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ON LOGICAL SENTENCES IN PA

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§ 1. A representation of PH

We give in this section a representation of Paris Harrington [PH] results, in a way which will be helpful later.

1.1. Definition: An (L,n)-model is a sequence $\bar{M} = \langle M_\ell : \ell \le n \rangle$ such that:

1) M_ℓ is an L-model except that functions are partial (so $M_\ell \neq \emptyset$).

2) M_ℓ is a submodel of $M_{\ell+1}$ (for $\ell + 1 \le n$) but for every function symbol $F \in L$, $F^{M_{\ell+1}} \restriction M_\ell$ is a total function (with range $\subseteq M_{\ell+1}$).

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1.1.A. Notation: $\bar{M}^- = \langle M_\ell : 1 \le \ell \le n\rangle$, $\bar{M}_i = \langle M_{i,\ell} : \ell \le n\rangle$. $\bar{M}^{[i,j]} = \langle M_\ell : i \le \ell \le j\rangle$. Let $dp(\varphi)$ be the quantifier depth of φ and $|\psi|$ is the length of ψ.

1.2. Definition: For a formula $\varphi(\bar{x}) \in L$ (with negation in front of atomic formulas only, and only atomic terms used, w.l.o.g.) and a sequence $\bar{a}$ from M_0, $\ell(\bar{a}) = \ell(\bar{x})$, and an (L,n)-model $\bar{M}$ of length $> dp(\varphi)$ (i.e. $n \ge dp(\varphi)$) we define when $\bar{M} \models \varphi[\bar{a}]$ by induction on the quantifier depth of φ:

if φ atomic : as usual (note that only atomic terms were used, and the length of $\bar{M}$ is $> 0 = dep(\varphi)$, hence we can compute the terms and check the satisfaction of the relation in M_1)

if $\varphi(\bar{x}) = \psi(\bar{x}) \wedge \theta(\bar{x})$: $M \models \varphi[\bar{a}]$ iff $\bar{M} \models \psi[\bar{a}]$ and $\bar{M} \models \theta[\bar{a}]$

if $\varphi(\bar{x}) = \psi(\bar{x}) \vee \theta(\bar{x})$: $\bar{M} \models \varphi[\bar{a}]$ iff $\bar{M} \models \psi[\bar{a}]$ or $M \models \theta[\bar{a}]$

if $\varphi(\bar{x}) = \neg\psi(\bar{x})$: $M \models \varphi[\bar{a}]$ iff not $\bar{M} \models \psi[\bar{a}]$

if $\varphi(\bar{x}) = \exists y\ \psi(y,\bar{x})$: $\bar{M} \models \varphi[\bar{a}]$ iff for some $b \in M_1$, $\bar{M}^- \models \varphi[b,\bar{a}]$

if $\varphi(\bar{x}) = \forall y\ \psi(y,\bar{x})$ iff for every $k \ge dp(\psi)$ $b \in M_{n-k}$, $\bar{M}^{[n-k,n]} \models \psi[b,\bar{a}]$

1.3. *Claim*: 1) If a sentence ψ has an n-model then

a) ψ has a $dp(\psi)$-model (in fact an m-model whenever $dp(\psi) \le m \le n$)

b) ψ has an n-model $\bar{M}$ satisfying

$$\|M_0\| \le |\psi|, \quad \|M_{\ell+1}\| \le |\psi|\ \|M_\ell\|^{dp(\psi)}$$

c) ψ has a model $\bar{M}$, $\|M_{dp(\psi)}\| \le |\psi|^{(|\psi|^n)}$

2) ψ has a model iff it has an n-model for every n.

3) If ψ has no n-model then there is a proof of $\neg\psi$ of length $2^{|\psi|^{|\psi|^n}}$ (the length of a proof is the number of symbols in it).

4) If ψ has an n-model then there is no cut free proof of $\neg\psi$ of length $\le n$.

Remark. We have not tried to minimize the function and numbers nor we shall do it elsewhere in this article.

Proof: 1) a) If $\bar{M}$ is an n-model of ψ then $\bar{M}^{[0,dp(\psi)]}$ is a $dp(\psi)$-model of ψ (check the definition).

2) Suppose $\bar{N}$ is an n-model of ψ. We define by induction on $\ell \le n$, $A_\ell \subseteq N_\ell$

such that:

a) let A_0 consist of all individual constants (w.l.o.g. every non logical symbol in L appears in ψ)

b) if A_ℓ is defined, $\exists y\, \varphi(y,\bar{x})$ is a subformula of ψ, $\bar{a} \subseteq A_\ell$, $\bar{N}^{[\ell,n]} \models (\exists y)\varphi(y,\bar{a})$, then for some $b \in N_{\ell+1}$, $\bar{N}^{[\ell+1,n]} \models \varphi[b,\bar{a}]$; we demand that there is such $b \in A_{\ell+1}$ and $A_{\ell+1}$ is the union of A_ℓ with all such b's.

