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Models of real-valued measurability

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Abstract

Solovay's random-real forcing [R.M. Solovay, Real-valued measurable cardinals, in: Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967), Amer. Math. Soc., Providence, R.I., 1971, pp. 397–428] is the standard way of producing real-valued measurable cardinals. Following questions of Fremlin, by giving a new construction, we show that there are combinatorial, measure-theoretic properties of Solovay's model that do not follow from the existence of real-valued measurability.

Keywords: Real-valued measurable

1. Introduction

Solovay [8] showed how to produce a real-valued measurable cardinal by adding random reals to a ground model which contains a measurable cardinal. (Recall that a cardinal κ is *real-valued measurable* if there is an atomless, κ-additive measure on κ that measures all subsets of κ. For a survey of real-valued measurable cardinals see Fremlin [4].)

The existence of real-valued measurable cardinals is equivalent to the existence of a countably additive measure on the reals which measures all sets of reals and extends Lebesgue measure (Ulam [9]). However, the existence of real-valued measurable cardinals, and particularly if the continuum is real-valued measurable, has an array of Set Theoretic consequences reaching beyond measure theory. For example: a real-valued measurable cardinal has the tree property (Silver [7]); if there is a real-valued measurable cardinal, then there is no rapid p-point ultrafilter on $\mathbb{N}$ (Kunen); the dominating invariant $\mathfrak{d}$ cannot equal a real-valued measurable cardinal (Fremlin). And further, if the continuum is real-valued measurable then $\diamondsuit_{2^{\aleph_0}}$ holds (Kunen); and for all cardinals λ between $\aleph_0$ and the continuum we have $2^\lambda = 2^{\aleph_0}$ (Prikry [6]); see [4].

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On the other hand, there are other properties of Solovay's model that have not been shown to follow from the mere existence of real-valued measurable cardinals: for example, the covering invariant for the null ideal $\mathrm{cov}(\mathcal{N})$ has to equal the continuum.

Thus, Fremlin asked [4, P1] whether every real-valued measurable cardinal can be obtained by Solovay's method (the precise wording is: suppose that κ is real-valued measurable; must there be an inner model $M \subset V$ such that κ is measurable in M and a random extension $M[G] \subset V$ of M which contains $\mathcal{P}\kappa$?). The question was answered in the negative by Gitik and Shelah [5]. The broader question remains: what properties of Solovay's model follow from the particular construction, and which properties are inherent in real-valued measurability?

In this paper we present a new construction of a real-valued measurable cardinal and identify a combinatorial, measure-theoretic property that differentiates between Solovay's model and the new one.

The property is the existence of what we call *general sequences* — Definition 4.5. A general sequence is a sequence which is sufficiently random as to escape all sets of measure zero. Standard definitions of randomness are always restricted, in the sense that the randomness has to be measured with respect to a specified collection of null sets (from effective Martin-Löf tests to all sets of measure zero in some ground model). Of course, we cannot simply remove all restrictions, as no real escapes all null sets. However, we are interested in a notion that does not restrict to a special collection of null sets but considers them all. One way to do this is to change the nature of the random object — here, from a real to a long sequence of reals, and to change the nature of escaping. We remark here that the definition echoes (in spirit) the characterization of (effective) Martin-Löf randomness as a string, each of whose initial segments have high Kolmogorov complexity.

We thus introduce a notion of forcing $\mathbb{Q}_\kappa$. We show that if κ is measurable (and $2^\kappa = \kappa^+$), then in $V^{\mathbb{Q}_\kappa}$, κ (which is the continuum) is real-valued measurable (Theorem 3.18). We then show that in Solovay's model, the generic (random) sequence is general (Theorem 4.6); and that in the new model, no sequence is general (Theorem 4.14).

1.1. Notation

$\mathcal{P}X$ is the power set of X. $A - B$ is set difference. $\subset$ denotes inclusion, not necessarily proper; $\subsetneq$ denotes proper inclusion.

The reals $\mathbb{R}$ are identified with Cantor space 2^ω. If $\sigma \in 2^{<\omega}$ then $[\sigma] = \{x \in \mathbb{R} : \sigma \subset x\}$ denotes the basic open set determined by σ. If $\lambda \in \mathrm{On}$ then $\mathbb{R}^\lambda$ is the λ-fold product of $\mathbb{R}$. If $\alpha < \lambda$ and B is a Borel subset of $\mathbb{R}$ then B^α denotes $\{\bar{x} \in \mathbb{R}^\lambda : x_\alpha \in B\}$.

If A is a Borel set (on some copy of Cantor space) and W is an extension of the universe V then we let A^W denote the interpretation in W of any code of A.

If $\mathbb{P} = (\mathbb{P}, \leqslant)$ is a partial ordering then we sometimes write $\leqslant_{\mathbb{P}}$ for $\leqslant$.

If $\alpha < \beta$ are ordinals then $[\alpha, \beta) = \{\gamma : \alpha \leqslant \gamma < \beta\}$.

If X and Y are sets and $B \subset X \times Y$, then for $x \in X$, $B_x = \{y \in Y : (x, y) \in B\}$ and $B^y = \{x : (x, y) \in B\}$ are the sections.

Suppose that $\langle X_\alpha \rangle_{\alpha<\delta}$ is an increasing sequence of things (ordinals, sets (under inclusion), etc.); for limit $\beta \leqslant \delta$ we let $X_{<\beta}$ be the natural limit of $\langle X_\alpha \rangle_{\alpha<\beta}$ (the supremum, the union, etc.), and for successor $\beta = \alpha + 1$ we let $X_{<\beta} = X_\alpha$.

1.1.1. Forcing

For notions of forcing, we use the notation common in the World — {Jerusalem}. Thus, $q \leqslant p$ means that q extends p. As far as $\mathbb{P}$-names are concerned, we often confuse between canonical objects and their names. Thus, G is both a generic filter but also the name of such a filter.

If $\mathbb{B}$ is a complete Boolean algebra and φ is a formula in the forcing language for $\mathbb{B}$, then we let $[\![\varphi]\!]_{\mathbb{B}}$ be the Boolean value of φ according to $\mathbb{B}$; this is the greatest element of $\mathbb{B}$ forcing φ. For a complete Boolean algebra the partial ordering corresponding to $\mathbb{B}$ is not $\mathbb{B}$ itself but $\mathbb{B} - \{0_{\mathbb{B}}\}$. Nevertheless we often think as if the partial ordering in the forcing were $\mathbb{B}$ and let $0 \Vdash_{\mathbb{B}} \varphi$ for all formulas φ in the forcing language.

If $\mathbb{P}$ is a partial ordering and $p \in \mathbb{P}$ then $\mathbb{P}(\leqslant p)$ is the partial ordering inherited from $\mathbb{P}$ on $\{q \in \mathbb{P} : q \leqslant p\}$.

$\mathbb{P} \lessdot \mathbb{Q}$ denotes the fact that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$. If $\mathbb{P} \lessdot \mathbb{Q}$ and G is the (name for the) $\mathbb{P}$-generic filter, then $\mathbb{Q}/G$ is the (name for the) quotient of $\mathbb{Q}$ by G: the collection of all $q \in \mathbb{Q}$ which are compatible with all $p \in G$.

If $\mathbb{P} \subset \mathbb{Q}$, a strong way of getting $\mathbb{P} \lessdot \mathbb{Q}$ is having a *restriction map* $q \mapsto q \restriction \mathbb{P}$ from $\mathbb{Q}$ to $\mathbb{P}$: a map which is order preserving (but does not necessarily preserve $\not\leqslant$), and such that for all $q \in \mathbb{Q}$, $q \restriction \mathbb{P} \Vdash_{\mathbb{P}} q \in \mathbb{Q}/G$. If $\mathbb{B}$ is a complete subalgebra of a complete Boolean algebra $\mathbb{D}$ then there is a restriction map from $\mathbb{D}$ to $\mathbb{B}$; $d \restriction \mathbb{B} = \prod^{\mathbb{B}}\{b \in \mathbb{B} : b \geqslant d\}$ is in fact the largest $b \in \mathbb{B}$ which forces that $d \in \mathbb{D}/G$; $\mathbb{D}/G = \{d \in \mathbb{D} : d \restriction \mathbb{B} \in G\}$.

If $\mathbb{B}$ is a complete subalgebra of a complete Boolean algebra $\mathbb{D}$ then we let $\mathbb{D} : G$ be the (name for the) quotient of $\mathbb{D}$ by the filter generated by the generic ultrafilter $G \subset \mathbb{B}$; $\mathbb{D} : G$ is the completion of the partial ordering $\mathbb{D}/G$.

1.2. Measure theory

Notation; recollection of basic notions

Recall that a *measurable space* is a set X together with a *measure algebra on* X: a countably complete Boolean subalgebra of $\mathcal{P}X$, that is some $\mathcal{S} \subset \mathcal{P}X$ containing 0 and X and closed under complementation and unions (and intersections) of countable subsets of $\mathcal{S}$. A *probability measure* on a measure space $(X, \mathcal{S})$ is a function $\mu : \mathcal{S} \to [0, 1]$ which is monotone and countably additive: $\mu(0) = 0$, $\mu(X) = 1$ and whenever $\{B_n : n < \omega\} \subset \mathcal{S}$ is a collection of pairwise disjoint sets, then $\mu(\cup B_n) = \sum \mu(B_n)$. All measures we encounter in this work are probability measures.

Let μ be a measure on a measurable space $(X, \mathcal{S})$. Then a *μ-null* set is a set $A \in \mathcal{S}$ such that $\mu(A) = 0$. We let $\mathcal{I}_\mu$ be the collection of μ-null sets; $\mathcal{I}_\mu$ is a countably complete ideal of the Boolean algebra $\mathcal{S}$; we can thus let $\mathbb{B}_\mu = \mathcal{S}/\mathcal{I}_\mu$; this is a complete Boolean algebra and satisfies the countable chain condition. For $A \in \mathcal{S}$, we let $[A]_\mu = A + \mathcal{I}_\mu \in \mathbb{B}_\mu$. We often confuse A and $[A]_\mu$, though. We let $\subset_\mu$, $=_\mu$ etc. be the pullback of the Boolean notions in $\mathbb{B}_\mu$. Namely: $A \subset_\mu B$ if $[A]_\mu \leqslant_{\mathbb{B}_\mu} [B]_\mu$ (iff $A - B \in \mathcal{I}_\mu$), etc. We also think of μ as measuring the algebra $\mathbb{B}_\mu$; we let $\mu([A]_\mu) = \mu(A)$.

Definition 1.1. Let $\mathcal{S} \subset \mathcal{R}$ be two measure algebras on a space X, and let μ be a measure on $\mathcal{S}$ and ν be a measure on $\mathcal{R}$. We say that ν is *absolutely continuous with respect to* μ (and write $\nu \ll \mu$) if $\mathcal{I}_\mu \subset \mathcal{I}_\nu$; that is, if for all $A \in \mathcal{S}$, if $\mu(A) = 0$ then $\nu(A) = 0$.

(Of course, if $\nu \ll \mu$, $A \in \mathcal{S}$ and $\mu(A) = 1$ then $\nu(A) = 1$.)

If $\nu \ll \mu$ then the identity $\mathcal{S} \subset \mathcal{R}$ induces a map $i : \mathbb{B}_\mu \to \mathbb{B}_\nu$ which is a complete Boolean homomorphism. If $\mathcal{I}_\mu = \mathcal{I}_\nu \cap \mathcal{S}$ then i is injective.

Definition 1.2. Let μ be a measure on $(X, \mathcal{S})$, and let $A \in \mathcal{S}$ be a μ-positive set. We let $\mu \| A$, the *localization* of μ to A, be μ restricted to A, recalibrated to be a probability measure: it is the measure on $(X, \mathcal{S})$ defined by $(\mu \| A)(B) = \mu(B \cap A)/\mu(A)$.

If $A =_\mu A'$ then $\mu \| A = \mu \| A'$ so we may write $\mu \| a$ for $a \in \mathbb{B}_\mu$. We have $\mu \| a \ll \mu$ and $\mathbb{B}_{\mu \| a} \cong \mathbb{B}_\mu(\leqslant a)$; under this identification, the natural map $i : \mathbb{B}_\mu \to \mathbb{B}_{\mu \| a}$ is given by $i(b) = b \cap a$. If $a \neq 1$ (so $\mu \neq \mu \| a$) then i is not injective.

Products of measures

If for $i < 2$, μ_i is a measure on a measurable space $(X_i, \mathcal{S}_i)$, then there is a unique measure $\mu_0 \mu_1 = \mu_0 \times \mu_1$ defined on the measure algebra on $X_0 \times X_1$ generated by the *cylinders*, i.e. the sets $A_0 \times A_1$ for $A_i \in \mathcal{S}_i$, such that $(\mu_0 \mu_1)(A_0 \times A_1) = \mu(A_0)\mu_1(A_1)$ for all cylinders $A_0 \times A_1$. We recall Fubini's theorem: For any measurable $A \subset X_0 \times X_1$, we have

$$(\mu_0 \mu_1)(A) = \int_{X_0} \mu_1(A_x)\, \mathrm{d}\mu_0(x),$$

where again for $x \in X_0$, $A_x = \{y \in X_1 : (x, y) \in A\}$ is the x-section of A.

We note that localization commutes with finite products:

$$(\mu_0 \| B_0)\, (\mu_1 \| B_1) = (\mu_0\, \mu_1) \| (B_0 \times B_1).$$

We can generalize the notion of absolute continuity.

Definition 1.3 (*Generalized Absolute Continuity*). Suppose that μ measures $(X, \mathcal{S})$ and ν measures $(Y, \mathcal{R})$, and further that there is a Boolean homomorphism $i : \mathcal{S} \to \mathcal{R}$. We say that $\nu \ll \mu$ if whenever $A \in \mathcal{S}$ and $\mu(A) = 0$ then $\nu(i(A)) = 0$.

