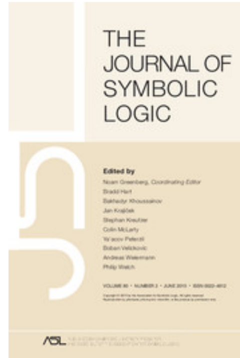

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A COMBINATORIAL FORCING FOR CODING THE UNIVERSE BY A REAL WHEN THERE ARE NO SHARPS

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Abstract. Assuming $0^\sharp$ does not exist, we present a combinatorial approach to Jensen's method of coding by a real. The forcing uses combinatorial consequences of fine structure (including the Covering Lemma, in various guises), but makes no direct appeal to fine structure itself.

§**0. Introduction.** In [5], S. Friedman calls the original proof in [1] of the Coding Theorem "one of the hardest in all of set theory. The technical considerations are extremely elaborate and the proof draws heavily on Jensen's profound fine structure theory." In addition to providing an excellent overview, both of the general approach and of the particulars of that proof, [5] presents certain simplifications. R. David, [2], [3], and, in subsequent work, Friedman, [7–10], [12], and David, [4], have shown how to integrate additional structure into the forcing conditions to obtain yet stronger results. In particular, [8] has generalized the coding method to coding over ground models where there are measurable cardinals, while preserving the measurability of a designated measurable cardinal (in the extension, $V = L[\mu^*, R]$, where R is a real and μ^* is a normal measure on κ, extending a designated normal measure μ of the ground model). Also, ignoring all references to measures and mice, pp. 1147–1154 of [8] provide a the skeleton of a highly general and concise version of coding over L (obtaining $V = L[R]$, in the extension, where R is a real), with no hypotheses on the ground model (other than GCH), which is fully developed in [11].

Nevertheless, the fundamental features of Jensen's approach remain unchanged: the obstacles (which we shall discuss shortly) in the path of a "naive" attempt to piece together the "building blocks" of Chapter 1 of [1] are overcome by integrating fine-structural considerations into the very definition of the forcing conditions. As a result, questions of uniformity, effectiveness, and absoluteness of notions involved in "locally defined" approximations to the forcing must be faced.

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This paper is number 340 in the list of Professor Shelah's publications.

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Our approach will be radically different, drawing upon the great simplification afforded by the hypothesis of the nonexistence of $0^\sharp$. One of the two main obstacles will be overcome by a preliminary forcing. The heart of the proof of Lemma 3 in [16], which states that this preliminary forcing behaves as needed, involves an appeal to the Covering Lemma.

The other main difficulty is to prove a strategic closure property of the class of coding conditions. This will be done in (5.1) below; in (5.2) and (5.3) it is shown how this yields distributivity properties of the class of coding conditions. The material of §5 (and the material of [17], upon which it draws) appeals to strong combinatorial properties of L, developed in [17]. A sketch of the results required is presented in (1.2), (1.4), (1.5) below. Here again, the Covering Lemma plays a role, this time by guaranteeing that the L-combinatorics give us a handle on the situation in V.

The obvious downside to our approach is the need for the hypothesis that $0^\sharp$ does not exist. The main advantage is that the role of fine structure is "modular": it is crystallized in the Covering Lemma itself, and in the L-combinatorics. This is quite analogous to the approach that the first author took in his proof of Strong Covering (an early version appears as [13], and a revised version will appear in [14]). Indeed, in some ways, this paper is an outgrowth of that work. One of the first author's motivations for this paper was to develop a family of forcing which is closed under certain kinds of iterations. The sort of iteration should be clear from (5.3); finer points will be worked out in a forthcoming paper, bearing number 295 in the first author's list of publications. This allows for a simpler definition of the coding conditions, involving the combinatorial apparatus, but making no direct reference to fine-structural nor definability notions. It is our hope that nonexperts will find this easier to use and to "customize" for particular applications they have in mind.

There are two other drawbacks to our approach. The first is that it seems to preclude obtaining sharp definability-type or minimal-degree type of results. In fact, this is a rather natural consequence of our seeking a combinatorial forcing, intended for use in obtaining combinatorial consequences. We recognize that this is a significant departure from the "tradition" created by Jensen's original treatment, and that, in the eyes of those steeped in that tradition, the entire approach may seem somewhat unattractive.

The second is a consequence of our treatment of inaccessible cardinals. We treat them in a way which is much closer to our treatment of successor cardinals than to our treatment of singular cardinals, in that, if κ is inaccessible, then, in any condition p, except for fewer than κ many $\alpha \in [\kappa, \kappa^+)$, if α is mentioned in p, then p says *nothing* about a tail of the coding area for α, whereas, if κ is singular, $\alpha \in [\kappa, \kappa^+)$, α is a multiple of κ^2 (nonmultiples of κ^2 are treated totally differently), and α is mentioned in p, then p says *something* about a tail of the coding area for α. Jensen has informed us that in early, unpublished versions of the coding paper (which evolved into [1]), he treated inaccessible cardinals as we do, but that he later shifted to treating them in a way similar to his treatment of singular cardinals, in order, e.g., to be able to prove the preservation of small large cardinals, as in §4.3 of [1]. Thus, though we have not yet investigated the question, our approach may preclude analogues of some of the results there.

We should point out that this way of dealing with inaccessibles (essentially by requiring that the set of cardinals mentioned in a condition is an Easton set) has two main uses. The first has to do with dealing with "contamination", see below, (2.3), (3.3)–(3.4), and the portion of the "Summary and Introduction" section, below, which deals with these items. The second involves the results of [17] and will be more fully discussed in (1.3). Apparently, the second use is really an essential feature of using ground model scales as the main coding areas at singular cardinals (see the discussion leading up to Lemma 5, below), while the *need* for the first use is a *result* of treating inaccessibles very differently than singulars.

Recently, S. Friedman has circulated a preprint (*A short proof of Jensen's coding theorem, assuming not* 0#) which draws, in part, on ideas of this paper and [17].

Finally, we would like to thank the referee for pointing out that the methods of this paper are compatible with the existence of generics, but that, unlike the more "classical" coding methods, this requires the techniques of [6].

Discussion. The coding theorem we prove is:

THEOREM 1. *If $V \models ZFC + GCH +$ "$0^\sharp$ does not exist", then there is a class forcing* $\mathbf{P}$ *which preserves ZFC, cofinalities, and GCH, and is such that, in $V^{\mathbf{P}}$, (there is a real r such that "$V = L[r]$") holds.*

We should immediately point out that the conditions, $\mathbf{P} = \mathbf{P}_{\aleph_2}$ of §3 add a subset $B_{\aleph_3} \subseteq \aleph_3$, rather than a real. However, since in the generic extension $V = L[B_{\aleph_3}]$ holds, it is an easy matter to code $B_{\aleph_3}$ into a subset of $\aleph_2$, which, in turn, is coded into a subset of $\aleph_1$, which, finally, is coded into a real, using, e.g., almost disjoint coding. It may be necessary to intersperse the forcings of §1.3 of [1] to reshape the intervals $(\aleph_i, \aleph_{i+1})$, for $i = 0, 1$, but this is not problematical, since when this is called for, the subset of $\aleph_{i+1}$ which we already have codes the universe.

Before embarking on the promised discussion of the obstacles to a naive attempt to piecing together the building blocks of [1], Chapter 1 (or some variation on them), and how these obstacles are overcome in Lemmas 3 and 5, below, we should note that we follow [5] for the general strategy for proving such a coding theorem, and especially pp. 1005–1006, middle. In particular, it will suffice, by the arguments presented in [5], to prove the four main properties of $\mathbf{P}$ presented there: Extendability, Distributivity, Factoring, and Chain Condition. The versions of these properties which we prove reflect the differences between the detailed definition of our $\mathbf{P}$ and that considered in [5], but they are sufficiently similar that the general arguments for their sufficiency go over to the setting of this paper. These properties are proved in (6.1), (5.3), (4.4), and (6.2) respectively.

The Factoring property states that for all regular $\theta > \aleph_2$, there are $\mathbf{P}_\theta$, $\dot{\mathbf{P}}^\theta$ such that $\mathbf{P} \cong \mathbf{P}_\theta * \dot{\mathbf{P}}^\theta$. The Distributivity property states that $\mathbf{P}_\theta$ is (θ, ∞)-distributive. The Chain Condition property states that, in $V^{\mathbf{P}_\theta}$, $\dot{\mathbf{P}}^\theta$ has the θ^+-chain condition. The proof of Distributivity in (5.3) is based on a strategic closure property of $\mathbf{P}_\theta$ established in (5.1), together with the result of (5.2), which proves that the BAD player need not lose the game for "trivial" reasons. Though the proof of (5.1) has been rendered rather short and easy, by the introduction of the "very tidy" conditions, and the preliminary results of (4.3) and (4.5), in many ways, this result is the main lemma of the entire paper. It shows that, by the use of "deactivators" and generic scales (in addition to the ground model scales used in setting up the main coding apparatus), we can overcome the second of the two main obstacles to a naive attempt to piece together the building blocks. We turn now to a discussion of these obstacles.

The first main obstacle simply involves the possibility of coding $R \subseteq \kappa^+$ into a subset of κ, when κ is regular. In order to use almost disjoint set coding (or, as below, in §§2 and 3, almost inclusion coding, a variant used in [15], (1.3)), we seem to need extra properties of the ground model, or of the set **R**, since, in order to carry out the *decoding* recursion across $[\kappa, \kappa^+)$, we need, e.g., an almost disjoint sequence satisfying:

$$(*) \qquad \begin{array}{c}\text{for all } \theta \in (\kappa, \kappa^+),\ (b_\alpha | \alpha \leq \theta) \in L[R \cap \theta], \\ \text{and is "canonically definable" there.}\end{array}$$

Such a $\vec{b}$ is called *decodable*. It is easy to obtain a decodable $\vec{b}$ if R satisfies:

$$(**) \qquad \text{for all } \theta \in (\kappa, \kappa^+),\ (card\,\theta)^{L[R\cap\theta]} = \kappa.$$

If $(**)$ holds, we say that R *promptly collapses fake cardinals.*

Of course, typically $(**)$ fails, and the "reshaping" conditions of 1.3 of [1], the F^B of [5], are introduced to obtain $(**)$ in a generic extension. Unfortunately, the distributivity argument for the F^B *seems* to require not merely that $H_{\gamma^+} = L_{\gamma^+}[B]$, but that $H_{\gamma^{++}} = L_{\gamma^{++}}[B]$, where $B \subseteq \gamma^+$. This will be the case if B is the result of coding as far as γ^+, but that is another story, which leads to the original approach to the Coding Theorem.

Instead, in [16], we showed, assuming GCH and that $0^\sharp$ does not exist:

PROPOSITION 2. *Let $\kappa > \aleph_1$ be a cardinal, let $Z \subseteq \kappa^{+\omega}$ be such that for all cardinals λ with $\kappa \leq \lambda \leq \kappa^{+\omega}$, $H_\lambda = L_\lambda[Z]$. Then, there is a cofinality-preserving, GCH-preserving forcing $\mathbf{S}(\kappa)$ which adds a $W \subseteq (\kappa, \kappa^+)$ such that $Z \in L[W, Z \cap \kappa]$ and, for all $\kappa \leq \theta < \kappa^+$, $(card\,\theta)^{L[W\cap\theta,\ Z\cap\kappa]} = \kappa$.*

Then, starting from $\hat{A} \subseteq OR$ such that $H_\kappa = L_\kappa[\hat{A}]$ for all infinite cardinals κ, and taking **S** to be the product, with Easton supports, of the $\mathbf{S}(\kappa)$ for $\kappa = \aleph_2$ or κ a limit cardinal, we have:

LEMMA 3. *In $V^{\mathbf{S}}$ there is an $A \subseteq OR$ such that, letting Λ be the class of limit cardinals together with $\aleph_2$:*

$$A = (A \cap \aleph_2) \cup \bigcup \{A \cap (\kappa, \kappa^+) : \kappa \in \Lambda\},$$

such that for all infinite cardinals κ, $H_\kappa = L_\kappa[A]$ and such that for $\kappa = \aleph_2$ or κ inaccessible, for all $\kappa \leq \theta < \kappa^+$, $(card\,\theta)^{L[A\cap\theta]} = \kappa$ (for singular κ, the last property is true with L in place of $L[A \cap \theta]$, by virtue of Covering).

By virtue of the preceding discussion, we clearly have:

COROLLARY 4. *In $V^{\mathbf{S}}$, letting A be as in Lemma 3, for all regular $\kappa \geq \aleph_2$, there is a decodable $\vec{b} = (b_\alpha | \alpha \in (\kappa, \kappa^+))$ of cofinal almost disjoint subsets of κ as above.*

In order to discuss the difficulty in proving the strategic closure properties of the $\mathbf{P}_\theta$, we need to say a bit about the coding apparatus for singular cardinals. This material is discussed at somewhat greater length in (1.2) and (2.2)–(2.4), and formally presented in (3.4), (3.5), so the reader who finds the present discussion insufficiently informative is encouraged to look ahead to these items.

If κ is singular and $\kappa < \alpha < \kappa^+$, α a multiple of κ^2, then the *main coding area* for α will be a cofinal subset of κ which is the range of a function f^*_α. and f^*_α is part of a *scale* between κ and κ^+. The domain of f^*_α is a fixed club subset D_κ of the cardinals below κ, and for each $\lambda \in D_\kappa$, $f^*_\alpha(\lambda)$ is of the form $\lambda^2\tau$, where τ is even, $0 < \tau < \lambda^+$. If κ is a limit of singular cardinals, then the $\lambda \in D_\kappa$ are all singular cardinals, while if κ is of the form $\mu^{+\omega}$, then the λ are all of the form $\aleph_\tau$, $\tau > 1$, τ odd, where $\kappa = \aleph_{\tau+\omega}$.

In a condition p in which κ is mentioned, an initial segment $(\kappa, \delta^p(\kappa))$ of ordinals from (κ, κ^+) will be mentioned, and a tail of $\lambda \in D_\kappa$ will be mentioned. We shall require that $\delta^p(\kappa)$ is a multiple of κ^2. If $\kappa < \alpha < \delta^p(\kappa)$, α a multiple of κ^2, then for a tail of $\lambda \in D_\kappa$, $f^*_\alpha(\lambda)$ is mentioned in p (i.e., $f^*_\alpha(\lambda) < \delta^p(\lambda)$). It is natural to expect, and will, in fact, be true in *tidy* conditions that

$$(**)\quad \begin{gathered}\text{if } \delta^p(\kappa) \le \alpha < \kappa^+, \text{ and } \alpha \text{ is a multiple of } \kappa^2, \text{ then, on a tail of } \lambda \in D_\kappa,\\ f^*_\alpha(\lambda) \text{ is } \textit{not} \text{ mentioned in } p \text{ (i.e. } \delta^p(\lambda) \le f^*_\alpha(\lambda)).\end{gathered}$$

If $(***)$ failed, then it might be impossible to extend p to a condition which mentions α and which "codes correctly" at α, since the portion of p below κ may have already imposed an unbounded amount of information on the main coding area for α. However, $(***)$ is quite hard to maintain when trying to construct an upper bound for an increasing sequence of length $\theta = \mathrm{cf}\,\kappa$ of conditions from $\mathbf{P}_\theta$.

So, rather than require the property, we drop the requirement that p has to code correctly at all α. Instead, we allow certain α to be "deactivated", not used for coding. We inherit another problem though: how to detect deactivated ordinals. For this we are led to introduce two auxiliary coding areas. The first, s_α, is simply the set of multiples of κ between α and $\alpha + \kappa^2$. This area is used for coding an ordinal $h^p(\alpha) \ge \alpha$. The idea is that not only α but all the ordinals in $[\alpha, h^p(\alpha))$ will also be deactivated.

For *singular* κ, we have, associated with each such α, a function $\sigma^{p,\alpha}$ with domain D_κ, and we have that $h^p(\alpha)$ is the least $\gamma \ge \alpha$ such that $f^*_\gamma \ge^* \sigma^{p,\alpha}$; in the notation introduced at the end of this section, $h^p(\alpha) = \mathrm{scale}(\sigma^{p,\alpha})$. The $\sigma^{p,\alpha}$ are the *generic* scale functions, as opposed to the ground model scale functions f^*_α. We thank the referee for insisting on the point of view that what we are really doing is forcing a generic scale, since the ground model scale is not adequate for dealing with deactivation. In fact, for singular κ, $h^p(\alpha)$ is decoded as $\mathrm{scale}(\sigma^{p,\alpha})$ rather than being read directly in s_α. For λ of the form $\aleph_\tau$ where $\tau > 1$ is odd, we also have $h^p(\eta)$ for multiples η of λ^2 which are mentioned in p. Here, however, there is no associated function and the $h^p(\alpha)$ are directly decoded from s_α.

When α is a limit of multiples of κ^2, the second auxiliary coding area will be a club subset $C_\alpha \subseteq \alpha$. This will be used to help us detect deactivation. The C_α will be part of a "square system" between κ and κ^2.

Returning to $(***)$, we have mentioned that we do require it in *tidy* conditions, and we require something even stronger in *very tidy* conditions. In (4.3), we show that the latter are dense. However, by dropping the requirement $(***)$, we make it easier to construct upper bounds which might not be tidy, as we do in (4.5).

In (1.1), we define games $G(\theta, \vec{\mathscr{N}}, p_0)$, where $\theta > \aleph_1$ is regular, $\vec{\mathscr{N}}$ is a certain kind of sequence (of length $\theta + 1$) of models, and $p_0 \in P_\theta$. The two players, GOOD and BAD, alternately pick conditions $p_i \in P_\theta$. GOOD plays at nonzero even stages, and BAD plays at odd stages. BAD also picks a subsequence of $\vec{\mathscr{N}}$ by choosing an increasing sequence $(\alpha(i) | i < \theta)$ from θ; of course, $\alpha(i)$ is chosen at stage $2i + 1$. We require that the p_i are increasing, and that $p_{2i}, p_{2i+1} \in |\mathscr{N}_{\alpha(i)}|$. In the cases of interest, $\vec{\mathscr{N}}$ will satisfy a more technical condition, introduced in (1.3), called supercoherence. This will guarantee that at limit stages, we will have the hypotheses of (4.5). GOOD wins if she succeeds in playing p_θ. BAD wins if at some even stage $j \leq \theta$, GOOD has no legal move.

In (5.1) we prove:

LEMMA 5. *For θ, p_0 as above, and for supercoherent $\vec{\mathscr{N}}$, GOOD has a winning strategy in $G(\theta, \vec{\mathscr{N}}, p_0)$.*

In (5.2), it is argued (using results [17]) that this gives that $\mathbf{P}_\theta$ is (θ, ∞)-distributive. What is at issue here is whether BAD always loses because of his inability to play supercoherent sequences. The results of [17], summarized in (1.4), below, show that this is *not* the case: there are enough supercoherent sequences. In (1.4), this is presented as a property of the combined squares and scales system, introduced in (1.2).

Summary and organization. In §1 we present the coding apparatus for singular cardinals and the related results from [17], notably the result about the existence of supercoherent sequences. In (1.1) we introduce the model sequences and the games $G(\theta, \vec{\mathscr{N}}, p_0)$. In (1.2) we introduce the combinatorial apparatus of squares and scales. In (1.3) we introduce the notion of supercoherence. In (1.4) we state the main result of [17], presented as an additional property of the combinatorial apparatus. In (1.5) we state a small combinatorial result about the system of scales which we use in (4.3). The result is clearly closely related to the definition of very tidy condition. This is also proved in [17].

