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# MORE ON POWERS OF SINGULAR CARDINALS

BY

SAHARON SHELAH†

*The Hebrew University of Jerusalem, Jerusalem, Israel;*
*Simon Fraser University, Burnaby, British Columbia, Canada;*
*and Rutgers University, New Brunswick, New Jersey, USA*

ABSTRACT

We give bounds for $\aleph_\delta^{\aleph_1}$ where $\mathrm{cf}\,\delta = \aleph_1$, $(\forall \alpha < \delta)\aleph_\alpha^{\aleph_0} < \aleph_\delta$, in cases which previously remained opened, including the first such cardinal: the $\omega_1$-th cardinal in $C_\omega = \bigcap_{n<\omega} C_n$ where $C_0$ is the cardinal and $C_{n+1}$ the set of fixed points of $C_n$. No knowledge of earlier results is required. A subsequent work generalizing this was applied to many more cardinals ([Sh 7]).

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## 0. Introduction

The problem of what $2^{\aleph_0}$ can be has been considered central in set theory for a long time. Scott [Sc] had proved that, e.g., $2^\kappa = \kappa^+$ if $\kappa$ is measurable and $(\forall \mu < \kappa)\, 2^\mu = \mu^+$. Solovay [So] proved that if $\kappa$ is strong limit singular larger than a supercompact (or even a compact) cardinal, then $2^\kappa = \kappa^+$. Magidor [Mg 1], confirming the general expectation, proved the consistency of "$\aleph_\delta$ strong limit, $2^{\aleph_\delta} \geqq \aleph_{\delta+\alpha+1}$" ($\alpha < \delta$ and even $\alpha = \delta$) for, e.g., $\delta = \omega, \omega_1$, using supercompact cardinals. Magidor then proved that if a certain filter exists on small cardinals then $2^{\aleph_{\omega_1}}$ is small (see [S]). Subsequently Silver [S] proved, contradicting the general expectation, that, e.g., if $\aleph_{\omega_1}$ is strong limit, $\{\delta < \omega_1 : 2^{\aleph_\delta} = \aleph_{\delta+1}\}$ is stationary, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

Immediately much activity follows (see on the history, e.g. [Sh 5], [Sh 6, Ch. XIII, §0]). We continue the chain: Galvin and Hajnal [GH], Shelah [Sh 2], [Sh 5]. Galvin and Hajnal proved, e.g., $2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$ when $\aleph_{\omega_1}$ is strong limit, and more generally $T_{D_{\omega_1}}(f) \leqq \aleph_{\|f\|_{D_{\omega_1}}}$ where: $D_{\omega_1}$ is the filter of closed unbounded subsets on $\omega_1$, $\|f\|_{D_{\omega_1}}$ is the reasonable rank function for $f \in {}^{\aleph_1}\mathrm{Ord}$, i.e., $f$ is a function from $\omega_1$ to ordinals, $\|f\|_{D_{\omega_1}} = \sup\{\|g\|_{D_{\omega_1}} : g <_{D_{\omega_1}} f\}$, and

$$T_{D_{\omega_1}}(f) = \sup\{|G| : G \subseteq {}^{\omega_1}\mathrm{Ord}, (\forall g \in G) g <_{D_{\omega_1}} f, (\forall g_1 \neq g_2 \in G) g_1 \neq_{D_{\omega_1}} g_2\}.$$

And when $\delta = \bigcup_{i<\omega_1} \alpha_i$, $\alpha_i$ increasing, $\aleph_\delta^{\aleph_1} = T_D(f)$ where $f(i) = \prod_{j<i} \aleph_{\alpha_j}$. Remember that when $\aleph_\delta$ is strong limit, $\aleph_\delta^{\mathrm{cf}\,\delta} = 2^{\aleph_\delta}$; so they get a bound to $2^{\aleph_\delta}$ for such $\aleph_\delta$ when $\delta < \aleph_\delta$. They bound $\|f\|_{D_{\omega_1}}$ by $(\prod_{i<\omega_1} f(i))^+$. The first cardinal $\lambda$, $\mathrm{cf}\,\lambda = \aleph_1 \wedge (\forall \mu < \lambda) \mu^{\aleph_1} < \lambda$, on which they do not get information was the $\omega_1$-th fixed point where $\lambda$ is a fixed point iff $\lambda = \aleph_\lambda$.