If i is injective then we do not really get anything new (we may identify $\mathcal{S}$ with its image). In any case, the map i induces a Boolean homomorphism from $\mathbb{B}_\mu$ to $\mathbb{B}_\nu$.

The standard example is of course if $\mathcal{S} = \mathcal{S}_0$ and $\mathcal{R}$ is the algebra generated by $\mathcal{S}_0 \times \mathcal{S}_1$ as above. We then let $i(A) = A \times X_1$ and get $\mu_0\,\mu_1 \ll \mu_0$. The map i is injective and induces a complete embedding

$$i_{\mu_0}^{\mu_0\mu_1} : \mathbb{B}_{\mu_0} \to \mathbb{B}_{\mu_0\,\mu_1}.$$

The following is an important simplification in notation.

Notation 1.4. Unless otherwise stated, we identify $\mathbb{B}_{\mu_0}$ with its image under $i_{\mu_0}^{\mu_0\mu_1}$. Thus $A \in \mathcal{S}_0$ is identified with $A \times X_1$.

Thus if $A_i \in \mathcal{S}_i$ then $A_0 \cap A_1 = A_0 \times A_1$.

The restriction map from $\mathbb{B}_{\mu_0\mu_1}$ onto $\mathbb{B}_{\mu_0}$ is nicely defined: for measurable $A \subset X_0 \times X_1$, we let

$$A \restriction \mu_0 = \{x \in X_0 : \mu_1(A_x) > 0\};$$

this is the measure-theoretic projection of A onto X_0. If $A =_{\mu_0\mu_1} A'$ then $A \restriction \mu_0 =_{\mu_0} A' \restriction \mu_0$, so we indeed get a map from $\mathbb{B}_{\mu_0\mu_1}$ onto $\mathbb{B}_{\mu_0}$, and $[A]_{\mu_0\mu_1} \restriction \mathbb{B}_{\mu_0} = [A \restriction \mu_0]_{\mu_0}$.

We make use of the following.

Lemma 1.5. *Let ν be a measure on X and for $i < 2$ let μ_i be a measure on Y_i. Let $B_i \subset X \times Y_i$ and let $A_i = B_i \restriction \nu$. Then $A_0 \cap A_1 =_\nu 0$ iff $B_0 \cap B_1 =_{\nu\mu_0\mu_1} 0$.*

Proof. To avoid confusion, in this proof we do not use the convention 1.4.

Suppose that A_0 and A_1 are ν-disjoint. Then $A_0 \times Y_0 \times Y_1 \cap A_1 \times Y_0 \times Y_1 =_{\nu\mu_0\mu_1} 0$. Also, $B_i \subset_{\nu\mu_i} A_i \times Y_i$ so $B_i \times Y_{1-i} \subset_{\nu\mu_0\mu_1} A_i \times Y_0 \times Y_1$; it follows that $B_0 \times Y_1$ and $B_1 \times Y_0$ are $\nu\mu_0\mu_1$-disjoint.

Suppose that $B_0 \times Y_1$ and $B_1 \times Y_0$ are $\nu\mu_0\mu_1$-disjoint. Consider $(B_0 \times Y_1) \restriction \nu\mu_1$; as $B_1 \times Y_0$ is a cylinder in the product $(X \times Y_1) \times Y_0$, we have $(B_0 \times Y_1) \restriction \nu\mu_1 \cap B_1 =_{\nu\mu_1} 0$. However, $(B_0 \times Y_1) \restriction \nu\mu_1 = A_0 \times Y_1$. Now reducing from $X \times Y_1$ to X we get $A_1 = B_1 \restriction \nu$ is ν-disjoint from A_0. □

Infinite products

Iterating the two-step product, we can consider products of finitely many measures. However, we need the more intricate notion of a product of infinitely many measures. Countable products behave much as finite products do. Let, for $n < \omega$, μ_n be a measure on a measurable space $(X_n, \mathcal{S}_n)$. Again, a *cylinder* is a set of the form $\prod_{n<\omega} A_n$ for $A_n \in \mathcal{S}_n$. There is a unique measure $\mu_\omega = \prod \mu_i$ on the measure algebra on $\prod X_n$ generated by the cylinders such that for a cylinder $\prod A_n$ we have $\mu_\omega(\prod A_n) = \prod_{n<\omega} \mu_n(A_n)$, where the infinite product is taken as the limit of the finite products.

Localization commutes with countable products: if $A_n \in \mathcal{S}_n$ is a sequence such that $\mu_\omega(\prod A_n) > 0$, then $\mu_\omega \| \prod A_n = \prod(\mu_n \| A_n)$. On the other hand, note that we can have a sequence of A_ns such that for each n, $\mu_n(A_n) > 0$, but $\mu_\omega(\prod A_n) = 0$; in this case we can use the measure $\prod(\mu_n \| A_n)$, but $\mu_\omega \| \prod A_n$ cannot be defined.

To better understand uncountable products, we notice that a countable product can be viewed as a direct limit of finite products. Namely, we let a *finite cylinder* be a set of the form $\prod_{n<k} A_n \times \prod_{n\geqslant k} X_n$ for some $k < \omega$ and $A_n \in \mathcal{S}_n$ for $n \leqslant k$. The finite cylinders are the cylinders of $\prod_{n<k} X_n$ under the standard identification of subsets of $\prod_{n<k} X_n$ with subsets of $\prod_{n<\omega} X_n$. The measure algebra on $\prod_{n<\omega} X_n$ generated by the finite cylinders is the same as the algebra generated by the infinite cylinders. Under the standard identifications, the finite product measures cohere and μ_ω is the measure generated by their union.

Let $\lambda > \aleph_0$ and suppose that for $\alpha < \lambda$, $\mu_{\{\alpha\}}$ is a measure on a measurable space $(X_{\{\alpha\}}, \mathcal{S}_{\{\alpha\}})$. For $u \subset \lambda$ let $X_u = \prod_{\alpha\in u} X_{\{\alpha\}}$. If $u \subset \lambda$ is countable, let $\mathcal{S}_u$ be the measure algebra on X_u generated by the cylinders, and let μ_u be the product $\prod_{\alpha\in u} \mu_{\{\alpha\}}$. As discussed, we can identify $\mathcal{S}_u$ with an algebra of subsets of X_λ (or more generally subsets of X_V for any $u \subset V \subset \lambda$) by considering the measure algebra generated by cylinders *with support in* u: subsets of X_V of the form $\prod_{\alpha\in u} A_\alpha \times X_{V-u}$, for $A_\alpha \in \mathcal{S}_{\{\alpha\}}$.

For any $V \subset \lambda$, we let $\mathcal{S}_V$ be the union of $\mathcal{S}_u$ for countable $u \subset V$ (we note that any cylinder $A \subset X_V$ has least support). The measures μ_u cohere (i.e. μ_u and μ_v agree on $\mathcal{S}_{u\cap v}$); thus the union of the μ_us is a measure μ_V on $\mathcal{S}_V$ (this is more immediate than the countable case because every countable subset of $\mathcal{S}_V$ lies in some $\mathcal{S}_u$). As for countable sets, we can view μ_V as measuring subsets of any X_W for $V \subset W \subset \lambda$. In fact, under this identification, $\mathcal{S}_V$ consists of those sets of $\mathcal{S}_W$ which have support in V, that is, sets of the form $A \times X_{W-V}$ for some $A \subset X_V$. [We note that unlike a cylinder, a set with infinite support may not have a minimal support: consider the set of all sequences in 2^ω which are eventually 0.] The measure μ_V is determined by its values on the cylinders with finite support; for any $V \subset W \subset \lambda$ we have $\mu_W = \mu_V \, \mu_{W-V}$.

General framework

For our work, we fix $\lambda > \aleph_0$. For all $u \subset \lambda$, we let $\mathbb{R}^u$ be the u-product of Cantor space. Elements of $\mathbb{R}^u$ are often written as $\bar{x} = \langle x_\alpha \rangle_{\alpha \in u}$. For *countable* $u \subset \lambda$, we let $\mathcal{S}_u$ be the collection of Borel subsets of $\mathbb{R}^u$, and let m_u be Lebesgue measure on $\mathbb{R}^u$. For countable and uncountable $u \subset \lambda$, $\mathcal{S}_u$ is the algebra generated by $\mathcal{S}_v$ for finite $v \subset u$ and m_u is the product $\prod_{\alpha \in u} m_{\{\alpha\}}$.

The measures we shall consider will all be localizations of products of localizations of the $m_{\{\alpha\}}$:

Definition 1.6. Let $u \subset \lambda$. A *pure local product measure on u* is a measure on $\mathcal{S}_u$ of the form $\prod_{\alpha \in u}(m_{\{\alpha\}} \| B_\alpha)$ for $B_\alpha \in \mathcal{S}_{\{\alpha\}}$. A *local product measure on u* is a measure on $\mathcal{S}_u$ of the form $\nu \| B$, where ν is a pure local product measure on u.

We will mention other measures (such as a measure witnessing that a cardinal is real-valued measurable); but when it is clear from context that we only mention local product measures, we drop the long name and just refer to "measures" and "pure measures".

If μ is a local product measure on u then we let $u^\mu = u$ and call u the *support* (or *domain*) of μ.

Topology

We note that every $\mathbb{R}^u$ is also a topological space (which can be viewed as the Tychonoff product of $\mathbb{R}^{\{\alpha\}}$ for $\alpha \in u$). However, when u is uncountable, then the Borel subsets of $\mathbb{R}^u$ properly extend $\mathcal{S}_u$. This is not a concern of ours because the completion of any local product measure measures the Borel subsets of $\mathbb{R}^u$. We thus abuse terminology and when we say "Borel" we mean a set in $\mathcal{S}_u$; so for us, every Borel set has countable support. [In some texts, sets in $\mathcal{S}_u$ are called *Baire sets*. We choose not to use this terminology to avoid confusion between measure and category.]

Recall that a measure μ which is defined on the Borel subsets of a topological space is *regular* if for all Borel A, $\mu(A)$ is both the infimum of $\mu(G)$ for open $G \supset A$ and the supremum of $\mu(K)$ for compact $K \subset A$. [Thus up to μ-measure 0, each Borel set is the same as a $\mathbf{\Sigma}^0_2$ (an F_σ) set and as a $\mathbf{\Pi}^0_2$ (a G_δ) set.] Lebesgue measure is regular, and a localization of a regular measure is also regular. Also, regularity is preserved under products; again note that even with uncountable products, every measurable set has countable support and so the closed sets produced by regularity have countable support.

Corollary 1.7. *Every local product measure is regular.*

Random reals

Let μ be a local product measure. Forcing with $\mathbb{B}_\mu$ is the same as forcing with $\mathcal{I}^+_\mu = \mathcal{S}_{u^\mu} - \mathcal{I}_\mu$, ordered by inclusion. The regularity of μ shows that the closed sets are dense in $\mathcal{I}^+_\mu$. It follows that a generic $G \subset \mathcal{I}^+_\mu$ is determined by

$$\{\bar{r}^G\} = \cap_{B \in G} B^{V[G]}.$$

We have $B \in G$ iff $\bar{r}^G \in B^{V[G]}$. We have $\bar{r}^G \in \bigcap\{A^{V[G]} : A \in V \text{ is co-null}\}$; and conversely, if W is an extension of V and $\bar{r} \in W$ lies in $\bigcap\{A^{V[G]} : A \in V \text{ is co-null}\}$, then $G = \{A \in \mathcal{I}^+_\mu : \bar{r} \in A^W\}$ is generic over V and $\bar{r} = \bar{r}^G$.

Suppose that ν, μ are local product measures and that $u^\nu \cap u^\mu = 0$. Then $\nu\mu$ is a local product measure. Recall that we have a complete embedding $i^{\nu\mu}_\nu : \mathbb{B}_\nu \to \mathbb{B}_{\nu\mu}$. Thus if $G \subset \mathbb{B}_{\nu\mu}$ is generic then $G_\nu = (i^{\nu\mu}_\nu)^{-1} G$ is generic for $\mathbb{B}_\nu$. In fact, $\bar{r}^{G_\nu} = \bar{r}^G \restriction u^\nu$.

Quotients are measure algebras

Let $V[G]$ be any generic extension of V. There is a canonical extension of μ to a measure on $\mathcal{S}^{V[G]}_u$, which we denote by $\mu^{V[G]}$. For if $\mu = (\prod_{\alpha \in u}(m_{\{\alpha\}} \| B_\alpha)) \| B$ then we can let $\mu^{V[G]} = (\prod_{\alpha \in u}(m_{\{\alpha\}} \| B^{V[G]}_\alpha)) \| B^{V[G]}$. The usual absoluteness arguments show that indeed $\mu^{V[G]}$ is an extension of μ, and does not depend on the presentation of μ.

Again let μ and ν be local product measures on disjoint $u = u^\nu$, $v = u^\mu \subset \lambda$. We make use of the following.

Fact 1.8. Let $G \subset \mathbb{B}_\nu$ be generic. Then the map $A \mapsto A^{V[G]}{}_{\bar{r}^G}$ induces an isomorphism from $\mathbb{B}_{\nu\mu} : G$ to $\mathbb{B}_{\mu^{V[G]}}$.

In particular, $\Vdash_{\mathbb{B}_\nu}$ "$\mathbb{B}_{\nu\mu} : G$ is a measure algebra".

Proof. Let $\pi_{\nu,\mu} : \mathbb{B}_{\nu\mu} \to \mathbb{B}_{\nu\mu} : G$ be the quotient map. We know that

$$\pi^{-1}_{\nu,\mu}(\mathbb{B}_{\nu\mu} : G - \{0\}) = \mathbb{B}_{\nu\mu}/G$$

(the partial ordering). Thus for $A \in \mathcal{S}_{u\cup v}$, we have

$$\begin{aligned}\pi_{\nu,\mu}([A]_{\nu\mu}) > 0 \iff [A \restriction \nu]_\nu \in G &\iff \\ \bar{r}^G \in (A \restriction \nu)^{V[G]} &\iff \mu^{V[G]}(A^{V[G]}{}_{\bar{r}^G}) > 0.\end{aligned}$$

The last equivalence follows from the fact that $(A \restriction \nu)^{V[G]} = A^{V[G]} \restriction \nu^{V[G]}$; again we use absoluteness. Thus we may define an embedding $\sigma_{\nu,\mu} : \mathbb{B}_{\nu\mu} : G \to \mathbb{B}_{\mu^{V[G]}}$ by letting $\sigma_{\nu,\mu}(\pi_{\nu,\mu}([A]_{\nu\mu})) = \left[A^{V[G]}{}_{\bar{r}^G}\right]_{\mu^{V[G]}}$. It is clear that $\sigma_{\nu,\mu}$ preserves the Boolean operations.