So clearly $|A_0| \leq$ [nr.of individual constants in ψ] $\leq |\psi|$ and

$$|A_{\ell+1}| \leq |A_\ell| + \sum_{\exists y \varphi(y,\bar{x})} |A_\ell|^{\ell(\bar{x})} \leq |\psi| \cdot \text{Max}\,\{1, |A_\ell|^{dp(\psi)}\}$$

(we can forget "$|A_\ell| +$ " as the multiplication by $|\psi|$ is more than needed). As models are non empty

$$|A_{\ell+1}| \leq |\psi| \quad |A_\ell|^{dp(\psi)} \quad .$$

Now let $M_\ell = N_\ell \restriction A_\ell$, and we can prove that for every subformula $\theta(\bar{x})$ of ψ, $\bar{a} \subseteq A_\ell$,

$$n - \ell \geq dp(\theta(\bar{x})) : \quad \bar{N}^{[\ell,n]} \models \theta[\bar{a}] \quad \text{iff} \quad \bar{M}^{[\ell,n]} \models \theta[\bar{a}]$$

(just like Tarski Vaught criterion).

2) If M is a model of ψ, then let (for all ℓ) $M_\ell = M$, so easily $\langle M_\ell : \ell \leq n\rangle$ is an n-model of ψ.

If for every n $\langle M_\ell^n : \ell \leq n\rangle$ is an n-model of ψ, by compactness there is $\langle M_\ell : \ell < \omega\rangle$ which is an ω-model of ψ. Easily $\bigcup_{\ell<\omega} M_\ell$ is a model of ψ.

3) Also immediate.

[There is a quite short proof (of length $< |\psi|^{|\psi|^n}$), showing $\psi \vdash \exists x_0^0 \ldots x_{m_0-1}^0$ $\exists x_0^1, \ldots, x_{m_1-1}^1, \exists x_0^2, \ldots, \exists x_1^n, \ldots, x_{d_n-1}^n\ \theta$ where $m_0 = |\psi|$, $m_{i+1} = |\psi|(m_i)^{dp(\psi)}$ and θ says that $\langle \{x_m^i : m \leq m_i, i \leq j\}; j \leq n\rangle$ is an n-model of ψ, θ is quantifier free. Now θ can be refuted by a truth table of size $2^{|\psi|^{|\psi|^n}}$].

4) Just like the proof that every model satisfies any provable sentence.

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Let PA be Peano arithmetic, and PA_k be Peano arithmetic when only instances of the induction scheme of length $\le k$ are included (but except the instances of the induction scheme there are only finitely many axioms, which are included, hence PA_k is finite). Let PA^{PL} be like PA, but we take only the instances of the induction hypothesis with no parameters, and PA_k^{PL} is the intersection of PA^{PL} and PA_k. It is known (Friedman thesis, I think) that PA, PA^{PL} are equi-consistent. It is clear that the consistency of PA is equivalent to "for every $n \ge k$, PA_k has an n-model "and to "for every $n \ge k$, PA_k^{PL} has an n-model" (by monotonicity we can take only $n = k$) .

What is the most natural n-model of PA_k^{PL}? I think that the obvious choice is $\bar{M} = \langle M_{r_\ell} : \ell \le n\rangle$ where M_r is the set $r = \{0,1,2,\dots,r-1\}$ with the usual addition and multiplication restricted to this, and where $r_0 < r_1 < r_2 < \dots < r_n$. If $r_{\ell+1} > r_\ell^2$, $r_\ell > 1$, the axioms of PA minus induction holds. So we want that for any formula $\varphi(x)$, if $\bar{M} \models (\exists x)\varphi(x)$ then $\bar{M} \models (\exists x)[\varphi(x) \wedge (\forall y)(\varphi(y) \to x \le y)]$, i.e. there is $x < r_0$ such that $\bar{M}^- \models \varphi[x] \wedge (\forall y)(\varphi(y) \to x \le y)$ ($\le$ is definable, or can be added as a relation).

This suggests defining

$$F_\varphi(r_1,\dots,r_n) = \min \{x < r_1 : \langle M_{r_\ell} : \ell = 0, n\rangle \models \varphi(x) \quad \text{letting } r_0 = 2\}$$

$$F'_\varphi(r_1,\dots,r_{n+1}) = \begin{cases} 0 & \text{if } F_\varphi(r_1,r_3,\dots,r_{n+1}) = F_\varphi(r_2,r_3,\dots,r_{n+1}) \\ 1 & \text{otherwise} \end{cases}$$

Let $r_0 < \dots < r_m$ be homogeneous for F'_φ (where $m > 2n$). If the constant value of F'_φ on increasing sequences from $\{r_0,\dots,r_m\}$ is 1, then necessarily $\langle F_\varphi(r_\ell, r_{m-n+2}, \dots, r_m) : \ell < m-n+2\rangle$ is strictly decreasing (in ℓ); hence $r_0 + n \ge m$, so if m is large enough compare to r_0, the constant value of F'_φ is 0, hence the instances of induction hold in the n-model $\langle M_{r_\ell} : \ell = 0, n\rangle$. We can work with all the formulas of length $\le k$, say φ_i $(i \le i_0)$ and define

$$F''_\varphi(r_1,\dots,r_{n+2}) = \begin{cases} 0 & \text{if } F_{\varphi_i}(r_1,\dots,r_{n+1}) = F_{\varphi_i}(r_2,\dots,r_{n+2}) \\ & \qquad \text{for every } i \le i_0 \\ 1 & \text{otherwise} \end{cases}$$

So if $\langle r_i : i = 1,m\rangle$ is indiscernible for F''_φ and large enough $\langle M_{r_i} : i \le n\rangle$

is an n-model of PA_k^{PL} .