In §2 we take care of some other preliminaries. In (2.1) we recapitulate some of the material of Lemma 3, above, and (1.2), by giving a complete discussion of coding areas for various kinds of ordinals. In particular, in (2.1.1) we cite an additional result from [17] which shows that, without loss of generality, we can assume that the system of b_η, for η such that *card*η is inaccessible, has an additional property called *tree-like*. In (2.2) we give a preliminary idea of the nature of conditions, by introducing the class $P(0)$ of "protoconditions". In (2.3) we discuss the phenomenon of "contamination" at limit cardinals, and the devices for dealing with it, namely the sets X_γ of "candidates" for coding γ which are not multiples of κ. We also introduce the weak deactivator, denoted by ?, and the component β^p of conditions which provides bounds for contamination. The X_γ are also useful in the context of the strong deactivator, denoted by !, which we discuss in (2.4), along with the generic scales. In (2.5) we give a very brief sketch of the decoding procedure, which we complete in (4.6).

In §3 we give the formal definition of the class of coding conditions. (3.1) recalls some notation, terminology and conventions. In (3.2) we define a sub-class

$\tilde{P}$ of P_0. These still incorporate none of the sophisticated properties intended to deal with contamination and deactivators. In (3.3) we formally define the notions associated with contamination, and in (3.4) we cut down $\tilde{P}$ still further by imposing five additional properties. The first four of these deal with contamination. The last deals with the use of the auxiliary coding areas, s_α, and thus foreshadows (3.5), where we deal with the strong deactivator ! and the generic scales $\sigma^{p,\alpha}$, and finally define the class of coding conditions by imposing four additional properties related to these. In (3.6) we give the (very simple) definition of the partial ordering of conditions.

In §4 we prove some basic lemmas which will greatly facilitate our work in §§5 and 6. In (4.1) we introduce the tidy and very tidy conditions. We develop some of their properties in (4.1) and (4.2), and in (4.3) we prove the crucial result that the very tidy conditions are dense. In (4.4) we develop the Factoring property. In (4.5) we show that certain increasing sequences have least upper bounds. Taken together, (4.3) and (4.5) provide most of the groundwork for (5.1). In (4.6) we provide a fully detailed discussion of the decoding procedure, completing the sketch of (2.4).

In §5, we first prove Lemma 5, above, in (5.1). In (5.2), we show that the results of [17] really do mean that BAD need not lose for trivial reasons, and that this yields the (θ, ∞)-distributivity of $\mathbf{P}_\theta$. We close with two remarks in (5.3). The first has to do with iterations of $\mathbf{P}_\theta$. The second concerns a variant of the games $G(\theta, \vec{\mathcal{N}}, p_0)$, which we use in case (c) of (6.1)(7). In (6.1), we establish the Extendability properties of $\mathbf{P}$, and in (6.2) we establish the Chain Condition property.

Notation and terminology. Our notation and terminology is intended to be standard, or have a clear meaning, e.g., o.t. for order type, *card* for cardinality. A catalogue of possible exceptions follows. Also, the index of notation at the end of this section summarizes not just what follows but also some of the important definitions and notation introduced in later sections. When forcing, $p \leq q$ means q gives more information. Closed unbounded sets are clubs. The set of limit points of a set X of ordinals is denoted by X'. $A \Delta B$ is the symmetric difference of A and B, and $A \setminus B$ is the relative complement of B in A. Notions like $=$, $\leq$, $\subseteq$, etc., when decorated with a superscript *, mean "on a tail". For ordinals $\alpha \leq \beta$, $[\alpha, \beta)$ is the half-open interval $\{\gamma : \alpha \leq \gamma < \beta\}$. The notation for the other three intervals is clear. It should be clear from context whether the open interval or the ordered pair is meant. For ordinals α, β, we write $\alpha \gg \beta$ to mean that α is MUCH greater than β; the precise sense of how much greater will be clear from context.

For infinite cardinals, κ, H_κ is the set of all sets hereditarily of cardinality $< \kappa$, i.e. those sets x such that if t is the transitive closure of x, then *card* $t < \kappa$. We regard ω as a successor cardinal, by ignoring the positive finite cardinals. Thus, for us, $\omega = 0^+$. We say that a cardinal κ is *s-like* if it singular or of the form $\aleph_\tau$ where $\tau > 1$ is odd, and that it is *i-like* if is inaccessible or of the form $\aleph_\tau$ where $\tau > 0$ is even. For s-like cardinals κ we define $U(\kappa)$ to be the set of multiples of κ^2 in (κ, κ^+), while for i-like cardinals κ we define $U(\kappa)$ to be the set of multiples of κ in (κ, κ^+). We define E to be the class of ordinals α such that, letting

$\kappa = card\,\alpha$, $\alpha \in U(\kappa)$, κ is regular and either κ is inaccessible or (κ is s-like and α is an even multiple of κ^2).

For models $\mathscr{M}$, $\mathrm{Sk}_{\mathscr{M}}$ denotes the Skolem hull operator for $\mathscr{M}$, where the Skolem functions are obtained in some reasonable fixed fashion. We often suppress mention of the membership relation as a relation of a model, but we usually intend that it is one. Thus, (M, A) frequently denotes the same model as $(M, \in, A)$.

When we have a $\leq^*$ -increasing sequence of functions $(\phi_\alpha | \alpha \in X)$, where X is a set of ordinals and ϕ is a function which is $\leq^*$ one of the ϕ_α, we let scale(ϕ) denote the least $\alpha \in X$ such that $\phi \leq^* \phi_\alpha$. All other notation is introduced as needed (we hope).

TABLE. Index of Notation

Symbol	First occurs	Brief definition	Where defined
b_α	$(*)$, §0	main coding area for α	(2.1)
f^*_α	§0, after Corollary 4	α^{th} scale function in V	(1.2)(C)
D_k	same	club of κ; domain of f^*_α	(1.2)(A)
$\delta^p(\kappa)$	same	sup of ordinals between κ and κ^+ mentioned in p	(2.2.1), (2.2.4)
$h^p(\alpha)$	§0, after $(***)$	in cases of interest, right endpoint of deactivated interval, starting from α	(3.4), (3.5.2)
$\sigma^{p,\alpha}$	same	α^{th} function in *generic* scale	(3.5.1)
C_α	same	club assigned to α by square system	(1.2)(B)
$G(\theta, \vec{\mathscr{N}}, p_0)$	§0, before Lemma 5	game used in proof of distributivity	(1.1)
$P(0)$	§0 in "Summary and organization"	class of "protoconditions"	(2.2)
X_γ	same	set of "surrogates" for coding γ	(2.3)
?, !	same	weak and strong deactivators, respectively	(2.3), (2.4)
$\beta^p(\kappa)$	same	bound on "contamination" between κ and κ^+ in any $q \geq p$	(2.2), (2.3), (3.4)
$\widetilde{P}$	same	subclass of $P(0)$	(2.2)
$\chi_{\mathscr{N}}$	(1.3)	"characteristic function" of $\mathscr{N}$; $\chi_{\mathscr{N}}(\kappa) = \sup \lvert\mathscr{N}\rvert \cap [\kappa, \kappa^+)$, when $\kappa \in \lvert\mathscr{N}\rvert$	(1.3)

TABLE. Index of Notation (continued)

Symbol	First Occurs	Brief Definition	Where Defined
$p\chi_{\mathscr{N}}$	(1.3)	"pseudo-characteristic function" for $\mathscr{N}$; defined for certain $\kappa \notin \lvert\mathscr{N}\rvert$	(1.3)
s_η	§0, after (∗∗∗)	set of multiples of κ between η and $\eta + \kappa^2$, where $card\eta = \kappa$	(2.1.3)
Ξ^p	(2.2.5)	set of "promises" made by p	(2.2.5)
g^p	(2.2.1)	p's contribution to the generic function, G	(2.2.1), (2.2.4), (2.2.5), (3.4), (3.5.2)
$P, P_\theta, \dot{P}^\theta$	(2.2.6)	class of coding conditions, and its upper and lower factors at regular θ	(3.5), (4.4)
$v^{p,\alpha}, \pi^{p,\alpha}$	(2.4.2)	functions involved in the definition of $\sigma^{p,\alpha}$	(3.5.1)
Z_α	(3.5.1)	stongly deactivated points in C_α	(3.5.1)
G_0, G_1	(2.5), (4.6)	intermediate stages in decoding $G\lvert[\kappa, \kappa^+)$	(4.6)
$\chi = \chi_A$	(2.5)	characteristic function of the class $A \subseteq OR$	(2.5)
Pr_i^p, Pr^p	(3.5.1)	proprties involved in the definition of strong p-deactivation	(3.5.1)

§1. Singular combinatorics: Results from [17].

1.1. Model sequences and the games $G(\theta,\ \mathscr{M},\ p_0)$. Let $\theta > \aleph_1$ be regular. Let $\mathscr{M} = (H_{\nu^+},\ \in,\ \cdots)$, where ν is a singular cardinal, $\nu \gg \theta$ and $(H_\nu,\ \in)$ models a sufficiently rich fragment of ZFC. Let $\sigma \leq \theta$ and let $(\mathscr{N}_i : i \leq \sigma)$ be an increasing continuous tower of elementary substructures of $\mathscr{M}$. We say that $(\mathscr{N}_i | i \leq \sigma)$ is $(\mathscr{M},\ \theta)$*-standard of length* $\sigma + 1$ if, letting $N_i\ :=\ |\mathscr{N}_i|$, for all $i \leq \sigma$, we have $card\, N_i = \theta$, $\theta + 1 \subseteq N_0$, for $i < \sigma$, $[N_{i+1}]^{<\theta} \subseteq N_{i+1}$, and, if i is even, $\mathscr{N}_i \in N_{i+1}$.

Let $\mathbf{Q}$ be a partial ordering (of course, in §5, $\mathbf{Q}$ will be $\mathbf{P}_\theta$, the upper part of $\mathbf{P}$ at θ). Let X be dense in $\mathbf{Q}$ (in §5, below, X will be the dense subclass of very tidy conditions, see (4.1) and (4.3)). Let $\vec{\mathscr{N}} = (\mathscr{N}_i | i \leq \sigma)$ be θ-standard with each $\mathscr{N}_i \prec \mathscr{M}$ (in §5, below, $\vec{\mathscr{N}}$ will be supercoherent (see below)), and let $q_0 \in Q\ (:=\ |\mathbf{Q}|) \cap M\ (:=\ |\mathscr{M}|)$. The game $G(\theta,\ \vec{\mathscr{N}},\ \mathbf{Q},\ X, q_0)$ is defined as follows.

Two players, GOOD and BAD, alternate plays. GOOD plays at positive even stages (including limit stages); BAD plays at odd stages. GOOD's moves are conditions, $q_{2i} \in Q \cap M$, where $0 < i \leq \theta$. For $0 \leq i < \theta$, BAD's move at stage $2i+1$ is a pair, $(q_{2i+1},\ \alpha(i))$, where $q_{2i+1} \in X$, $q_{2i} \leq q_{2i+1}$, $\alpha(i) > \sup\{\alpha(j) | j < i\}$, q_{2i}, $q_{2i+1} \in N_{\alpha(i)}$. We require that at all stages $\sigma \leq \theta$ the sequence $(q_i : i \leq \sigma)$ is increasing. BAD loses if GOOD succeeds in playing q_θ. GOOD loses if at some stage $i \leq \theta$, she has no legal move, i.e., there is no upper bound to the sequence $(q_j : j < i)$. Of course, this can only occur if i is a limit ordinal.

We have already hinted at the difficulty for GOOD at limit stages in the discussion in the Introduction, preceding the statement of Lemma 5.

1.2. The squares and scales. From [17] (and, for singular cardinals of the form $\aleph_{\tau+\omega}$, using Lemma 3 of the Introduction, above, as well), we have the following combinatorics for singular cardinals.

(A) *A Square on singular limits of limit cardinals.* We have a sequence, $(D_\mu : \mu$ is a singular cardinal), where D_μ is a club subset of the singular cardinals below μ satisfying the following conditions:

(1) o.t. $D_\mu < min\ D_\mu$.
(2) If λ is a limit point of D_μ, then $D_\lambda = D_\mu \cap \lambda$.
(3) If $\lambda \in D_\mu$ is not a limit point of D_μ, then λ is not a limit of limit cardinals.
(4) Suppose that $\lambda \in D_{\kappa_i}$, $i = 1, 2$, and let j_i be such that λ is the j_ith member of D_{κ_i}. Then $j_1 = j_2$.

If τ is not a successor ordinal, and $\kappa = \aleph_{\tau+\omega}$, conventionally we let $D_\kappa := \{\aleph_{\tau+n} | n$ is odd, $\tau + n \neq 1\}$. For such κ we set $\Delta_\kappa := D_\kappa$. For κ which are singular limits of limit cardinals we set $\Delta_\kappa := \bigcup\{\{\lambda\} \cup D_\lambda | \lambda \in D_\kappa\}$.

(B) *Squares on* $(U(\kappa))' \cap \kappa^+$, *where* κ *is a singular cardinal.* For each such κ, we have a sequence $(C_\alpha | \alpha \in (U(\kappa))' \cap \kappa^+)$ such that each C_α is a club subset of the set of even multiples of κ^2 below α, of order type less than κ, and such that if $\beta \in C_\alpha$ but is not a limit point of C_α, then β is not a limit point of $U(\kappa)$, and with the usual coherence property: if β is a limit point of C_α, then $C_\beta = C_\alpha \cap \beta$.

(C) *Scales on* (κ, κ^+), *where* κ *is a singular cardinal.* For such κ, we have a sequence $(f^*_\alpha : \alpha \in U(\kappa))$, where dom $f^*_\alpha = D_\kappa$, for $\lambda \in D_\kappa$, $f^*_\alpha(\lambda)$ is an even multiple of λ^2, and:

(1) If $\kappa < \alpha < \beta$, α, $\beta \in U(\kappa)$, then $f^*_\alpha <^* f^*_\beta$, i.e., for some $\lambda_0 < \kappa$, whenever $\lambda \in D_\kappa \setminus \lambda_0$, we have $f^*_\alpha(\lambda) < f^*_\beta(\lambda)$; further, if $\alpha \in C_\beta$, then the preceding holds for *all* $\lambda \in D_\kappa$.
(2) Whenever g is a function with dom $g = D_\kappa$ and $g(\lambda) < \lambda^+$ for all $\lambda \in D_\kappa$, then $g <^* f^*_\alpha$ for some $\alpha \in U(\kappa)$.
(3) If κ is a singular limit of limit cardinals, $\lambda \in D_\kappa$, $\alpha \in U(\kappa)$, $\alpha' = f^*_\alpha(\lambda)$, and $\lambda' \in D_\kappa \cap \lambda$, then $f^*_\alpha(\lambda') = f^*_{\alpha'}(\lambda')$; and if κ is not a limit of limit cardinals and α, $\beta \in U(\kappa)$, $\lambda \in D_\kappa$ and $f^*_\alpha(\lambda) = f^*_\beta(\lambda)$, then $f^*_\alpha | \lambda = f^*_\beta | \lambda$.
(4) For limit points α of $U(\kappa)$ and $\lambda \in D_\kappa$, the set $\Phi(\alpha, \lambda) := \{f^*_\beta(\lambda) | \beta \in C_\alpha\}$ is a final segment of $C_{f^*_\alpha(\lambda)}$.

Regarding (3), the property given in the second clause follows from the property given in the first. Unfortunately, we needed two different clauses, since we do not have any f^*_α where *card* α is a successor cardinal. However, the property of the second clause of (3) in fact allows us to define these according to the following convention. Once this is done, in virtue of this definition, we will have the property of the first clause of (3) even for κ which are not limits of limit cardinals:

> Suppose that $\lambda = \aleph_\tau$, where $\tau > 1$ is odd. Let $\kappa = \aleph_{\tau+\omega}$. Suppose that $\alpha' = f^*_\alpha(\lambda)$ for some $\alpha \in U(\kappa)$. For $\lambda' \in D_\kappa \cap \lambda$, we *define* $f^*_{\alpha'}(\lambda')$ to be $f^*_\alpha(\lambda')$. By the second clause of (3), this does not depend on our choice of α.

Property (4) is the crucial condensation coherence property. It plays an important role in the proof, in [17], of the existence of supercoherent sequences. We state this in (1.4), below, as an additional property of the above combinatorial system (A)–(C). Strictly speaking, we never appeal directly to (4), only to the property of (1.4), but, when C_α is cofinal in α, we do appeal to the following more obvious consequence of (4):

(4$^-$) On a tail of D_κ, Φ is cofinal in $f^*_\alpha(\lambda)$.

We close by stating the decodability property of the above system. As usual, A is as given by Lemma 3 of the Introduction.

(D) Decodability of (A)–(C): For all singular κ, D_κ and the systems $(C_\alpha | \alpha < \kappa^+$ is a limit point of $U(\kappa))$ and $(f^*_\alpha | \alpha \in U(\kappa))$ are canonically definable in $L[A \cap \kappa]$.

The decodability property is an easy consequence of the fact that the systems of (B) and (C) are rather simple modifications of systems which are canonically constructed in L, for singular limits of limit cardinals, and for κ which are not limits of limit cardinals, in $L[A \cap \kappa]$, while the system of (A) is a simple modification, also given in [17], of a constructible system.

1.3. Coherence and supercoherence. Let $\theta > \aleph_1$ be regular. Let $\nu > \mathrm{cf}\,\nu \gg \theta$ be such that $(H_\nu, \in) \models$ a sufficiently rich fragment of ZFC. Let $\mathcal{M} = (H_{\nu^+}, \in, \cdots)$. Suppose that $\mathcal{N} \prec \mathcal{M}$, where *card* $N = \theta$ for $N := |\mathcal{N}|$, and let κ be a cardinal with $\theta \leq \kappa$, $\kappa \in N$. Let $\chi_{\mathcal{N}}(\kappa) = \sup(N \cap (\kappa, \kappa^+))$.

Recall that an *Easton set of ordinals* is one which is bounded below any inaccessible cardinal. For such $\mathcal{N}$ and singular cardinals κ with $\theta < \kappa \leq \nu$, we say that κ is *$\mathcal{N}$-controlled* if there is an Easton set d with $\kappa \in d \in N$. The Easton sets we have in mind are those consisting of the sets of cardinals mentioned in some condition in N.

We define $p\chi_{\mathcal{N}}$, an analogue of $\chi_{\mathcal{N}}$, defined on all singular cardinals κ which are $\mathcal{N}$-controlled. The definition makes sense for all cardinals $\kappa \in [\theta, \nu]$, but we will only use it for the singulars which are $\mathcal{N}$-controlled. If $\kappa \in N$, then of course κ is $\mathcal{N}$-controlled, and in this case $p\chi_{\mathcal{N}}(\kappa) := \chi_{\mathcal{N}}(\kappa)$. Otherwise, $p\chi_{\mathcal{N}}(\kappa) := \sup(\kappa^+ \cap \mathrm{Sk}_{\mathcal{M}}(\{\kappa\} \cup N))$.

The reason that we only consider controlled κ is that one of the results of [17] gives an alternative characterization of $p\chi_{\mathcal{N}}(\kappa)$ which is central in proving the main result about the existence of supercoherent sequences (see below). The alternative characterization is equivalent only for controlled κ. The restriction to such κ is benign, for our purposes, since it allows us to handle any cardinal mentioned in any condition in $\mathcal{N}$. This is the essential use, alluded to in the Introduction, just prior to the "Discussion" section, of the fact that the set of cardinals mentioned in any condition is an Easton set.

Now, let $(\mathcal{N}_i | \ i \leq \theta)$ be $(\mathcal{M}, \ \theta)$-standard of length $\theta + 1$. For $i \leq \theta$, let $\chi_i = \chi_{\mathcal{N}_i}, \ p\chi_i = p\chi_{\mathcal{N}_i}$. Let $\mathcal{N} = \mathcal{N}_\theta = \bigcup\{\mathcal{N}_i : i < \theta\}$, and let $\chi = \chi_\theta, \ p\chi = p\chi_\theta$, so dom $\chi = \bigcup\{\text{dom } \chi_i : i < \theta\}$, and, for $\kappa \in$ dom χ, $\chi_{(\kappa)} = sup \ \{\chi_i(\kappa) : \kappa \in N_i\}$. Also, for singular cardinals $\kappa \in [\theta, \ \nu]$ which are $\mathcal{N}$-controlled,

$$p\chi(\kappa) = \sup\{p\chi_i(\kappa) : i < \theta \ \& \ \kappa \text{ is } N_i\text{-controlled}\}.$$

Let κ be a singular cardinal, $\kappa \in$ dom χ. Note that since cf $\theta = \theta > \omega$, there is a club $D \subseteq \theta$ such that for all $i \in D, \ \chi_i(\kappa) \in C_{\chi(\kappa)}$. This motivates the following.

