $\alpha$  and let  $P_n A = H$  for all  $n$ . Then  $(P_n A) \in \mathcal{M}$  with cardinality  $\alpha$ .

(III) Take, for instance,  $P_n C = P_n A_1 \oplus P_n A_2$  ( $n < \omega$ ) and the natural injections  $\varphi_i: P_0 A_i \rightarrow P_0 C$  ( $i = 1, 2$ ).

(IV) Consider the free product with amalgamation

$$P_0 C = P_0 B_1 *_{P_0 A} P_0 B_2$$

and the identity map  $id_i: P_0 B_i \rightarrow P_0 C$ . Then  $id_1 \circ \varphi_1 = id_2 \circ \varphi_2$ , since  $\varphi_1(a) = \varphi_2(a)$  ( $a \in P_0 A$ ) when viewed as elements of  $P_0 C$ . Define  $P_n C = \langle P_n B_1, P_n B_2 \rangle^{P_0 C}$ , the normal subgroup generated by  $\langle P_n B_1, P_n B_2 \rangle$  in  $P_0 C$ . Clearly  $(P_n C) \in \mathcal{M}$ . It suffices to verify that  $P_n C \cap P_0 B_i = P_n B_i$  ( $n < \omega$ ). Let

$$G_n^* = P_0 B_1 / P_n B_1 *_{P_0 A / P_n A} P_0 B_2 / P_n B_2$$

be a free product with amalgamation and

$$\varphi_{ni}: P_0 A / P_n A \hookrightarrow P_0 B_i / P_n B_i \quad (i = 1, 2)$$

be the homomorphisms defined by  $\varphi_{ni}(a P_n A) = \varphi_i(a) P_n B_i$ . Consider the canonical epimorphisms

$$\pi_{ni}: P_0 B_i \rightarrow P_0 B_i / P_n B_i \quad (n < \omega, i = 1, 2).$$

Then the map

$$\pi_{n1} \cup \pi_{n2}: P_0 B_1 \cup P_0 B_2 \rightarrow G_n^*$$

extends to a homomorphism  $\pi: P_0 C \rightarrow G_n^*$  such that  $\pi \upharpoonright_{P_0 B_i} = \pi_{ni}$ . If  $x \in P_0 B_i \setminus P_n B_i$ , then  $\pi_{ni}(x) \neq 1$  and so  $\pi(x) \neq 1$ . If  $x \in P_n B_1 \cup P_n B_2$ , then  $\pi(x) = 1$ . Thus  $\ker \pi = \langle P_n B_1, P_n B_2 \rangle^{P_0 C} = P_n C$  and so  $P_n C \cap P_0 B_i = P_n B_i$  ( $i = 1, 2$ ). Thus  $(P_n B_i) \leq (P_n C)$ .

(VI) $_{\kappa}$  Note that  $(P_n C)$  is itself in  $\mathcal{M}$ , hence take  $(P_n B) = (P_n C)$ .  $\blacksquare$

The preceding properties (I)–(VI) $_{\kappa}$  and the countability of the language  $L$  show that  $\mathcal{M}$  is a  $\kappa$ -class, where  $\kappa$  is an uncountable cardinal. Note that if  $\mathcal{M}^n$

is defined to consist of all elements in  $\mathcal{M}$  such that the first  $n + 1$  terms of the sequence are all equal (for a fixed  $n$ ), then  $\mathcal{M}^n$  is still a  $\kappa$ -class.