So PH (Paris Harrington partition theorem) is enough to prove the consistency of PA.

2. On the π_1^1-comprehension axiom.

Simpson and Schmerl [SS] following Macintyre proved that the consistency strength of PA augumented by the Ramsey quantifier, $PA(Q^{MM})$, and of π_1^1-CA_0 (see 2.2) are the same, in fact they are biinterpretable. Macintyre's aim was to strengthen PA so as to eliminate the well-known incompleteness for finite combinatorics. Here we suggest an answer to a question from [M] and [SS], i.e. present a finitary π_2^0 combinatorial principle capturing π_1^1-CA_0, based on end-homogeneity instead of homogeneity (see the book [EHMR]).

This principle was constructed from the proof and not as usual by minituarizing infinitary ones. It was clear that its infinitary analogy is "parametrized Galvin-Prikry" so we have its consistency strength equivalent to that of π_1^1-CA_0. This naturally hints that they are actually equivalent. Simpson takes it on himself to prove this and succeeds.

This work was done in summer 1979, after hearing on the results of Friedman [F1] and just after Simpson has explained us [M] and [SS] (and before Friedman and Simpson started investigating the quasi-order of trees), and we would like to thank Simpson for the conversation.

2.1. Notation: For a set A of natural numbers, we call f a k-colouring of A if it is a function from increasing sequences from A of length k (or subsets of A of power k) into $\{0,1\}$.

For f, A as above

$B \subseteq A$ is f-homogeneous if f is constant on B.

2.2. Notation: We call H a hereditary function for n if for $p < n$, m, $A_1, \ldots, A_m \subseteq n$, $H(p,A_1,\ldots,A_m)$ belongs to $\{0,1\}$, and if it is zero then $H(p,A_1,\ldots,A_{m-1},\mathcal{D}) = 0$ for every $\mathcal{D} \subseteq A_m$ (i.e. H is monotonic in the last variable). As H is always like this we would not require hereditarity explicitely.

2.2A. Remark: π_1^1-CA_0 is a theory "speaking" on natural numbers and reals (i.e.

sets of natural numbers) satisfying: if $\psi(Y,x)$ is a formula, possibly with real parameters but all quantification are on natural numbers, then $\{x : (\exists Y)\ \psi(Y,x)\}$ is a real (x varies on natural numbers, Y on reals).

2.3. Definition: We define a combinatorial principle which corresponds to the existence of "end homogeneous sequence" in Erdos-Rado terminology:

$CP_{eh}(n,k,\ell)$: For any function F, k-place, defined on subsets on n, and whose values are k-colouring of n, and function H, there are $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots \supseteq A_\ell$ subsets of n such that (a) $|A_\ell| \geq \text{Min}\, A_\ell$ and (b) each A_m is $F(A_{i_1},\dots,A_{i_k})$-homo when $1 \leq i_1 < \dots < i_k < m \leq \ell$, (c) $\langle A_i : i = 1,\ell\rangle$ is H-end-homo, i.e. for $m < r \leq \ell$, $p < \text{Min}\, A_{r-1}$, $H(p,A_1,\dots,A_{m-1},A_m/p) = H(p,A_1,\dots,A_{m-1},A_r)$ where A_m/p is $A_m - \{i : (p,i) \cap A_m = \emptyset \text{ or } i \leq p\}$.

Remark: So the combinatorial statement we are interested in is: for every k, ℓ for some n $CP_{eh}(n,k,\ell)$ holds.

2.4. Definition: $CP'_{eh}(n,k)$ means: for every function F defined on sequences of subsets of $n = \{0,\dots,n-1\}$ whose range are k-colouring of n and function H there are $A_1 \supseteq A_2 \supseteq \dots \supseteq A_k$, $A_1 \subseteq n$ such that (a) $|A_k| \geq \text{Min}\, A_k$ and (b) each A_m is $F(A_1,\dots,A_{m-1})$-homogeneous and (c) $\langle A_i : i = 1,k\rangle$ is H-end-homo.

2.5. Claim: The statements

$$\forall k\ \forall \ell\ \exists n\quad CP_{eh}(n,k,\ell)$$
$$\forall k\ \exists n\quad CP'_{eh}(n,k)$$

are equivalent.

Proof: Immediate.