$\sigma_{\nu,\mu}$ is onto: every set in the random extension is determined by a set in the plane in the ground model (see [2, 3.1]). For any countable $v' \subset v$, every $\mathbb{B}_\nu$-name y for an element of $\mathbb{R}^{v'}$ corresponds to a $\mathbf{\Pi}^0_3$ function $f_y : \mathbb{R}^u \to \mathbb{R}^{v'}$ defined by $f_y(x)(i)(n) = k \iff x \in [\![y(i)(n) = k]\!]_{\mathbb{B}_\nu}$ (f_y can be taken to be $\mathbf{\Pi}^0_3$ because $[\![y(i)(n) = k]\!]$ can be taken to be either a $\mathbf{\Sigma}^0_2$ or a $\mathbf{\Pi}^0_2$ set). The function f_y has the property that $f_y^{V[G]}(\bar{r}^G) = y^G$. Let C be a $\mathbb{B}_\nu$-name for a Borel subset of $\mathbb{R}^v$. The algebra $\mathbb{B}_\nu$ is c.c.c., so in V, there is some countable $v' \subset v$ and some $\mathbb{B}_\nu$-name C' for a Borel subset of $\mathbb{R}^{v'}$ such that $\Vdash_{\mathbb{B}_\nu} C = C' \times \mathbb{R}^{v-v'}$. We can let

$$A = \{(x, f_y(x)) : x \in [\![y \in C']\!]_{\mathbb{B}_\nu}\} \times \mathbb{R}^{v-v'}$$

where f_y ranges over $\mathbf{\Pi}^0_3$ functions from $\mathbb{R}^u$ to $\mathbb{R}^{v'}$; thus A is Borel and $(A^G)_{\bar{r}^G} = C^G$. [However, in what follows, we do not use the fact that $\sigma_{\nu,\mu}$ is onto.] □

Commuting diagrams

We thus have the following diagram:

$$\mathbb{B}_{\nu\mu} \xrightarrow{\pi_{\nu,\mu}} \mathbb{B}_{\nu\mu} : G \xrightarrow{\sigma_{\nu,\mu}} \mathbb{B}_{\mu^{V[G]}}.$$

Suppose now that ν is a local product measure on u; μ, ϱ are local product measures on v_0, v_1, and u, v_0, v_1 are pairwise disjoint. Let $\upsilon = \mu\varrho$. Let $G \subset \mathbb{B}_\nu$ be generic. For the rest of the section, we retract our convention 1.4. We thus have a complete embedding $i^{\nu\upsilon}_{\nu\mu} : \mathbb{B}_{\nu\mu} \to \mathbb{B}_{\nu\upsilon}$. This embedding induces a complete embedding $\iota^{\nu\upsilon}_{\nu\mu} : \mathbb{B}_{\nu\mu} : G \to \mathbb{B}_{\nu\upsilon} : G$.

Lemma 1.9. *The following diagram commutes.*

$$\begin{array}{ccccc}
\mathbb{B}_{\nu\mu} & \xrightarrow{\pi_{\nu,\mu}} & \mathbb{B}_{\nu\mu} : G & \xrightarrow{\sigma_{\nu,\mu}} & \mathbb{B}_{\mu^{V[G]}} \\
\downarrow{\scriptstyle i^{\nu\upsilon}_{\nu\mu}} & & \downarrow{\scriptstyle \iota^{\nu\upsilon}_{\nu\mu}} & & \downarrow{\scriptstyle i^{\upsilon^{V[G]}}_{\mu^{V[G]}}} \\
\mathbb{B}_{\nu\upsilon} & \xrightarrow{\pi_{\nu,\upsilon}} & \mathbb{B}_{\nu\upsilon} : G & \xrightarrow{\sigma_{\nu,\upsilon}} & \mathbb{B}_{\upsilon^{V[G]}}
\end{array}$$

Proof. Let $A \in \mathcal{S}_{u\cup v_0}$, and let

$$\begin{aligned}
&a = \pi_{\nu,\mu}([A]_{\nu\mu});\\
&a' = \iota^{\nu\upsilon}_{\nu\mu}(a) = \pi_{\nu,\upsilon}([A \times \mathbb{R}^{v_1}]_{\nu\upsilon});\\
&b = \sigma_{\nu,\mu}(a) = \left[A^{V[G]}{}_{\bar{r}^G}\right]_{\mu^{V[G]}}; \text{ and}\\
&b' = \sigma_{\nu,\upsilon}(a') = \left[(A \times \mathbb{R}^{v_1})^{V[G]}{}_{\bar{r}^G}\right]_{\upsilon^{V[G]}}.
\end{aligned}$$

Sh:763

The desired equation $i^{\upsilon^{V[G]}}_{\mu^{V[G]}}(b) = b'$ follows from the fact that

$$A^{V[G]}{}_{\bar{r}^G} \times \mathbb{R}^{\upsilon_1\, V[G]} = \left(A \times \mathbb{R}^{\upsilon_1}\right)^{V[G]}{}_{\bar{r}^G}. \quad \square$$

Note that $i^{\upsilon^{V[G]}}_{\mu^{V[G]}}$ is measure-preserving.

Next, suppose that ς, ϱ are local product measures on u_0, u_1 and that μ is a local product measure on v; and that u_0, u_1, v are pairwise disjoint. We let $\nu = \varsigma\varrho$.

As i^{ν}_{ς} is a complete embedding, we know that if $G_\nu \subset \mathbb{B}_\nu$ is generic, then $G_\varsigma = (i^{\nu}_{\varsigma})^{-1} G_\nu$ is also generic. The map $i^{\nu\mu}_{\varsigma\mu}$ induces a complete embedding $\iota^{\nu\mu}_{\varsigma\mu}$ from $\mathbb{B}_{\varsigma\mu} : G_\varsigma$ to $\mathbb{B}_{\nu\mu} : G_\nu$.

Also, as $V[G_\varsigma] \subset V[G_\nu]$ we have (relying on absoluteness) a *measure-preserving* embedding $\imath^{\nu,\mu}_{\varsigma,\mu} : \mathbb{B}_{\mu^{V[G_\varsigma]}} \to \mathbb{B}_{\mu^{V[G_\nu]}}$, given by $[B]_{\mu^{V[G_\varsigma]}} \to \left[B^{V[G_\nu]}\right]_{\mu^{V[G_\nu]}}$.

Lemma 1.10. *The following diagram commutes:*

$$\begin{array}{ccccc}
\mathbb{B}_{\varsigma\mu} & \xrightarrow{\pi_{\varsigma,\mu}} & \mathbb{B}_{\varsigma\mu} : G_\varsigma & \xrightarrow{\sigma_{\varsigma,\mu}} & \mathbb{B}_{\mu^{V[G_\varsigma]}} \\
\downarrow{\scriptstyle i^{\nu\mu}_{\varsigma\mu}} & & \downarrow{\scriptstyle \iota^{\nu\mu}_{\varsigma\mu}} & & \downarrow{\scriptstyle \imath^{\nu,\mu}_{\varsigma,\mu}} \\
\mathbb{B}_{\nu\mu} & \xrightarrow{\pi_{\nu,\mu}} & \mathbb{B}_{\nu\mu} : G_\nu & \xrightarrow[\sigma_{\nu,\mu}]{} & \mathbb{B}_{\mu^{V[G_\nu]}}
\end{array}$$

Proof. Let $A \in \mathcal{S}_{u_0 \cup v}$. We let:

$$\begin{aligned}
a &= \pi_{\varsigma,\mu}\left([A]_{\varsigma\mu}\right); \\
a' &= \iota^{\nu\mu}_{\varsigma\mu}(a) = \pi_{\nu,\mu}\left(\left[A \times \mathbb{R}^{u_1}\right]_{\nu\mu}\right); \\
b &= \sigma_{\varsigma,\mu}(a) = \left[A^{V[G_\varsigma]}{}_{\bar{r}^{G_\varsigma}}\right]_{\mu^{V[G_\varsigma]}}; \text{ and} \\
b' &= \sigma_{\nu,\mu}(a') = \left[(A \times \mathbb{R}^{u_1})^{V[G_\nu]}{}_{\bar{r}^{G_\nu}}\right]_{\mu^{V[G_\nu]}}.
\end{aligned}$$

We want to show that $\imath^{\nu,\mu}_{\varsigma,\mu}(b) = b'$. Letting $B = A^{V[G_\varsigma]}{}_{\bar{r}^{G_\varsigma}}$ and $B' = (A \times \mathbb{R}^{u_1})^{V[G_\nu]}{}_{\bar{r}^{G_\nu}}$, we show that $B' = B^{V[G_\nu]}$. We know, though, that $\bar{r}^{G_\nu} = \bar{r}^{G_\varsigma} {}^\frown \bar{r}^{G_\varrho}$, from which we deduce that $B' = A^{V[G_\nu]}{}_{\bar{r}^{G_\varsigma}}$. The conclusion follows from absoluteness. $\square$

In our third scenario, we have ϱ, μ which are local product measures on disjoint v, u; and we let $\nu = \varrho \| B$ be some localization of ϱ. In this case we have a projection $i^{\nu}_{\varrho} : \mathbb{B}_\varrho \twoheadrightarrow \mathbb{B}_\nu$. Let $G_\nu \subset \mathbb{B}_\nu$ be generic; then $G_\varrho = (i^{\nu}_{\varrho})^{-1} G_\nu$ is generic, but in fact contains no less information; so we denote the extension by $V[G]$.

Lemma 1.11. *The following diagram commutes:*

$$\begin{array}{ccccc}
\mathbb{B}_{\varrho\mu} & \xrightarrow{\pi_{\varrho,\mu}} & \mathbb{B}_{\varrho\mu} : G_\varrho & & \\
 & & & \searrow{\scriptstyle \sigma_{\varrho,\mu}} & \\
\downarrow{\scriptstyle i^{\nu\mu}_{\varrho\mu}} & & \downarrow{\scriptstyle \iota^{\nu\mu}_{\varrho\mu}} & & \mathbb{B}_{\mu^{V[G]}} \\
 & & & \nearrow{\scriptstyle \sigma_{\nu,\mu}} & \\
\mathbb{B}_{\nu\mu} & \xrightarrow{\pi_{\nu,\mu}} & \mathbb{B}_{\nu\mu} : G_\nu & &
\end{array}$$

Proof. As usual we take $A \in \mathcal{S}_{v \cup u}$ and follow $[A]_{\varrho\mu}$ along the diagram. We have $\iota^{\nu,\mu}_{\varrho,\mu}(\pi_{\varrho,\mu}([A]_{\varrho\mu})) = \pi_{\nu,\mu}([A]_{\nu\mu})$; and $\sigma_{\varrho,\mu}(\pi_{\varrho,\mu}([A]_{\varrho\mu})) = \left[A^{V[G]}{}_{\bar{r}^{G_\varrho}}\right]_{\mu^{V[G]}}$ and $\sigma_{\nu,\mu}(\pi_{\nu,\mu}([A]_{\nu\mu})) = \left[A^{V[G]}{}_{\bar{r}^{G_\nu}}\right]_{\mu^{V[G]}}$; the latter two are equal because $\bar{r}^{G_\varrho} = \bar{r}^{G_\nu}$. $\square$

Our last case is perhaps the easiest (in fact we do not use it later but we include it for completeness). Suppose that u and v are disjoint and that ν, μ are local product measures on u, v respectively. Suppose that $C \in \mathcal{B}_\mu$ is positive; let $\upsilon = \mu \| C$. Let $G \subset \mathbb{B}_\nu$ be generic. Then by absoluteness $\upsilon^{V[G]} = \mu^{V[G]} \| C^{V[G]}$. Note that unlike the previous cases, the Boolean homomorphism $i^{\upsilon^{V[G]}}_{\mu^{V[G]}}$ is not measure-preserving.

Lemma 1.12. *The following diagram commutes.*

$$\begin{array}{ccccc}
\mathbb{B}_{\nu\mu} & \xrightarrow{\pi_{\nu,\mu}} & \mathbb{B}_{\nu\mu} \colon G & \xrightarrow{\sigma_{\nu,\mu}} & \mathbb{B}_{\mu^{V[G]}} \\
\downarrow{\scriptstyle i^{\nu\upsilon}_{\nu\mu}} & & \downarrow{\scriptstyle \iota^{\nu\upsilon}_{\nu\mu}} & & \downarrow{\scriptstyle i^{\upsilon^{V[G]}}_{\mu^{V[G]}}} \\
\mathbb{B}_{\nu\upsilon} & \xrightarrow{\pi_{\nu,\upsilon}} & \mathbb{B}_{\nu\upsilon} \colon G & \xrightarrow{\sigma_{\nu,\upsilon}} & \mathbb{B}_{\upsilon^{V[G]}}
\end{array}$$

Proof. Immediate, because for $A \in \mathcal{S}_{u \cup v}$, $i^{\nu\varsigma}_{\nu\mu}([A]_{\nu\mu}) = [A]_{\nu\upsilon}$ (which is the same as $[A \cap (\mathbb{R}^u \times C)]_{\nu\upsilon}$). □

2. Solovay's construction

We hope that the gentle reader will not be offended if we repeat a proof of Solovay's original construction of a real-valued measurable cardinal, starting from a measurable cardinal. The exposition which we give is different from the one found in most textbooks, indeed from the one given by Solovay in his paper; since in the rest of this paper we shall elaborate on this proof, we thought such an exposition may be useful.