Definition. Let $\mathcal{M}, \ \theta$ be as above, and let $(\mathcal{N}_i | \ i \leq \theta)$ be $(\mathcal{M}, \ \theta)$-standard of length $\theta + 1$. Let $\mathcal{N} = \mathcal{N}_\theta$. Let $N, \ N_i, \ \chi, \ p\chi, \ \chi_i, \ p\chi_i$ be as above.

Let $\kappa \geq \theta$ be a singular cardinal, $\kappa \in N$. $(\mathcal{N}_i : i \leq \theta)$ is *$\mathcal{M}$-coherent at κ* iff, for all limit ordinals $\delta \leq \theta$ with $\kappa \in N_\delta$, there is a club $D \subseteq \delta$ such that $\chi_i(\kappa) \in C_{\chi_\delta(\kappa)}$ for all $i \in D$. $(\mathcal{N}_i : i \leq \theta)$ is *$\mathcal{M}$-coherent* iff, for all singular cardinals $\kappa \in N \setminus \theta$, $(\mathcal{N}_i : i \leq \theta)$ is $\mathcal{M}$-coherent at κ. $(\mathcal{N}_i : i \leq \theta)$ is *strongly $\mathcal{M}$-coherent* iff for all $i < \theta$ and all singular cardinals $\kappa \in N_i, \ \chi_i(\kappa) \in C_{\chi(\kappa)}$. Finally, $(\mathcal{N}_i : i \leq \theta)$ is *super $\mathcal{M}$-coherent* iff $(\mathcal{N}_i : i \leq \theta)$ is strongly $\mathcal{M}$-coherent and, for all limit ordinals $\sigma \leq \theta$ and all singular cardinals κ which are $\mathcal{N}_\sigma$-controlled, for sufficiently large $i < \sigma$, we have $p\chi_i(\kappa) \in C_{p\chi_\sigma(\kappa)}$.

1.4. The existence of supercoherent sequences. Here is the statement of the main result of [17], which is the crucial additional property of the combinatorial system of (1.2).

Lemma. *Let $\theta, \ \nu, \ \mathcal{M}$ be as in* (1.3). *Let $C \subseteq [H_{\nu^+}]^\theta$ be club. Then there is a super $\mathcal{M}$-coherent $(\mathcal{N}_i|i \leq \theta)$ with each $|\mathcal{N}_i| \in C$.*

1.5. An additional result about the scales. The following small combinatorial result concerning the scales of (1.2) is also proved in [17] and will be quite useful in (4.3), below.

Proposition. *Let $\theta > \aleph_1$ and let $\nu, \ \mathcal{M}$ be as in* (1.4). *Let $d \subseteq [\theta, \nu)$ be an Easton set of cardinals, and let γ be a function with domain d such that $\gamma(\kappa) < \kappa^+$, for all $\kappa \in d$. Then there is a function γ^* with domain d such that $\gamma^*(\kappa) > \gamma(\kappa)$ for all s-like $\kappa \in d$, and such that for all singular $\kappa \in d$, if we let $\alpha = \gamma^*(\kappa)$, then $f^*_\alpha =^* \ \gamma^*|D_\kappa$. Further, if $\mathcal{N} \prec \mathcal{M}$ with $(\theta + 1) \cup \{\gamma\} \subseteq |\mathcal{N}|$, then $\gamma^* \in |\mathcal{N}|$.*

§2. Preliminaries about conditions.

2.1. Coding areas for $\eta \in (\kappa, \ \kappa^+)$. We recapitulate here some of what we have done in Corollary 4 of the Introduction and (1.2), and provide some insight into how the coding will work. First, suppose that $\eta, \ \kappa$ fall under one of the following cases.

(1) $\kappa \geq \aleph_2$ is a successor cardinal, $\eta \in (\kappa, \ \kappa^+)$,
(2) κ is inaccessible, $\eta \in U(\kappa)$,
(3) κ is a singular cardinal, $\eta \in U(\kappa)$.

Then, by Corollary 4 of the Introduction for cases (1) and (2), and by (1.2) for case (3), we have associated to η an unbounded subset $b_\eta \subseteq \kappa$. In cases (1) and (2), this is the coding area for η. In case (3), it is the main coding area for

η, but we also have one, and sometimes two, auxiliary coding areas for η as well, (see below).

2.1.1. In case (2), we shall need an additional property of the b_η. So, let $\mathscr{U} := \bigcup\{U(\kappa)|\kappa$ is inaccessible$\}$. We say that the system $(b_\eta|\eta \in \mathscr{U})$ is *tree-like* iff whenever $\eta_1, \eta_2 \in \mathscr{U}$, if $\xi \in b_{\eta_1} \cap b_{\eta_2}$, then $b_{\eta_1} \cap \xi = b_{\eta_2} \cap \xi$. In [17] we also prove the rather simple observation that without loss of generality we can assume that $(b_\eta|\eta \in \mathscr{U})$ is tree-like and has the following additional property: $b_\eta =$ range g_η, where g_η is a funtion, dom $g_\eta = \{\aleph_\tau|\aleph_\tau < card\,\eta$ & $\aleph_\tau$ is an i-like successor cardinal$\}$; further, for all $\xi \in b_\eta$, ξ is a multiple of 4 but not of 8.

In case (1), if $\kappa = \mu^+$, it is easy to see that we can, without loss of generality, assume that the b_η have the following additional properties: $b_\eta \cap \mu = \varnothing$ and the members of b_η are even ordinals but not multiples of 4; further, if μ is s-like, then the members of b_η are never of the form $\alpha + 2$, where $\alpha \in E$.

2.1.2. In case (3), (1.2) already gives us b_η which have the following properties. Once again, the b_η are ranges of functions f^*_η with domain D_κ. Case (3) subdivides according to whether κ is a limit of singular cardinals, or of the form $\aleph_{\tau+\omega}$. In the first subcase, D_κ is a club subset of singular cardinals below κ, whose order-type is less than its least element. In the second subcase, D_κ is the set of $\aleph_\tau$ such that $\tau > 1$ is odd and such that $\kappa = \aleph_{\tau+\omega}$. In both subcases of case (3), the $f^*_\eta(\lambda)$ are even multiples of λ^2, i.e., they are of the form $\lambda^2\imath$, where $\imath > 0$ is even.

2.1.3. Recall that a cardinal κ is *s-like* if κ is singular or $\kappa = \aleph_\tau$, where $\tau > 1$ and τ is odd. If κ is s-like and $\eta \in U(\kappa)$, we set $s_\eta :=$ the set of multiples of κ in $(\eta, \eta + \kappa^2)$; s_η is an auxiliary coding area for η discussed in in the Introduction, above, and at greater length in (2.4), (3.4)(E), and (3.5), below. Finally, if κ is singular and $\eta \in (U(\kappa))'$, we have an additional auxiliary coding area for η, namely C_η, from (1.2) (and so also, implicitly, all of the s_α for $\alpha \in C_\eta$). This will be used for detecting deactivation of η in a way that is rather important for the limit case of GOOD's winning strategy in the game of (1.1), and for determining $\sigma^{p,\eta}$. This will also be discussed more fully in (2.4), below. It should be noted that, unlike the previous coding areas, which are essentially unique to η, this last is not, since if $\alpha \in (C_\eta)'$ then this coding area for α is an initial segment of this coding area for η, and if $\eta \in (C_\alpha)'$ then this coding area for η is a subset of this coding area for α.

It is worth recalling that if κ is a limit of singular cardinals and $\lambda \in D_\kappa$ is not a limit point of D_κ, then λ is not a limit of singular cardinals, while if λ is a limit point of D_κ then $D_\lambda = D_\kappa \cap \lambda$. It is also worth recalling (3) of (1.2), and the related convention whereby we regard f^*_ν as defined when $\nu = f^*_\alpha(\lambda)$, $\alpha \in U((card\,\nu)^{+\omega})$, and $card\,\nu$ is s-like but regular.

2.1.4. For regular κ and η as in (1) or (2), above, b_η will be used for coding η as follows. If, on a tail of b_η, we read value 1, then we will decode value 1 for η. Any condition will mention at most a bounded subset of b_η, so we will guarantee a tail of 1's on b_η by making "promises" of the form (η, ξ), where $\xi < \kappa$. Such a pair is a promise to have value 1 at all members of b_η above ξ. By a density argument, (6.1) (2), except when κ is inaccessible and η in a bounded subset of $U(\kappa)$ (the bound is $\beta^p(\kappa)$, see (2.2), (2.3)), we will have made such a promise *whenever* η gets value 1.

If κ is a successor cardinal, and we do not have value 1 on a tail of b_η, then we will have value 0 on an unbounded subset of b_η. This is also by a density argument, (6.1) (5). We then decode value 0 for η. This will essentially be the procedure when κ is inaccessible, again, except for a bounded subset of $U(\kappa)$. The situation regarding the η in the bounded set will be discussed more fully in (2.3), and (3.3), (3.4).

2.1.5. For singular cardinals κ, and η as in case (3), the situation is more complicated. Here, any condition which mentions η will mention a tail of b_η. In the simplest situation, we will have an $i \in \{0, 1\}$ and a tail of b_η on which we have value i. It is natural to expect that when this occurs, we will decode value i for η. However, it could still occur that η is deactivated, and that we therefore decode value !, or sometimes value ? for η. We discuss this in (2.4) and (3.3) - (3.5). If there is no such tail, then we will decode either value ? or value ! for η. Again this will be discussed more fully in (2.3) and (3.3)–(3.5).

2.2. Protoconditions. We define $P(0)$, the class of "protoconditions".

DEFINITION. $p \in P(0)$ iff $p = (g, \beta, \Xi) = (g^p, \beta^p, \Xi(p))$, and (2.2.1)–(2.2.5), below, hold; g is the "main component", the approximation to the class function G which we are seeking to add (and code down to a subset of $\aleph_3$).

2.2.1. There is an Easton set $d = d^p$ of cardinals $\geq \aleph_2$, and a function $\delta = \delta^p$ with $\mathrm{dom}\,\delta = d$ such that $\kappa < \delta(\kappa) < \kappa^+$ for all $\kappa \in d$, and we will have $g : \mathrm{dom}\, g \longrightarrow \{0, 1, ?, !\}$, with $\mathrm{dom}\, g = \bigcup\{(\kappa, \delta(\kappa))|\kappa \in d\}$. In addition, d will have the following property: for singular cardinals $\kappa \in d$, there is a tail of $D_\kappa \subseteq d$. In addition to the usual characters, 0, 1, we have the strong deactivator ! and the weak deactivator ?, whose roles will be discussed in (2.3), (2.4), (3.4), and (3.5).

2.2.2. $g(\alpha) \in \{0, 1\}$ *unless* κ is singular and $\alpha \in U(\kappa)$. However, we have a convention for systematic abuse of notation for certain α.

2.2.3. Recall that $\alpha \in E$ iff (letting $\kappa = \mathit{card}\,\alpha$) $\kappa > \aleph_1$ is regular, $\alpha \in U(\kappa)$, and either κ is inaccessible or κ is s-like and α is an even multiple of κ^2. In either case, for $i = 0, 1$, we take "$g(\alpha) = i$" as an abbreviation for $(g(\alpha), g(\alpha+2)) = (0, i)$, and we take "$g(\alpha) =$?", as an abbreviation for $(g(\alpha), g(\alpha+2)) = (1, 0)$. In the second case *only*, we take "$g(\alpha) =$!" as an abbreviation for $(g(\alpha), g(\alpha+2)) = (1, 1)$; thus it is our intent that we can have "$g(\alpha) =$?" for *any* $\alpha \in E$, but that we have "$g(\alpha) =$!" only for those $\alpha \in E$ whose cardinalities are s-like.

2.2.4. β is a function with $\mathrm{dom}\,\beta = d$, such that $\kappa < \beta(\kappa) \leq \delta(\kappa)$ for $\kappa \in d$. For successor cardinals $\kappa \in d$ we have $\beta(\kappa) = \kappa + 1$. The role of $\beta(\kappa)$ for limit cardinals will be made clearer in (2.3) when we discuss "contamination". For now, we will just say that $\beta(\kappa)$ is a bound on the contamination in (κ, κ^+), not only in p, but in all stronger conditions q. For cardinals $\kappa \in d$ which are either s-like or inaccessible, we will have that $\delta(\kappa), \beta(\kappa) \in U(\kappa)$, and if κ is inaccessible, we will have that $\beta(\kappa) \geq \kappa^2$.

2.2.5. Finally, Ξ is the system of "promises" which we discussed in (2.1), above. Ξ is a set of ordered pairs (α, ξ) such that $g(\alpha) = 1$, $\mathit{card}\,\alpha$ is regular, $\aleph_2 < \xi < \mathit{card}\,\alpha$ and if $\mathit{card}\,\alpha$ is inaccessible then $\alpha \geq \beta(\mathit{card}\,\alpha)$ and $\alpha \in U(\mathit{card}\,\alpha)$.) We shall also require that if $\mathit{card}\,\alpha = \lambda^+$ then $\xi > \lambda$. Let $W(p) = \mathrm{dom}\,\Xi(p)$. We let $R(p) := \bigcup\{(b_\alpha \setminus \xi)|(\alpha, \xi) \in \Xi(p)\}$. We then require $g(\zeta) = 1$ for all $\zeta \in R(p)$;

thus $(\alpha,\ \xi) \in \Xi$ is the "promise" to put all 1's in b_α from ξ on.

2.2.6. In §3 we will build to the definition of P, by imposing additional restrictions on the protoconditions. If $\theta > \aleph_2$ and θ is regular, then we shall define P_θ in (4.4). It is only slightly inaccurate, and not at all misleading, at this point, to say that the main idea is that $d \cap \theta = \varnothing$. The real point is that $\mathbf{P}_\theta$ is the class of conditions for coding down to a subset of θ^+.

The partial ordering of protoconditions is defined in the most obvious way: $p \leq q$ iff $g^p \subseteq g^q$, $\beta^p \subseteq \beta^q$, and $\Xi(p) \subseteq \Xi(q)$. This is identical to the definition of the partial ordering of conditions, in §3.

2.3. X_γ, "Contamination", $\beta(\kappa)$, and the deactivator ?. In (2.1), no coding areas were defined for γ such that $\kappa = card\,\gamma$ is a limit cardinal and γ is not a multiple of κ. Strictly speaking, for singular κ and α which are multiples of κ but which are not in $U(\kappa)$, there was also no coding area defined, but, except for the multiples of $\kappa \in [\kappa,\ \kappa^2)$, these ordinals are in s_η, where η is the largest member of $U(\kappa)$ below α. The multiples of κ in $[\kappa,\ \kappa^2)$ are simply ignored.

This is because such ordinals γ are not coded directly. Instead, each such γ has a set X_γ of "surrogates" for coding γ. Each of the surrogates α will have a coding area b_α associated with it. X_γ will have size κ^+, so we have many "tries" at coding γ correctly. When κ is singular, this is not entirely unexpected, since possibly some of the surrogates have been deactivated with the strong deactivator, !, as in the discussion in the Introduction leading up to Lemma 5. Here we discuss the weak deactivator, ?, and the phenomenon of "contamination" which is one of the contexts in which it arises. The reasons for calling ? weak and ! strong are discussed at the beginning of (2.4) and in (2.3.5), below, where we also discuss another context in which ? arises, for singular κ. Before doing this, we present the X_γ.

2.3.1. For limit cardinals κ and $\alpha \in (\kappa,\ \kappa^+)$ which are not multiples of κ, we have sets $X_\alpha \in [U(\kappa)]^{\kappa^+}$. If κ is singular, the $\xi \in X_\alpha$ are all odd multiples of κ^2, i.e., of the form $\xi = \kappa^2\iota$, where ι is odd. If κ is singular, the system $(X_\alpha : \alpha \in (\kappa,\ \kappa^+),\ \alpha \not\equiv 0 \pmod{\kappa}) \in L$, while if κ is inaccessible, each $X_\alpha = \tilde{X}_\alpha \cap \kappa^+$, where the $\tilde{X}_\alpha$ are classes of ordinals and the relation "$\xi \in \tilde{X}_\alpha$" is canonically Σ_1-definable over L. When κ is inaccessible, we shall also require the following property of the X_γ:

(*) If $\kappa < \gamma < \zeta < \kappa^+$ and ζ is a cardinal in L, then $\zeta = \sup(X_\gamma \cap \zeta)$.

Finally, for inaccessible κ, we take the X_γ to partition $U(\kappa)$, while if κ is singular, we take the X_γ to partition the set of odd multiples of κ^2 in $(\kappa,\ \kappa^+)$.

2.3.2. "Contamination" is most easily understood in the context of inaccessible κ. For such κ and $\alpha \in U(\kappa)$, it can occur that, for some inaccessible $\kappa' > \kappa$ and some $\alpha' \in U(\kappa')$, $b_\alpha \cap b_{\alpha'}$ is unbounded in κ. Further, it could also occur that $g^p(\alpha') = 1$ in some condition p, and in fact that $(\alpha',\ \xi) \in \Xi(p)$ for some $\xi < \kappa$. This will prevent us either from having $g^p(\alpha) = 0$ or from coding this correctly. When, for other reasons, we are *required* to have $g^p(\alpha) = 0$, α is said to be "contaminated" (by α') in p. We cannot prevent such contamination, but we will define conditions in such a way (see (3.2)(A), below) that

($*$) Fewer than κ many $\alpha \in (\kappa,\ \kappa^+)$ are contaminated.

Typically, contamination occurs here because κ was added to d^p after the promise (α', ξ) had already been made.

2.3.3. When κ is singular, we shall also have the phenomenon of contaminated ordinals. It may occur, for singular κ and conditions p with $\kappa \in d^p$, that for some $\xi \in U(\kappa) \cap \delta^p(\kappa)$, one of the following holds:

(1) There are $x_1 \neq x_2$ and cofinal subsets Y_1, Y_2 of b_ξ such that for $i = 1, 2$ and $\zeta \in Y_i$ we have $g^p(\zeta) = x_i$,

(2) $i \in \{0, 1\}$, and for other reasons we are required to have $g^p(\xi) = i$, but on a cofinal set of $\zeta \in b_\xi$, we have $g^p(\xi) = 1 - i$ (see (3.3) below, the definition of "forced to be i", for what these "other reasons" are).

If $g^q(\xi) = !$, then ξ will be deactivated anyway. If, however, this fails, then ξ is contaminated in q. Once again, contamination will occur only for a bounded set of ξ, though here this is a simple observation which does not require a special property of the conditions, as in the inaccessible case. Here again, typically, contamination arises due to the fact that an unbounded set of information below κ was part of a condition before κ was mentioned.

In both the inaccessible and the singular case, $\beta^p(\kappa)$ is the supremum of the contaminated ordinals $\xi \in (\kappa, \kappa^+)$ (see (3.4)(C)). Once $\beta^p(\kappa)$ has been specified in a condition, no further contamination is allowed in any stronger condition q, since $\beta^q(\kappa) = \beta^p(\kappa)$. In both the inaccessible and the singular case, we require (see (3.4)(B)), that if α is contaminated then $g^p(\alpha) = ?$.