2.6. Claim: Suppose $PA + \forall k \exists n\ CP'_{eh}(n,k)$ is consistent. Then $\pi^1_1\text{-}CA_0$ is consistent.

Proof: So there is a non standard model $\mathcal{C}$ of $PA + \forall k \exists n\ CP'_{eh}(n,k)$. Choose a non-standard $k \in \mathcal{C}$, and choose $n \in \mathcal{C}$ such that

$$\mathcal{C} \models CP'_{eh}(n,3k)\ .$$

Now we shall define F, so let $\bar{A}$ be a sequence of subsets of n, and we shall define a $(3k)$-colouring of n, $F(\bar{A})$.

For this for every natural numbers $r \le n$, let $\mathfrak{A}_r = \mathfrak{A}_r[\bar{A}]$ be the model with universe $r = \{0,1,2,3,\dots,r-1\}$, functions: addition and multiplication (restricted to r), individual constants 0,1 and relations $<$ and A_ℓ $(\ell < \ell(\bar{A}))$ (i.e. A_ℓ is a one place relation). The language L depends on $\ell(\bar{A})$ only.

We can code the sentences of this language in many ways, choose one so that a sentence of length m_1 involving only A_i, $i < m_2$, is not too large compare to $m_1 + m_2$. Now we define $F(\bar{A})\ (i_0, i_1, \dots, i_{3k-1})$: it is zero

if $\quad G(\bar{A})\ (i_0,\dots,i_{2k-1}) = G(\bar{A})\ (i_k,\dots,i_{3k-1})$

and one otherwise, where

$G(\bar{A})\ (i_0,\dots,i_{2k-1})$ is the first $p \le i_0$ which is i_0 or which is an L-sentece ψ and

$$<\mathfrak{A}_{i_0}[\bar{A}], \mathfrak{A}_{i_1}[\bar{A}], \dots, \mathfrak{A}_{i_{k-1}}[\bar{A}]> \models \psi$$

$$<\mathfrak{A}_{i_k}[\bar{A}], \mathfrak{A}_{i_{k+1}}[\bar{A}], \dots, \mathfrak{A}_{i_{2k-1}}[\bar{A}]> \models \neg\psi$$

We now will define a function H.

i.e. we will define $H(p, A_1,\dots,A_{m+1})$:

<u>it is zero</u> if p codes a first order formula $\theta(-, -, A_1,\dots,A_{m-1})$, and whenever $p < \beta < i_0 < i_1 < \dots < i_m$, $\quad m = dp(\theta) + \beta + 8$

$i^*, i_0, \dots, i_m \in A_{m+1}$.

$<\mathfrak{A}_{i_\beta}[\bar{A}], \dots, A_{i_m}[\bar{A}]> \models$ "θ define a tree and for some t^* for every $x, y < i_\beta$ $\quad x,y \in A_{m+1}$, $\quad x < y$, the interval (x,y) has a member below t^* (in the tree)"

<u>otherwise the value is one</u>

(so if A_{m+1} is empty the value is zero, note H is hereditary).

As $CP'_{eh}(n,3k)$ holds, there are $A_1 \supseteq A_2 \supseteq \dots \supseteq A_k$, subsets of n, such that (for each m) A_m is $F(A_1,\dots,A_{m-1})$-homogeneous, and $<A_\ell : \ell = 1,m>$ is H-end-homo.

Let for $\ell < k$

$$n_\ell = \text{Min}\ A_\ell\ .$$

Now we define a model $\mathfrak{B}$ which will be a model of $\pi^1_1\text{-}CA_0$: the natural numbers of $\mathfrak{B}$ are

$$\mathfrak{B}_N = \{p \in \mathfrak{C} : \text{ for some standard } \ell \in \mathfrak{C},\ p < n_\ell\}$$

(remember k is not standard).

Addition, multiplication 0, 1, < are inherited from $\mathfrak{C}$.

The family of reals (= sets of natural numbers) is the family

$$\mathfrak{B}_R = \{B \subseteq \mathfrak{B}_N : \text{ for some standard } \ell,\ B \text{ is first order definable (with parameters) in the model } (\mathfrak{B}_N, +, \times, A_1, \dots, A_\ell)\}$$

The only non-trivial part is π^1_1-CA_0.
So we have to prove that

(F0) $\{p \in \mathfrak{B}_N : \mathfrak{B} \models (\exists Y)\ \psi(Y,p,\bar{B})\} \in \mathfrak{B}_R$

where $\bar{B}$ is a finite sequence of sets from $\mathfrak{B}_R$, ψ first order, Y varies on $\mathfrak{B}_R$.

It is well known that w.l.o.g. ψ "says" that Y is an infinite branch in a tree $\theta(-,-;\ p, \bar{B})$ (whose order is included in the natural order of $\mathfrak{B}_N$).

By the definition of $\mathfrak{B}_R$ we can replace $\bar{B}$ by $\langle q\rangle^\wedge\langle A_1,\dots,A_m\rangle$ for some truely finite m, $q \in \mathfrak{B}_N$. Suppose for some Y

$$\mathfrak{B} \models \psi[Y,p,q,A_1,\dots,A_m]$$

Then for some truely finite $m(*)$, Y is first order definable in $(\mathfrak{B}_N, A_1, \dots, A_{m(*)})$ using a parameter $q(*) \in \mathfrak{B}_N$.