In [Sh 2] we consider $\|f\|_D$ for all normal $D$ getting better bounds for $\|f\|_D$ (hence $\aleph_\delta^{\aleph_1}$) when, e.g., $\beth_\omega \leqq f(i) < \beth_\omega^+$ (i.e. $\beth_\omega^+$ rather than $((\beth_\omega)^{\aleph_1})^+ = \beth_{\omega+1}^+$). This is represented in [EHMR]. We get also a bound for $\aleph_\lambda^{\aleph_1}$ for $\lambda$ the $\omega_1$-th fixed point (and $\aleph_\alpha^{\aleph_0} < \aleph_\lambda$ for $\alpha < \lambda$): the $\omega_2$-th fixed point *but* only provided that Chang's conjecture holds.

Finishing to prepare the final version of [Sh 2], we succeeded in eliminating Chang's conjecture (at the expense of using the $\beth_2(\aleph_1)^+$-th fixed point). We use a different rank (alternatively, games) $\mathrm{rk}_D(f)$, $\mathrm{rk}'_D(f)$ ($D$ a filter on $\omega_1$) which are $< \infty$, if the covering lemma for $K[A]$ ($A \subseteq \beth_2(\aleph_1)^+$, $K$ standing for the core model of Dodd and Jensen) fails. By this we prove the existence of (normal) filters $D$ (on $\omega_1$) such that

(a) $\mathrm{Ord}^{\omega_1}/D$ has $\lambda$-like initial segment (for each regular $\lambda > \beth_2(\aleph_1)$ there is such $D$);

(b) $D$ is nice: in the following game Player II has a winning strategy: in the $n$-th move Player I chooses $A_n \subseteq \omega_1$ and $f_n \in {}^{\omega_1}\mathrm{Ord}$ such that $\bigwedge_{m<n} f_m <_{D_n + A_n} f_n$ ($D_0 = D$) and $A_n \neq \emptyset \bmod D_n$ and Player II chooses $D_{n+1}$, $D_n \cup \{A\} \subseteq D_{n+1}$, $D_{n+1}$, a normal filter on $\omega_1$ and ordinal $\alpha_n$, $\bigwedge_{m<n} \alpha_n < \alpha_m$. Player II loses if he has no legal move and wins otherwise.

This was used to prove, e.g., for appropriate $\aleph_\delta$, if there is no weakly inaccessible $\lambda < \aleph_\delta$ then there is no weakly inaccessible $\lambda < \aleph_\delta^{\aleph_1}$. See [Sh 5] for the details.

We then even claim ([Sh 3]) that the method gives:

Smallness Thesis. If $\delta$ is "small", $\mathrm{cf}\,\delta = \aleph_1$, $(\forall \alpha < \delta)\ \aleph_\alpha^{\aleph_0} < \aleph_\delta$, *then* $\aleph_\delta^{\aleph_1}$ is "small" (see more in [Sh 5]).

Hajnal pointed out that the proof does not work for the $\omega_1$-th member of $C_\omega$ where

$$C_0 = \{\aleph_\alpha : \alpha < \infty\},$$

$$C_{n+1} = \{\aleph_\alpha : |\aleph_\alpha \cap C_n| = \aleph_\alpha\},$$

$$C_\omega = \bigcap_{n<\omega} C_n.$$

Now, finishing to prepare the final version of [Sh 5] we have proved the smallness thesis in this case.

Making the cofinality $\aleph_1$ (and the filters on $\omega_1$) is just to save a parameter, any uncountable regular cardinal $\kappa$ will do, we can use fine (normal) filters on $\mathcal{P}_{<\kappa}(\lambda)$, and in the definition of nice filters we can use many functions.

