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SINGULAR COHOMOLOGY IN L

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ABSTRACT

We prove, that in the world of constructible sets, there does not exist a space X with $H^n(X, \mathbf{Z})$ isomorphic to the rational numbers. The proof requires a result about the growth of $\mathrm{Ext}^1_{\mathbf{Z}}(-, \mathbf{Z})$ inside of Gödel's constructible universe L.

We present here a result concerning the growth of the functor $\mathrm{Ext}^1_{\mathbf{Z}}(-, \mathbf{Z})$ inside of Gödel's constructible universe L [4], and then discuss an interesting topological consequence. These ideas are, of course, directly inspired by Shelah's recent work on Whitehead's problem. Recall Shelah, [11], [12], (see also [2], [3], [10]), has proven:

THEOREM A. ($V = L$) *Every Whitehead group is free.*

(G is a Whitehead group if $\mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z}) = 0$, i.e. all short exact sequences $0 \to \mathbf{Z} \to E \to G \to 0$ split.)

To bridge the gap between regular and singular cardinals, he proves a general compactness result for singular cardinals.

THEOREM B. *If λ is a singular cardinal and G is λ-free then G is λ^+-free.*

(For any infinite cardinal κ, G is κ-free if all subgroups of cardinality less than κ are free; by convention, $\aleph_0$-free means torsion-free, i.e. all finitely-generated subgroups are free.) The theorem we refer to above is closely related to Shelah's Theorem A and has a curious consequence for homological algebra. We refer to [5] for basic facts about the Ext functors and other homological algebraic machinery. As usual, we denote by $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ the integers, the rational

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numbers, and the real numbers respectively. Also ($V = L$) denotes the axiom of set theory asserting all sets are constructible.

Kan and Whitehead in [8] mention that the status of the following "elementary" proposition in algebraic topology is unclear.

(1) *There does not exist a topological space X and integer $n \geqq 2$ such that $H^n(X, \mathbf{Z}) \cong Q$, the rational numbers. ($H^n(-\mathbf{Z})$ denotes ordinary singular cohomology with integral coefficients.)*

They prove though, using elementary homological algebra, that if $H^{n-1}(X, \mathbf{Z}) = 0$, then $H^n(X, \mathbf{Z}) \not\cong \mathbf{Q}$. Nunke and Rotman [9] improve this result slightly by weakening the hypothesis to $H^{n-1}(X, \mathbf{Z})$ is countable. Proposition (1) admits the following equivalent algebraic formulation:

(2) *There does not exist an abelian group G such that* $\mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z}) = \mathbf{Q}$.

The study of this proposition led to the following:

THEOREM 1. *Suppose ($V = L$). If κ is a regular cardinal and G is a κ-free group of cardinality κ, that is not free, then $\mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z})$ has cardinality κ^+. (Actually for the following we need only bound the above Ext group below by $\aleph_1$.)*

REMARK 1. The theorem for $\kappa = \aleph_0$ is a well-known result of homological algebra [5], [15]. Historically this was precisely the status of Whitehead's problem before Shelah's work; we think of this as circumstantial evidence.

REMARK 2. The assertion in Theorem 1 cannot be proved or disproved from the usual axioms of set theory. The latter assertion follows immediately from Theorem 1 and Gödel [4]. To see it cannot be proven, one exhibits, as in [11], a certain $\aleph_1$-free abelian group G of cardinality $\aleph_1$ which is not free. Under the assumption of Martin's Axiom and the negation of the continuum hypothesis, it is shown that $\mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z}) = 0$. The construction of the Solovay–Tennenbaum model [14] completes the argument.

It is also easy to see that κ^+ is, in general, a crude upper bound for the above Ext group, so the assertion of the theorem is that this upper bound is frequently realized. We prove firstly that (1) is equivalent to (2), then that Theorem 1 implies (2), and finally Theorem 1.

PROPOSITION 1. (1) *is equivalent to* (2).