As $A_{m(*)+1}$ is $F(A_1,\dots,A_{m(*)})$-homogeneous it is easy to see that

(F1) $(\forall x,y)\ [x \in A_{m(*)+1} \wedge y \in A_{m(*)+1} \wedge x < y \rightarrow (\exists z \in Y)\ (x < z < y)]$

i.e. the branch has a member in each interval of $A_{m(*)+1}$. Hence

(F2) The model $(\mathfrak{B}_N, A_1,\dots,A_{m(*)+1})$ satisfies "for every $a \in A_{m(*)+1}$ there is t^* such that for every $x < y < a$ $x \in A_{m(*)+1}$, $y \in A_{m(*)+1}$, in the interval (x,y) there is an element below t^* (in the tree $\theta(-,-,p,q,A_1,\dots,A_m)$".

By the H-end-homo, in (F2) we can replace $m(*)+1$ by $m+1$, hence clearly (remembering König lemma) the set

$\{p \in \mathcal{B}_N : \mathcal{B} \models (\exists Y)\ \psi(Y, p, \bar{B})\} =$

$\{p \in \mathcal{B}_N :$ for every $a \in A_{m+1}$ there is t^* such that for every $x < y < a$, $x \in A_{m+1}$, $y \in A_{m+1}$ in the interval (x,y) there is an element below t^* (in the tree $\theta(-,-,p,q, A_1,\dots,A_m)$ $\}$.

which belongs to $\mathcal{B}_R$ as required.

2.7. Claim: For every natural number k we can prove in $PA(Q^{MM})$ the statement $\exists n\ CP'_{eh}\ (n, k)$.

Proof: First we prove that the conclusion is true, i.e. true in the universe $\mathcal{B}$ if it is a model of say second order Peano arithmetic.

Suppose that the conclusion fails for k. Then for every n, there is a pair of functions (F_n, H_n) which forms a counterexample to $CP'_{eh}\ (n,k)$.

We now define by induction on $i \le k$ infinite sets A_i, so that $A_{i+1} \subseteq A_i$. For $i = 0$ there is no problem. Let $A_0 = B_0 =$ the set of all natural numbers. So suppose we have defined A_j for $j \le i$ and we shall define A_j for $j = i+1$.

By the infinite Ramsey theorem we can get an infinite $A_i^1 \subseteq A_i$ such that:

(a) for every $y < z$ in A_i^1, $f_{y,z}$ is the function $F_z(A_1 \cap y, A_2 \cap y, \dots, A_i \cap y)$ restricted to y.

Now $f_{y,z}$ does not depend on z, i.e. if $y < z_1, z_2 \in A$ then $f_{y,z_1} = f_{y,z_2}$.
So let $f_y = f_{y,z}$ for $y < z \in A_i^1$.

(b) For $y_0 < y_1 < y_2$ in A_i^1, $f_{y_1} \restriction y_0 = f_{y_2} \restriction y_0$.

[Simply apply Ramsey theorem to the natural three place colouring, first for (a) then for (b)].

So $f = \bigcup \{f_y \restriction z : z < y$ are in $A_i^1\}$ is a k-colouring (of the natural numbers). By the infinite Ramsey theorem there is in $\mathcal{B}_R$ an infinite set $A_i^2 \subseteq A_i^1$, which is f-homogenous.

Now we shall deal with H. Again there is an infinite $A_i^3 \subseteq A_i^2$ ($A_i^3 \in \mathcal{B}_R$, of course) such that:

(c) if $y < z$ are in A_i^3, $\mathcal{D} \subseteq A_i \cap y$, $p < y$ then the value of $H_z(p, A_1 \cap y, \dots, A_i \cap y, \mathcal{D})$ does not depend on z.

Moreover

(d) if $x < y < z$ are in A_i^3, $\mathcal{D} \subseteq A_i \cap x$, $p < x$ then the value of

$H_z(p, A_1 \cap y, \ldots, A_i \cap y, \mathcal{D})$ does not depend on y (and on z).

For every $p \in \mathcal{B}_N$ and infinite set $\mathcal{D} \subseteq A_i^3$ we define $T(p, \mathcal{D})$: it is zero iff for arbitrarily large $q \in \mathcal{B}_N$ for every $y < z$ in A_i^3/q the value of $H_z(p, A_1 \cap y, \ldots, A_i \cap y, \mathcal{D} \cap y - q)$ is zero. Clearly a finite change in $\mathcal{D}$ does not change the value of H_z, and "for arbitrarily large q" can be replaced by "for every large enough q" (see 2.2).

Now we want to apply the Galvin Prikry theorem to T, more exactly to a parametrized version of it (p as the parameter). Simply iterate the usual Galvin-Prikry theorem on the natural numbers, and take the diagonal intersection. What we get we call A_i^4.

Again remembering 2.2, and the conclusion of the Galvin-Prikry theorem, we can find $A_{i+1} \subseteq A_i^4$, so that for every p, if for some $q > p$, for every $y < z$ in A_i^3/q the value of $H_z(p, A_1 \cap y, \ldots, A_i \cap y, A_{i+1} \cap y - q)$ is zero, then $q = p$ will serve.