2.3.4. We can now specify how the $\alpha \in X_\gamma$ are used to code γ. If $\gamma \in \operatorname{dom} g^p$, then we will have $g^p(\gamma) \in \{0, 1\}$. In the inaccessible case, we will have that if $\alpha \in X_\gamma \setminus \beta^p(\kappa)$, then $g^p(\alpha) = g^p(\gamma)$ (see (3.4)(A), and one clause of the definition of "forced to be i"). In the singular case, things are somewhat more complicated, since even for $\alpha \in X_\gamma \setminus \beta^p(\kappa)$, we can have $g^p(\alpha) \in \{?, !\}$. However, as part of the definition of condition (3.4)(A), again, we will have that $g^p(\alpha) \neq 1 - g^p(\gamma)$ for such α. We will show, by a density argument (see (6.1)(4)), that when $\mathcal{I}$ is generic and G is the union of the g^p for $p \in \mathcal{I}$, there will be a cofinal set of $\alpha \in X_\gamma$ such that $G(\alpha) = G(\gamma)$. Thus, in decoding, there is a common definition: decode for γ the unique value $i \in \{0, 1\}$ such that we have value i on a cofinal subset of X_γ.

2.3.5. To conclude, we should mention the other way the weak deactivator ?, can occur. For singular κ, in addition to occurring at contaminated α, it can occur at other $\alpha \in U(\kappa)$, but only if on a tail of b_α the value ? occurs. The reason for this has to do with the density argument we just mentioned. As will become clearer in (2.4), the strong deactivator at α can contribute to deactivating larger ordinals. This is not the case for the weak deactivator ?. Thus, the weak deactivator ? can play the role of "safe, neutral filler", and does not present the "potential danger" of forcing us to deactivate ordinals we want to preserve as "active", to get value in $\{0, 1\}$, such as the cofinally many $\alpha \in X_\gamma$ we need for the preceding.

2.4. The strong deactivator ! and the generic scales. Suppose that κ is singular and $\alpha \in U(\kappa)$. We have already mentioned most of the elements of this discussion:

(1) When $g^p(\alpha) = !$, not only α, but also all members of $U(\kappa)$ in the open interval $(\alpha, h^p(\alpha))$ are deactivated (if $g^p(\alpha) = !$ and $h^p(\alpha) = \alpha$, α

itself is still deactivated); this is one of the senses in which ! is the strong deactivator.

(2) $h^p(\alpha) = \mathrm{scale}(\sigma^{p,\alpha})$.

(3) If $g^p(\alpha) = !$ this can contribute to making $g^p(\nu) = !$, for certain larger $\nu \in U(\kappa)$; this is the other sense in which ! is the strong deactivator.

(4) When α is a limit point of $U(\kappa)$, C_α is an additional, auxiliary coding area used for detecting strong deactivation.

(5) Detecting strong deactivation and lengths of deactivated intervals (by (2), above, this amounts to the same thing as decoding the $\sigma^{p,\alpha}$) is one of our major preoccupations.

2.4.1. We now put these elements together and lay the groundwork for (3.5), omitting, for now, some of the finer points related to certain $\nu \geq \delta^p(\kappa)$ for which $\sigma^{p,\nu}$ will nevertheless be defined. We should say, at the outset, that $\sigma^{p,\alpha}$ will be defined whether or not we end up having $g^p(\alpha) = !$, but that this is just for convenience, since the only case in which it has any significance is when this occurs; when $g^p(\alpha) \neq !$, we ignore $\sigma^{p,\alpha}$ and take $h^p(\alpha)$ to be α. As we have already mentioned, we are grateful to the referee for emphasizing the point of view that the $\sigma^{p,\alpha}$ are really potential members of generic scales which we are forcing as we do the coding. In almost all cases, we will have $\sigma^{p,\alpha} \geq^* f^*_\alpha$; the exceptions are discussed in (3.5) and (3.6).

2.4.2. We will have two other functions, $\upsilon^{p,\alpha}$ and $\pi^{p,\alpha}$, and, in most cases, for $\lambda \in D_\kappa \cap d^p$, we take $\sigma^{p,\alpha}(\lambda) := \max(\upsilon^{p,\alpha}(\lambda), \pi^{p,\alpha}(\lambda))$. Looking at $\upsilon^{p,\alpha}$ amounts to considering what happens "from below", on b_α. Looking at $\pi^{p,\alpha}$ amounts to considering what happens "to the left", on C_α. These are two of the ways in which α could be strongly deactivated, and are two of the places we have to look to detect strong deactivation.

2.4.3. Before developing this, however, there is a third way in which α can be strongly deactivated, and we deal with this first, since it is simplest, and directly related to (1), above. α is p-interval-strongly-deactivated if it is in a deactivated interval $(\nu, h^p(\nu))$ for some $\beta^p(\kappa) \leq \nu < \alpha$. When this occurs, we take ν least possible and set $\sigma^{p,\alpha} := \sigma^{p,\nu}$, without considering the $\upsilon^{p,\alpha}$ and $\pi^{p,\alpha}$. Thus, when α is p-interval-stronly-deactivated, "only this counts", even if it turns out that it is also deactivated in one of the two other ways we now discuss. In terms of Lemma 4.3, this corresponds to the α between δ and the $t^p_2(\kappa)$ of (4.2), and, roughly speaking, to limit stages of GOOD's winning strategy.

2.4.4. α is p-strongly deactivated on b_α ("from below") iff on a tail of $\xi \in b_\alpha$ we have $g^p(\xi) = !$. Typically, this occurs when α *did not have to be deactivated,* when we are strongly deactivating α intentionally, to be sure that we are able to strongly deactivate other, larger, α, which will be more problematical; see the discusssion of $(*)$ in the Introduction. This corresponds to some of the work in (4.3) (beyond the $t^p_2(\kappa)$ of (4.3)) and all of the work of (6.1), and, roughly speaking, to successor stages in GOOD's winning strategy.

2.4.5. α is p-strongly deactivated on C_α iff it is a limit point of $U(\kappa)$ and on a cofinal subset of $\nu \in C_\alpha$ we have $g^p(\nu) = !$. Typically, this occurs in situations where we *really needed to deactivate* α and we are happy to find that we prepared for this by strongly deactivating enough members of C_α. This corresponds to the portion of the work in (4.3) dealing with $\delta^p(\kappa)$ and to the situation of $\alpha = \delta(\kappa)$ in (4.5), and, roughly speaking, to limit stages in GOOD's winning strategy.

2.4.6. It remains only to give the main idea of the definitions of the $\upsilon^{p,\alpha}$ and the $\pi^{p,\alpha}$ (there are some fine points which can be deferred until the official definition in (3.5)). The main idea for the $\upsilon^{p,\alpha}(\lambda)$ is that this should be $h^p(f^*_\alpha(\lambda))$. The fine points arise when $f^*_\alpha(\lambda) \geq \delta^p(\lambda)$. The main idea for the $\pi^{p,\alpha}(\lambda)$ is that this should be sup $\{\sigma^{p,\nu}(\lambda) | \nu \in C_\alpha\}$. The fine points arise because we want this supremum to be $\geq f^*_\alpha(\lambda)$, but $\leq \delta^p(\lambda)$.

2.5. Overview of the decoding procedure. Let $\chi = \chi_A$ be the (class) characteristic function of A. Our forcing will produce a generic class function G with domain $\subseteq OR$ and range $\subseteq \{0, 1, ?, !\}$. We will code A into G on odd ordinals, i.e., for nonsuccessor ordinals δ and $n < \omega$ we shall have $G(\delta + 2n + 1) = \chi(\delta + n)$.

Of course, we want to recover G from $G|\aleph_3$ by *decoding*. This is done by recursion on cardinals κ. The basic recursion step is to go from $G|\kappa$ to $G|(\kappa, \kappa^+)$, when $\kappa \in card$. This will involve a nested recursion across (κ, κ^+). The procedure for obtaining $G|(\kappa, \kappa^+)$ from $G|\kappa$ will be uniform within each of the following classes of cardinals: inaccessibles, singulars, and successors. Thus, at limit cardinals μ, we can piece together $G|\mu$ from the $G|\kappa$, $\kappa < \mu$, and continue. The recursion step for successor cardinals is provided by (2.1.4). As noted there, for inaccessibles, this also *essentially* gives the way we obtain G_0, which we now discuss.

For limit cardinals κ, it will simplify matters if, in decoding $G|(\kappa, \kappa^+)$, we have available not only $G|\kappa$, but also an auxiliary function G_0, which represents the first stage in defining $G|(\kappa, \kappa^+)$. The role of G_0 can best be understood by discussing the broad outline of how we finally obtain $G|(\kappa, \kappa^+)$. For inaccessible cardinals, this is a"two-pass" process. For singular cardinals, it is a "three-pass" process.

The first pass involves decoding the information provided by $G|\kappa$ on b_η without regard to the analogous information for the $\nu \in (\kappa, \eta)$. G_0 represents the outcome of this "first pass". For inaccessibles, even this first pass involves a recursion, since we have to decode the b_η as we go. For singulars, however, there is no recursion involved in the first pass, but there definitely is a recursion involved in the second pass for singulars, where we deal with the strong deactivator ! and the generic scales. The second pass for inaccessibles and the third pass for singulars are analogous in that this is where we deal with contamination, and define G on the nonmultiples of $\kappa \in (\kappa, \kappa^+)$.

§3. The coding conditions: definitions. We build to the definition of the class **P** of coding conditions in (3.5)–(3.6). In our original treatment we had stronger properties, which appear below as (4.1)(A) and (B+), in place of (3.5)(C) and (D). The latter are technical weakenings of the properties of (4.1), which are designed to allow us to prove, in (4.3), that the very tidy conditions, those with the properties of (4.1), are dense.

3.1. We recall some terminology and conventions from the Introduction and (2.2). A cardinal κ is *s-like* if it is singular or of the form $\aleph_\tau$, with $\tau > 1$ and odd. Next, let κ be a regular uncountable cardinal and let $\alpha \in (\kappa, \kappa^+)$. Recall that $\alpha \in E$ if $\alpha \in U(\kappa)$ and either κ is inaccessible or κ is s-like. Formally, for

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conditions p and $\alpha \in E$ we shall have $g^p(\alpha) \in \{0, 1\}$, but recall the convention from (2.2.3) involving the use of $\alpha + 2$ as an "extra bit" for $\alpha \in E$. Naturally, we have taken care not to assign any other "coding duties" to the $\alpha + 2$ when $\alpha \in E$.

DEFINITION 3.2. Suppose $p = (g, \beta, \Xi) = (g^p, \beta^p, \Xi(p)) \in P(0)$, where $P(0)$ is as in (2.2). Then $p \in \tilde{P}$ iff the following properties, (A) and (B) are satisfied.

(A) For all regular κ', there are fewer than κ' many α such that $(\alpha, \xi) \in \Xi(p)$ for some $\xi < \kappa$ (note: κ' need not be a member of d).

(B) If $\alpha \in \operatorname{dom} g^p$, $\kappa = card\, \alpha$ is singular, and α is a multiple of κ^2, then

(1) $f^*_\alpha <^* \delta^p | D_\kappa$,

(2) if $g^p(\alpha) \in \{0, 1\}$, then, on a tail of b_α, $g^p(\zeta) = g^p(\alpha)$,

(3) if $\beta^p(\kappa) \leq \alpha$ and $g^p(\alpha) \ = ?$, then, on a tail of b_α, $g^p(\zeta) \ = ?$.

3.3. Suppose $p \in \tilde{P}$. First, consider α such that $\kappa = card\, \alpha$ is a limit cardinal, and suppose that $\alpha \in X_\gamma$. We say that $g^p(\alpha)$ *is forced to be* 0 (*resp.* 1) if $g^p(\gamma) = 0$ (resp. 1). We also say that $g^p(\alpha)$ *is forced to be 1* if $\alpha \in X_\gamma$ for some $\gamma \in R(p)$. Finally, drop the restriction on κ. If $\alpha = 2\alpha' + 1$, then we say that $g^p(\alpha)$ *is forced to be* 0 (*resp.* 1) if $\alpha' \notin A$ (resp. $\alpha' \in A$).

If $\kappa = card\, \alpha$ is inaccessible and α is a multiple of κ, then α is *contaminated by* τ if $\tau \in W(p)$, $(\tau, \xi) \in \Xi(p)$ for some $\xi < \kappa$, $b_\tau \cap b_\alpha$ is cofinal in κ, and $g^p(\alpha)$ is forced to be 0; α is *contaminated* iff for some τ it is contaminated by τ. Because the system of b_α is tree-like for $\alpha \in \mathscr{U}$ (see (2.1.1)), it is easy to see that any $\tau \in W(p)$ contaminates at most one $\alpha \in (\kappa, \kappa^+)$. Therefore, (3.2)(A) says that there are fewer than κ many $\alpha \in (\kappa, \kappa^+)$ which are contaminated.

If $\kappa = card\, \alpha$ is singular and $d^p \cap D_\kappa$ is cofinal in κ, then α is *contaminated* iff α is a multiple of κ^2, $g^p(\alpha) \neq\, !$, and one of the following holds:

(1) There are $x_1 \neq x_2$ and cofinal subsets Y_1, $Y_2 \subseteq b_\alpha \cap \operatorname{dom} g^p$ such that $g^p(\xi) = x_i$ for $\xi \in Y_i$.

(2) $g^p(\alpha)$ is forced to be 0 (resp. 1), but on a cofinal subset of $b_\alpha \cap \operatorname{dom} g^p$ we have $g^p(\zeta) = 1$ (resp. 0).

Here, it is easy to see that at most κ many $\alpha \in (\kappa,\ \kappa^+)$ are contaminated, since if $\alpha > \operatorname{scale}(\delta^p | D_\kappa)$ then α cannot be contaminated. Also, note that α which are contaminated because of (2) are *odd* multiples of κ^2, since they are members of some X_γ.

DEFINITION 3.4. If $p \in \tilde{P}$, then $p \in P^*$ iff the following properties (A)–(E) hold.

(A) If $g^p(\alpha)$ is forced to be i and $g^p(\alpha) \in \{0, 1\}$, then $g^p(\alpha) = i$,

(B) If $\alpha \in \operatorname{dom} g^p$ and α is contaminated, then $g^p(\alpha) = ?$,

(C) For limit $\kappa \in d$, $\beta^p(\kappa) = \sup\, \{\alpha \in (\kappa, \kappa^+) | \alpha \text{ is contaminated}\}$,

(D) If $\kappa \in d^p$ and κ is inaccessible, we also define $\beta^p_1(\kappa) \ := \ \sup\, \{\alpha \in (\kappa, \kappa^2) | \alpha \text{ is contaminated}\}$ and we require:

(1) $g^p(\beta^p_1 + \kappa\omega) = 0$, and $g^p(\beta^p_1 + \kappa\sigma) = 1$ for all limit ordinals $\sigma < \kappa$,

(2) $g^p | (\beta^p_1 + \kappa\omega,\ \kappa^2)$ codes a well-ordering of κ in type $\beta^p(\kappa)$ on odd successor multiples of κ in $(\beta^p_1(\kappa) + \kappa\omega,\ \kappa^2)$, and codes $A \cap \beta^p(\kappa)$ on even successor multiples of κ in $(\beta^p_1(\kappa) + \kappa\omega,\ \kappa^2)$.

(E) Suppose $\kappa \in d$ is s-like. Suppose that $\beta^p(\kappa) \leq \alpha < \delta^p(\kappa)$, with α a multiple of κ^2. Let $\Gamma(\ ,\ )$ denote the Gödel pairing function. Let

$H^p(\alpha) := \{(\xi, \zeta) \in \kappa \times \kappa | g^p(\alpha + \kappa(1 + \Gamma(\xi, \zeta))) = 1\}$. We then require that $H^p(\alpha)$ is a well-ordering of a subset of κ which lies in $L[A \cap \kappa]$, and, if κ is singular, we also require that it is the $<_{L[A\cap\kappa]}$-least well-ordering of a subset of κ in its order type.

We let $h^p(\alpha) :=$ the least multiple of $\kappa^2 \geq$ the order type of $H^p(\alpha)$. We further require:

(1) $h^p(\alpha) \geq \alpha$.

(2) If κ is singular, $\beta^p(\kappa) \leq \beta < \alpha$, and β is a multiple of κ^2, then $h^p(\alpha) \geq h^p(\beta)$, and if $h^p(\beta) \geq \alpha$, then $h^p(\alpha) = h^p(\beta)$.

(3) If $g^p(\alpha) \neq\ !$, then we require that $h^p(\alpha)$ has the smallest value consistent with (1) and (2) (which, it will be easy to see from (3.5)(A), will be α).

3.5.

DEFINITION 3.5.1. Fix $p \in P^*$ and singular $\kappa \in d^p$. Suppose $\alpha \in U(\kappa) \setminus \beta^p(\kappa)$. We are mainly interested in the case where $\alpha \leq \delta^p(\kappa)$, but it will be useful to have the definition in the more general context. This results in somewhat more complicated definitions; we will also give the simpler definitions that result when we restrict to $\alpha \leq \delta^p(\kappa)$.

Let $g = g^p$, $h = h^p$, $f = f^*_\alpha$, $\beta = \beta^p(\kappa)$. We first define the additional functions $\upsilon^{p,\alpha}$, $\pi^{p,\alpha}$, $\sigma^{p,\alpha}$, with domain $D_\kappa \cap d^p$.

First, for $\lambda \in \operatorname{dom} \upsilon^{p,\alpha}$, if $f(\lambda) \in \operatorname{dom} g$, we set $\upsilon^{p,\alpha}(\lambda) = h^p(f(\lambda))$; otherwise, $\upsilon^{p,\alpha}(\lambda) = \delta^p(\lambda)$. Note that if $\alpha \in \operatorname{dom} g$, then on a tail of $\lambda \in D_\kappa \cap d$ we have $\upsilon^{p,\alpha}(\lambda) = h^p(f(\lambda)) \geq f(\lambda)$. Thus, if it is *not* the case that $f \leq^* \upsilon^{p,\alpha}$, then $\alpha \geq \delta^p(\kappa)$ and there is no tail of $b_\alpha \subseteq \operatorname{dom} g$.

We now define $\pi^{p,\alpha}$, $\sigma^{p,\alpha}$ by simultaneous recursion on α; at the same time we define three properties, Pr^p_1, Pr^p_2, Pr^p_3, by recursion on α when $\mathrm{Pr}^p_i(\alpha)$ holds. We use $\mathrm{Pr}^p(\alpha)$ as an abbreviation for $\mathrm{Pr}^p_1(\alpha)$ or $\mathrm{Pr}^p_2(\alpha)$ or $\mathrm{Pr}^p_3(\alpha)$. We say that α is *p-interval-strongly deactivated* iff $\alpha \leq \delta^p(\kappa)$ and $\mathrm{Pr}^p_1(\alpha)$ holds. We say that α is *p-strongly deactivated on* b_α iff $\alpha \leq \delta^p(\kappa)$ and $\mathrm{Pr}^p_2(\alpha)$ holds. Finally, we say that α is *p-strongly deactivated on* C_α iff $\alpha \leq \delta^p(\kappa)$ and $\mathrm{Pr}^p_3(\alpha)$ holds. We say that α is *p-strongly deactivated* iff it is p-strongly deactivated on b_α or it is p-interval-strongly deactivated or it is p-strongly deactivated on C_α. Thus, α is p-strongly deactivated iff $\alpha \leq \delta^p(\kappa)$ and $\mathrm{Pr}^p(\alpha)$ holds.

We now turn to the recursive definition of the two above-mentioned functions, and the three properties. $\mathrm{Pr}^p_1(\alpha)$ holds just in case there is $\nu \in U(\kappa) \cap [\beta, \alpha)$, such that $\mathrm{Pr}^p(\nu)$ holds and $\mathrm{scale}(\sigma^{p,\nu}) > \alpha$. If $\mathrm{Pr}^p_1(\alpha)$ holds, let ν be the least witness to this. In this case, we set $\pi^{p,\alpha} := \pi^{p,\nu}$, and $\sigma^{p,\alpha} := \sigma^{p,\nu}$.