PROOF. ($\Leftarrow$) Assume (2) and suppose X is such that $H^n(X, \mathbf{Z}) \cong \mathbf{Q}$. Let $G = H_{n-1}(X, \mathbf{Z})$ and observe by the universal coefficient formula:

$$\mathbf{Q} \cong H^n(X, \mathbf{Z}) \cong \mathrm{Hom}(H_n(X, \mathbf{Z}), \mathbf{Z}) \oplus \mathrm{Ext}^1_{\mathbf{Z}}(H_{n-1}(X, \mathbf{Z}), \mathbf{Z}).$$

Since $\mathbf{Q}$ is indecomposable and the Hom group could not be divisible: $\mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z}) = \mathbf{Q}$, contradicting (2).

($\Rightarrow$) Similarly, assume (1) and let G be such that $\mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z}) \cong \mathbf{Q}$. Let X be the Moore space $M(G, 2)$, i.e. a topological space whose homology satisfies:

$$\tilde{H}_*(M(G,2), \mathbf{Z}) = \begin{cases} G & \text{if } * = 2, \\ 0 & \text{otherwise}. \end{cases}$$

Such spaces are easily constructed, see [1, p. 31]. Then:

$$\begin{aligned} H^3(M(G,2), \mathbf{Z}) &= \mathrm{Hom}(H_3(M(G,2), \mathbf{Z}), \mathbf{Z}) \oplus \mathrm{Ext}^1_{\mathbf{Z}}(H_2(M(G,2), \mathbf{Z}), \mathbf{Z}) \\ &= \mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z}) = \mathbf{Q}, \end{aligned}$$

contradicting (1).

We now prove a stronger form of (2) for torsion-free abelian groups, assuming Theorem 1, of course.

Proposition 2. ($V = L$) *There does not exist a torsion-free abelian group G such that* $|\mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z})| = \aleph_0$.

Proof. Choose a group G contradicting the assertion and suppose the cardinality of G, say κ, is regular. We can certainly assume G is not free, since $\mathrm{Ext}^1_{\mathbf{Z}}(F, \mathbf{Z}) = 0$, for F a free abelian group. We also can assume $\kappa > \aleph_0$, for otherwise Remark 1 above allows us to use the torsion-free hypothesis to conclude $|\mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z})| = \aleph_1$. Similarly if G is κ-free, Theorem 1 gives us $|\mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z})| = \kappa^+ > \aleph_0$. So we can suppose G is not κ-free and then pick a non-free subgroup H of G of minimal cardinality. If β is the cardinality of H, then H is β-free, by our minimality assumption. We claim β is regular. Otherwise Shelah's Theorem B above would imply that H is β^+-free, hence free, contradicting the choice of H. If β is uncountable, Theorem 1 allows us to conclude $|\mathrm{Ext}^1_{\mathbf{Z}}(H, \mathbf{Z})| = \beta^+ > \aleph_0$. If β is $\aleph_0$, then the torsion-free hypothesis gives us this same strict lower bound. Now consider the short exact sequence:

$$0 \to H \to G \to G/H \to 0$$

which induces the long exact sequence:

$$\cdots \to \mathrm{Ext}^1_{\mathbf{Z}}(G/H, \mathbf{Z}) \to \mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z}) \to \mathrm{Ext}^1_{\mathbf{Z}}(H, \mathbf{Z}) \xrightarrow{\delta} \mathrm{Ext}^2_{\mathbf{Z}}(G/H, \mathbf{Z}) \to \cdots$$

Since $\mathrm{Ext}^2_{\mathbf{Z}} = 0$, $|\mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z})| \geqq |\mathrm{Ext}^1_{\mathbf{Z}}(H, \mathbf{Z})| > \aleph_0$, contradicting our original choice of G.

Now suppose the counterexample G has singular cardinality λ. Then let

$G' = G \oplus (\bigoplus_{\lambda^+} \mathbf{Z})$. Thus $\mathrm{Ext}^1_{\mathbf{Z}}(G', \mathbf{Z}) \cong \mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z})$ and G' has regular cardinality λ^+, contradicting the first part of the proof. The proof is now complete.

The case of torsion groups can be treated easily within ZFC:

LEMMA. *There is no torsion group T such that* $|\mathrm{Ext}^1_{\mathbf{Z}}(T, \mathbf{Z})| = \aleph_0$.