If $A_0, A_1, \ldots, A_{k+1}$ are defined, choose $y < z$ in A_{k+1} and $\langle A_1 \cap y, \ldots, A_k \cap y\rangle$ contradicts the choice of F_z, H_z.

However we want to prove this in $PA(Q^{MM})$ (and not in ZFC or in second order number theory).

The proof is the same, replacing "a set of natural numbers" by "a definable set of natural numbers".

Why is $\langle\langle F_n, H_n\rangle : n < \omega\rangle$ definable? We can choose for each n, a minimal pair $\langle F_n, H_n\rangle$ (by a simple enough coding). We can apply the infinite Ramsey theorem as Macintyre [M] proves the parallel statement holds (from $PA(Q^{MM})$, of course the proof depends on the formula defining the colouring and the infinite set). More exactly he proves that if A is infinite and definable,

$$\varphi(x_1, \ldots, x_n, \bar{p}, z) \quad \text{is such that} \quad \forall x_1, \ldots, x_n \ (\exists z < c)\ \varphi(x_1, \ldots, x_n, p, z)$$

then there is a definable infinite $B \subseteq A$ and $c_0 < c$ such that

$$(\forall x_1, \ldots, x_n)\ [\wedge x_i \in \mathcal{B} \wedge \wedge x_i < x_{i+1} \rightarrow \varphi(x_1, \ldots, x_n, \bar{p}, c_0)].$$

We are left with the "parametrized Galvin-Prikry".

Let $a <^* b$ mean: a, b code finite increasing sequences, $\langle a(m) : m < \ell(a)\rangle$, $\langle b(m) : m < \ell(a)\rangle$ respectively, and $\langle a(m) : m < \ell(a)\rangle$ is a proper initial segment of $\langle b(m) : m < \ell(b)\rangle$.

We define A_i^4 (after A_i^3 has been defined) by defining by induction on ℓ, k_ℓ, ξ_ℓ such that

(a) $k_0 < \dots < k_{\ell-1}$ and $\xi_0, \xi_1, \dots, \xi_{\ell-1} \in \{0,1\}$

(b) $(Q^{MM} a,b)\,[\langle k_0, \dots, k_{\ell-1}\rangle <^* a < b$ and for every $p < \ell$ for some q, for every y, z if $q < y < z$, $y \in A_i^3$, $z \in A_i^3$ then $H_z(p, A_1 \cap y, \dots, A_i \cap y, \{a(m) : m < \ell(a)\}/q) = \xi_p]$.

(c) if compatible with (a)+(b), $\xi_{\ell-1} = 0$.

Now we can carry the definition (as in [M] noting the formulas we use have bounded complexity). So we finish the proof of 2.7.

§ 3. A true π_1^0-sentence of PA not provable in PA

In summer '80 Friedman and Harrington offered hotly their view that it is one of the main problems of contemporary logic to find mathematical sentences as mentioned in the tytle, as well as to find natural theories with incomparable consistency strength. The "technical difficulty" is immaterial; in fact the easiness of the proof may indicate the profoundness and naturality of the sentence. Now an answer to such question is naturally more open to debate than the usual mathematical problem.

As the autor did not want to go into such discussion, and Harrington wanted a solution, an agreement was reached: if the author could find a solution which Harrington would think is O.K., he would write it up, discuss it and publish it. The content of this section was done in spring '81, Harrington O.K.ed it, as well as § 4 (which was done in summer '80) but was too lazy to fulfill his promise.

3.1. Context: 1) Let N be a natural number $\bar{r} = \langle r_\ell : \ell < \ell(\bar{r})\rangle$ a finite sequence of natural numbers.

2) Let $K = K_N^{\bar{r}} = \{(A, <, \bar{R})$: A a subset of N, $<$ a linear order of A, $\bar{R}$ a sequence of relations over A, R_ℓ an r_ℓ-place $\ell(\bar{R}) = \ell(\bar{r})\}$.

3) Members of K are denoted by $\mathcal{A}, \mathcal{B}$, letting $A = |\mathcal{A}|$, so $\|\mathcal{A}\|$ is the power of the set of the set of elements of $\mathcal{A}$.

4) In K we define

(4a) $\mathcal{A} <_{en} \mathcal{B}$ ($\mathcal{B}$ an end extension of $\mathcal{A}$) if $\mathcal{A}$ is a submodel of $\mathcal{B}$ and

$x \in |B| - |A|$, $y \in |A|$ implies $y < x$.

(4b) $A < B$ (B an universal end extension of A) if $A <_{en} B$ and for any A' satisfying $A <_{en} A'$, $\|A'\| \leq \|A\| + 1$, there is an embedding of A' into B over A .

5) A function G from $\mathcal{P}(N)$ into $\mathcal{P}(N)$ is f-small if $|G(A)| \leq f(|A|)$.

3.2. The $(N, \bar{r}, k, n)$-principle: Let F be a k-place function satisfying

1) Domain: F is defined on increasing sequences of length k from K .

2) Choice Function: $F(A_1, \ldots, A_k) \in |A_1|$.