Let κ be a measurable cardinal; let $j \colon V \to M$ be an elementary embedding of V into a transitive class model M with critical point κ, such that $M^\kappa \subset M$.

We move swiftly between M, V, $M[G]$ and $V[G]$. Whenever necessary we indicate where we work, but many notions are absolute and there is not much danger of confusion.

The forcing Solovay uses is $\mathbb{P} = \mathbb{B}_{m_\kappa}$, i.e. forcing with Borel subsets of $\mathbb{R}^\kappa$ of positive Lebesgue measure. We show that after forcing with $\mathbb{P}$, κ is real-valued measurable.

We have $j(\mathbb{P}) = \left(\mathbb{B}_{m_{j(\kappa)}}\right)^M = \mathbb{B}_{m_{j(\kappa)}}$. Also, $\mathbb{P} \in M$ and $\mathbb{P} = \left(\mathbb{B}_{m_\kappa}\right)^M$.

Let $G \subset \mathbb{P}$ be generic over V. Then G is generic over M. We have the following diagram:

$$j(\mathbb{P}) \xrightarrow{\pi_{m_\kappa, m_{[\kappa, j(\kappa))}}} j(\mathbb{P}) \colon G \xrightarrow{\sigma_{m_\kappa, m_{[\kappa, j(\kappa))}}} \left(\mathbb{B}_{m_{[\kappa, j(\kappa))}}\right)^{M[G]}.$$

For shorthand, we let $\pi = \pi_{m_\kappa, m_{[\kappa, j(\kappa))}}$ and we let ν be the pullback to $j(\mathbb{P}) \colon G$ of $m_{[\kappa, j(\kappa))}{}^{M[G]}$ by $\sigma_{m_\kappa, m_{[\kappa, j(\kappa))}}$.

Let A be a $\mathbb{P}$-name for a subset of κ. In M, $j(A)$ is a $j(\mathbb{P})$-name for a subset of $j(\kappa)$. Let $b_A = \left([\![\kappa \in j(A)]\!]_{j(\mathbb{P})}\right)^M$ (note $A \mapsto b_A$ is in V) and in $V[G]$ let $\mu(A) = \nu(\pi(b_A))$. We now work in V so we refer to the objects defined as names.

Lemma 2.1. *Suppose that $a \in \mathbb{P}$, that A, B are $\mathbb{P}$-names for subsets of κ, and that $a \Vdash_{\mathbb{P}} A \subset B$. Then $a \Vdash_{\mathbb{P}} \mu(A) \leqslant \mu(B)$.*

Proof. The point is that $j(a) = a$. Let $G \subset \mathbb{P}$ be generic over V such that $a \in G$. In M, $a \Vdash_{j(\mathbb{P})} j(A) \subset j(B)$. Let $b = b_A \cap a$. As $a \in G$ we have $\pi(a) = 1_{j(\mathbb{P}) \colon G}$ so $\pi(b) = \pi(b_A)$. However, in M, $b \Vdash_{j(\mathbb{P})} \kappa \in j(B)$ (as it forces that $j(A) \subset j(B)$ and that $\kappa \in j(A)$) and so $b \leqslant b_B$. It follows that $\pi(b_A) \leqslant \pi(b_B)$ so $\mu(A) \leqslant \mu(B)$. As G was arbitrary, $a \Vdash_{\mathbb{P}} \mu(A) \leqslant \mu(B)$. □

It follows that μ, rather than being defined on names for subsets of κ, can be well-defined on subsets of κ in $V[G]$. The following lemmas ensure that μ is indeed a (non-trivial) κ-complete measure.

Lemma 2.2. *Let $A \subset \kappa$ be in V and let $G \subset \mathbb{P}$ be generic over V. If $\kappa \in j(A)$ then $\mu(A) = 1$ and if $\kappa \notin j(A)$ then $\mu(A) = 0$.*

Proof. Suppose that $\kappa \in j(A)$. Then in M, $1_{j(\mathbb{P})} \Vdash \kappa \in j(A)$. Thus $b_A = 1_{j(\mathbb{P})}$ so $\pi(b_A) = 1_{j(\mathbb{P}):\,G}$. Thus $\mu(A) = 1$. On the other hand, if $\kappa \notin j(A)$ then in M, no $b \in j(\mathbb{P})$ forces that $\kappa \in j(A)$, so $b_A = 0$; it follows that $\mu(A) = 0$. □

Lemma 2.3. *Suppose that $\langle A_n \rangle_{n<\omega}$ is a sequence of $\mathbb{P}$-names for subsets of κ. Suppose that $a \in \mathbb{P}$ forces that $A = \bigcup_{n<\omega} A_n$ is a disjoint union. Then $a \Vdash_{\mathbb{P}} \mu(A) = \sum_{n<\omega} \mu(A_n)$.*

Proof. We have $j(a) = a$ and $j(\langle A_n \rangle_{n<\omega}) = \langle j(A_n) \rangle_{n<\omega}$; so in M, a forces (in $j(\mathbb{P})$) that $j(A) = \cup_{n<\omega} j(A_n)$ is a disjoint union. Again let G be generic such that $a \in G$.

Let $l, k < \omega$ and $l \neq k$. In M, $a \Vdash_{j(\mathbb{P})} j(A_k) \cap j(A_l) = 0$ so $b_{A_k} \cap a$ and $b_{A_l} \cap a$ are disjoint in $j(\mathbb{P})$. As $a \in G$ it follows that in $j(\mathbb{P}):\,G$, $\pi(b_{A_k}) \wedge \pi(b_{A_l}) = 0$. We thus have $\nu(\sum_n^{j(\mathbb{P}):\,G} \pi(b_{A_n})) = \sum_n \mu(A_n)$. It thus suffices to show that $\sum_n^{j(\mathbb{P}):\,G} \pi(b_{A_n}) = \pi(b_A)$.

For any $n < \omega$, $a \Vdash A_n \subset A$ so as we saw before, $\pi(b_{A_n}) \leqslant \pi(b_A)$. To show the other inclusion, let $b = b_A - \sum_n^{j(\mathbb{P})} b_{A_n}$. Then in M, $b \Vdash_{j(\mathbb{P})} \kappa \in j(A) - \cup_n j(A_n)$. Since $a \Vdash_{j(\mathbb{P})} j(A) \subset \cup_n j(A_n)$ we must have $a \cap b = 0$, which implies that $\pi(b) = 0$. The equality follows. □

Lemma 2.4. *Suppose that $\gamma < \kappa$ and that $\langle A_\alpha \rangle_{\alpha<\gamma}$ is a sequence of $\mathbb{P}$-names for subsets of κ. Suppose that $a \in \mathbb{P}$ and $a \Vdash_{\mathbb{P}} \forall \alpha < \gamma\ (\mu(A_\alpha) = 0)$. Then $a \Vdash_{\mathbb{P}} \mu(\cup_\alpha A_\alpha) = 0$.*

Proof. Let A be a $\mathbb{P}$-name for a subset of κ such that $a \Vdash_{\mathbb{P}} A = \cup_{\alpha<\gamma} A_\alpha$. Then in M, $a \Vdash_{j(\mathbb{P})} j(A) = \cup_{\alpha<\gamma} j(A_\alpha)$ and also $a \Vdash_{j(\mathbb{P})} \forall \alpha < \gamma\ \left[\pi(b_{A_\alpha}) = 0\right]$, that is, $a \Vdash \forall \alpha < \gamma\ [\kappa \notin j(A_\alpha)]$. Thus $a \Vdash_{j(\mathbb{P})} \kappa \notin j(A)$ so $a \cap b_A =_{j(\mathbb{P})} 0$ so $a \Vdash_{j(\mathbb{P})} \pi(b_A) = 0$ so $a \Vdash_{\mathbb{P}} \mu(A) = 0$. □

3. A new construction of a real-valued measurable cardinal

Definition 3.1. A set of ordinals u is of *Easton type* if whenever θ is an inaccessible cardinal, $u \cap \theta$ is bounded below θ.

Let $\mathbb{Q}$ consist of the collection of all local product measures (Definition 1.6) whose support is of Easton type. If u is a set of ordinals, then we let $\mathbb{Q}_u$ be the collection of measures in $\mathbb{Q}$ whose support is contained in u.

Let $\mu, \nu \in \mathbb{Q}$. We say that ν is a *pure* extension of μ (and write $\nu \leqslant_{\text{pur}} \mu$) if $\nu = \mu\,\varsigma$ where ς is a *pure* local product measure.

We say that ν is a *local* extension of μ (and write $\nu \leqslant_{\text{loc}} \mu$) if ν is a localization of μ (in particular μ and ν have same support u).

We let ν extend μ ($\nu \leqslant \mu$) if there is some ς such that $\nu \leqslant_{\text{loc}} \varsigma \leqslant_{\text{pur}} \mu$. It is not hard to verify that $\leqslant$ is indeed a partial ordering on $\mathbb{Q}$, and in fact on every $\mathbb{Q}_u$.

Lemma 3.2. *Suppose that $\nu \leqslant_{\text{pur}} \varsigma$ and $\varsigma \leqslant_{\text{loc}} \mu$. Then $\nu \leqslant \mu$.*

Proof. Let υ be a pure measure such that $\nu = \varsigma\,\upsilon$. Let $B \in \mathcal{S}_{u^\mu}$ such that $\varsigma = \mu \| B$. Then $\nu = (\mu\,\upsilon) \| B$, so $\mu\,\upsilon$ witnesses $\nu \leqslant \mu$. □

Note that if $\nu \leqslant \mu$ then $\nu \ll \mu$.

3.1. Characterization of a generic

We wish to find some characterization of a generic filter of $\mathbb{Q}_u$, analogous to the description of a generic for random forcing in terms of a random real. We need to discuss compatibility in $\mathbb{Q}$.

3.1.1. Compatibility in $\mathbb{Q}$

Definition 3.3. Let $\mu, \nu \in \mathbb{Q}$. We say that μ and ν are *explicitly incompatible* (and write $\mu \perp_{\text{exp}} \nu$) if there is some $B \in \mathcal{S}_{u^\mu \cap u^\nu}$ such that $\mu(B) = 0$ but $\nu(B) = 1$.

It is clear that if $\mu \perp_{\text{exp}} \nu$ then $\mu \perp \nu$ (in $\mathbb{Q}$ and in every $\mathbb{Q}_u$); because we cannot have some $\varsigma \ll \mu, \nu$.

Lemma 3.4. *Suppose that $u^\mu \cap u^\nu \neq 0$ and $\mu \not\perp_{\text{exp}} \nu$. Then there is some pure measure ς on $u^\mu \cap u^\nu$ such that $\mu, \nu \leqslant \varsigma$.*

Sh:763

Proof. Let μ_0, ν_0 be pure local product measures such that $\mu = \mu_0 \| C^\mu$, $\nu = \nu_0 \| C^\nu$ for some positive sets C^μ, C^ν. Pick sequences $\langle B^\mu_\alpha \rangle_{\alpha \in u^\mu}$, $\langle B^\nu_\alpha \rangle_{\alpha \in u^\nu}$ which define μ_0, ν_0 (i.e. $\mu_0 = \prod_{\alpha \in u^\mu} (m_{\{\alpha\}} \| B^\mu_\alpha)$ and similarly for ν_0). Note that μ_0 and ν_0 are not by any means unique, but that the B_αs are determined (up to Lebesgue measure) by μ_0, ν_0.

Let $v = u^\mu \cap u^\nu$. Let $\varsigma = \prod_{\alpha \in v} (m_{\{\alpha\}} \| (B^\mu_\alpha \cup B^\nu_\alpha))$.

First we show that for all but countably many $\alpha \in v$ we have $B^\mu_\alpha =_{m_{\{\alpha\}}} B^\nu_\alpha$. Suppose not; then for some $\epsilon < 1$ we have some countable, infinite $w \subset v$ such that for all $\alpha \in w$, $(m_{\{\alpha\}} \| B^\mu_\alpha)(B^\nu_\alpha) < \epsilon$ (or the other way round). Let $A = \prod_{\alpha \in w} B^\nu_\alpha$. Then $\nu(A) = 1$ but $\mu(A) = 0$.

Let $w = \{\alpha \in v : B^\mu_\alpha \neq_{m_{\{\alpha\}}} B^\nu_\alpha\}$. We assume that $w \neq 0$ for otherwise we are done. Let $A^\mu = \prod_{\alpha \in w} B^\mu_\alpha$ (and similarly define A^ν). It is sufficient to show that $\varsigma(A^\mu), \varsigma(A^\nu) > 0$; it will then follow that μ_0 is a pure extension of $\varsigma \| A^\mu$, and similarly for ν_0. Suppose that $\varsigma(A^\mu) = 0$. Let $\alpha \in w$; let $a_\alpha = m_{\{\alpha\}}(B^\nu_\alpha - B^\mu_\alpha)$, $c_\alpha = m_{\{\alpha\}}(B^\mu_\alpha - B^\nu_\alpha)$ and $b_\alpha = m_{\{\alpha\}}(B^\mu_\alpha \cap B^\nu_\alpha)$. The assumption is that $\prod_{\alpha \in w} \frac{b_\alpha + c_\alpha}{a_\alpha + b_\alpha + c_\alpha} = 0$. However, for each $\alpha \in w$, $\frac{b_\alpha}{a_\alpha + b_\alpha} \leqslant \frac{b_\alpha + c_\alpha}{a_\alpha + b_\alpha + c_\alpha}$, which means that $\prod_{\alpha \in w} (m_{\{\alpha\}} \| B^\nu_\alpha)(B^\mu_\alpha) = 0$, so $\nu(A^\mu) = 0$ (and of course $\mu(A^\mu) = 1$). □

For $\mu, \nu \in \mathbb{Q}$, if $u^\mu \cap u^\nu = 0$ then $\mu\nu \leqslant \mu, \nu$ and so μ and ν are compatible. The following is the generalization we need:

Lemma 3.5. *Let u be a set of ordinals and let $\mu, \nu \in \mathbb{Q}_u$. Then $\mu \perp_{\mathbb{Q}_u} \nu$ iff $\mu \perp_{\exp} \nu$.*

Proof. Suppose that $\mu \not\perp_{\exp} \nu$. We may assume that $v = u^\mu \cap u^\nu \neq 0$. By Lemma 3.4, find some pure ς on v, some pure μ_1, ν_1 and some C^μ, C^ν such that $\mu = (\varsigma\, \mu_1) \| C^\mu$, $\nu = (\varsigma\, \nu_1) \| C^\nu$. Let $\upsilon = \varsigma\, \mu_1\, \nu_1$. We have $\upsilon(C^\mu \cap C^\nu) > 0$ for otherwise, by Lemma 1.5, $C^\mu \upharpoonright \varsigma$, $C^\nu \upharpoonright \varsigma$ are ς-disjoint and would witness that $\mu \perp_{\exp} \nu$. Then $\upsilon \| (C^\mu \cap C^\nu)$ is a common extension of μ and ν; for example, $\upsilon \| (C^\mu \cap C^\nu) = (\mu\, \nu_1) \| C^\nu$. □

Remark 3.6. If $\mu \not\perp \nu$ then there is some υ on $u^\mu \cup u^\nu$ which is a common extension of μ and ν. In fact, the common extension constructed in the proof of Lemma 3.5 is the greatest common extension of μ and ν in $\mathbb{Q}$ (thus this extension does not depend on the choice of ς).