Thus, for the definition of the two functions, we can assume $\mathrm{Pr}^p_1(\alpha)$ fails. In this case, for $\lambda \in \operatorname{dom} \pi^{p,\alpha}$ we set $\pi^{p,\alpha}_1(\lambda) := \min(\delta^p(\lambda), f(\lambda))$, $\pi^{p,\alpha}_2(\lambda) := \sup\{\sigma^{p,\nu}(\lambda) | \nu \in C_\alpha\}$ and $\pi^{p,\alpha}(\lambda) := \max(\pi^{p,\alpha}_1(\lambda), \pi^{p,\alpha}_2(\lambda))$. Finally, for $\lambda \in \operatorname{dom} \sigma^{p,\alpha}$, we set $\sigma^{p,\alpha}(\lambda) = \max(\upsilon^{p,\alpha}(\lambda), \pi^{p,\alpha}(\lambda))$.

We conclude by defining when the other two properties, $\mathrm{Pr}^p_2(\alpha)$ and $\mathrm{Pr}^p_3(\alpha)$, hold. $\mathrm{Pr}^p_2(\alpha)$ holds iff $\mathrm{Pr}^p(f(\lambda))$ holds on a tail of $\lambda \in D_\kappa$. $\mathrm{Pr}^p_3(\alpha)$ holds iff α is a limit of multiples of κ^2 and Z_α is cofinal in α, where $Z_\alpha = \{\nu \in C_\alpha | \mathrm{Pr}^p(\nu) \text{ holds}\}$.

Of course, more than one of these may be true for α. However, if $\mathrm{Pr}^p_1(\alpha)$, "only this counts", in terms of how $\sigma^{p,\alpha}$ is defined. It is also possible that, letting $\delta =$

$\delta^p(\kappa)$, δ is p-strongly-deactivated. This is clear in the case of interval deactivation and deactivation on C_δ. Deactivation on b_δ is only possible if a tail of $b_\alpha \subseteq \operatorname{dom} g$.

The simplifications which arise when we restrict to $\alpha \le \delta^p(\kappa)$ are mainly that we can remove the definitions of $\mathrm{Pr}_2^p(\kappa)$ and $\mathrm{Pr}_3^p(\alpha)$ from the recursion which gives us the definitions of $\pi^{p,\alpha}$ and $\sigma^{p,\alpha}$, by changing the definition of $\mathrm{Pr}_2^p(\alpha)$ to be: "on a tail of $\lambda \in D_\kappa$, $f(\lambda) \in \operatorname{dom} g$ & $g(f(\lambda)) = !$", and for $\mathrm{Pr}_3^p(\alpha)$, by changing the definition of Z_α to: "$\{\nu \in C_\alpha | g(\nu) = !\}$." The reasons will be clear from (A), below. We can also drop from the definition of $\mathrm{Pr}_1^p(\alpha)$ the requirement that $\mathrm{Pr}^p(\nu)$ holds.

A disquieting possibility is that $\mathrm{Pr}^p(\alpha)$ holds for *all* $\alpha \in U(\kappa) \setminus \beta^p(\kappa)$. In Remark 2 of (4.3) we shall show that this cannot occur. We are now ready for the definition of P.

DEFINITION 3.5.2. $p \in P$ iff $p \in P^*$, property (D), below, holds, and whenever κ, α, etc., are as above, and $\delta = \delta^p(\kappa)$, the following properties (A)–(C) hold:

(A) If $\alpha < \delta$, then $g(\alpha) = !$ iff α is p-strongly-deactivated,

(B) If $\alpha < \delta$ and $g(\alpha) = !$, then $h^p(\alpha) = \mathrm{scale}(\sigma^{p,\alpha})$,

(C) If $\neg(\delta^p | D_\kappa \le^* f_\delta^*)$, then δ is p-strongly-deactivated, $\delta^p|D_\kappa \le^* \sigma^{p,\delta}$; further, letting $\gamma = \mathrm{scale}(\sigma^{p,\delta})$, whenever $\eta \in U(\kappa)$ with $\delta < \eta < \gamma$, if $b_\eta \cap \operatorname{dom} g^p$ is cofinal in κ, then on a tail of $\xi \in b_\eta \cap \operatorname{dom} g^p$ we have $h^p(\xi) \le f_\gamma^*(\mathit{card}\,\xi)$,

(D) If $\lambda \in d^p$ is s-like, the set $\{\sigma^{p,\alpha}(\lambda) | \sigma^{p,\alpha}(\lambda)$ is defined$\}$ has power $\le \lambda$.

REMARK. The substantive part of (D) concerns those $\sigma^{p,\alpha}(\lambda)$ which are $> \delta^p(\lambda)$. As indicated at the beginning of this section, our original definition of P required that all the $\sigma^{p,\alpha}(\lambda) \le \delta^p(\lambda)$ and that, with the notation of (C), above, $\delta^p|D_\kappa =^* f_\delta^*$. Instead, we have opted to relax this requirement and show, in §4, that these properties hold on a dense set. With this in mind, condition (D) is clearly necessary for us to be able to extend p to a condition with these properties. (C) is a technical property, formulated with the same aim.

DEFINITION 3.6. If p, $q \in P$, we set $p \le q$ iff $g^p \subseteq g^q$, $\beta^p \subseteq \beta^q$, $\Xi(p) \subseteq \Xi(q)$.

REMARK 1. Note that if $p \le q$ then $h^p \subseteq h^q$. Note, also, that if $\alpha \ge \beta^p(\kappa)$ and $\alpha \in \operatorname{dom} g^p \cap U(\kappa)$, then $\upsilon^{p,\alpha} =^* \upsilon^{q,\alpha}$, $\pi_i^{p,\alpha} =^* \pi_i^{q,\alpha}$, $i = 1, 2$, and therefore $\pi^{p,\alpha} =^* \pi^{q,\alpha}$ and $\sigma^{p,\alpha} =^* \sigma^{q,\alpha}$. It is also easy to see that if $\delta = \delta^p(\kappa)$, then $\pi_2^{p,\delta} =^* \pi_2^{q,\delta}$. It is possible that $\upsilon^{q,\delta}(\lambda) > \upsilon^{p,\delta}(\lambda)$; this will occur exactly when $\delta^p(\lambda) < f_\delta^*(\lambda) < \delta^q(\lambda)$. Similarly, $\pi_1^{p,\delta}(\lambda) < \pi_1^{q,\delta}(\lambda)$ just in case $\delta^p(\lambda) < f_\delta^*(\lambda) < \delta^q(\lambda)$. Thus, we could have $\upsilon^{q,\delta}(\lambda) > \upsilon^{p,\delta}(\lambda)$ on a tail of λ. It is also clear that δ is p-interval-strongly deactivated just in case it is q-interval-strongly deactivated, and similarly for deactivation on C_δ. However, it is possible that δ is q-strongly deactivated on b_δ without being p-strongly deactivated on b_δ.

The situation is similar for $\alpha \in U(\kappa) \setminus \delta + 1$. It is easy to see that if $\mathrm{Pr}_i^p(\alpha)$ holds then $\mathrm{Pr}_i^q(\alpha)$ holds, and that $\upsilon^{p,\alpha} \le^* \upsilon^{q,\alpha}$, $\pi_i^{p,\alpha} \le^* \pi_i^{q,\alpha}$, and therefore $\pi^{p,\alpha} \le^* \pi^{q,\alpha}$, $\sigma^{p,\alpha} \le^* \sigma^{q,\alpha}$.

REMARK 2. In virtue of (3.5)(B), above, letting $\delta = \delta^p(\kappa)$, we define $h^p(\delta)$ by $h^p(\delta) := \mathrm{scale}(\sigma^{p,\delta})$.

REMARK 3. Let $\delta = \delta^p(\kappa)$, $\gamma = \mathrm{scale}(\delta^p|D_\kappa)$, and suppose that $\alpha \in U(\kappa) \cap$

(α, γ). Note that this occurs exactly when $\alpha \in U(\kappa)$, $\delta < \alpha$, and $b_\alpha \cap \operatorname{dom} g^p$ is cofinal in κ. Thus, for such α there is already an unbounded set of information imposed by p on b_α, which might require us to deactivate α, and the question arises of how far this deactivation should go. However, if this occurs, then we have the hypotheses of (3.5)(C), above, and so δ is p-strongly deactivated. Further, since (3.5)(C) gives us that $\delta^p|D_\kappa \leq^* \sigma^{p,\delta}$, and $\alpha < \gamma$, i.e. $\alpha < \operatorname{scale}(\sigma^{p,\delta})$. Thus, $\Pr_1^p(\alpha)$ holds. Suppose now that $q \geq p$ and $\delta < \delta^q(\kappa)$. By Remark 1, above, $\sigma^{p,\delta} \leq^* \sigma^{q,\delta}$. Thus, in such $q \geq p$, α will already be q-interval-strongly deactivated by δ. Finally, since $\alpha < \operatorname{scale}(\delta^p|D_k)$ and, by hypothesis, $b_\alpha \cap \operatorname{dom} g^p$ is cofinal in κ, it follows that $h^p(\xi) \leq f_\gamma^*(\mathit{card}\ \xi)$ on a tail of $\xi \in b_\alpha \cap \operatorname{dom} g^p$. Thus, as far as such α are concerned, δ already provides the essentials of the deactivation information.

§4. The coding conditions: Basic lemmas.

Definition 4.1. If $p \in P$, p is *tidy* iff for all s-like $\kappa \in d^p$ the following condition (A) holds and for all singular $\kappa \in d^p$, (B) holds.

(A) if $\alpha \in U(\kappa)$ with $\beta^p(\kappa) \leq \alpha < \delta^p(\kappa)$, then $h^p(\alpha) \leq \delta^p(\kappa)$.

(B) $\delta^p|D_\kappa \leq^* f_\delta^*$, where $\delta = \delta^p(\kappa)$.

If $p \in P$, then p is *very tidy* iff (A) holds for all s-like $\kappa \in d^p$, and (B+), below, holds for all singular $\kappa \in d^p$.

(1) (B+) $\delta^p|D_\kappa =^* f_\delta^*$, where $\delta = \delta^p(\kappa)$.

Remark 1. Suppose that p is tidy. We argue that for all singular $\kappa \in d^p$, all $\alpha \in U(\kappa) \cap [\beta^p(\kappa), \delta^p(\kappa)$, and all $\lambda \in D_\kappa \cap d^p$, $\sigma^{p,\alpha}(\lambda) \leq \delta^p(\lambda)$. It suffices, of course, to prove this for the $h^{p,\alpha}$ and the $\pi^{p,\alpha}$. For the $h^{p,\alpha}$, if $f_\alpha^*(\lambda) \notin \operatorname{dom} g$, then $h^{p,\alpha}(\lambda) = \delta^p(\lambda)$, so suppose that $f_\alpha^*(\lambda) \in \operatorname{dom} g$. Then, $h^{p,\alpha}(\lambda) = h^p(f_\lambda^*(\alpha))$, and by (A) above (with λ in place of κ and $f_\alpha^*(\lambda)$ in place of α), the latter is $\leq \delta^p(\lambda)$, as required. For the $\pi^{p,\alpha}$, we work by induction on α, with the induction hypothesis being the statement of the remark, i.e., the statement for the $\sigma^{p,\nu}$, with $\nu < \alpha$. But then the conclusion is immediate by the definition of $\pi^{p,\alpha}$: clearly, $\pi_1^{p,\alpha}(\lambda) \leq \delta^p(\lambda)$, and $\pi_2^{p,\alpha}(\lambda)$ is the supremum of things all $\leq \delta^p(\lambda)$ and so the conclusion is clear.

Remark 2. If p is very tidy, then for singular $\kappa \in d^p$ it is easy to see that, with the convention of Remark 2 of (3.6), $h^p(\delta^p(\kappa)) \leq \delta^p(\kappa)$. This is clear from Remark 1 and the fact (which is just a restatement of (B+)) that $\delta^p(\kappa) = \operatorname{scale}(\delta^p|D_\kappa)$.

Remark 3. If p is very tidy, $p \leq q$, and, for all s-like, regular $\lambda \in d^p$, $h^q(\delta^p(\lambda)) = \delta^p(\lambda)$, then $h^q(\delta^p(\kappa)) = \delta^p(\kappa)$ for all s-like $\kappa \in d^p$. This is easily argued by induction on the rank of κ in the well-founded relation "$\lambda \in D_\kappa$" . The basis is the hypothesis. Let $\eta := \delta^p(\kappa)$. By the induction hypothesis, we have that $h^{q,\eta} =^* \delta^p|D_\kappa$. Clearly $\pi_2^{q,\eta} =^* \pi_2^{p,\eta}$, and by Remark 2, $\pi_2^{p,\eta} =^* \delta^p|D_\kappa$. Clearly, $\pi_1^{q,\eta} =^* \delta^p|D_\kappa$, and the conclusion is then immediate.

4.2. The following material will be helpful in both (4.3) and §5. If $p \in P$, and t is a function with $d^p \subseteq \operatorname{dom} t$, we say that t *covers* p iff whenever $\kappa \in d^p$ is singular, $\alpha \in U(\kappa) \cap [\beta^p(\kappa), \delta^p(\kappa)]$, and $\lambda \in D_\kappa \cap d^p$, we have $\sigma^{p,\alpha}(\lambda) < t(\lambda)$. If $q \in P$, we we say that q *covers* p iff $p \leq q$ and δ^q covers p. If $q \in P$ and t is a function with $\operatorname{dom} t = d^q$, we say that q *dominates* t iff for all s-like $\lambda \in d^q$ we have $t(\lambda) < \delta^q(\lambda)$.

Next, we define still more functions associated with a $p \in P$. For singular $\kappa \in d^p$, and $\lambda \in D_\kappa \cap d^p$, we let $t_{\kappa,1}^p(\lambda) := \sup\{\sigma^{p,\alpha}(\lambda) | \alpha \in U(\kappa) \cap [\beta^p(\kappa), \delta^p(\kappa))\}$.

We note that, by (3.5)(D), $t^p_{\kappa,1}(\lambda) < \lambda^+$. For regular, s-like $\kappa \in d^p$, we let $t^p_1(\kappa) := \sup\{h^p(\alpha) | \alpha \in U(\kappa) \;\&\; \alpha < \delta^p(\kappa)\}$. Again, by (3.5)(D), for regular, s-like $\kappa \in d^p$ we have $t^p_1(\kappa) < \kappa^+$. For singular $\kappa \in d^p$, we let $t^p_1(\kappa) := \mathrm{scale}(t^p_{\kappa,1})$. Clearly, for singular $\kappa \in d^p$ we have $t^p(\kappa) < \kappa^+$. We also define the $t^p_{\kappa,2}$ and the t^p_2 analogously, but based on the function δ^p; thus, $t^p_{\kappa,2} := \delta^p | D_\kappa \cap d^p$, and $t^p_2(\kappa) := \mathrm{scale}(t^p_{\kappa,2})$, for singular $\kappa \in d^p$, while for s-like, regular $\kappa \in d^p$ we have $t^p_2(\kappa) := \delta^p(\kappa)$. Note that whenever these functions are defined, we have $t^p_{\kappa,2}(\lambda) \leq t^p_{\kappa,1}(\lambda)$ and $t^p_2(\kappa) \leq t^p_1(\kappa)$.

Now, let θ, $\mathscr{M}$, $\mathscr{N}$, etc., be as in (1.2), (1.3), and suppose that $p \in N$. Then it is obvious that the $t^p_i \in \mathrm{Sk}_{\mathscr{M}}(N)$ and that the $t^p_{\kappa,i} \in \mathrm{Sk}_{\mathscr{M}}(N \cup \{\kappa\})$, and therefore, that

$$(*) \qquad t^p_1(\kappa) < p\chi_{\mathscr{N}}(\kappa) \quad \text{for all s-like } \kappa \in d^p,$$

since $(*)$ holds for *any* $t \in \mathrm{Sk}_{\mathscr{M}}(N \cup \{\kappa\})$ in place of t^p_1.

LEMMA 4.3. *If $p \in P$, t is a function with* dom $t = d^p$, *and $t(\kappa) < \kappa^+$ for all $\kappa \in d^p$, then there is a very tidy $q \in P$ with $p \leq q$ and such that $t(\kappa) < \delta^q(\kappa)$ for all s-like $\kappa \in d^p$.*

PROOF. We shall prove this in the way that will be most useful for (5.1). Choose regular $\theta \geq \aleph_2$, and let $\mathscr{M}$, $\mathscr{N}$ be as above for this θ. We have just observed that since $p \in N$, $p\chi_{\mathscr{N}}$ is everywhere $\geq t^p_1$; similarly, since $t \in N$, if we construct very tidy $q \geq p$ such that

$$(*) \qquad \delta^q(\kappa) > p\chi_{\mathscr{N}}(\kappa) \quad \text{for all s-like } \kappa \in d^p,$$

then q will be as required. This is the approach we shall take; we shall choose such an $\mathscr{N}$, and construct q satisfying $(*)$ and such that, whenever $\kappa \in d^p$ is s-like, $g^q(\delta^p(\kappa)) = !$. Our approach to this will be to take $\gamma = p\chi_{\mathscr{N}}$, and to let γ^* be as given by (1.5) for this γ, and to take $d^q = d^p$, $\Xi(q) = \Xi(p)$, $\beta^q = \beta^p$, $\delta^q(\lambda) = \gamma^*(\lambda)$ for all s-like $\lambda \in d^p$, and $\delta^q(\lambda) = \delta^p(\lambda)$ for all other $\lambda \in d^p$. Recall that, for all singular $\kappa \in d^p$, letting $\nu = \gamma^*(\kappa)$, we have $\gamma^* | D_\kappa =^* f^*_\nu$. This makes it clear that we will have (B+) of (4.1) and that q will satisy $(*)$. Thus, in order to complete the proof, it will suffice to verify that (A) of (4.1) holds and that $q \in P$, since it will then be clear that $q \geq p$.

Before going further, it will be useful to exploit (4.1)(B+) some more. Suppose that u is a function with domain $=^* D_\kappa$. We make the following observations:

(A) Suppose that $u(\lambda) \leq \gamma^*(\lambda)$ for all $\lambda \in \mathrm{dom}\ u$. Then, clearly, $\mathrm{scale}(u) \leq \nu$.

(B) Suppose further that b is a cofinal subset of D_κ, and that $u(\lambda) = \gamma^*(\lambda)$ for $\lambda \in b$. Then, again clearly, $\mathrm{scale}(u) = \nu$.

An important property of the way we will define q is:

(C) $g^q(\eta) = !$, for all $\delta^p(\kappa) \leq \eta < \delta^q(\kappa)$, whether or not κ is singular.

By (C) and the definition of t^p_2 if κ is singular, then,

(D) for $t^p_2(\kappa) \leq \eta < \delta^q(\kappa)$, with $\eta \in U(\kappa)$, η will be q-strongly deactivated on b_η, if for no other reason.

It remains to see that for singular κ and $\delta^p(\kappa) \leq \eta < t^p_2(\kappa)$, with $\eta \in U(\kappa)$, we will still have that η is q-strongly deactivated. It is here that we will appeal to (3.5)(C). Let $\delta := \delta^p(\kappa)$. If $\delta^p | D_\kappa \leq^* f^*_\delta$, then $t^p_2(\kappa) = \delta$, so there is nothing to verify in this case. If, on the other hand, the above fails, then, by (3.5)(C), δ is p-strongly deactivated and $\delta^p|D_\kappa \leq^* \sigma^{p,\delta}$. Since it is clear that $\sigma^{p,\delta} \leq^* \sigma^{q,\delta}$, this means that δ will be q-strongly deactivated and that we will have $h^q(\delta) \geq \mathrm{scale}(\delta^p|D_\kappa)$, and $\mathrm{scale}(\delta^p|D_\kappa)$ is just $t_2^p(\kappa)$. This in turn means that if $\delta < \eta < t_2^p(\kappa)$, with $\eta \in U(\kappa)$, then η will be q-interval-strongly deactivated, as required.