PROOF. Consider the short exact sequence:

$$0 \to \mathbf{Z} \to \mathbf{R} \to \mathbf{R}/\mathbf{Z} \to 0$$

which induces:

$$\cdots \to \mathrm{Hom}(T, \mathbf{R}) \to \mathrm{Hom}(T, \mathbf{R}/\mathbf{Z}) \to \mathrm{Ext}^1_{\mathbf{Z}}(T, \mathbf{Z}) \to \mathrm{Ext}^1_{\mathbf{Z}}(T, \mathbf{R}) \to \cdots$$

The extreme terms vanish since R is both torsion-free and divisible. Hence we obtain the well-known isomorphism:

$$\mathrm{Ext}^1_{\mathbf{Z}}(T, \mathbf{Z}) \cong \mathrm{Hom}(T, \mathbf{R}/\mathbf{Z}).$$

But the character group $\mathrm{Hom}(T, \mathbf{R}/\mathbf{Z})$ is a compact group in the topology of pointwise convergence, so is either finite or uncountable.

THEOREM 2. $(V = L)$ *There is no abelian group G such that* $|\mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z})| = \aleph_0$.

PROOF. Let G be a counterexample and consider the short exact sequence:

$$0 \to \mathrm{tor}(G) \to G \to G/\mathrm{tor}(G) \to 0$$

where $\mathrm{tor}(G)$ denotes the torsion subgroup of G. This induces:

$$\cdots \to \mathrm{Hom}(\mathrm{tor}(G), \mathbf{Z}) \to \mathrm{Ext}^1_{\mathbf{Z}}(G/\mathrm{tor}(G), \mathbf{Z}) \to \mathrm{Ext}^1_{\mathbf{Z}}(G, \mathbf{Z}) \to \mathrm{Ext}^1_{\mathbf{Z}}(\mathrm{tor}(G), \mathbf{Z}) \to 0.$$

(This short exact sequence even splits since $\mathrm{Ext}^1_{\mathbf{Z}}(G/\mathrm{tor}(G), \mathbf{Z})$ is divisible.) This now contradicts either Proposition 2 or the Lemma.

REMARK. Using a result of C. U. Jensen [6, p. 64] one can obtain a similar "non-realizability" result for the functor $\varprojlim^1$, the first right derived functor of inverse limit.

COROLLARY. $(V = L)$ *implies* (2).

PROOF. Immediate, since $\mathbf{Q}$ is countable.

Finally we come to the proof of Theorem 1. It is essentially a slight refinement of Shelah's original argument [11], using a splitting theorem due to R. Solovay [13]. Namely:

Solovay's Theorem. *If κ is a regular cardinal and Δ is a stationary subset of κ, then $\Delta = \bigcup_{\alpha<\kappa} \Delta_\alpha$ where each Δ_α is a stationary subset of κ, and the collection $\{\Delta_\alpha\}_{\alpha<\kappa}$ is pairwise disjoint.*

The proof below has been obtained independently by P. Eklof and A. Mekler.

Proof of Theorem 1. Suppose G is a counterexample and list representative short exact sequences for all the elements of $\mathrm{Ext}^1_{\mathbf{Z}}(G,\mathbf{Z})$ (maybe with repetition), say:

$$(\mathrm{S}_\gamma) \qquad 0 \to \mathbf{Z} \to E_\gamma \xrightarrow{\pi_\gamma} G \to 0 \qquad (\gamma < \kappa)$$

By the usual argument, see [10, p. 18], we can assume that each E_γ is $G \times \mathbf{Z}$ as a set and π_γ is projection on the first factor. Working in the framework of [2], we can express $G = \bigcup_{\alpha<\kappa} G_\alpha$ with each G_α free and of cardinality less than κ, and

$$\Delta = \{\alpha < \kappa : G_{\alpha+1}/G_\alpha \text{ is not free}\}$$

a stationary subset of κ [2, theor. 5.3]. Now Solovay's theorem provides us with a partition $\Delta = \bigcup_{\beta<\kappa} \Delta_\beta$, with each Δ_β stationary. Define $E_{\alpha,\gamma} = \pi_\gamma^{-1}(G_\alpha)$.