3.1.2. Characterization of the generic

Let u be a set of ordinals, and let $G \subset \mathbb{Q}_u$ be generic over V. Let

$$A_G = \cap \{B^{V[G]} : \text{for some } \mu \in G,\ \mu(B) = 1\}.$$

Lemma 3.7. *A_G is not empty.*

Proof. Let $\mathcal{F}_G = \{B^{V[G]} : B \text{ is closed and for some } \mu \in G,\ \mu(B) = 1\}$, and let $B_G = \cap \mathcal{F}_G$. We show that $B_G = A_G$ and that B_G is not empty.

For the first assertion, recall (Corollary 1.7) that every $\mu \in \mathbb{Q}$ is a regular measure. Let $\mu \in \mathbb{Q}_u$ and let B be of μ-measure 1. There is some closed $A \subset B$ of positive measure, so $\mu \| A \in \mathbb{Q}_u$. Thus by genericity, for every B such that $\mu(B) = 1$ for some $\mu \in G$, there is some closed $A \subset B$ and some $\nu \in G$ such that $\nu(A) = 1$. This shows that $B_G = A_G$.

Next, we note that $\mathcal{F}_G$ has the finite intersection property. Let $F \subset \mathcal{F}_G$ be finite. For $B \in F$ let $\nu_B \in G$ witness $B \in \mathcal{F}_G$. There is some $\mu \in G$ which extends all ν_B for $B \in F$. Then $\mu(\cap F) = 1$ which implies that $\cap F \neq 0$. As $\mathbb{R}^{u\, V[G]}$ is compact, $B_G \neq 0$. □

In fact,

Lemma 3.8. *A_G is a singleton $\{\bar{s}^G\}$.*

Proof. Let $\alpha < \lambda$ and let $n < \omega$. There is some $\mu \in G$ and some $\sigma \in 2^n$ such that $\mu([\sigma]^\alpha) = 1$. For given any μ we can extend it to some ν such that $\alpha \in u^\nu$ and then extend ν locally to some ς such that $\varsigma([\sigma]^\alpha) = 1$ for some $\sigma \in 2^n$. □

As usual,

Lemma 3.9. *$V[G] = V[\bar{s}^G]$.*

Proof. In fact, G can be recovered from $\bar{s}^G$ because for all $\mu \in \mathbb{Q}_u$, $\mu \in G$ iff for all B such that $\mu(B) = 1$ we have $\bar{s}^G \in B^{V[G]}$. For if $\mu \notin G$ then there is some $\nu \in G$ such that $\nu \perp \mu$. By Lemma 3.5, there is some B such that $\nu(B) = 0$ and $\mu(B) = 1$. Then $\bar{s}^G \notin B^{V[G]}$. □

3.1.3. The size of the continuum

Here is an immediate application:

Lemma 3.10. *$\mathbb{Q}_u$ adds at least $|u|$ reals.*

Proof. Let G be generic and let $\bar{s}^G$ be the generic sequence. We want to show that for distinct $\alpha, \beta \in u$ we have $\bar{s}^G_\alpha \neq \bar{s}^G_\beta$. Let $\mu \in G$ be such that $\alpha, \beta \in u^\mu$. $\mu \leqslant m_{\{\alpha,\beta\}}$ so $\mu \ll m_{\{\alpha,\beta\}}$. Let $A = \{(x, y) \in \mathbb{R}^{\{\alpha\}} \times \mathbb{R}^{\{\beta\}} : x \neq y\}$ be the complement of the diagonal. Then $m_{\{\alpha,\beta\}}(A) = 1$ so $\mu(A) = 1$. Thus $\mu \Vdash \bar{s}^G \in A^{V[G]}$. But $A^{V[G]} = \{(x, y) \in {\mathbb{R}^{\{\alpha\}}}^{V[G]} \times {\mathbb{R}^{\{\beta\}}}^{V[G]} : x \neq y\}$. Thus $\bar{s}^G_\alpha \neq \bar{s}^G_\beta$. □

3.2. More on local and pure extensions

Let $\mu \in \mathbb{Q}$. The collection of local extensions of μ (ordered by $\leqslant$) is isomorphic to $\mathbb{B}_\mu$, so we identify the two.

Lemma 3.11. *Let $\mu \in \mathbb{Q}_u$. Then $\mathbb{B}_\mu \lessdot \mathbb{Q}_u(\leqslant\mu)$.*

Proof. Let $A, B \in \mathbb{B}_\mu$. Then A and B are compatible in $\mathbb{B}_\mu$ iff $\mu(A \cap B) > 0$ iff $\mu\|A$, $\mu\|B$ are compatible in $\mathbb{Q}$.

Let $\langle A_n \rangle_{n<\omega}$ be a maximal antichain of $\mathbb{B}_\mu$. Let $\nu \in \mathbb{Q}_u$, $\nu \leqslant \mu$. Since $\mu(\cup A_n) = 1$ we have $\nu(\cup A_n) = 1$ and so for some $n < \omega$ we have $\nu(A_n) > 0$. Then $\nu\|A_n$ is a common extension of ν and $\mu\|A_n$. □

It follows that $\bar{s}^G$ is a string of random reals.

Remark 3.12. For all $u \subset v$ we have $\mathbb{Q}_u \lessdot \mathbb{Q}_v$; we do not need this fact.

Definition 3.13. Let $\mu \in \mathbb{Q}_u$ and let $\mathbb{U} \subset \mathbb{Q}_u$. We say that μ *determines* $\mathbb{U}$ if $\mathbb{U} \cap \mathbb{B}_\mu$ is dense in $\mathbb{B}_\mu$.

We say that $\mu \in \mathbb{Q}_u$ *determines* a formula φ of the forcing language for $\mathbb{Q}_u$ if μ determines $\{\nu \in \mathbb{Q}_u : \nu \text{ decides } \varphi\}$. Of course, this depends on u, so if not clear from the context we will say "u-determines". Informally, μ determining φ means that φ is transformed to be a statement in the random forcing $\mathbb{B}_\mu$, which is a simple notion, compared to formulas of $\mathbb{Q}_u$. If μ determines pertinent facts about a $\mathbb{Q}_u$-name then that name essentially becomes a $\mathbb{B}_\mu$-name.

For a formula φ of the forcing language for $\mathbb{Q}_u$ and $\mu \in \mathbb{Q}_u$ we let

$$[\![\varphi]\!]^u_\mu = \sum\nolimits^{\mathbb{B}_\mu}\{b \in \mathbb{B}_\mu : \mu\|b \Vdash_{\mathbb{Q}_u} \varphi\}.$$

Then μ u-determines φ iff $[\![\varphi]\!]^u_\mu \vee [\![\neg\varphi]\!]^u_\mu = 1_{\mathbb{B}_\mu}$. Recall that if $\nu \leqslant \mu$ then $\nu \ll \mu$ so there is a natural map $i^\nu_\mu : \mathbb{B}_\mu \to \mathbb{B}_\nu$ (which is a measure-preserving embedding if ν is a pure extension of μ). For all $a \in \mathbb{B}_\mu$, if $i^\nu_\mu(a) \neq 0$ then $\nu\|i^\nu_\mu(a) \leqslant_{\mathbb{Q}} \mu\|a$, so for all φ, $[\![\varphi]\!]^u_\nu \geqslant_{\mathbb{B}_\nu} i^\nu_\mu([\![\varphi]\!]^u_\mu)$. Thus if μ determines φ then so does ν and in this case $[\![\varphi]\!]^u_\nu = i^\nu_\mu([\![\varphi]\!]^u_\mu)$. If also $\nu \leqslant_{\text{pur}} \mu$ then these Boolean values have the same measure: $\mu([\![\varphi]\!]^u_\mu) = \nu([\![\varphi]\!]^u_\nu)$.

We now prove that determining a formula is prevalent. Here and in the rest of the paper we often make use of sequences of pure extensions. This gives us some closedness that the forcing as a whole does not have; the situation is similar to that of Prikry forcing. We should think of pure extensions as mild ones.

A *pure sequence* is a sequence $\langle \mu_i \rangle_{i<\delta}$ such that for all $i < j < \delta$, $\mu_j \leqslant_{\text{pur}} \mu_i$. If δ is limit, then such a sequence has a natural limit (which by our notational conventions we usually denote by $\mu_{<\delta}$). For all $i < \delta$ we have $\mu_{<\delta} \leqslant_{\text{pur}} \mu_i$. However we note that it may be that $\mu_{<\delta}$ is not a condition in $\mathbb{Q}$ as its support may be too large. If $\delta = \gamma + 1$ then we let $\mu_{<\delta} = \mu_\gamma$.

Lemma 3.14. *Let $\mu_0 \in \mathbb{Q}_u$ and let $\mathbb{U} \subset \mathbb{Q}_u$ be dense and open. Then there is some $\mu \leqslant_{\text{pur}} \mu_0$ in $\mathbb{Q}_u$ which determines $\mathbb{U}$.*

Proof. We construct a pure sequence $\langle \mu_i \rangle$, starting with μ_0. If μ_j is defined then we also pick some $a_j \in \mathbb{B}_{\mu_j}$ such that $\mu_j \| a_j \in \mathbb{U}$ and for all $i < j$ we have $\mu_j(a_j \cap a_i) = 0$. (Note that for all $i < j$, $\mu_j(a_i) = \mu_i(a_i)$.)

We keep constructing until we get stuck: we get some δ such that $\mu_{<\delta}$ is defined but $\mu_{<\delta}$ does not have any pure extension ς such that there is some $a \in \mathbb{B}_\varsigma \cap \mathbb{U}$ which is ς-disjoint from all a_i for $i < \delta$.

We get stuck at a countable stage. For if not, $\langle a_i \rangle_{i<\omega_1}$ are pairwise $\mu_{<\omega_1}$-disjoint which is impossible. This shows that at limit stages i we indeed have $\mu_{<i} \in \mathbb{Q}_u$ so the construction can continue.

Suppose that we got stuck at stage δ; let $\mu = \mu_{<\delta} \in \mathbb{Q}_u$. We show that μ is as desired. $\{\mu \| a_i : i < \delta\} \subset \mathbb{U}$; we claim that this is a maximal antichain in $\mathbb{B}_\mu$. If not, find some $a \in \mathbb{B}_\mu$ which is μ-disjoint from all a_i. Now there is some extension of $\mu \| a$ in $\mathbb{U}$; it is of the form $\varsigma \| b$ where $\varsigma \leqslant_{\mathrm{pur}} \mu$ and $b \subset a$. But then we can pick ς for μ_δ and b for a_δ. □

Lemma 3.15. *Suppose that $\kappa < \delta$ are inaccessible. Let $\mu_0 \in \mathbb{Q}_{[\kappa,\delta)}$ and let $\mathbb{U} \subset \mathbb{Q}_\delta$ be dense and open. Then there is some $\mu \leqslant_{\mathrm{pur}} \mu_0$ in $\mathbb{Q}_{[\kappa,\delta)}$ such that*

$$\{\nu \in \mathbb{Q}_\kappa : \nu\mu \text{ determines } \mathbb{U}\}$$

is dense in $\mathbb{Q}_\kappa$.

Proof. We construct a pure sequence $\langle \mu_i \rangle$ of elements of $\mathbb{Q}_{[\kappa,\delta)}$ of length below κ^+, starting with μ_0. Together with this sequence we enumerate an antichain $\mathcal{A} \subset \mathbb{Q}_\kappa$. At stage i, we search for a pure extension ϱ of $\mu_{<i}$ in $\mathbb{Q}_\delta$ which determines $\mathbb{U}$ and is of the form $\varrho = \nu'\mu'$ where $\nu' \in \mathbb{Q}_\kappa$, $\mu' \in \mathbb{Q}_{[\kappa,\delta)}$ and ν' is incompatible with all elements enumerated so far into $\mathcal{A}$. If such exist, then we pick one, enumerate ν' into $\mathcal{A}$ and let $\mu_i = \mu'$. If none such exist then we stop the construction and let $\varsigma = \mu_{<i}$.

We must stop at some stage $i^* < \kappa^+$ because $|\mathbb{Q}_\kappa| = \kappa$.