3) Isomorphism Invariancy: if $A_1 < \ldots < A_k \in K$, $B_1 < \ldots < B_k \in K$, and there is an isomorphism g from A_k onto B_k mapping A_ℓ onto B_ℓ then F and g commute, i.e.

$$F(B_1, \ldots, B_k) = g(F(A_1, \ldots, A_k)) .$$

4) Weak Hereditarity: For every $A_1 < A_2 < \ldots < A_k$, there is an x^k-small function f such that: if B_ℓ is a submodel of A_ℓ, $f(B_{\ell-1}) \cap A_\ell \subseteq B_\ell$ (stipulating $B_{-1} = \emptyset$) then $F(A_1, \ldots, A_k) = F(B_1, \ldots, B_k)$.

Then there is an increasing sequence $\langle A_\ell : \ell < n \rangle$ on which F depends on the first structure only.

3.3. Fact if $N > 2^{2^{k+n+\Sigma(r(i)+1)}}$, F as in 3.2 and $N' > N$, then we can extend F to F' a k-place function satisfying 1)-4) of 3.2 for N', in one and only one way.

Proof. By 3.2 (4) (3) (as in the proof of 1.3 (1) (b)).

3.4. Claim: In PA + PH we can prove

$$\psi^* = (\forall \bar{r}, k, \ell) \quad (\text{the } (N, \bar{r}, k, n)\text{-principle holds for e.g. } N = 2^{2^{2^{k+n+\sum_i [r(i)+1]}}}).$$

Proof: Define by induction on ℓ, a model $A_\ell \in K$ with universe ℓ, so that if m is smaller enough than ℓ, $A_m <^* A_\ell$. (just take care of 3.1 (4)(b), easy computation). Applying PH we get a set C of natural numbers $(\forall x < y \in C)[2^{2^x} < y]$, and indiscernible for (F' as in 3.2 for N' large enough)

$$F'(A_{i_1}, \ldots, A_{i_k}) = F'(A_{j_1}, \ldots, A_{j_k}) , \text{ and } |C| > 4 \text{ Min } C$$

(we use an equivalent variant of P.H.). As in § 1 easily for subsequences of $\langle A_i : i \in C \rangle$, F depends on the first variable only. Now as in the proof of 1.3 (1) (b) we collapse the solution below N.

3.5. Claim: In $PA + \psi^*$ we can prove the consistency of PA (hence $PA \nvdash \psi^*$).

Proof. We can build a non-standard model of $PA_8 + \psi^*$, M, choose k, n non standard, N large enough, and $\bar{r} = \langle 4 \rangle$ define F :

$F(A_1, \ldots, A_k)$ is defined as follows: let $\varphi(x,\bar{y})$ be a formula with minimal code for which $\langle A_1, \ldots, A_k \rangle \models$ "induction fails, i.e.

$$\neg (\forall \bar{y}) \ \{\exists x \varphi(x,\bar{y}) \rightarrow \exists x[\varphi(x,\bar{y}) \wedge (\forall z < x) \neg \varphi(z,\bar{y})]\}"$$

and then take minimal $\bar{y}$ (by the lexicographic order of $\langle \mathrm{Max}\ \bar{y}, y_0, y_1, \ldots \rangle$) and then $x \in A_0$, $\varphi(x,\bar{y})$ (if there is one). The rest is as in § 1 the only additional point is why are addition and multiplication definable? This is by 3.1 (4) (b).

§ 4. On consistency strength

Let T denote here a (recursive) theory (with finite-sort, finite-language) extending PA but it may also "speak" on reals and even arbitrary sets. Let CON(T) be the sentence (in PA language) saying T is consistent.

Definition: We say $T_1 \leq_{cs} T_2$ (the consistency strength of T_1 is smaller (or equal) than the consistency strength of T_2) if $PA \vdash CON(T_2) \rightarrow CON(T_1)$.

It was observed that essentially "large cardinal axioms are linearly ordered" (though in some cases this "has not yet been proved"). More exactly it seems that all set theories which has been considered so far, are linearly ordered by $\leq_{cs}$. Solovay (I think) has found T's which are $\leq_{cs}$-incomparable, but they were "paradoxial" (i.e. have self-referential sentences). We shall try to get more reasonable ones.

* * * * * * *

Let PA^+ be $PA + CON(PA)$. We work inside PA. A model will mean one which is definable.

Let T_1, T_2 be consistent theories (in our "universe" which satisfies PA).

T_2 "saying" there is a model of T_1, T_1 "saying" there is a model of PA (think of PA^+, $PA^+ + \pi_1^1\text{-}CA_0$; or use ATR_0 see Friedman, McAloon and Simpson [FMS], or of course ZFC, or ZFC+large cardinals).