0.1. Problem. Is the role of $\beth_2(\aleph_1)$ in [Sh 5] and $\beth_3(\aleph_1)$ here really necessary?

0.2. PROBLEM. Is there a bound for $\aleph_\delta^{\aleph_0}$ when, e.g., $\aleph_\delta$ is minimal such that $\aleph_\delta = \delta$, $\mathrm{cf}\,\delta = \aleph_0$? Even being smaller than the first (weakly) inaccessible.

The work was announced in [Sh 5].

However in the summer of '86 we strengthened it considerably. After some considerations we revised it by adding the parameter $\sigma$, originally it was $\sigma = 1$, and the reader may want to read it that way. In particular, in our conclusion $\beth_3(\aleph_1)$ was replaced by $\beth_2(\aleph_1)$ thus partially solving 0.1. On the new results see [Sh 7].

NOTATION. We do not always distinguish strictly between a filter $D$ on $I$ and $\{x \subseteq A : x \cup (I - A) \in D\}$ where $A \in D$.

$m, n, l, k$ are natural numbers;

$\alpha, \beta, \gamma, \delta, \xi, \zeta$ are ordinals ($\delta$ a limit ordinal);

$\lambda, \mu, \kappa, \chi$ are cardinals (usually infinite);

${}^B A$ denotes the family of functions from $B$ to $A$;

Ord is the class of ordinals.

So ${}^{\aleph_1}\mathrm{Ord}$ is the class of functions from $\aleph_1$ ( = set of countable ordinals) to ordinals;

$f, g, h$ denote functions from $\aleph_1$ to ordinals;

$f \leqq_D g$ means $\{i < \aleph_1 : f(i) \leqq g(i)\} \in D$ (similarly for $<_D$, $=_D$, $\neq_D$) so $f \neq_D g$, $f <_D g$ are *not* the negations of $f =_D g$, $g \leqq_D f$, respectively, as $D$ is not an ultrafilter (but see 0.B);

$f \leqq g$ means $(\forall i < \omega_1) f(i) \leqq g(i)$;

$P$ denotes a forcing notion, and we assume it has a minimal element which we denote by $\varnothing_P$, and sometimes $\varnothing$;

$\underset{\sim}{G}_P$ denotes the $P$-name of the generic subset of $P$;

$\underset{\sim}{x}[G]$ denotes the interpretation of the $P$-name $\underset{\sim}{x}$ when $G$ is a subset of $P$ generic over $V$;

$\mathscr{P}(A) = \{B : B \subseteq A\}$ is the power set of $A$.

* * *

If the reader is not happy with the definitions below, for the sake of this paper alone, he can think systematically as follows: Let $D$ be a normal filter on $\omega_1$; we identify it with $(D \restriction A)^+$ for any $A \in D$ where $D \restriction A = \{X \cap A : X \in D\}$,

$$D^+ = \{X : X \subseteq \bigcup\{A : A \in D\}, \text{ and } \bigcup\{A : A \in D\} - X \notin D\}.$$

We let $E$ denote a set of normal filters on $\omega_1$, with a minimal one $\mathrm{Min}\, E$. We let $\mathbb{E}$ be a set of $E$'s.

Let $D+A \overset{\text{def}}{=} \{X : X \subseteq \cup \{B : B \in D\}, A - X \notin D^+\}$,

$$(D \restriction B)^+ + A = ((D+A) \restriction B)^+.$$

0.A. DEFINITION. We define by induction on $\sigma$ (an ordinal) a set $OB_\sigma$, and for $X \in D \in OB_\sigma$ a set $D_{[X]} \in OB_\sigma$ and Min $D$ for $D \in OB_\sigma$ such that $OB_\sigma \cap OB_\theta = \emptyset$ for $\theta < \sigma$, and we let lev$(E)$ be the unique $\sigma$ such that $E \in OB_{\text{lev}(E)}$.

*Case* 1. $\sigma = 0$: we let $OB_\sigma = \{A : A \subseteq \omega_1\}$.