Let $\nu \in \mathcal{A}$. If ν is enumerated into $\mathcal{A}$ at stage $i < i^*$ then $\nu\mu_i$ determines $\mathbb{U}$; as $\varsigma \leqslant_{\mathrm{pur}} \mu_i$ we have $\nu\varsigma \leqslant_{\mathrm{pur}} \nu\mu_i$ so $\nu\varsigma$ determines $\mathbb{U}$. It thus remains to show that $\mathcal{A}$ is a maximal antichain of $\mathbb{Q}_\kappa$. Suppose not; let $\upsilon \in \mathbb{Q}_\kappa$ be incompatible with all elements of $\mathcal{A}$. By Lemma 3.14, we can find some $\varrho \leqslant_{\mathrm{pur}} \upsilon\varsigma$ which determines $\mathbb{U}$. We can write ϱ as $\nu'\mu'$ where $\nu' \leqslant_{\mathrm{pur}} \upsilon$ is in $\mathbb{Q}_\kappa$ and $\mu' \leqslant_{\mathrm{pur}} \varsigma$ is in $\mathbb{Q}_{[\kappa,\delta)}$. But ν' is incompatible with all elements of $\mathcal{A}$ so we can pick $\mu_{i^*} = \mu'$, which we did not. □

Scenario 3.16. Suppose now that $\kappa < \delta$ are both inaccessible. Let $\bar{\mu} = \langle \mu_\alpha \rangle_{\alpha<\alpha^*}$ be a pure sequence of measures in $\mathbb{Q}_{[\kappa,\delta)}$.

Let $G \subset \mathbb{Q}_\kappa$ be generic over V. For all $\nu \in G$, by Lemma 3.11, $G_\nu = G \cap \mathbb{B}_\nu$ is generic for $\mathbb{B}_\nu$ over V. The system $\langle G_\nu \rangle_{\nu \in G}$ coheres: if $\nu \leqslant \varrho$ then $G_\varrho = (i^\nu_\varrho)^{-1} G_\nu$.

Let $\mathbb{D}_G = \{\nu\mu_\alpha : \nu \in G \ \&\ \alpha < \alpha^*\}$. This is a directed system (under $\geqslant_\mathbb{Q}$). Note that from $\varsigma \in \mathbb{D}_G$ we can recover ν and μ_α. We thus let, for $\varsigma = \nu\mu_\alpha \in \mathbb{D}_G$, $\tau_\varsigma = \sigma_{\nu,\mu_\alpha} \circ \pi_{\nu,\mu_\alpha} : \mathbb{B}_\varsigma \to \mathbb{B}_{\mu_\alpha^{V[G_\nu]}}$ be the quotient by G_ν (this of course depends on G_ν and not on ς alone, but we suppress its mention). Lemmas 1.9–1.11 and the discussion between them show that for any $\varsigma = \nu\mu_\alpha \geqslant \varsigma' = \nu'\mu_\beta$ in $\mathbb{D}_G$ and any $a \in \mathbb{B}_\varsigma$ we have $\mu_\alpha^{V[G_\nu]}(\tau_\varsigma(a)) = \mu_\beta^{V[G_{\nu'}]}(\tau_{\varsigma'}(i^{\varsigma'}_\varsigma(a)))$.

Let φ be a formula of the forcing language for $\mathbb{Q}_\delta$. For $\varsigma = \nu\mu_\alpha \in \mathbb{D}_G$ we let $\xi_\varsigma(\varphi) = \mu_\alpha^{V[G_\nu]}(\tau_\varsigma([\![\varphi]\!]^\delta_\varsigma))$. The analysis above shows that if $\varsigma \geqslant \varsigma'$ are in $\mathbb{D}_G$ then $\xi_\varsigma(\varphi) \leqslant \xi_{\varsigma'}(\varphi)$, and that if ς δ-determines φ then $\xi_\varsigma(\varphi) = \xi_{\varsigma'}(\varphi)$ for all $\varsigma' \leqslant \varsigma$. We therefore let $\xi_G(\varphi) = \sup_{\varsigma \in \mathbb{D}_G} \xi_\varsigma(\varphi)$. To calculate $\xi_G(\varphi)$ it is sufficient to take the supremum of $\xi_\varsigma(\varphi)$ over a final segment of $\varsigma \in \mathbb{D}_G$ (or in fact any cofinal subset of $\mathbb{D}_G$). If some $\varsigma \in \mathbb{D}_G$ determines φ then $\xi_\varsigma(\varphi)$ is eventually constant and we get $\xi_G(\varphi) = \max_{\varsigma \in \mathbb{D}_G} \xi_\varsigma(\varphi)$ which equals $\xi_\varsigma(\varphi)$ for any ς which determines φ.

Remark 3.17. This is important. Suppose that M is an inner model of V. Then we can work with this scenario "mostly in M": we will have all the ingredients in M (so κ, δ are inaccessible in M, and $\mathbb{Q}_\kappa, \mathbb{Q}_\delta$ are in the sense of M) but the sequence $\bar{\mu}$ will not be in M. Thus if $G \subset \mathbb{Q}_\kappa^M$ is generic *over V* then the entire system $(\mathbb{D}_G, \tau_\varsigma, \xi_\varsigma(\varphi), \ldots)$ will be in $V[G]$ but not in $M[G]$ (of course G is generic over M too). We can still make, in $V[G]$, the above calculations of $\xi_G(\varphi)$ for $\varphi \in M$ (although "determining" and the calculation of $[\![\varphi]\!]^\delta_\varsigma$ and $\xi_\varsigma(\varphi)$ for each particular ς will be done in M or $M[G]$).

3.3. Real-valued measurability

In this section we prove the following:

Theorem 3.18. *Suppose that there is an elementary $j : V \to M$ with critical point κ such that $M^{<2^\kappa} \subset M$ (for example, if κ is measurable and $2^\kappa = \kappa^+$). Then in $V^{\mathbb{Q}_\kappa}$, κ is real-valued measurable.*

Let j be as in the theorem.

Let $\mathbb{P} = \mathbb{Q}_\kappa$. Then $\mathbb{P} \in M$ and $\mathbb{P} = (\mathbb{Q}_\kappa)^M$; more importantly, $j(\mathbb{P}) = \left(\mathbb{Q}_{j(\kappa)}\right)^M$ (note that this is not absolute; we do not have $j(\mathbb{P}) = \mathbb{Q}_{j(\kappa)}$). Let $\mathbb{P}' = \left(\mathbb{Q}_{[\kappa, j(\kappa))}\right)^M$.

What we do now is construct a pure sequence $\bar{\mu} = \langle \mu_\alpha \rangle_{\alpha<2^\kappa}$ of elements of $\mathbb{P}'$. We start with a list $\langle \mathbb{U}_\alpha \rangle_{\alpha<2^\kappa}$ of dense subsets of $j(\mathbb{P})$ each of which is in M (note that this sequence is not in M). Rather than specify now which dense sets we put on this list, we will, during the verifications that κ is real-valued measurable in $V^{\mathbb{Q}_\kappa}$, list dense sets that are necessary for the proofs, making sure that we never put more than 2^κ sets on the list.

Given $\langle \mathbb{U}_\alpha \rangle$, we construct $\bar{\mu}$ as follows. For μ_0 we pick any element of $\mathbb{P}'$. At stage $\alpha < 2^\kappa$, we note that by the closure property of M, $\langle \mu_\beta \rangle_{\beta<\alpha} \in M$ and so $\mu_{<\alpha} \in M$. As $2^\kappa \leqslant (2^\kappa)^M$ is less than the least inaccessible beyond κ in M, $\mu_{<\alpha} \in \mathbb{P}'$. We now apply Lemma 3.15 in M, with κ standing for κ, $j(\kappa)$ standing for δ, $\mu_{<\alpha}$ for μ_0 and $\mathbb{U}_\alpha$ for $\mathbb{U}$. The resulting measure is μ_α.

If $G \subset \mathbb{P}$ is generic over V then we find ourselves in Scenario 3.16 (as modulated by Remark 3.17). For every $\mathbb{U}$ on our list, we know that some $\varsigma \in \mathbb{D}_G$ determines $\mathbb{U}$.

Let $\mathcal{N}$ be the set of all $\mathbb{P}$-names for subsets of κ (up to equivalence); note that because $|\mathbb{Q}_\kappa| = \kappa$, $|\mathcal{N}| = 2^\kappa$. If G is generic over V then for every $A \in \mathcal{N}$ we let $f_G(A) = \xi_G(\kappa \in j(A))$. For the rest of this section, let $\delta = j(\kappa)$.

Lemma 3.19. *Let $\nu \in \mathbb{P}$ and $A, B \in \mathcal{N}$. Suppose that $\nu \Vdash_{\mathbb{P}} A \subset B$. Then $\nu \Vdash_{\mathbb{P}} f_G(A) \leqslant f_G(B)$.*

Proof. As we had in our discussion of Solovay's construction, $j \restriction \mathbb{P}$ is the identity. So in M, $\nu \Vdash_{j(\mathbb{P})} j(A) \subset j(B)$. Let $G \subset \mathbb{P}$ be generic and suppose that $\nu \in G$. For any $\varsigma = \nu' \mu_\alpha \in \mathbb{D}_G$ such that $\nu' \leqslant \nu$ we have $\varsigma \leqslant \nu$ so in M, $\varsigma \Vdash_{j(\mathbb{P})} j(A) \subset j(B)$ so $[\![\kappa \in j(A)]\!]^\delta_\varsigma \leqslant_{\mathbb{B}_\varsigma} [\![\kappa \in j(B)]\!]^\delta_\varsigma$ so $\xi_\varsigma(\kappa \in j(A)) \leqslant \xi_\varsigma(\kappa \in j(B))$. As this is true for a final segment of $\varsigma \in \mathbb{D}_G$ we have $\xi_G(\kappa \in j(A)) \leqslant \xi_G(\kappa \in j(B))$. [Note that in this proof we did not need any particular $\mathbb{U}$.] $\square$

It follows that f_G induces a function on subsets of κ in $V[G]$ (rather than only on their names). We show this function is the desired measure on κ.

Lemma 3.20. *Let $A \subset \kappa$ be in V. If $\kappa \in j(A)$ then $\Vdash_{\mathbb{P}} f_G(A) = 1$ and if $\kappa \notin j(A)$ then $\Vdash_{\mathbb{P}} f_G(A) = 0$.*

Proof. Suppose that $\kappa \in j(A)$. Then in M, every condition in $j(\mathbb{P})$ forces this fact. Let $G \subset \mathbb{P}$ be generic. It follows that for all $\varsigma \in \mathbb{D}_G$, $[\![\kappa \in j(A)]\!]^\delta_\varsigma = 1_{\mathbb{B}_\varsigma}$ so $\xi_G(\kappa \in j(A)) = 1$.

We get a similar argument if $\kappa \notin j(A)$. $\square$

Lemma 3.21. *Let $\langle B_n \rangle_{n<\omega}$ be a sequence of names in $\mathcal{N}$. Suppose that $\nu \in \mathbb{P}$ forces that B_n are pairwise disjoint. Then $\nu \Vdash_{\mathbb{P}} f_G(\cup_n B_n) = \sum_{n<\omega} f_G(B_n)$.*

Proof. Let $B \in \mathcal{N}$ be such that $\nu \Vdash_{\mathbb{P}} B = \cup_n B_n$.

We have $j(\nu) = \nu$ and $j(\langle B_n \rangle_{n<\omega}) = \langle j(B_n) \rangle_{n<\omega}$; so in M, ν forces (in $j(\mathbb{P})$) that $j(B) = \cup_{n<\omega} j(B_n)$ is a disjoint union. Again let G be generic such that $\nu \in G$.

Let $\mathbb{U}$ be the collection of $\mu \in j(\mathbb{P})$ extending ν such that in M, if $\mu \Vdash_{j(\mathbb{P})} \kappa \in j(B)$ then for some $n < \omega$, $\mu \Vdash_{j(\mathbb{P})} \kappa \in j(B_n)$. $\mathbb{U} \in M$ and $\mathbb{U}$ is dense in $j(\mathbb{P})$. We assume that some $\varsigma \in \mathbb{D}_G$ (and so a final segment of $\varsigma \in \mathbb{D}_G$) determines $\mathbb{U}$. [Note that the number of such sequences $\langle B_n \rangle$ is $|\mathcal{N}|^{\aleph_0} = 2^\kappa$ so we may put all the associated $\mathbb{U}$'s on the list.]

If ς determines $\mathbb{U}$ then $[\![\kappa \in j(B)]\!]^\delta_\varsigma = \sum^{\mathbb{B}_\varsigma}_{n<\omega} [\![\kappa \in j(B_n)]\!]^\delta_\varsigma$. Also, if $\varsigma \in \mathbb{D}_G$ extends ν then for $n \neq m$ we have $[\![\kappa \in j(B_n)]\!]^\delta_\varsigma \wedge_{\mathbb{B}_\varsigma} [\![\kappa \in j(B_m)]\!]^\delta_\varsigma = 0$. It follows that in addition, if ς determines $\kappa \in j(B)$ then it determines $\kappa \in j(B_n)$ for every $n < \omega$ (again only 2^κ many $\mathbb{U}$'s to add).

Thus, for plenty $\varsigma \in \mathbb{D}_G$ we have $f_G(B) = \xi_\varsigma(\kappa \in j(B)) = \sum_{n<\omega} \xi_\varsigma(\kappa \in j(B_n)) = \sum_{n<\omega} f_G(B_n)$ as required. $\square$

Lemma 3.22. *Suppose that $\gamma < \kappa$ and that $\langle B_\alpha \rangle_{\alpha<\gamma}$ is a sequence of names in $\mathcal{N}$. Suppose that $\nu \in \mathbb{P}$ forces that for all $\alpha < \gamma$, $f_G(B_\alpha) = 0$. Then $\nu \Vdash_{\mathbb{P}} f_G(\cup_\alpha B_\alpha) = 0$.*

Proof. Let $B \in \mathcal{N}$ be such that $\nu \Vdash_{\mathbb{P}} B = \cup_{\alpha<\gamma} B_\alpha$. Then in M, $\nu \Vdash_{j(\mathbb{P})} j(B) = \cup_{\alpha<\gamma} j(B_\alpha)$. Let $G \subset \mathbb{P}$ be generic over V and suppose that $\nu \in G$. For all $\alpha < \gamma$, $f_G(B_\alpha) = 0$ so for all $\varsigma \in \mathbb{D}_G$, $[\![\kappa \in j(B_\alpha)]\!]^\delta_\varsigma = 0_{\mathbb{B}_\varsigma}$.