A structure  $(P_n A) \in \mathcal{M}$  is said to be  $(\mathcal{M}, \kappa)$ -**homogeneous** if given any two  $(P_n B_1), (P_n B_2) \in \mathcal{M}$  of cardinality  $< \kappa$  which are substructures of  $(P_n A)$  and an isomorphism  $\varphi: (P_n B_1) \rightarrow (P_n B_2)$ , then there is an isomorphism  $\gamma: (P_n A) \rightarrow (P_n A)$  such that  $\gamma \upharpoonright_{(P_n B_1)} = \varphi$ . A structure  $(P_n A)$  is said to be  $\mathcal{M}$ -**homogeneous** if  $(P_n A)$  is of cardinality  $\kappa$  and  $(\mathcal{M}, \kappa)$ -homogeneous.  $(P_n A)$  is said to be  $\mathcal{M}$ -**universal**, if  $(P_n A)$  is of cardinality  $\kappa$  and, given any  $(P_n B) \in \mathcal{M}$  of cardinality  $< \kappa^+$ , there is an embedding of  $(P_n B)$  into  $(P_n A)$ . By Jónsson's Theorem (see [1], p. 213 or [2]), if  $\kappa$  is chosen such that  $\kappa = \kappa^{<\kappa}$  (e.g.,  $\kappa$  is regular and either  $\kappa$  is a limit beth number or the GCH holds), then  $\mathcal{M}$  contains an  $\mathcal{M}$ -homogeneous,  $\mathcal{M}$ -universal structure of cardinality  $\kappa$  which is unique up to isomorphism. We denote this structure by  $(P_n G)$ .

### 3. Normal subgroups of $P_0 G$

We state and verify here special properties of  $P_0 G$  and use them to obtain the required group.

**PROPOSITION 3.1:** *Let  $(P_n G)$  be  $\mathcal{M}$ -homogeneous,  $\mathcal{M}$ -universal of cardinality  $\kappa$ . Then  $P_n G \cong P_0 G$  ( $n < \omega$ ).*

*Proof:* Let  $M = (P_n G)$ . Define

$$M^n = (P_n G, P_n G, \dots, P_n G, P_{n+1} G, P_{n+2} G, \dots),$$

where the first  $n + 1$  terms in the sequence are all equal to  $P_n G$  and the succeeding terms agree with the corresponding terms in  $M$ , i.e., if  $k > 1$ , the  $(n + k)$ th term of  $M^n$  is the  $(n + k)$ th term of  $M$ . Define

$$\mathcal{M}^n = \{(P_n A) \in \mathcal{M}: \text{the first } n + 1 \text{ terms of } (P_n A) \text{ are equal}\},$$

which is a  $\kappa$ -class containing  $M^n$ . Since  $M$  is  $\mathcal{M}$ -homogeneous,  $\mathcal{M}$ -universal and  $M^n \leq M$ , it follows that  $M^n$  is  $\mathcal{M}^n$ -homogeneous and  $\mathcal{M}^n$ -universal.

Consider  $M^- = (P_n G, P_{n+1} G, \dots)$ . We show that  $M^-$  is  $\mathcal{M}$ -homogeneous and  $\mathcal{M}$ -universal. For then, by Jónsson's Theorem,  $M$  must be isomorphic to  $M^-$  and so we have shown that  $P_0 G \cong P_n G$ .

Suppose  $X = (P_n A) \in \mathcal{M}$  is of cardinality  $< \kappa^+$ . Define

$$X^0 = (P_0 A, P_0 A, \dots, P_0 A, P_0 A, P_1 A, P_2 A, \dots)$$

to be an element of  $\mathcal{M}^n$ , where the first  $n$  terms are all equal to  $P_0A$  and the succeeding terms of the chain  $X^0$  are  $P_nA$  ( $n < \omega$ ). Since  $M^n$  is  $\mathcal{M}^n$ -universal, there exists an embedding  $\varphi: X^0 \hookrightarrow M^n$ . Hence  $\varphi \upharpoonright_X: X \hookrightarrow M^-$  is also an embedding, and so  $M^-$  is  $\mathcal{M}$ -universal.