*Case* 2. $\sigma = 1$: we let

$$OB_\sigma = \{D\text{: for some } A \in D, D \subseteq \mathscr{P}(A) \text{ and}$$
$$\{x \subseteq \omega_1 : x \cap A \notin D\} \text{ is a normal ideal on } \omega_1\}$$

for $D \in OB_1$, Min $D$ is the $A$ mentioned above, which is $\bigcup_{x \in D} X$ and for $y \in D$, $D_{[y]} \overset{\text{def}}{=} \{x \subseteq y : x \in D\}$.

*Case* 3. $\sigma = \theta + 1$, $\theta > 0$,

$$\begin{aligned} OB_\sigma = \{E : {} & E \text{ is a subset of } \textstyle\bigcup_{i<\sigma} OB_i, \text{ such that: } E \cap OB_\theta \text{ has a minimal} \\ & \text{element under inclusion, Min } E, \\ & (\forall D \in E)(\forall y \in D)[D_{[y]} \in E] \text{ and} \\ & E \cap OB_{<\theta} = \textstyle\bigcup\{A : (\exists D \in OB_\theta)(A \in D \in E)\}\} \end{aligned}$$

for $E \in OB_\sigma$, $x \in E$,

$$E^0_{[x]} \overset{\text{def}}{=} \{D : D \in E \cap OB_\theta, [\text{lev}(x) < \theta \rightarrow x \in D], [\text{lev}(x) = \theta \rightarrow x \subseteq D]\},$$

$$E[x] \overset{\text{def}}{=} E_{[x]} \overset{\text{def}}{=} E^0_{[x]} \cup \bigcup\{D : D \in E^0_{[x]}\}.$$

*Case* 4. $\sigma$ limit,

$$OB_\sigma = \{E : E \subseteq OB_{<\sigma}, \text{ and } E \cap OB_{\leqq \theta} \in OB_{\theta+1} \text{ for } \theta < \sigma\}$$

if $E \in OB_\sigma$, $x \in E$,

$$E_{[x]} = \{D\text{: for some } \theta, \text{lev}(x) < \theta, \text{lev}(D) < \theta \text{ and } D \in (E \cap OB_{\leqq\theta})_{[x]}\}.$$

0.B. DEFINITION.

(1) For $f, g \in {}^{\aleph_1}\text{Ord}$, $D \in OB_1$, $f \leqq_D g$ iff Min $D - \{i : f(i) \leqq g(i)\} \notin D$.

(2) For $E \in OB_\sigma$, $\sigma > 1$, $f \leqq_E g$ means that for every $D \in E \cap OB_1$, $f \leqq_D g$.

(3) $E_1 \leqq E_2$ if lev$(E_1) = $ lev$(E_2)$ and $E_2 \subseteq E_1$.

(4) For $E \in OB_\sigma$, let fil$(E) = \{A \subseteq \omega_1 : O_{\omega_1} <_E O_{\omega_1 - A} \cup 1_A\}$ where $i_A$ is a function with domain $A$ and constant value $i$.

(5) $\mathrm{Fil}(E) = \{\mathrm{fil}(E_{[D]}) : D \in E\}$.

(6) $f <_D g$ for $f, g \in {}^{\aleph_1}\mathrm{Ord}$, $D \in OB_1$ means: $\mathrm{Min}\, D - \{i : f(i) < g(i)\}$; for $f, g \in {}^{\aleph_1}\mathrm{Ord}$, $E \in OB_\sigma$, $\sigma > 1$ let $f <_E g$ mean: $f <_D g$ for every $D \in E \cap OB_1$.

0.C. Fact.

(1) $OB_\sigma$ are really pairwise disjoint and $[E_1 \in OB_{\sigma_1}, E_2 \in OB_{\sigma_2}, E_1 \subseteq E_2 \Rightarrow \sigma_1 < \sigma_2]$.

(2) If $X \in E \in OB_\sigma$ then $E_{[X]} \in OB_\sigma$, $E_{[X]} \subseteq E$.