Let $\mathbb{U}$ be the collection of conditions $\mu \in j(\mathbb{P})$ extending ν such that in M, if $\mu \Vdash_{j(\mathbb{P})} \kappa \in j(B)$ then for some $\alpha < \gamma$, $\mu \Vdash_{j(\mathbb{P})} \kappa \in j(B_\alpha)$. Then $\mathbb{U} \in M$ and $\mathbb{U}$ is dense below ν in $j(\mathbb{P})$. We assume that some $\varsigma \in \mathbb{D}_G$ determines $\mathbb{U}$. If ς determines $\mathbb{U}$ then $[\![\kappa \in j(B)]\!]^\delta_\varsigma = \sum^{\mathbb{B}_\varsigma}_{\alpha<\gamma} [\![\kappa \in j(B_\alpha)]\!]^\delta_\varsigma = 0$. Thus on a final segment of $\varsigma \in \mathbb{D}_G$ we have $[\![\kappa \in j(B)]\!]^\delta_\varsigma = 0$ so $\xi_G(\kappa \in j(B)) = 0$.

Now there are $|\mathcal{N}|^{<\kappa} = 2^\kappa$ such sequences $\langle B_\alpha \rangle$ so we only need 2^κ many such $\mathbb{U}$ on our list of dense sets to determine. □

4. General sequences

To facilitate the definition, we introduce some notation. Suppose that $w \subset \mathrm{On}$ and that $\bar{x} = \langle x_\alpha \rangle_{\alpha\in w}$ is a sequence of reals. Suppose that $B \subset \mathbb{R}^{\mathrm{otp}\, w}$. Then we say that $\bar{x} \in B$ if $\langle x_{f(\xi)} \rangle \in B$, where $f : \mathrm{otp}\, w \to w$ is order-preserving. If $\bar{B} = \langle B_i \rangle_{i<\sigma}$ is a sequence of sets such that for all $i < \sigma$, $B_i \subset \mathbb{R}^i$, then we say that $\bar{x} \in \bar{B}$ if $\bar{x} \in B_{\mathrm{otp}\, w}$.

Let $\sigma \leqslant \kappa$ be regular, uncountable cardinals.

Definition 4.1. A *κ-null* set is a union of fewer than κ null sets.[1]

Definition 4.2. A *κ-null sequence* is a sequence $\bar{B} = \langle B_i \rangle_{i<\sigma}$ such that for each $i < \sigma$, B_i is a κ-null subset of $\mathbb{R}^i$.

Definition 4.3. Let A be any set. A *noncountable club* on $[A]^{<\sigma}$ is some $\mathcal{C} \subset [A]^{<\sigma}$ which is cofinal in $([A]^{<\sigma}, \subseteq)$ and is closed under taking unions of increasing chains of uncountable cofinality.

Definition 4.4. Let $\bar{B} = \langle B_i \rangle_{i<\sigma}$ be such that for all $i < \sigma$, $B_i \subset \mathbb{R}^i$. Let $\bar{x} = \langle x_\alpha \rangle_{\alpha\in U}$ be a sequence of reals (where $U \subset \mathrm{On}$). We say that $\bar{x}$ *escapes* $\bar{B}$ if there is some noncountable club $\mathcal{C}$ on $[U]^{<\sigma}$ such that for all $w \in \mathcal{C}$, $\bar{x} \restriction w \notin \bar{B}$.

Definition 4.5. A sequence $\bar{x} = \langle x_\alpha \rangle_{\alpha<\kappa}$ of reals is *σ-general* if for every κ-null sequence $\bar{B} = \langle B_i \rangle_{i<\sigma}$, there is some final segment W of κ such that $\bar{x} \restriction W$ escapes $\bar{B}$.

4.0.1. Justifying the definition

Naïve approaches might have liked to strengthen the above definition. However, it is fairly straightforward to see that expected modes of strengthening result in empty notions. For example, one would like to eliminate the restriction to a final segment of κ. But given a sequence $\bar{x} = \langle x_\alpha \rangle_{\alpha<\kappa}$, we can let, for $i < \sigma$,

$$B_i = \{x_0\} \times \mathbb{R}^{i-\{0\}} = \{\bar{y} \in \mathbb{R}^i \ : \ \bar{y}(0) = x_0\}.$$

Then whenever $w \subset \kappa$ such that $0 \in w$, $\bar{x} \restriction w \in B_{\mathrm{otp}\, w}$ (and every noncountable club on $[\kappa]^{<\sigma}$ contains such a w).

Accepting the restriction to a final segment, we may ask why we need to restrict to a club — why we cannot have $\bar{x} \restriction w \notin \bar{B}$ for all $w \in [W]^{<\sigma}$. But consider

$$B_\omega = \{\bar{y} \in \mathbb{R}^\omega \ : \ \forall n < \omega \ \ \bar{y}(2n)(0) = \bar{y}(2n+1)(0)\}.$$

Given a final segment W of κ, we can always choose some ω-sequence $w \subset W$ such that $\bar{x} \restriction w \in B_\omega$.

4.1. General sequences in Solovay's model

Theorem 4.6. *Let κ be inaccessible. Then in $V^{\mathbb{B}_{m_\kappa}}$, the random sequence is σ-general for all regular, uncountable $\sigma < \kappa$.*

This relies on the following well-known fact:

[1] Let $\mathcal{N}$ be the ideal of null sets. If κ is real-valued measurable, we have $non(\mathcal{N}) = \aleph_1$ and $cov(\mathcal{N}) \geqslant \kappa$ [4]. Hence, for a real-valued measurable κ, κ-null sets form a proper ideal extending $\mathcal{N}$ properly. By the inequality above we have $cov(\mathcal{N}) = \kappa$ in Solovay's model as well as in the new model. The existence of a σ-general sequence, which separates between the models, can be viewed as a strengthening of the equation $cov(\mathcal{N}) = \kappa$.

Fact 4.7. Let $\mathbb{P}$ be a notion of forcing which has the λ-Knaster condition for all regular uncountable $\lambda < \sigma$, and let $A \in V$. Then (in $V^{\mathbb{P}}$), $([A]^{<\sigma})^V$ is a noncountable club of $[A]^{<\sigma}$.

(Recall that $\mathbb{P}$ has the λ-Knaster condition if for all $A \subset \mathbb{P}$ of size λ, there is some $B \subset A$ of size λ such that all elements of B are pairwise compatible in $\mathbb{P}$.)

Proof. Let $\mathcal{A} = ([A]^{<\sigma})^V$. To see that $\mathcal{A}$ is cofinal in $[A]^{<\sigma}$, let u be a name for an element of $[A]^{<\sigma}$. Let $p \in \mathbb{P}$ force that $\{t_i : i < \lambda\}$ is an enumeration of u (for some $\lambda < \sigma$). For $i < \lambda$, let $P_i \subset \mathbb{P}(\leqslant p)$ be a maximal antichain of elements q which force that $t_i = a_{i,q}$ for some $a_{i,q} \in A$. Then p forces that $w = \{a_{i,q} : i < \lambda, q \in P_i\}$ (which is in $\mathcal{A}$) contains u.

Now suppose that $p \in \mathbb{P}$ forces that $\langle u_i \rangle_{i<\lambda}$ is an increasing sequence in $\mathcal{A}$, for some regular uncountable $\lambda < \sigma$. For every $i < \lambda$ pick some $p_i \leqslant p$ and some $w_i \in \mathcal{A}$ such that $p_i \Vdash u_i = w_i$. Note that if p_i and p_j are compatible and $i < j$ then $w_i \subset w_j$. Let $X \in [\lambda]^\lambda$ be such that for $i, j \in X$, p_i and p_j are compatible. Without loss of generality, assume that $\mathbb{P}$ is a complete Boolean algebra. For $i \in X$ let $q_i = \sum^{\mathbb{P}}_{j>i, j\in X} p_j$. Then $\langle q_i \rangle_{i\in X}$ is decreasing and so halts at some q_{i^*}. Then q_{i^*} forces that for unboundedly many $i \in X$, $p_i \in G$, and so that $\bigcup_{i<\lambda} u_i = \bigcup_{i<\lambda} w_i$ which is in $\mathcal{A}$. $\square$

Fact 4.8. A measure algebra $(\mathbb{B}, \mu)$ has the λ-Knaster condition for all regular uncountable λ.

Proof. This is well-known; see, for example, [1]. We give a proof for the sake of completeness.

Suppose that $\{b_i : i < \lambda\} \subset \mathbb{B}$. Let $X_0 \in [\lambda]^\lambda$ such that for all $i \in X_0$, $\mu(b_i) > 1/n$. Inductively define X_{m+1} from X_m: if there is some $i \in X_m$ such that for λ many $j \in X_m$, $b_i \cap b_j = 0$, then let i be minimal such and let $X_{m+1} = \{j \in X_m : j > i \; \& \; b_j \cap b_i = 0\}$. This process has to terminate with some X_{m^*} because $\sum_{i\in X_m} b_i - \sum_{i\in X_{m+1}} b_i$ has measure $>1/n$. We can now find $Y \in [X_{m^*}]^\lambda$ which indexes a set of pairwise compatible conditions by inductively winnowing all j such that b_j is disjoint from something we put into Y so far. $\square$

Proof of Theorem 4.6. Let $G \subset \mathbb{B}_{m_\kappa}$ be generic over V, and let $\bar{r} = \langle r_\alpha \rangle_{\alpha<\kappa}$ be the random sequence obtained from G. In $V[G]$, let $\bar{B} = \langle B_i \rangle_{i<\sigma}$ be a κ-null sequence of length σ. For $i < \sigma$ choose null sets $B^i_\alpha \subset \mathbb{R}^i$ for $\alpha < \alpha_i < \kappa$ such that $B_i = \bigcup_{\alpha<\alpha_i} B^i_\alpha$.

A code for each B^i_α is a real, together with some countable subset of i. It follows that there is some $\theta < \kappa$ such that each B^i_α is defined in $V' = V[G \cap \mathbb{B}_{m_\theta}]$. Let $W = [\theta, \kappa)$. Then $\bar{r} \restriction W$ is random (for $\mathbb{B}_{m_W}$) over V'. Let $w \in ([W]^{<\sigma})^{V'}$, and let $i = \operatorname{otp} w$. The collapse $h : w \to i$ induces a bijection $h : \mathbb{R}^w \to \mathbb{R}^i$. Let $\alpha < \alpha_i$ and consider $h^{-1} B^i_\alpha$; this is a null subset of $\mathbb{R}^w$ defined in V' and so $\bar{r} \restriction w$, being random over V', is not in $h^{-1} B^i_\alpha$. Which means, in our notation, that $\bar{r} \restriction w \notin B^i_\alpha$ and so $\bar{r} \restriction w \notin \bar{B}$. The noncountable club $\mathcal{C} = ([W]^{<\sigma})^{V'}$ thus witnesses that $\bar{r} \restriction W$ escapes $\bar{B}$.

4.2. *Some necessary facts about $\mathbb{Q}_\kappa$*

The following information will be useful in showing the lack of general sequences. From now, assume that $\aleph_2 \leqslant \sigma < \kappa$, both σ and κ are regular, and that σ is at most the least inaccessible; this is a convenience, since then $\mathbb{Q}_\kappa$ is purely σ-closed.

4.2.1. *Cardinal preservation*

Lemma 4.9. *All cardinals and cofinalities below the least inaccessible are preserved by $\mathbb{Q}_\kappa$.*

This is important; if σ is not regular in the extension then $[W]^{<\sigma}$ ceases to be interesting.

Proof. Let θ be a regular, uncountable cardinal below the least inaccessible cardinal. Let $\lambda < \theta$ and suppose that $\mu_0 \in \mathbb{Q}_\kappa$ forces that $f : \lambda \to \theta$ is a function. By Lemma 3.14 construct a pure sequence $\langle \mu_i \rangle_{i<\lambda}$ in $\mathbb{Q}_\kappa$ starting with μ_0 such that for each $i < \lambda$, μ_i determines the value of $f(i)$ (that is, the collection of $a \in \mathbb{B}_{\mu_i}$ such that for some $\gamma < \theta$, $\mu_i \| a \Vdash_{\mathbb{Q}_\kappa} f(i) = \gamma$ is dense in $\mathbb{B}_{\mu_i}$). For $i < \lambda$ let A_i be the (countable) set of such values γ. For every i, $\mathbb{B}_{\mu_i} \lessdot \mathbb{Q}_\kappa(\leqslant\mu_i)$ and so $\mu_{<\lambda}$ forces that the range of f is contained in $\bigcup_{i<\lambda} A_i$. $\square$

As for preservation of cardinals beyond the least inaccessible, we mention that the only cardinals that may be collapsed by $\mathbb{Q}_\kappa$ are those that lie between δ and 2^δ, where $\delta < \kappa$ is a singular limit of inaccessible cardinals. We omit the proof as it does not involve new techniques (and we do not use this fact). As to whether $\mathbb{Q}_\kappa$ does collapse any cardinals, it seems this may be independent. The general results of [3] may be relevant here.

4.2.2. Finding elements of clubs

Lemma 4.10. *Let $A \in V$. Suppose that $\mu \in \mathbb{Q}_\kappa$ forces that $\mathcal{C}$ is a noncountable club on $[A]^{<\sigma}$. Then there is some (pure) extension ν of μ and some $w \in [A]^{<\sigma}$ (in V) such that $\nu \Vdash w \in \mathcal{C}$.*

Proof. We show the following claim: given $\mu \in \mathbb{Q}_\kappa$ forcing that $\mathcal{C}$ is a noncountable club on $[A]^{<\sigma}$ and given some $w \in [A]^{<\sigma}$, there is some ν purely extending μ and some $w' \in [A]^{<\sigma}$ containing w such that ν forces that there is some $v \in \mathcal{C}$, $w \subset v \subset w'$.