The preceding guarantees that we can carry out our plan of making (C) hold, while respecting (3.5). It remains to complete the definition of the $g^q|[\delta^p(\kappa), \delta^q(\kappa))$ and to define the $h^q(\eta)$, for $\delta^p(\kappa) \leq \eta \leq \delta^q(\kappa)$, with $\eta \in U(\kappa)$. This will be done by recursion on the rank of κ in the well-founded relation "$\lambda \in D_\kappa$", so the basis is when κ is regular. In view of (C), we must carry out the following:

(1) Define $h^q(\eta)$ and $g^q|s_\eta$ for η as above.

(2) Define $g^q(\xi)$ for $\delta^p(\kappa) < \xi < \delta^q(\kappa)$ which are not covered by (C) nor by (1).

As far as (2) is concerned, in all cases, we shall have $g^q(\kappa) := 1$ unless it is forced to be 0. As far as (1) is concerned, once we have computed $h^q(\eta)$, (3.4)(E) tells us how to define $g^q|s_\eta$. Thus, it remains to compute the $h^q(\eta)$ and verify that the computed value is consistent with (3.4), (3.5) and (4.1)(A). It will then be clear that q is a very tidy condition, and, of course, that $p \leq q$, completing the proof of the lemma. Of course, (3.5)(B) tells us that for singular κ, in order to compute the $h^p(\eta)$, it suffices to compute the $\sigma^{p,\eta}$. This will be done by recursion on η, within the recursion on κ.

We now appeal to (A) and (B). Our induction hypotheses are

(E) For all relevant $\lambda < \kappa$ and $\delta^p(\lambda) \leq \eta' < \delta^q(\lambda)$, we have $h^q(\eta') \leq \gamma^*(\lambda)$.

(F) For all relevant $\delta^p(\kappa) \leq \eta' < \eta$ and *all*, (not just on a tail of) $\lambda \in D_\kappa \cap d^p$, we have $\sigma^{q,\eta'}(\lambda) \leq \gamma^*(\lambda)$.

Now, (E) guarantees that $h^{q,\eta}(\lambda) \leq \gamma^*(\lambda)$ for all $\lambda \in D_\kappa \cap d^p$. Similarly, (F) guarantees that $\pi_2^{q,\eta}(\lambda) \leq \gamma^*(\lambda)$ for all $\lambda \in D_\kappa \cap d^p$ and therefore, for all such λ, $\sigma^{q,\eta}(\lambda) \leq \gamma^*(\lambda)$. This preserves the induction hypothesis (F), and then, by (A), $\mathrm{scale}(\sigma^{q,\eta}) \leq \gamma^*(\kappa)$, which preserves the induction hypothesis (E), at least as far as η. As indicated above, this completes the proof that $q \in P$ and $p \leq q$. □

REMARK 1. The q we obtain depends, obviously, on p and on the $\mathscr{N}$ we choose, so we naturally denote it by $q(p, \mathscr{N})$. A very plausible choice for $\mathscr{N}$ is to take it to be some sort of Skolem hull in $\mathscr{M}$ of p, t, and possibly some additional elements, e.g., all the members of θ, for some cardinal θ. We return to this in (5.1) and (5.2), below. In this connection, it is also worth pointing out that if $p \in |\mathscr{N}|$, $\mathscr{N}^* \prec \mathscr{M}$, and, letting $N^* := |\mathscr{N}^*|$, if $[N^*]^{<\theta} \cup N \cup \{\mathscr{N}\} \subseteq |N^*|$, $card\, N^* = \theta$, then clearly $q \in N^*$.

REMARK 2. We are now in a position to show, as promised at the end of (3.5.1), that in no condition $q \in P$ does $\mathrm{Pr}^q(\alpha)$ hold for a tail of $\alpha \in U(\kappa) \setminus \delta^q(\kappa) + 1$, where $\kappa \in d^q$ is singular. What we show, in fact, is that if $p \in P$ is very tidy, $\kappa \in d^p$ is singular, and $\alpha \in U(\kappa) \setminus \delta^p(\kappa) + 1$, then $\mathrm{Pr}^p(\alpha)$ fails. This suffices, since, as noted in the last sentence of Remark 1 of (3.6), if $q \leq p$, then for $\alpha \in U(\kappa)$, if $\mathrm{Pr}_i^q(\alpha)$ holds then $\mathrm{Pr}_i^p(\alpha)$ holds.

This will also complement Remark 3 of (3.6), since if $p \in P$ is very tidy and $\kappa \in d^p$ is singular, then, letting $\delta = \delta^p(\kappa)$, we have that $\delta = \mathrm{scale}(\delta^p|D_\kappa)$, so there are no ordinals α of the sort dealt with in Remark 3 of (3.6). We would like to know that no others share the property pointed out there, of being q-strongly deactivated in *any* $q \geq p$ with $\delta^q(\kappa) \geq \alpha$. Actually, this will follow from the extendability properties developed in (6.1), but showing that if $\Pr^p(\alpha)$ holds then $\alpha \leq \delta^p(\kappa)$ will in fact be quite useful for (6.1).

If $\alpha \in U(\kappa) \setminus \delta + 1$, since p is very tidy, there is a tail of b_α disjoint from dom g^p, so $\Pr_2^p(\alpha)$ fails. Again, since p is tidy we cannot have $h^p(\eta) > \alpha$, for any $\eta < \delta^p(\kappa)$, and by Remark 3 of (4.1) we cannot have $h^p(\delta) > \alpha$. We conclude by induction on α, so suppose that $\Pr^p(\gamma)$ fails for all $\gamma \in (\delta, \alpha) \cap U(\kappa)$. Clearly, then $\Pr_3^p(\alpha)$ cannot hold, and $\Pr_1^p(\alpha)$ cannot be witnessed by any $\gamma \in (\delta, \alpha)$. But we have already argued that $\Pr_1^p(\alpha)$ cannot be witnessed by any $\eta \leq \delta$, so the proof is complete.

4.4. Factoring. Let θ be a regular cardinal, $\theta \geq \aleph_2$, and let $p \in P$. We set $W_\theta(p) := W(p) \setminus \theta^+$ and $\Xi_\theta(p) := \{(\alpha, \max(\xi, \theta)) | (\alpha, \xi) \in \Xi(p) \;\&\; \alpha \in W_\theta(p)\}$. We let $W^\theta(p) := W(p) \cap \theta^+$, $\Xi^\theta(p) := \Xi(p) \cap W^\theta(p) \times \theta$, and $R^\theta(p) := \bigcup\{b_\alpha \cap [\xi, \theta) | (\alpha, \xi) \in \Xi(p) \;\&\; \xi^p(\alpha) < \theta,\ \theta^+ \leq \alpha\}$. We are now ready to define the upper and lower parts of $\mathbf{P}$, relative to θ, which give the Factoring property of $\mathbf{P}$.

DEFINITION. For p and θ as above, we set $(p)_\theta = (g^p | d^p \setminus \theta,\ \beta^p | d^p \setminus \theta,\ \Xi_\theta(p))$. Note that $(p)_\theta \in P$. We let $\mathbf{P}_\theta = \{(p)_\theta | p \in P\}$, with the restriction of $\leq$. Thus, $\mathbf{P}_\theta$ is the class of conditions for coding down to a subset of θ^+. Note that $\mathbf{P} = \mathbf{P}_{\aleph_2}$, since $(p)_{\aleph_2} = p$ for $p \in P$.

We also define $(p)^\theta$, for $p \in P$: $(p)^\theta = (g^p|\theta,\ \beta^p|\theta,\ \Xi^\theta(p),\ R^\theta(p))$, and we let $\dot{P}^\theta := \{((p)^\theta, (p)_\theta) | p \in P\}$. Thus, $\dot{P}^\theta$ is a (proper class) $\mathbf{P}_\theta$-name for a subset of $\{(p)^\theta | p \in P\}$. We have guaranteed that the latter is a set by replacing $\{(\alpha, \xi) | (\alpha, \xi) \in \Xi(p) \;\&\; \xi < \theta,\ \theta^+ < \alpha\}$ by $R^p(\theta)$. Of course, our intention is to have $\dot{P}^\theta$ be the name of the underlying set of a partial subordering of $(\{(p)^\theta | p \in P\}, S)$, where $(p)^\theta\ S\ (q)^\theta$ iff $g^p|\theta \subseteq g^q$, $\beta^p|\theta \subseteq \beta^q$, $R^\theta(p) \subseteq R^\theta(q)$, $\Xi^\theta(p) \subseteq \Xi^\theta(q)$. We let $\dot{\mathbf{P}}^\theta$ be the name for this subordering.

In fact we can cut $\dot{\mathbf{P}}^\theta$ down to a set name, as follows. Let $n < \omega$ be such that all relevant notions about $\mathbf{P}$ are Σ_n. Let $\chi \gg \theta$ be such that $(H_\chi, \in)$ reflects all Σ_n formulas. The set name is then simply
$\{((p)^\theta, (p)_\theta) | p \in P \cap H_\chi\}$, and whenever $p \in P$, there is $q \in H_\chi$ such that $(p)^\theta = (q)^\theta$. What makes $\dot{P}$ a name is the linkage between the "top" and the "bottom", which is what guarantees that $\dot{\mathbf{P}}^\theta$ will code down to a subset of $\aleph_3$ the subset $\dot{B}$ of θ^+ added by $\mathbf{P}_\theta$. The $R^\theta(p)$ is one feature of this linkage. The following is then clear:

LEMMA (Factoring). $\mathbf{P} \cong \mathbf{P}_\theta * \dot{\mathbf{P}}^\theta$.

LEMMA 4.5. *Let θ be as in (4.4), and suppose that $\sigma \leq \theta$ is a limit ordinal and $(p_i | i < \sigma)$ is an increasing sequence from P_θ. Let $d := \bigcup\{d^{p_i} | i < \sigma\}$, $g := \bigcup\{g^{p_i} | i < \sigma\}$, $\beta := \bigcup\{\beta^{p_i} | i < \sigma\}$, $\Xi := \bigcup\{\Xi(p_i) | i < \sigma\}$. For $\kappa \in d$, let $\delta(\kappa) := \bigcup\{\delta^{p_i}(\kappa) | i < \sigma \;\&\; \kappa \in d^{p_i}\}$. If $\kappa \in d$ is singular, set $\eta \in Z(\kappa)$ iff $\eta \in C_{\delta(\kappa)}$ and $g(\eta) = !$. Let $i \in I(\kappa)$ iff, for some $i \leq j < \sigma$ and some $\eta \in Z(\kappa)$, $\delta^{p_i}(\kappa) \leq \eta$, and, for all $\lambda \in D_\kappa \cap d^{p_i}$, $\delta^{p_i}(\lambda) \leq \sigma^{p_i,\eta}(\lambda)$.*

Suppose, further, that $(p_i | i < \sigma)$ has the following properties.

(1) *$\{i < \sigma | p_i$ is tidy$\}$ is cofinal.*

(2) *For all singular* $\kappa \in d$, $I(\kappa)$ *is cofinal in* σ (*this, of course, implies that* $Z(\kappa)$ *is cofinal in* $\delta(\kappa)$).

Then, $p := (g, \beta, \Xi) \in P_\theta$ *and is the least upper bound for* $(p_i | i < \sigma)$.

PROOF. We will concentrate on showing that (3.5)(C) holds. This is the heart of the matter for verifying that $p \in P_\theta$, as verifying that the other clauses hold is totally routine, and once we know that $p \in P_\theta$ it is clear that it is the least upper bound.

So, let κ be as in the statement of the lemma, and adopt the other notation there. Also, let $\alpha := \delta(\kappa)$, and note that $Z(\kappa)$ is just the Z_α of (3.5). As observed in the parenthetical remark to hypothesis (2) in the statement of the lemma, Z_α is therefore cofinal in α, which means that α is p-strongly deactivated on C_α. So, it remains to verify the last clause of (3.5)(C).

For this, we first note that for all $\lambda \in D_\kappa \cap d$ we have

$$(*) \qquad \pi_2^{p,\alpha}(\lambda) \geq \delta(\lambda).$$

This is because, since $I(\kappa)$ is cofinal in σ, if $\lambda \in D_\kappa \cap d$, then $\delta(\lambda) = \sup \Delta$, where $\Delta := \{\delta(i)(\lambda) | i \in I(\kappa) \ \&\ \lambda \in d^{p_i}\}$. Now, let $i \in I(\kappa)$ with $\lambda \in d^{p_i}$. Let $i \leq j < \sigma$ and $\eta \in Z(\kappa) \setminus \delta^{p_i}(\kappa)$ be as guaranteed by the fact that $i \in I(\kappa)$. Then, $\pi^{p,\alpha}(\lambda) \geq \sigma^{p_j,\eta}(\lambda) \geq \delta^{p_i}(\lambda)$.

We now complete the proof by verifying the last clause of (3.5)(C). So, let $\gamma := \text{scale}(\sigma^{p,\alpha})$. Note that by virtue of $(*)$, and the fact that $\sigma^{p,\alpha}(\lambda) \geq \pi_2^{p,\alpha}(\lambda)$ for all relevant λ, we have that $f_\gamma^* \geq^* \sigma^{p,\alpha} \geq^* \pi_2^{p,\alpha} \geq^* \delta | D_\kappa$. Now, if $\eta \in U(\kappa)$ with $\alpha < \eta < \gamma$, if $b_\eta \cap \operatorname{dom} g^p$ is cofinal in κ, and if $\xi \in b_\eta \cap \operatorname{dom} g^p$, then $\xi < \delta(\mathit{card}\, \xi)$. But then there is $i < \sigma$ with $\xi < \delta^{p_i}(\mathit{card}\, \xi)$, and by virtue of (1), we can assume that p_i is tidy. But then $h^{p_i}(\xi) < \delta^{p_i}(\mathit{card}\, \xi)$, by (4.1)(A), and since $h^{p_i}(\xi) = h^p(\xi)$, the conclusion is clear. □

REMARK. In addition to the hypotheses of the lemma, suppose that $\sigma < \theta$, that $\mathscr{M}$ is as in (1.2) and (1.3), that $\mathscr{N} \prec \mathscr{N}^* \prec \mathscr{M}$ are such that, letting $N := |\mathscr{N}|$ and $N^* := |\mathscr{N}^*|$, we have $p_i \in N$ for all $i < \sigma$, and that $[N^*]^{<\theta} \cup N \cup \{\mathscr{N}\} \subseteq N^*$. Then $p \in N^*$.

4.6. Decoding. We now complete the sketch of the decoding procedure given in (2.5) by supplying the details of the decoding for limit cardinals. For singulars, in addition to the case of decoding the generic, we also treat the case of decoding $g | [\kappa, \delta^*)$ from a g defined on a large enough domain below κ. We will give the details of the situation when we encounter it. This case is needed in (6.1)(7)(b), below. We treat the inaccessible case first.

4.6.1. Assume that κ is inaccessible, and that we are given G, a function with range $G \subseteq \{0, 1, ?, !\}$ and $\operatorname{dom} G = \bigcup\{(\lambda, \lambda^+) | \aleph_2 \leq \lambda < \kappa,\ \lambda \text{ a cardinal}\}$, such that for some generic ideal $\mathscr{I}$ in $\mathbf{P}$, $G = \bigcup\{g^p | \kappa \ | p \in \mathscr{I}\}$.

We will first obtain an ordinal β which will be the common value of the $\beta^p(\kappa)$ for $\kappa \in d^p$, $p \in \mathscr{I}$. Recall that, by Lemma 3 of the Introduction and the discussion preceding it, for all $\alpha \in (\kappa, \kappa^+)$, if $\kappa \leq \nu \leq \alpha$ and in $L[A \cap \nu]$, $\mathit{card}\, \alpha = \kappa$, then we obtain $(b_\eta | \kappa < \eta \leq \alpha)$ canonically from α in $L[A \cap \nu]$. In particular, in L we have $(b_\alpha : \alpha < \kappa^2)$. So, we first decode G on $U(\kappa) \cap \kappa^2$. For such α, we let $G_0(\alpha) = 1$ iff $G(\xi) = 1$ on a tail of $\xi \in b_\alpha$. Otherwise, we set $G_0(\alpha) = 0$. Now, by (3.4)(D)(1), there is a largest $\nu \in (U(\kappa))' \cap (\kappa, \kappa^2)$ such that $G_0(\nu) = 0$ and, further, this ν is of the form $\eta + \kappa\omega$, where $\eta \in \{\kappa\} \cup U(\kappa)$. Also, by (3.4)(D)(2), we have that $G_0|(\eta, \kappa^2)$ codes a well-ordering of κ on odd successor multiples of κ in this interval. We take $\beta :=$ the order-type of this well-ordering. By (3.4)(D)(2), again, we have that β is the common value of $\beta^p(\kappa)$ for $\kappa \in d^p$, $p \in \mathcal{J}$ and that $A \cap \beta$ is coded by G_0 on even successor multiples of κ in this interval.

4.6.2. We can now define $G(\alpha)$ by recursion on α for $\alpha \in U(\kappa) \setminus \beta$. We first define $\nu(\alpha)$ and obtain $A \cap \nu(\alpha)$. We shall have *card* $\alpha = \kappa$ in $L[A \cap \nu(\alpha)]$, so that we have b_α available. If α is not a cardinal in L, we let $\nu(\alpha) := \mathit{card}^L\ \alpha$. Otherwise, we let $\nu(\alpha) = \alpha$. Recall from (2.3) that if $\kappa < \zeta < \kappa^+$ is a cardinal in L, then, for all nonmultiples of κ, γ with $\kappa < \gamma < \zeta$, $X_\gamma \cap \zeta$ is cofinal in ζ. This allows us to define $A \cap \nu(\alpha)$ as follows. If $\nu(\alpha) \leq \beta$, then we already have $A \cap \beta$. Otherwise, if $\kappa < \xi < \nu(a)$, let $\gamma := 2\xi + 1$. Then, $\gamma < \nu(\alpha)$, so $X_\gamma \cap (\beta, \nu(\alpha)) \neq \varnothing$, and we have $\xi \in A$ iff, $G(\eta) = 1$ for some (all) $\eta \in X_\gamma \cap (\beta, \nu(\alpha))$. Thus, we have $A \cap \nu(\alpha)$, and therefore in $L[A \cap \nu(\alpha)]$ we have b_α. We set $G(\alpha) := 1$ iff $G(\xi) = 1$ on a tail of $\xi \in b_\alpha$; otherwise, $G(\alpha) = 0$. This completes the recursion.

4.6.3. We can then define G on $(\kappa, \kappa^+) \setminus U(\kappa)$ by $G(\gamma) := i$ iff $G(\alpha) = i$ for some (all) $\alpha \in X_\gamma \setminus \beta$. Finally, we can go back and define G on $U(\kappa) \cap \beta$ as follows: Let $\gamma \notin U(\kappa)$ be such that $\alpha \in X_\gamma$. If $G(\gamma) = 1$ and, on a tail of $\xi \in b_\alpha$, $G(\xi) = 1$, define $G(\alpha) := 1$. If $G(\gamma) = 0$ and, on a cofinal subset of $\xi \in b_\alpha$, $G(\xi) = 0$, then $G(\alpha) := 0$. Otherwise, we set $G(\alpha) := ?$. This completes the decoding procedure in the inaccessible case.