We construct inductively a short exact sequence:

$$(\mathrm{S}) \qquad 0 \longrightarrow \mathbf{Z} \longrightarrow \bigcup_{\alpha<\kappa} B_\alpha \xrightarrow{\pi} G \longrightarrow 0$$

distinct from all the (S_γ)'s as an element of the group of extensions. As sets each B_α will be $G_\alpha \times \mathbf{Z}$. For each $\beta < \kappa$, we have by $\diamondsuit_\kappa(\Delta_\beta)$ a collection of Jensen functions (see [2], [7], [10]):

$$\{f_{\alpha,\beta} : E_{\alpha,\beta} \longrightarrow B_\alpha : \alpha \in \Delta_\beta\}.$$

The critical case in the inductive construction is when $\alpha \in \Delta$, say $\alpha \in \Delta_\beta$, for some unique $\beta < \kappa$, and $f_{\alpha,\beta}$ produces an isomorphism of the extensions:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Z} & \longrightarrow & E_{\alpha,\beta} & \xrightarrow{(\pi_\beta \mid E_{\alpha,\beta})} & G_\alpha & \longrightarrow & 0 \\
 & & \| & & \wr\downarrow f_{\alpha,\beta} & & \| & & \\
0 & \longrightarrow & \mathbf{Z} & \longrightarrow & B_\alpha & \xrightarrow[(\pi \mid B_\alpha)]{} & G_\alpha & \longrightarrow & 0
\end{array}$$

Since $G_{\alpha+1}$ is free we can find a splitting $\rho : G_{\alpha+1} \to E_{\alpha+1,\beta}$ for $(\pi_\beta \mid E_{\alpha+1,\beta})$. Then $(\rho \mid G_\alpha)$ is a splitting for $(\pi_\beta \mid E_{\alpha,\beta})$, and also $f_{\alpha,\beta} \circ (\rho \mid G_\alpha)$ splits $(\pi \mid B_\alpha) : B_\alpha \to G_\alpha$. By assumption $G_{\alpha+1}/G_\alpha$ is not free, hence not a Whitehead group by Shelah's

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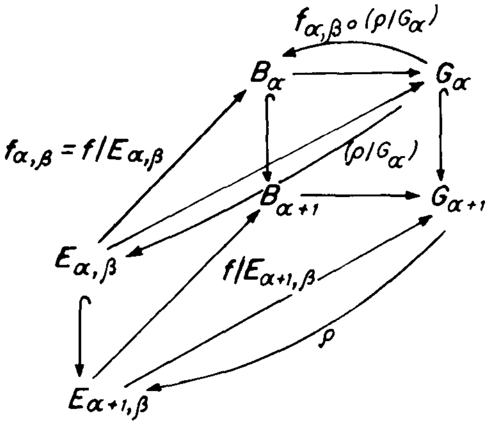
Theorem A, so we can find a group structure on $B_{\alpha+1}$ such that $f_{\alpha,\beta}\circ(\rho \mid G_\alpha)$ does not extend to a splitting of $(\pi \mid B_{\alpha+1}): B_{\alpha+1}\to G_{\alpha+1}$, [2, lemma 4.3]. The other cases in the construction are handled in the routine fashion [2, p. 28]. It is now the usual diamond-argument [2, proof of 6.3] to see that our new exact sequence (S) is "diagonalized" out of the original listing $(S_\gamma)_{\gamma<\kappa}$. Indeed suppose $f: E_\beta \to B$ is an isomorphism commuting with the respective projections to the first factor. By $\diamondsuit_\kappa(\Delta_\beta)$, there exists an $\alpha<\kappa$ such that $f_{\alpha,\beta}=f \mid E_{\alpha,\beta}$. Then we have a commutative diagram:



But then $(f \mid E_{\alpha+1,\beta})\circ\rho$ provides a splitting extending $f_{\alpha,\beta}\circ(\rho \mid G_\alpha)$ contradicting our construction. This contradiction completes the proof.

Conclusion. Of course, it is our eventual hope that an independence proof will be found of (2). The results above seem at least to be an indication of the possible variety of independence problems in the realm of algebraic topology. In any case we hope that this paper will further stimulate interest in the beautiful interplay between homological algebra and set theory.

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