This suffices: given μ as in the lemma, we construct a pure sequence $\langle \mu_\alpha \rangle_{\alpha<\omega_1}$ starting with μ and an increasing sequence of $w_\alpha \in [A]^{<\sigma}$ such that $w_0 = 0$, and μ_α forces that there is some $v_\alpha \in \mathcal{C}$, $w_{<\alpha} \subset v_\alpha \subset w_\alpha$. Then $\mu_{<\omega_1}$ forces that $\cup_{\alpha<\omega_1} w_\alpha = \cup_{\alpha<\omega_1} v_\alpha$ is in $\mathcal{C}$.

So let μ, w be as in the claim. Let w^* be a name such that $\mu \Vdash w^* \in \mathcal{C}$ and $w \subset w^*$ ($\mathcal{C}$ is cofinal). First, let μ' be a pure extension of μ such that there is an antichain $\langle a_n \rangle_{n<\omega}$ of $\mathbb{B}_{\mu'}$ and cardinals $\lambda_n < \sigma$ such that $\mu' \| a_n \Vdash |w^*| = \lambda_n$; for every n, let $\langle x_i^n \rangle_{i<\lambda_n}$ be a list of names such that $\mu' \| a_n \Vdash w^* = \{x_i^n : i < \lambda_n\}$. We now construct a pure sequence μ_i for $i < \lambda = \sup_n \lambda_n$; for each $i < \lambda$ and each n such that $i < \lambda_n$, the collection of $b \in \mathbb{B}_{\mu_i}$ such that $b \subset a_n$ and for some $a \in A$, $\mu_i \| b \Vdash a = x_i^n$ is dense below a_n; there are only countably many such a. Then $\mu_{<\lambda}$ forces that w^* is contained in w', the collection of all such a's which appeared in the construction (σ is regular, so $\lambda < \sigma$ and so w' has size $< \sigma$). □

In fact, for every $w \in [A]^{<\sigma}$ and such μ, there is a pure extension forcing that some w' containing w is in $\mathcal{C}$. This is immediate from the proof, or from the fact that $\{v \in \mathcal{C} : v \supset w\}$ is also a noncountable club of $[A]^{<\sigma}$ (in the extension).

4.2.3. Approximating measures by pure measures

The following is an easy fact which follows from regularity of our measures:

Lemma 4.11. *Let μ be a pure measure, and let $B \in \mathbb{B}_\mu$. Then for all $\epsilon < 1$ there is some* pure *measure ν which is a localization of μ such that $\nu(B) > \epsilon$.*

Proof. This follows from regularity of μ. There is some open set $U \supset B$ such that $\mu(U - B) < (1-\epsilon)\mu(B)/\epsilon$; so $\mu(B)/\mu(U) > \epsilon$. We can present U as a disjoint union of cylinders U_n; for some n we must have $\mu(U_n \cap B)/\mu(U_n) > \epsilon$. Then $\mu \| U_n$ is a pure measure and is as required. □

We need a certain degree of uniformity.

Lemma 4.12. *Let $B \subset \mathbb{R}^2$ be a positive Borel set. Then there is some positive $A \subset \mathbb{R}$ such that for all $\epsilon < 1$ there is some positive $C \subset \mathbb{R}$ such that $m_2 \| (A \times C)(B) > \epsilon$.*

Proof. Let $X \prec V$[2] be countable such that $B \in X$. Let C_0 be the measure-theoretic projection of B onto the y-axis (of course $C_0 \in X$). C_0 is positive, so we can pick some $r^* \in C_0$ which is random over X. Let $A = B^{r^*} = \{x \in \mathbb{R} : (x, r^*) \in B\}$ be the section defined by r^*; since $r^* \in C_0$, A is positive. Note that in X there is a name for B^{r^*}, where r^* is a name for the generic random real.

Let $\delta > 0$ be in X. By regularity of Lebesgue measure, there is some clopen set $U \subset \mathbb{R}$ such that $m(U \triangle A) < \delta$.[3] Of course $U \in X$. Then there is some positive $C \subset C_0$ in X such that $C \Vdash_{\mathbb{B}_m} m(U \triangle B^{r^*}) < \delta$.

[2] Yes, we mean $X \prec H(\chi)$. Complaints are to be lodged with set models of ZFC.

[3] Let $V \supset A$ be open such that $m(V - A) < \delta/2$; and recall that every open set is an increasing union of clopen sets.

For almost all $r \in C$ (those that are random over X), we have $m(U \triangle B^r) < \delta$. For such r, $m(A - B^r) \leqslant m(A - U) + m(U - B^r) \leqslant 2\delta$. So by Fubini's theorem, $m_2(A \times C - B) = \int_C m(A - B^r)\,\mathrm{d}r \leqslant 2\delta m(C)$; we get that $m_2 \| (A \times C)(\neg B) \leqslant 2\delta / m(A)$.[4] □

Corollary 4.13. *Let $\varrho, \mu \in \mathbb{Q}_\kappa$ and let μ be pure; assume $u^\varrho \cap u^\mu = 0$. Let $B \in \mathbb{B}_{\varrho\mu}$. Then there is a localization ϱ' of ϱ such that for all $\epsilon < 1$ there is some pure μ' which is a localization of μ and such that $\varrho'\mu'(B) > \epsilon$.*

Proof. What we need to note is that the proof of the previous lemma holds for $\varrho \times \mu$ (in place of $m \times m$) (we just use the relevant measure algebra); we get a set $A \in \mathbb{B}_\varrho$ such that for all $\delta > 0$ there is some $C \in \mathbb{B}_\mu$ such that $\varrho\mu \| (A \times C)(B) > 1 - \delta$.

Fix some $\delta > 0$. Get the appropriate C; we have

$$\frac{\varrho\mu(A \times C - B)}{\varrho\mu(A \times C)} < \delta.$$

By Lemma 4.11, we can find some cylinder $\tilde{C} \in \mathbb{B}_\mu$ sufficiently close to C so that both $\mu \| \tilde{C}(C) > 1 - \delta$ and $\mu(C)/\mu(\tilde{C}) < 1 + \delta$; from the first we get

$$\frac{\mu(\tilde{C} - C)}{\mu(\tilde{C})} < \delta.$$

Note that $A \times \tilde{C} - B \subset A \times (\tilde{C} - C) \cup (A \times C - B)$. Combining everything, we get

$$\begin{aligned}\frac{\varrho\mu(A \times \tilde{C} - B)}{\varrho\mu(A \times \tilde{C})} &\leqslant \frac{\varrho\mu(A \times (\tilde{C} - C)) + \varrho\mu(A \times C - B)}{\varrho\mu(A \times \tilde{C})} \\ &= \frac{\varrho(A)\mu(\tilde{C} - C)}{\varrho(A)\mu(\tilde{C})} + \frac{\varrho\mu(A \times C - B)}{\varrho\mu(A \times C)} \cdot \frac{\varrho(A)\mu(C)}{\varrho(A)\mu(\tilde{C})} \leqslant \delta + \delta(1 + \delta).\end{aligned}$$

We can thus let $\varrho' = \varrho \| A$ and $\mu' = \mu \| \tilde{C}$; the latter is pure because $\tilde{C}$ is a cylinder. We get $\varrho'\mu'(B) \geqslant 1 - 2\delta - \delta^2$ which we can make sufficiently close to 1. □

4.3. In the new model

Theorem 4.14. *Suppose that κ is Mahlo for inaccessible cardinals, and that $\sigma \geqslant \aleph_2$ is at most the least inaccessible (and is regular). Then in $V^{\mathbb{Q}_\kappa}$, there are no σ-general sequences.*

In fact, we prove something stronger:

Theorem 4.15. *Suppose that κ is Mahlo for inaccessible cardinals, and that $\sigma \geqslant \aleph_2$ is regular, and is at most the least inaccessible. Then there is a κ-null sequence $\bar{B}$ of length σ such that in $V^{\mathbb{Q}_\kappa}$, no κ-sequence of reals $\bar{r}$ escapes $\bar{B}$.*

(We have here identified $\bar{B}$ as it is interpreted in V and in $V^{\mathbb{Q}_\kappa}$. Of course, for every κ-null B, if $B = \cup_{i<i^*} B_i$ for some $i^* < \kappa$ then for any $W \supset V$ we let $B^W = \cup_{i<i^*} B_i^W$.)

Proof. Work in V. We define $\bar{B}$ as follows: for $i < \sigma$, an increasing ω-sequence $\bar{j} = \langle j_n \rangle_{n<\omega}$ from i, and $k < 2$, we let

$$B^i_{\bar{j},k} = \cap_{n<\omega} [\langle k \rangle]^{j_n} = \{\bar{x} \in \mathbb{R}^i : \forall n < \omega \ \ (\bar{x}(j_n)(0) = k)\};$$

and we let B_i be the union of the $B^i_{\bar{j},k}$ for all increasing $\bar{j}$ from i and $k < 2$. Each $B^i_{\bar{j},k}$ is null, and κ is inaccessible, so B_i is κ-null. As κ remains a cardinal in $V^{\mathbb{Q}_\kappa}$, $B_i^{V^{\mathbb{Q}_\kappa}}$ is also κ-null in $V^{\mathbb{Q}_\kappa}$.

[4] We glossed over uses of the forcing theorem over X, which is not transitive. We really work with X's collapse and use absoluteness. For example, we got $C \in X$ such that $C \Vdash_{\mathbb{B}_m} m(U \triangle B^{r^*}) < \delta$. Let $\pi : X \to M$ be X's transitive collapse. Then in M, $\pi(C)$ forces (in $\mathbb{B}_m^M$) that $m(\pi(U) \triangle \pi(B)^{r^*}) < \delta$. If $r \in C$ is random over M, then in $M[r]$, $m(\pi(U) \triangle \pi(B)^r) < \delta$. But $\pi(B)^{M[G]} = B \cap M[G]$ and similarly for U. Thus indeed $m(U \triangle B^r) < \delta$ as we claimed.

Let $\mu^* \in \mathbb{Q}_\kappa$ force that $\bar{r} = \langle r_\alpha \rangle_{\alpha<\kappa}$ is a sequence of reals and that $\mathcal{C}$ is a noncountable club on $[\kappa]^{<\sigma}$.

For every $\gamma < \kappa$, find some μ_γ extending μ^* and some $k(\gamma) \in 2$ such that $\mu_\gamma \Vdash r_\gamma(0) = k(\gamma)$.

Suppose that $\gamma > \sup u^{\mu^*}$. Then we can find $\varpi_\gamma \in \mathbb{Q}_\gamma$ which is an extension of μ^*, a pure measure $\nu_\gamma \in \mathbb{Q}_{[\gamma,\kappa)}$, and some Borel B_γ, such that $\mu_\gamma = (\varpi_\gamma \nu_\gamma) \| B_\gamma$. Let $u_\gamma \subset u^{\mu_\gamma}$ be a countable support for B_γ.

We now winnow the collection of μ_γ's. Let S_0 be the set of inaccessible cardinals below κ (but greater than $\sup u^{\mu^*}$); for $\gamma \in S_0$ we have $\mathbb{Q}_\gamma \subset V_\gamma$. Thus on some stationary $S_1 \subset S_0$, the function $\gamma \mapsto \varpi_\gamma$ is constant. Next, we find $S_2 \subset S_1$ such that on S_2:

- $u_\gamma \cap \gamma$ and $\operatorname{otp} u_\gamma$ are constant;
- under the identification of one $\mathbb{R}^{u_\gamma - \gamma}$ to the other by the order-preserving map, $\nu_\gamma \restriction (u_\gamma - \gamma)$ is constant;
- under the identification of one $\mathbb{R}^{u_\gamma}$ to the other by the order-preserving map, B_γ is constant;
- $k(\gamma)$ is a constant k^*.

By these constants, and using Corollary 4.13, we can find some μ^{**}, a localization of ϖ_γ for $\gamma \in S_2$, such that for all $\epsilon < 1$ and all $\gamma \in S_2$, there is some pure ς which is a localization of ν_γ, such that $(\mu^{**}\varsigma)(B_\gamma) > \epsilon$.

We now amalgamate countably many μ_γ's in the following way. Pick an increasing sequence $\langle \gamma_n \rangle_{n<\omega}$ from S_2. For each $n < \omega$, let ς_n be a pure measure, which is a localization of ν_{γ_n}, such that $\mu^{**}\varsigma_n(B_{\gamma_n}) > q_n$, where $\langle q_n \rangle$ is a sequence of rational numbers in (0, 1) chosen so that $\sum_{n<\omega}(1 - q_n) < 1$. We note that u^{ς_n} are pairwise disjoint, and so we can take their product $\varsigma^* = \mu^{**}\varsigma_0\varsigma_1 \ldots$. We now let $\varsigma^{**} = \varsigma^* \| \bigcap_{n<\omega} B_{\gamma_n}$; the ς_n were chosen so that this is indeed a measure.

The point is that for all n, $\varsigma^{**} \leqslant (\mu^{**}\varsigma_n)\|B_{\gamma_n} \leqslant (\mu^{**}\nu_{\gamma_n})\|B_{\gamma_n} \leqslant \mu_{\gamma_n}$. Thus for all n, $\varsigma^{**} \Vdash r_{\gamma_n}(0) = k^*$.

Finally, by Lemma 4.10, let ϱ be some extension of ς^{**} which forces that some $w \in \mathcal{C}$, where $w \in V$ and $w \supset \{\gamma_n : n < \omega\}$. Let $i = \operatorname{otp} w$ and let $h\colon w \to i$ be the collapse. Define $\bar{j}$ by letting $j_n = h(\gamma_n)$. Then ϱ forces that $\bar{r} \restriction w \in B^i_{\bar{j},k^*}$ so that $\bar{r} \restriction w \in \bar{B}$. Thus μ^* could not have forced that $\mathcal{C}$ witnesses that $\bar{r}$ escapes $\bar{B}$. □

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