4.6.4. So, assume next that κ is singular. We first treat the generic case. Assume that G and $\mathcal{J}$ are as above. This time there are "three passes" in the definition of G. The first is relatively straightforward: we ignore deactivated intervals, deactivation on C_α, we ignore the $\pi^{p,\alpha}$ and the $\sigma^{p,\alpha}$, and just organize the information "from below" provided by G. This is done simultaneously for all members of $U(\kappa)$. No recursion is involved. Recall here that we have $A \cap \kappa$ at our disposal, and that all of the coding apparatus is present in $L[A \cap \kappa]$. The second pass proceeds by recursion on $\alpha \in U(\kappa)$, and takes into account all of the above; we also define the $G|s_\alpha$. At the end of the second pass, we will have defined a function G_1 on all multiples of κ in (κ, κ^+). In the third pass we go back, define G on the nonmultiples of κ, detect contamination, define β (which, once again, will be the common value of the $\beta^p(\kappa)$ for $p \in \mathcal{J}$ with $\kappa \in d^p$), and revise the definition of G_1 below β.

For the first pass, if $\alpha \in U(\kappa)$ and there is no tail of b_α on which G is constant, set $G_0(\alpha) := ?$; otherwise, if $x \in \{0, 1, ?, !\}$ and G has constant value x on a tail of b_α, set $G_0(\alpha) := x$. We also define Υ^α at this time, by letting $\Upsilon^\alpha(\lambda) :=$ the order-type of the well-ordering coded by $G|s_{f^*_\alpha(\lambda)}$, for $\lambda \in D_\kappa$.

4.6.5. For the second pass, we assume that we have defined G_1 on all multiples of κ below α in such a way that for $\kappa < \nu < \alpha$, with $\nu \in U(\kappa)$, $G_1|s_\nu$ codes a well ordering of κ in order type $\geq \nu$. We let $H(\nu) :=$ the order type of this well ordering. We also assume that we have defined functions π^ν, σ^ν with domain D_κ, for such ν, "correctly" (i.e., according to (3.5) and what now follows), so far. If there is such a $\nu < \alpha$ with $H(\nu) > \alpha$, then we take the least such ν and set $G_1(\alpha) := !$, $\sigma^\alpha(\lambda) = \sigma^\nu(\lambda)$, for all $\lambda \in D_\kappa$, and we define $G_1|s_\alpha$ to code the $<_{L[A\cap\kappa]}$-least well ordering of κ in type $H(\nu)$.

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So, assume there is no such ν. Let $f := f^*_\alpha$ and define $\pi_1^\alpha := f$, and, for $\lambda \in D_\kappa$, define $\pi_2^\alpha(\lambda) := \sup\{\sigma^\nu(\lambda) | \nu \in C_\alpha\}$ and $\pi^\alpha(\lambda) := \max(\pi_1^\alpha(\lambda), \pi_2^\alpha(\lambda))$, $\sigma^\alpha(\lambda) = \max(H^\alpha(\lambda), \pi^\alpha(\lambda))$. If $G(\xi) = !$ on a tail of $\xi \in b_\alpha$, then we already had $G_0(\alpha) = !$ and we maintain $G_1(\alpha) := !$. However we also set $G_1(\alpha) := !$ if α is a limit of multiples of κ^2 and Z_α is cofinal in α, where $Z_\alpha := \{\nu \in C_\alpha | G_1(\nu) = !\}$. In all other cases, we maintain $G_1(\alpha) := G_0(\alpha)$. If $G_1(\alpha) = !$, we define $G_1|s_\alpha$ to code the $<_{L[A\cap\kappa]}$ -least well ordering of κ in type scale(σ^α). In all other cases we define $G_1|s_\alpha$ to code the $<_{LA\cap\kappa]}$ -least well ordering of κ in type α. This completes the recursive definition of G_1 on the multiples of κ. The following statement (whose verification is now totally straightforward and is left to the reader) makes precise the claim that this decoding procedure decodes correctly on a tail of $U(\kappa)$:

($*$) If $p \in \mathcal{S}$, $\kappa \in d^p$, $\beta^p(\kappa) \le \alpha < \delta^p(\kappa)$, and $\alpha \in U(\kappa)$, then $G_1(\alpha) = g^p(\alpha)$.

4.6.6. Before turning to the remainder of the definition in the generic case, we turn to the decoding of $g|[\kappa, \delta^*)$ on the multiples of κ only, from a g defined on a large enough domain below κ, since in (6.1)(7)(b) we only need this for the multiples of κ. Here, rather than having G defined on $\bigcup\{(\lambda, \lambda^+) | \aleph_2 \le \lambda = card\, \lambda < \kappa\}$, we have a tail t of D_κ and a function δ with domain t such that $\delta(\lambda) \in U(\lambda)$ for $\lambda \in t$ with $\lambda < \delta(\lambda) < \lambda^+$, such that g is defined on $\bigcup\{(\lambda, \delta(\lambda)) | \lambda \in t\}$ (of course, g will also be defined elsewhere, but only this is relevant for our decoding). We take $\delta^* = $ scale(δ). Finally, to complete the description of the context of (6.1)(7)(b), below, we have that there is a very tidy $p \in P$ such that $t = d^p \cap D_\kappa$ and that for $\lambda \in t$,

$$\delta^p(\lambda) \le \delta(\lambda), \qquad g^p|(\lambda, \delta^p(\lambda)) = g|(\lambda, \delta^p(\lambda)),$$

and that for $\alpha \in U(\lambda) \cap [\delta^p(\lambda), \delta(\lambda))$ we will have $g|s_\alpha$ coding the $<_{L[A\cap\kappa]}$ -least well ordering of λ in type α, and $g(\alpha) = ?$ except possibly in the case of $\alpha = \delta^p(\lambda)$ when it is also possible that $g(\delta^p(\lambda)) = !$.

Now that we have described the context, the procedure is essentially identical to the above, with the obvious notational analogies, so we limit ourselves to describing the more substantial differences. These deal only with the definitions of the functions analogous to the H^α and π_1^α, above. In both cases, we impose the requirement that $h^\alpha(\lambda)$, $\pi_1^\alpha(\lambda) \le \delta(\lambda)$ by taking them as defined to be the minimum of $\delta(\lambda)$ and the value defined as above. This completes the treatment of the singular nongeneric case.

4.6.7. We complete the description of the decoding procedure by returning to the final phase of the singular generic case: defining G on the nonmultiples of κ in (κ, κ^+), detecting contamination, and revising the definition of G_1 on the bounded initial segment of multiples of κ where there is contamination. In several places our argument will appeal to density arguments from (6.1). We should emphasize that *there is no circularity here*, since we have already completed the portion of the argument (the singular nongeneric case) needed in (6.1)(7)(b).

So, let $\kappa < \eta < \kappa^+$ with η not a multiple of κ. We argue that there is a tail T of X_η and an $i \in \{0, 1\}$ such that $1 - i \notin G_1[T]$ and, for a cofinal set of $\alpha \in T$, $G_1(\alpha) = i$. We first show that $X_\eta \cap G_1^{-1}[\{0\}]$, and $X_\eta \cap G_1^{-1}[\{1\}]$ cannot both be cofinal. To this end, let $p \in \mathcal{S}$ be such that $\kappa \in d^p$ and $\eta < \delta^p(\kappa)$ (such a p exists, by (6.1)(4) and (7)). Suppose, now, towards a contradiction, that $\beta^p(\kappa) \leq \alpha_0$, $\alpha_1 \in X_\eta$, and that $G_1(\alpha_j) = j$. Let $i := g^p(\eta)$. By (6.1)(4), again, there are q, $r \in \mathcal{J}$ with $\delta^q(\kappa) > \alpha_0$, $\delta^r(\kappa) > \alpha_1$. Clearly we can assume that $p \leq q \leq r$. By $(*)$, above, $g^r(\alpha_j) = j$. But this contradicts (3.4)(A), since $g^r(\alpha_j)$ is forced to be i. Finally, from $(*)$, above and (6.1)(4), it is immediate that if $\alpha < \kappa^+$ there is $\alpha < \alpha'$ and $p \leq q \in \mathcal{J}$ with $\alpha' \in X_\eta \cap \delta^q(\kappa)$ such that $g^q(\alpha') = i$ and $\beta^q(\kappa) \leq \alpha'$. Then, by $(*)$, again, $G_1(\alpha') = i$ and we are finished.

So, we define $G(\eta)$:= that $i \in \{0, 1\}$ such that $G_1(\alpha) = i$ for a cofinal set of $\alpha \in X_\eta$. Finally, for $\alpha \in U(\kappa)$, we say that α is G_1-contaminated iff (following (3.3)) $G_1(\alpha) \neq\, !$ and either there are cofinal subsets, $Y_i \subseteq b_\alpha$, $i = 1, 2$, and $x_1 \neq x_2$ such that $G(\xi) = x_i$ for $\xi \in Y_i$ or, if α is an odd multiple of κ^2, letting γ be such that $\alpha \in X_\gamma$, we have $G_1(\alpha) \in \{0, 1\}$ but $G_1(\alpha) \neq G(\gamma)$. We let $\beta := \sup\{\alpha | \alpha \text{ is } G_1\text{-contaminated}\}$. It is totally straightforward (and left to the reader to verify) that if $p \in \mathcal{J}$ and $\kappa \in d^p$, then $\beta = \delta^p(\kappa)$. Then we define $G(\alpha)$ for $\kappa < \alpha < \kappa^+$, α a multiple of κ, by setting $G(\alpha) := G_1(\alpha)$ if $\beta \leq \alpha$, while for $\alpha \in U(\kappa) \cap \beta$ we set $G(\alpha) :=\, ?$ and define $G|s_\alpha$ to code the $<_{L[A\cap\kappa]}$ -least well-ordering of κ in type α. This completes the decoding procedure.

§5. Strategic closure and distributivity.

5.1. The winning strategy. In this subsection we prove Lemma 5 of the Introduction. So, let $\theta \geq \aleph_2$ be regular and fix $p_0 \in P_\theta$, $\mathcal{M}$, and $\vec{\mathcal{N}}$ as in (1.1). Further, assume that $\vec{\mathcal{N}}$ is super $\mathcal{M}$-coherent, and recall the definition of the game $G(\theta, \vec{\mathcal{N}}, p_0)$ in (1.1) (where $\mathbf{Q} = \mathbf{P}$ and X is the class of very tidy conditions). Recall that BAD must play very tidy conditions.

GOOD's strategy will be to use (4.3) at successor stages, so that, in the notation of (4.3), she will have $p_{2\alpha+2} := q(p_{2\alpha+1}, \mathcal{N})$, for an $\mathcal{N}$ which we shall describe below. This will be chosen so as to guarantee that at limit stages we have the hypotheses of (4.5), so that, at limit stages σ, GOOD will take p_σ to be given by (4.5). The other implicit assumption is that at all stages so far, BAD has succeeded in "catching" p_{2i}, p_{2i+1} inside $|\mathcal{N}_{\alpha(i)}|$.

For the successor step, we take $\mathcal{N}' := \mathcal{N}_{\alpha(i)}$, $p := p_{2\alpha+1}$, $\mathcal{N} :=$ the Skolem hull in $\mathcal{M}$ of $N' \cup \{\mathcal{N}'\}$, and $q := q(p, \mathcal{N})$. Note that we easily have that $p\chi_{\mathcal{N}'} \in N$, so that, for all s-like $\kappa \in d(p)$, $\delta^q(\kappa) > p\chi_{\mathcal{N}'} > \delta^p(\kappa)$. This guarantees that we have $g^q(p\chi_{\mathcal{N}'}(\kappa)) =\, !$.

Now, let $\sigma \leq \theta$ be a limit ordinal and let $\kappa \in d(p)$ be singular, where p is as in (4.5). Let i_0 be the least $i < \sigma$ such that $\kappa \in d^{p_i}$. Since $p_{i_0} \in |N_{\alpha(j_0)}|$, where j_0 is least such that $i \leq 2j + 1$, κ is $\mathcal{N}_{\alpha(j)}$- controlled for all $j_0 \leq j < \sigma$. Then, letting $\delta := \delta^p(\kappa)$, we get that $\{p\chi_{\mathcal{N}_{\alpha(j)}}(\kappa) | j_0 \leq j < \sigma\}$ is a subset of $(g^p)^{-1}[\{!\}]$ which is cofinal in δ. Finally, the supercoherence of the model sequence $\vec{\mathcal{N}}$ guarantees that it is also a subset of C_δ, which means that it is a subset of Z_δ. It is then routine to see that we have the hypotheses of (4.5), and therefore that the strategy for GOOD is winning. To obtain the distributivity properties, we must see that BAD need not lose due to inability to "catch" p_{2i}, p_{2i+1} inside $|\mathcal{N}_{\alpha(i)}|$, and, more importantly, that there are enough supercoherent sequences. These points are addressed in the next item; the latter draws on the work of our companion paper [17].

COROLLARY 5.2. $\mathbf{P}_\theta$ *is* $(\theta,\ \infty)$*-distributive.*

PROOF. If $p_0 \in P_\theta$ and $(\tilde{D}_i|i<\theta)$ is a definable-in-parameters sequence of open dense subclasses of P_θ, begin by picking a singular ν with cf $\nu \gg \theta$, such that all parameters in the definition of $(\tilde{D}_i|i<\theta)$ lie in H_ν and such that $(H_\nu,\ \in)$ reflects all Σ_n-formulas, where the definitions of $(\tilde{D}_i|i<\theta)$ and $\mathbf{P}_\theta$ are Σ_n and n is larger than the number of quarks in the physical universe. Let $D_i = \tilde{D}_i \cap H_\nu$. We take $\mathscr{M} = (H_{\nu^+},\ \in,\ \cdots)$, we let $\mathscr{N}_0 \prec \mathscr{M}$, with $\theta+1 \cup \{(D_i|i<\theta),\ \mathbf{P}_\theta|H_\nu\} \subseteq N_0$ $(:= |\mathscr{N}_0|)$ and *card* $N_0 = \theta$, $[N_0]^{<\theta} \subseteq N_0$. By the main result of our companion paper [17] (see (1.4)) we can find super $\mathscr{M}$-coherent $(\mathscr{N}_i|i\leq\theta)$ starting from $\mathscr{N}_0$. We then play a run of the game $G(\theta,\ \vec{\mathscr{N}},\ p_0)$, where GOOD plays by her winning strategy.

We argue that BAD can produce a subsequence of $\vec{\mathscr{N}}$ which "catches" $p_{2i},\ p_{2i+1}$ inside $N_{\alpha(i)}$. For successor i, given that $p_{2i-2},\ p_{2i-1} \in N_{\alpha(i-1)}$, the last sentence of Remark 1 of (4.3) immediately gives that if BAD chooses $\alpha(i) > \alpha(i-1)$, then $p_{2i} \in N_{\alpha(i)}$. Then, if he chooses p_{2i+1} from $N_{\alpha(i)}$, this is as required. For limit i, letting $\alpha^* := \sup\{\alpha(j)|j<i\}$, given that all the $p_j \in N_{\alpha^*},\ j<i$, the remark of (4.5) immediately gives that if BAD chooses $\alpha(i) > \alpha^*$, then p_i, the p of (4.5), will lie in $N_{\alpha(i)}$. Once again, if he then chooses p_{i+1} from $N_{\alpha(i)}$, this will be as required.

Thus, in a such a play, we will actually produce a p_θ. If, in addition to the above, BAD chooses $p_{2i+1} \in D_i \cap N_{\alpha(i)}$, then clearly we will have $p_\theta \in \bigcap\{D_i|i<\theta\}$, as required. □

REMARK 5.3.

(1) It follows from (5.1) and (5.2) that any iteration of $\mathbf{P}_\theta$ with supports of cardinality $\leq\theta$ is (θ,∞)-distributive. The reason is quite simply that (5.1), (5.2), and the results of [17] depend only on ground model properties. In fact, S. Friedman has informed us that prior to our work similar observations had been made about Jensen's original coding and the variants of it mentioned in the Introduction.

(2) As mentioned in the Introduction, by a simple variant of the proof of (5.1), we can show that GOOD has a winning strategy for the version of the game where we start from a p_0 in P, which is not necessarily in P_θ. We "freeze" $g^{p_0}|\theta$, (and therefore, also $\beta^{p_0}|\theta$) and never add any $(\nu,\ \xi)$ with $\xi<\theta$ to any $\Xi(p_i)$. We use this in (6.1)(7)(c).

§6. Extendability and chain condition.

LEMMA 6.1. *Let* $\kappa > \aleph_1$ *be a cardinal, and let* $p \in P$. *Then, in each of the cases* $1 \leq i \leq 7$, *below, there is a* $q \in P,\ p \leq q$, *satisfying the conclusion of* (i) *(which follows the colon, in each case).*

(1) κ *is a successor cardinal,* $\kappa \notin d^p : \kappa \in d^q$,

(2) κ *is a regular cardinal,* $\kappa \in d^p,\ \kappa \leq \alpha < \delta^p(\kappa),\ g^p(\alpha)=1,\ \alpha \notin W(p)$, *and if* κ *is inaccessible, then, letting* β_1 *be as in* (3.4)(D), *either* $\beta^p(\kappa) \leq \alpha < \delta^p(\kappa)$ *or* $\beta_1 \leq \alpha < \kappa^2 : \alpha \in W(q)$,

(3) κ *is regular,* $\kappa \in d^p,\ \delta^p(\kappa) \leq \alpha < \kappa^+ : \alpha < \delta^q(\kappa)$,

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(4) *κ is singular, $\kappa \in d^p$, $\kappa < \gamma < \delta^p(\kappa)$, γ is not a multiple of κ, $\delta^p(\kappa) < \alpha < \kappa^+$: there is $\alpha' \in X_\gamma$ with $\alpha < \alpha' < \delta^q(\kappa)$ and $g^q(\alpha') = g^p(\gamma)$,*

(5) *κ is regular, $\kappa \in d^p$, $\kappa \leq \alpha < \delta^p(\kappa)$, $g^p(\alpha) = 0$, $\sigma < \kappa$ and if $\kappa = \lambda^+$, where λ is a limit cardinal, then $\lambda \in d^p$: there is $\zeta \in b_\alpha \setminus \sigma$ with $card\,\zeta \in d^p$, $\beta^q(card\,\zeta) \leq \zeta < \delta^q(card\,\zeta)$, and $g^q(\zeta) = 0$,*

(6) *κ is inaccessible, $\kappa \notin d^p : \kappa \in d^q$,*

(7) *κ is singular, $\kappa \notin d^p : \kappa \in d^q$.*

PROOF. The properties are given in order of increasing difficulty and/or dependence on earlier properties. The analogue of (5), where $\kappa = \lambda^+$, λ a limit cardinal, $\lambda \notin d^p$, is achieved by first adding λ to d^p, via (6) or (7), as appropriate, then using (5). We deal with the cases in the given order. In all cases, by virtue of (4.3), we can assume that p is very tidy. In all cases except (5), when we have to define $g^q(\alpha)$, we shall always make $g^q(\alpha) = 1$ unless it is forced to be 0, in which case we make it 0, *except* in the following cases:

(a) *card* α is s-like and α is a multiple of *card* α,

(b) $\kappa =$ *card* α is inaccessible, $\alpha < \beta^q(\kappa)$, and α is a multiple of κ.

Thus, in what follows, except in case (5) we shall limit ourselves to treating the above cases. In case (5), we will find a ζ *which is not forced to be* 1 *and which is not in* $R(p)$, and for this ζ we shall make $g^p(\zeta) = 0$. ζ will not be a multiple of *card*$\,\zeta$. This will be the only exception to our general procedure, even in case (5). □

For (1), if $\kappa = \lambda^+$, we simply set $d^q = d^p \cup \{\kappa\}$ and $\Xi(q) = \Xi(p)$; then we set $\delta^q(\kappa) = \kappa^2$ and $\beta^q(\kappa) = \kappa + 1$.

For (2), we shall have $g^q = g^p$. If $\kappa = \lambda^+$, where $\lambda \notin d^p$, let $\xi = \lambda$. If $\kappa = \lambda^+$, where $\lambda \in d^p$, let $\xi = \delta^p(\lambda)$. If κ is inaccessible and $\kappa \cap d^p = \varnothing$, let $\xi = \aleph_2$. Finally, if κ is inaccessible and $\kappa \cap d^p \neq \varnothing$, let $\xi = \sup(\kappa \cap \operatorname{dom} g^p)$. In the last two cases, if $\theta < \kappa$ and θ is regular, we can always take $\xi \geq \theta$ as well. Then we let $\Xi(q) = \Xi(p) \cup \{(\alpha, \xi)\}$.