Let  $X = (P_nA)$  and  $Y = (P_nB)$  be substructures of  $M^-$  and  $\varphi: (P_nA) \rightarrow (P_nB)$  be an isomorphism. We show that there exists an automorphism  $\gamma$  of  $M^-$  such that  $\gamma \upharpoonright_X = \varphi$ . As in the preceding paragraph, define the elements  $X^0$  and  $Y^0$ , which are substructures of  $M^n$ . Then  $\varphi: X^0 \rightarrow Y^0$  is also an isomorphism. Since  $M^n$  is  $\mathcal{M}^n$ -homogeneous, there exists an automorphism  $\rho$  of  $M^n$  such that  $\rho \upharpoonright_{X^0} = \varphi$ . It is clear that  $\gamma = \rho \upharpoonright_{M^-}$  is the required automorphism of  $M^-$  which extends  $\varphi$ . Thus we have shown that  $M^-$  is  $\mathcal{M}$ -homogeneous. ■

**THEOREM 3.2:** For all  $w \in P_nG \setminus P_{n+1}G$ ,  $\langle w \rangle^{P_0G} = P_nG$ .

*Proof:* By Proposition 3.1, it is enough to prove that for all  $w \in P_0G \setminus P_1G$ ,  $\langle w \rangle^{P_0G} = P_0G$ . If  $g \in P_1G$ , then  $g = (gw^{-1})w$  and  $gw^{-1} \in P_0G \setminus P_1G$ . Thus in order to prove that  $\langle w \rangle^{P_0G} = P_0G$ , we need only show that  $P_0G \setminus P_1G \subseteq \langle w \rangle^{P_0G}$ .

Suppose that  $w, a \in P_0G \setminus P_1G$ . Let  $H_1 = \langle w, a \rangle$  and  $N_1 = H_1 \cap P_1G$ . Let  $H_2 = H_1 * \langle x \rangle$  be the free product and let  $N_2 = N_1^{H_2}$ . Let  $b = wxw^{-1}x^{-1}$  and  $c = axw^{-1}x^{-1}$ . By considering the canonical homomorphism

$$H_2 \rightarrow (H_1/N_1) * \langle x \rangle$$

we see that

- (i)  $N_2 \cap H_1 = N_1$ ,
- (ii)  $b^n, c^n \notin N_2$  for all  $n > 0$ .

Let  $H_3 = \langle H_2, t \mid tbt^{-1} = c \rangle$  and let  $N_3 = N_2^{H_3}$ . Let  $\bar{b}, \bar{c}$  be the elements of  $H_2/N_2$  corresponding to  $b, c$ . By considering the canonical homomorphism

$$H_3 \rightarrow \langle H_2/N_2, t \mid t\bar{b}t^{-1} = \bar{c} \rangle$$

we see that  $N_3 \cap H_2 = N_2$ . Note that in  $H_3$  we have that

$$twxw^{-1}x^{-1}t^{-1} = axw^{-1}x^{-1}$$

and so  $a \in \langle w \rangle^{P_0G}$ . ■

**THEOREM 3.3:** Let  $I = \bigcap_{n < \omega} P_nG$ . If  $H$  is a normal subgroup of  $P_0G$  such that  $H$  is not contained in  $I$ , then  $H = P_nG$  for some  $n < \omega$ . Thus every non-trivial normal subgroup of  $P_0G/I$  is equal to  $P_kG/I$  for some  $k < \omega$ .

*Proof:* Let  $H$  be a normal subgroup of  $P_0G$  such that  $H$  is not contained in  $I$ . Then there exists a least  $n$  such that  $H \not\subseteq P_nG$ . So  $H \subseteq P_{n-1}G$  and there exists  $x \in H \setminus P_nG$ . By Theorem 3.2, it follows that  $H = P_{n-1}G$ . ■

Let  $F_\kappa$  be the free group of infinite rank  $\kappa$  and define  $P_nA = F_\kappa^{(n)}$  to be the  $n$ th derived group of  $F_\kappa$ . Then  $(P_nA) \in \mathcal{M}$  with cardinality  $\kappa$ . Since  $(P_nG)$  is  $\mathcal{M}$ -universal of cardinality  $\kappa$ , the free generators of  $F_\kappa$  correspond to  $\kappa$  distinct elements of  $P_0G$ . Since  $\bigcap_{n < \omega} P_nA = 1$ ,  $P_0G/I$  has cardinality  $\kappa$ . The proof of the main theorem will now follow, when we take  $G = P_0G/I$  and  $G_n = P_nG/I$  ( $0 < n < \omega$ ).

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