As T_2 is consistent, $T_1 + CON(T_1)$ is consistent, hence (by Godel incompleteness theorem) $T_1 + CON(T_1) + \neg CON(T_1 + CON\, T_1)$ has a model M_1. By the requirement on T_2 $M_1 \models \neg CON(T_2)$, hence there is $n \in M_1$ such that (n is a non-standard integer)

$$M_1 \models \quad "\psi_2^f(n)" \quad \text{where}$$

(1) $\psi_\ell(n)$ says n is a natural number and there is a proof of size n of $PA \vdash \neg CON(T_\ell)$

(2) $\psi_\ell^f(n)$ says $\psi_\ell(n)$ but n is the first such number.

As we have assumed that T_1 says that PA has a model, clearly

(3) $T_a = PA + (\exists n)\,[\psi_2^f(n) + \neg\psi_1(2^{2^n})]$

is consistent with PA^+ (as even $PA + \exists n \psi_2^f(n) + (\forall m)\neg\psi_1(m)$ is consistent with PA^+).

By a theorem of Friedman (based on analyzing Godel incompletness theorem)

(4) $T_b = PA + (\exists n)\,[\psi_2^f(n) + \psi_1(2^{2^n})]$

is consistent with PA^+.

We shall prove that T_a, T_b are $\le_{cs}$-incomparable. Let us prove e.q. $T_a \not\le_{cs} T_b$.

(5) $PA^+ + T_a \vdash CON(T_a)$

[Because for any model N_0 of $PA^+ + T_a$, being a model of PA^+ has a definition of a model N_1 of PA (i.e. $N_0 \models$ "N_1 is a model of PA") and an isomorphism from N_0 into a proper initial segment of N_1.

Clearly N_1 satisfies $T_a - PA$, as end extensions preserve the satisfaction of bounded formulas and $\psi_\ell(x)$, $\psi_\ell^f(x)$ are such formulas.]

Note also that

(6) $PA + T_a \vdash \neg CON(T_b)$.

[Otherwise there is a model N_0 of $PA + T_a + CON(T_b)$ hence in N_0 there is a definition of a model N_1 of T_b and an isomorphism g of N_0 onto a proper initial segment of N_1 (i.e. N_0 satisfies the sentences saying those things). As $N_0 \models T_a$, $N_0 \models$ "$\psi_2^f(n)$ and $\neg\psi_1(2^{2^n})$" for some n, but as ψ_2^f, ψ_1 are bounded formulas, and f commute with exponentiation, clearly $N_1 \models$ "$\psi_2^f(g(n))$ and

$\neg\psi_1(2^{2^{g(n)}})$". But $N_1 \models T_b$ hence for some m, $N_1 \models$ "$\psi_2^f(m)$ and $\psi_1(2^{2^m})$". So $N_1 \models$ "$\psi_2^f(m)$ and $\psi_2^f(g(n))$", hence by ψ_2^f's definition $g(n)$, m have to be equal. But $N_1 \models$ "$\psi_1(2^{2^m})$ and $\neg\psi_1(2^{2^{g(m)}})$" hence $g(n)$, m should be unequal, contraddiction].

By (5) and (6) clearly $PA^+ + CON(T_a) \nvdash CON(T_b)$ (as $PA^+ + T_a$ is consistent (by (3));this implies

(7) $PA \nvdash CON(T_a) \to CON(T_b)$.

So $T_a \not\leq_{cs} T_b$. The proof of $T_b \not\leq_{cs} T_a$ is totally analogous.

Now (Woodin suggests) we can replace the sentences $(\exists n)[\psi_2^f(n) + \neg\psi_2(2^{2^n})]$ and $(\exists n)[\psi_2^f(n) + \psi_2(2^{2^n})]$ by inequalities of the indicator functions corresponding to T_1 and T_2 (i.e. the function f_1, f_2 exhibiting the π_2^0 sentences corresponding to the consistency of T_1, T_2). E.g. $(\forall n)$ [if $f_2(n)$ is defined then so is $f_1(f_1(f_2(n)))$] and its negation. So if we accept those functions as "mathematical" (not just reasonable methamathematical) we get mathematical theories of incomparable consistency strength. (Originally we have used three theories).

We arrive to the dangerous question of which T's have mathematical indicator function: on PA see Paris and Harrington [PH], on many theories (like ZFC+large cardinal) see Friedman [F1] on ATR_0 see Friedman McAloon and Simpson [FMS] on π_1^1-CA_0 see § 2.

Alternatively for an indicator function f define f^* by $f^*(0) = 0$ $f^*(n+1) = f(f^*(n)+1)$, and use ψ_1 ψ_3 where ψ_ℓ = "the first n for which $f(n)$ is $\equiv \ell$ mod 4".

* * * * * * * *

Notice the following two phenomena

(A) For any two natural set theories, not only they are $\leq_{cs}$-comparable, but one is interpretable in the other (or expected to be so).

(B) Similarly, for any theories, e.q. undecidability results are gotten by interpretation.

Friedman [Fr2] proved a theorem saying (A) is really true. Concerning (B) however, in [Sh], the monadic theory of the real order is proven undecidable (under CH) without the usual interpretation. In Gurevich and Shelah [GS1] this is explained it is a Boolean-valued interpretation, and by [GS2] the usual interpretation is impossible. Now we can translate it to (A): list the reasonable axioms for the monadic theory of the real order (considered as a two-sort model).

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