For (3), we will have $d^q = d^p$, $\beta^q = \beta^p$, $\Xi(q) = \Xi(p)$ and for all $\mu \in d^p \setminus \{\kappa\}$, $\delta^q(\mu) = \delta^p(\mu)$. We set $\delta^q(\kappa) =$ the least $\theta > \alpha$ which is a multiple of κ^2. Now, suppose $\delta^p(\kappa) \leq \xi < \delta^q(\kappa)$. The only case we have to treat is when κ is s-like and regular, and ξ is a multiple of κ^2; we set $g^q(\xi) = ?$, and we take $g^q|s_\xi$ to be as required by (3.4)(E). Clearly q is as required.

For (4), pick $\alpha' \in X_\gamma$, $\alpha' > \alpha$. We shall have $d^q = d^p$, $\beta^q = \beta^p$, $\Xi(q) = \Xi(p)$. If $\mu \in d^p$, $\mu \notin \{\kappa\} \cup \Delta_\kappa$, we shall also have $\delta^q(\mu) = \delta^p(\mu)$.

We set $\delta^q(\kappa) = \alpha' + \kappa^2$. For a tail of $\lambda \in D_\kappa$ we have $\lambda \in d^p$ and $\delta^p(\lambda) \leq f^*_{\delta^p(\kappa)}(\lambda) < f^*_{\alpha'}(\lambda) < f^*_{\delta^q(\kappa)}(\lambda)$. So let $\lambda_0 = \lambda_0(\alpha')$ be sufficiently large so that the preceding holds for $\lambda \in D_\kappa \setminus \lambda_0$. Clearly, we may assume $\lambda_0 \notin D_\kappa$. For $\lambda \in D_\kappa \setminus \lambda_0$, let $\delta^q(\lambda) = f^*_{\delta^q(\kappa)}(\lambda)$. If κ is a limit of singular cardinals and $\tau > \lambda_0$ is a successor point of D_κ, let $\nu := \delta^q(\tau)$ and $\nu' = f^*_{\alpha'}(\tau)$. Then, as for κ and $\delta^p(\kappa)$, there is $\lambda_0(\nu') \geq \lambda_0$ such that for all $\lambda_0 \leq \lambda \in D_\tau$ we have $\delta^p(\lambda) \leq f^*_{\delta^p(\tau)}(\lambda) < f^*_{\nu'}(\lambda) < f^*_\nu(\lambda)$. For such λ we set $\delta^q(\lambda) := f^*_\nu(\lambda)$. For all other $\lambda \in \Delta_\kappa \cap d^p$ we set $\delta^q(\lambda) = \delta^p(\lambda)$.

Suppose now that $\lambda \in d^p$ and $\delta^p(\lambda) < \delta^q(\lambda)$ (so, in particular $\lambda \in \{\kappa\} \cup \Delta_\kappa$). We deal first with defining $g^q(\delta^p(\lambda))$, so let $\xi = \delta^p(\lambda)$. If ξ is p-deactivated, we set $g^q(\xi) = !$; otherwise, we set $g^q(\xi) = ?$. In both cases we take $h^q(\xi) = \xi$, and define $g^q|s_\xi$ to satisfy (3.4)(E). By virtue of the rest of the definition of g^q, Remark 2 of (4.3) will guarantee that this is as required (recall that p is very tidy!).

Next, suppose that ξ is a multiple of λ^2 with $\delta^p(\lambda) < \xi < \delta^q(\lambda)$. We shall have that $g^q(\xi) = ?$ *unless* one of the following occurs:

(c) $\lambda = \kappa$ and $\xi = \alpha'$,

(d) $\lambda \in D_\kappa \setminus \lambda_0(\alpha')$ and $\xi = f^*_{\alpha'}(\lambda)$,

(e) κ is a limit of singular cardinals and there is a successor point τ of $D_\kappa \setminus \lambda_0(\alpha')$ such that, letting $\nu' := f^*_{\alpha'}(\tau)$, we have $\lambda \in D_\tau \setminus \lambda_0(\nu')$ and $\xi = f^*_{\nu'}(\lambda)$.

In these cases, we set $g^q(\xi) = g^p(\gamma)$. In all cases we have $h^q(\xi) = \xi$, and we define $g^q|s_\xi$ to satisfy (3.4)(E). By Remark 2 of (4.3), and the fact that if $\lambda \in d^p$ is singular and $\delta^p(\lambda) < \nu$ then $f^*_\nu >^* \delta^p|D_\lambda$, it is then clear that $q \in P$ and is as required. In fact, it is easily verified that q is very tidy, though we do not need this.

For (5), suppose first that $\kappa = \lambda^+$ and $\lambda \in d^p$. In this case we have $d^q = d^p$, $\beta^q = \beta^p$, $\Xi(q) = \Xi(p)$. By (3.2)(A), for p and $\kappa' = \kappa$, we can find $\zeta_0 < \kappa$ such that whenever $(\eta, \xi) \in \Xi(p)$ and $\xi < \kappa$, then $b_\alpha \cap b_\eta \subseteq \zeta_0$. Without loss of generality, $\zeta_0 < \delta^p(\lambda)$. Now let $\zeta \in b_\alpha \setminus \max(\sigma, \zeta_0)$. By our choice of ζ_0, $\zeta \notin R(p)$. Also, since $\zeta \in b_\alpha$ and $\alpha \in (\kappa, \kappa^+)$, where κ is a successor cardinal, ζ is even, but not a multiple of λ. Thus, ζ is not forced to be 1. Accordingly, we set $g^q(\zeta) = 0$. The remainder of the construction of q divides into cases, according to whether λ is regular or singular.

If λ is regular, we proceed as in (3), with λ in place of κ and ζ in place of α and with the already-noted difference that $g^q(\zeta) = 0$. If λ is singular, we proceed as in (4), with λ in place of κ and ζ in place of α, with the already-noted difference that $g^q(\zeta) = 0$; the argument here is simpler than in (4) since there are no γ nor α' involved. Then q is as required.

If $\kappa = \lambda^+$, $\lambda \notin d^p$, then, by hypothesis, λ is a successor cardinal, so we can use case (1) to obtain $p \leq q'$ with $\lambda \in d^{q'}$, and then apply the immediately preceding argument to q' instead of p, to obtain the required q. Finally, suppose κ is inaccessible. As above, we can find $\zeta_0 < \kappa$ such that whenever $(\eta, \xi) \in \Xi(p)$ and $\xi < \kappa$, then $b_a \cap b_\eta \subseteq \zeta_0$. Pick $\kappa' \geq \max(\sigma, \zeta_0)$, $\kappa' = \aleph_\tau$, and τ an even successor. We take $\zeta := f^*_\alpha(\kappa')$. We note, once again, that, by our choice of ζ_0, $\zeta \notin R(p)$, and that, since ζ is even and *card* ζ is an even successor, ζ is not forced to be 1. Then we can proceed, as in (1) and (3), to add κ' to d^p, and make $\delta^q(\kappa') > \zeta$, *except* that, as above, we can also make $g^q(\zeta) = 0$. Clearly q is as required.

For (6), we take $d^q = d^p \cup \{\kappa\}$, $\Xi(q) = \Xi(p)$, and, for $\mu \in d^p$, $\delta^q(\mu) = \delta^p(\mu)$. By (3.2)(A), for p with $\kappa' = \kappa$, $d = d^p$, we can compute $\beta^q(\kappa)$ according to (3.4)(C) and β_1 according to (3.4)(D), and they will be bounded in κ^+ and κ^2, respectively. We take $\delta^q(\kappa) =$ the least multiple of κ greater than $\beta^q(\kappa)$. If $\kappa < \alpha < \beta_1$ or $\kappa^2 \leq \alpha < \beta^q(\kappa)$ and α is a multiple of κ, we set $g^q(\alpha) = ?$ iff α is contaminated. If it is not contaminated, we set $g^q(\alpha) = 1$, unless it is forced to be 0, in which case we make $g^q(\alpha) = 0$. If $\beta^q(\kappa) \leq \alpha < \delta^q(\kappa)$ and α is a multiple of κ, we set $g^q(\alpha) = 1$ unless it is forced to be 0; in this case we set $g^q(\alpha) = 0$. Finally, we define g^q on the multiples of κ in $[\beta_1, \kappa^2)$ to satisfy (3.4)(D), (1) and (2). Clearly this q is as required.

Case (7) divides into subcases, as follows:

(a) $D_\kappa \cap d^p$ is bounded in κ (in some sense, the simplest subcase: we must add to d^p a tail of Δ_κ, but there is no contamination).

(b) $D_\kappa \subseteq^* d^p$ (κ will be the only new member of d^q, but we must deal with contamination); we shall use the decoding procedure of (4.6).

(c) (a) and (b) both fail (the most complicated case: we must combine the methods used for (a) and (b), and also appeal to (5.1) and (5.2)).

In case (a), let $\lambda_0 < \kappa$ be such that $D_\kappa \cap d^p \subseteq \lambda_0$. Clearly, we may assume $\lambda_0 \notin D_\kappa$, and, anticipating the argument for (c), if $\theta < \kappa$, θ regular, we can take $\lambda_0 \geq \theta$. We shall have $\Xi(q) = \Xi(p)$ and $d^q = d^p \cup \{\kappa\} \cup (\Delta_\kappa \setminus \lambda_0)$. For $\mu \in d^p$, we take $\delta^q(\mu) = \delta^p(\mu)$. For $\lambda \in \{\kappa\} \cup (\Delta_\kappa \setminus \lambda_0)$, we take $\beta^q(\lambda) = \lambda + 1$ if λ is a successor cardinal, and $\beta^q(\lambda) = \lambda^2$ if λ is singular.

We set $\delta^q(\kappa) = \kappa^2$. For $\lambda \in D_\kappa \setminus \lambda_0$, we set $\delta^q(\lambda) = f^*_{\kappa^2}(\lambda)$. If κ is a limit of singular cardinals and λ a successor point of D_κ, $\lambda > \lambda_0$, let $\delta = \delta^q(\lambda)$. Then, if $\tau \in D_\lambda \setminus \lambda_0$, we set $\delta^q(\tau) = f^*_\delta(\tau)$. Then, for $\lambda \in \{\kappa\} \cup (\Delta_\kappa \setminus \lambda_0)$, if $\lambda < \alpha < \delta^q(\lambda)$ and α is a multiple of κ^2, we set $g^q(\alpha) = ?$, we take $h^q(\alpha) = \alpha$, as required by (3.4)(E), and we define $g^q|s_\alpha$ to code this, as required by (3.4)(E) (we can always find $R \in L[A \cap \lambda]$ as required, since either λ is singular, in which case $(\lambda^+)^L = \lambda^+$, or $\lambda \notin \Lambda$, in which case $(\lambda^+)^{L[A\cap\lambda]} = \lambda^+$). If λ is s-like and regular, recall that f^*_α was defined at the end of (1.2). This completes the proof in case (a) of (7).

In case (b), we will have $d^q = d^p \cup \{\kappa\}$, $\Xi(q) = \Xi(p)$, and for $\mu \in d^p$ also $\beta^q(\mu) = \beta^p(\mu)$. If $\mu \in d^p$, $\mu \notin \Delta_\kappa$, we shall also have $\delta^q(\mu) = \delta^p(\mu)$.

We set $\delta^q(\kappa) = \delta^* = \mathrm{scale}(\delta^p|D_\kappa)$. Let $\lambda_0 < \kappa$ be such that if $\lambda \in D_\kappa \setminus \lambda_0$, then $\lambda \in d^p$ & $\delta^p(\lambda) \leq f^*_{\delta^*}(\lambda)$. Clearly, we may assume $\lambda_0 \notin \Delta_\kappa$ and, anticipating the argument for (c) when κ is a limit of singular cardinals, below, if θ is regular, $\theta < \kappa$, we can take $\lambda_0 \geq \theta$. For $\lambda \in D_\kappa \setminus \lambda_0$ we set $\delta^q(\lambda) = f^*_{\delta^*}(\lambda)$. If κ is a limit of singular cardinals, for such λ, if λ is a successor point of D_κ and $\delta_0 = \delta^p(\lambda) < \delta^q(\lambda) = \delta_1$, then, on a tail of $\eta \in D_\lambda$, $\eta \in d^p$ and $\delta^p(\eta) = f^*_{\delta_0}(\eta) < f^*_{\delta_1}(\eta)$. So, let $\eta_0 = \eta_0(\lambda)$ be such that, whenever $\eta_0 \leq \eta \in D_\lambda$, $\eta \in d^p$ and $\delta^p(\eta) = f^*_{\delta_0}(\eta) < f^*_{\delta_1}(\eta)$. For such η, set $\delta^q(\eta) = f^*_{\delta_1}(\eta)$. If $\tau \in D_\kappa \cap \lambda_0$, or (if κ is a limit of singular cardinals), for some successor point λ of $D_\kappa \cap \lambda_0$, $\tau \in D_\lambda$, or (if κ is a limit of singular cardinals) for some successor point λ of $D_\kappa \setminus \lambda_0$, $\tau \in D_\lambda \cap \eta_0(\lambda)$, set $\delta^q(\tau) = \delta^p(\tau)$.

For $\lambda \in \Delta_\kappa$ such that $\delta^p(\lambda) < \delta^q(\lambda)$, we handle the definition of $g^q|[\delta^p(\lambda), \delta^q(\lambda))$ as we did in case (4), except that, here again, as in (5), the argument is simpler since there are no γ, α' involved.

Thus, it remains to define $\beta^q(\kappa)$ and $g^q|(\kappa, \delta^q(\kappa))$. We define g^q in the usual way on the nonmultiples of κ in (κ, δ^*). We shall define $\beta^q(\kappa)$ to satisfy (3.4)(C) with $d = d^p$ and $q|\kappa$ in place of $p|\kappa$. Note that conceivably $\overline{\delta} < \beta^q(\kappa) < \delta^*$, where $\overline{\delta}$ is the least multiple of κ^2, δ, with $\kappa < \delta \leq \delta^*$ such that $\neg(f^*_\delta \leq^* \delta^p|D_\kappa)$, since instances of contamination could arise due to the definition of the $g^q(\lambda)$ for those $\lambda \in D_\kappa \setminus \lambda_0$ with $\delta^p(\lambda) < \delta^q(\lambda)$ (if there are cofinally many such).

For α a multiple of κ^2, $\kappa < \alpha < \beta^q(\kappa)$, we set $g^q(\alpha) = ?$, and we define $g^q|s_\alpha$ to satisfy (3.4)(E). For $\beta^q(\kappa) \leq \alpha < \delta^*$, α a multiple of κ^2, we define $g^q|(\{\alpha\} \cup s_\alpha)$ by recursion on α, following the singular nongeneric case of the decoding procedure of (4.6), with $g = g^q|\kappa$. This completes the construction of q in case (b).

For case (c), our strategy is to obtain $p \leq p'$, $p' \in P$, such that the hypothesis of case (b), above, holds for p' and κ, and then apply (b) to p'. The construction of p' differs according to whether $\kappa = \lambda^{+\omega}$ or κ is a limit of singular cardinals. The former case is *much* easier, and we consider it first. Here, we obtain p' by *simultaneously* adding λ to d^p, following the procedure of (1), for $\lambda \in$ any final segment of $D_\kappa \setminus d^p$. In particular, anticipating the argument when κ is a limit of singular cardinals, the final segment can be taken to lie above θ, if θ is regular, $\theta < \kappa$. The simultaneity is emphasized to make clear that we *are not yet appealing to any strategic closure properties*. We then proceed as in (b) with p' in place of p.

When κ is a limit of singular cardinals, as a first step toward obtaining the desired p', we first simultaneously add to d^p all the $\tau \in D_\lambda$, for λ a successor point of D_κ, $\tau \notin d^p$, according to the procedure for (1). This is a condition p_0, intermediate between p and p'.

To obtain p', we let $(\lambda_i : i < \sigma)$ increasingly enumerate $D_\kappa \setminus d^p$. We let $\theta > \aleph_2$ be regular, $\sigma \leq \theta < \kappa$.

Let $\mathscr{M}$ be a master model, $\mathscr{M} = (H_{\nu^+}, \in, \cdots)$, ν singular, $\nu \gg \kappa$, such that $p_0 \in H_\nu$ and $(H_\nu, \in)$ models a sufficiently rich fragment of ZFC, etc., as in (5.1). As in (5.1), we can assume that we have $(\mathscr{N}_i : i \leq \theta)$ which is super $\mathscr{M}$-coherent, with $p_0 \in |\mathscr{N}_0|$. So, fix such $(\mathscr{N}_i : i \leq \theta)$. We then play the following run of the variant of $G(\theta, \vec{\mathscr{N}}, p_0)$, mentioned in (5.3). GOOD plays by the winning strategy of (5.1). BAD chooses $\alpha(i)$ as in (5.2), and obtains $p_{2i+1} \in |\mathscr{N}_{\alpha(i)}|$, by adding λ_i to $d^{p_{2i}}$, following the procedure of case (a) of (7) (note that the hypothesis of case (a) will always hold for λ_i and p_{2i}). Then, p' can be taken to be p_σ. This completes the proof for (c), when κ is a limit of singular cardinals, and therefore completes the proof of (7) and the lemma.

6.2. $\dot{\mathbf{P}}^\theta$ has the θ^+-chain condition. Let $\theta > \aleph_2$ be regular. The crucial observation is:

PROPOSITION 6.2.1. *Suppose $p, q \in P$, $(p)_\theta$, $(q)_\theta$ are compatible in $\mathbf{P}_\theta$, $g^p|\theta = g^q|\theta$, and $\beta^p|\theta = \beta^q|\theta$. Then p and q are compatible in P.*

PROOF. Let $r \in P_\theta$ with $(p)_\theta, (q)_\theta \leq r$. Note that, without loss of generality, we may assume that $W(r) = W_\theta(p) \cup W_\theta(q)$. We shall show that $r^* \in P$, $p, q \leq r^*$, where $r^* = (g^r \cup g^p|\theta, \beta^r \cup \beta^p|\theta, \Xi(r^*))$, where $\Xi(r^*) = \Xi(p) \cup \Xi(q) \cup \Xi(r)$. Of course, $p, q \leq r^*$ is clear, once we have verified that $r^* \in P$.

For this, all clauses of (2.2) and (3.2) are clear, as are (3.4)(E) and all clauses of (3.5). We argue that there is no new contamination in r^*, from which it will follow readily that we also have all (3.4)(A)–(D). This will complete the proof. Clearly there is no new contamination at singulars, and there is no new contamination at inaccessibles above θ. So, suppose that κ is inaccessible, $\kappa \leq \theta$. Suppose that $\alpha \in (\kappa, \kappa^+)$ and that α is contaminated by α'. Let $(\alpha', \xi) \in \Xi(r^*)$ witness this, as in (3.3). Then $\alpha' \in W(p) \cup W(q)$, and since $\xi < \kappa \leq \theta$, clearly $(\alpha', \xi) \in \{\Xi(p), \Xi(q)\}$. But then, by the hypotheses of the proposition, α must be contaminated by α' either in p or in q according to whether (α', ξ) is in $\Xi(p)$ or $\Xi(q)$. This completes the proof. □

COROLLARY 6.2.2. *In $V^{\mathbf{P}_\theta}$, $\dot{\mathbf{P}}^\theta$ has the θ^+-chain-condition.*

PROOF. This is clear from (6.2.1) and the easy computation that $\{(g^p|\theta, \beta^p|\theta)| p \in P\}$ has power θ, for all regular $\theta > \aleph_2$. $\square$

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