(3) $\leqq_E$ is transitive.

(4) If $f \leqq_E g$, $D \in E$ or $D \subseteq E$ (and $D, E \in \bigcup_\sigma OB_\sigma$), then $f \leqq_D g$.

(5) Every $E \in OB_\sigma$ has cardinality $\leqq \beth_\sigma(\aleph_1)$ so $|OB_\sigma| \leqq \beth_{\sigma+1}(\aleph_1)$.

(6) For $E \in OB_\sigma$, $\sigma > 0$, $\mathrm{fil}(E)$ is a normal filter on $\omega_1$.

0.D. Lemma.

(1) *If* $f_\alpha \in {}^{\aleph_1}\mathrm{Ord}$ *for* $\alpha < \lambda$, $\lambda > 2^{\aleph_1}$ then *for some* $\alpha < \beta$, $f_\alpha \leqq f_\beta$, *i.e.* $(\forall i < \omega_1)[f_\alpha(i) \leqq f_\beta(j)]$ (*really if* $\lambda = \mathrm{cf}\,\lambda \wedge (\forall \mu < \lambda)\mu^{\aleph_1} < \lambda$ *there is* $A \subseteq \lambda$, $|A| = \lambda$ *such that for* $\alpha < \beta$ *from* $A$, $f_\alpha \leqq f_\beta$, $\{i : f_\alpha(i) < f_\beta(i)\}$ *constant*).

(2) *If* $D$ *is a filter on* $\omega_1$, $f_\alpha \in {}^{\aleph_1}\mathrm{Ord}$ *for* $\alpha < \delta$, $[\alpha < \beta < \delta \Rightarrow f_\alpha \leqq_D f_\beta]$ *and* $\mathrm{cf}\,\delta > 2^{\aleph_1}$ then $\{f_\alpha/D : \alpha < \delta\}$ *has a least upper bound* $f/D$, *i.e.* $(\forall \alpha < \delta) f_\alpha \leqq_D g$ *and if* $(\forall \alpha < \delta) f_\alpha \leqq_D g' \Rightarrow g \leqq_D g'$ (see [Sh 2] or [Sh 5]).

## §1. Existence of nice $t$'s

Here we repeat some material from [Sh 5]:

1.1. Definition. We say $t = (P, \underset{\sim}{D})$ is pre-nice if:

(a) $P$ is a forcing notion (i.e., a partially ordered set).

(b) $\underset{\sim}{D}$ is a $P$-name of an ultrafilter on the Boolean algebra

$$\mathscr{P}(\omega_1)^V \stackrel{\mathrm{def}}{=} \{A : A \subseteq \omega_1^V, A \in V\}.$$

(c) For each $p \in P$, $D_p^t \stackrel{\mathrm{def}}{=} \{A : A \subseteq \omega_1, A \in V, p \Vdash_P \text{“} A \in \underset{\sim}{D} \text{”}\}$ is a normal filter on $\omega_1$.

1.1A. Remark. (1) Condition (c) does not seem essential.

(2) Note that $A \neq \emptyset \mod D_p^t$, $A \subseteq \omega_1$, $p \in P$ implies that for some $q$, $p \leqq q \in P$, $D_p^t + A \subseteq D_q^t$.

(3) Note that for $p \leqq q$ in $P$, $D_p^t \subseteq D_q^t$.

1.2. Definition. We say $t = (P, \underset{\sim}{D})$ is nice to $g \in {}^{\aleph_1}\mathrm{Ord}$ if $t$ is pre-nice and

(d) $\Vdash_P$ "$\{f: f\in V, f\leqq g\}$ is well ordered by $\leqq_D$" (so for $G\subseteq P$ generic over $V$, $(\{f/\underset{\sim}{D}[G]: f\in V, f\leqq g\}, \leqq_{\underset{\sim}{D}[G]})$ is isomorphic to an ordinal).

1.3. FACT. If $t$ is nice to $f$, $g\leqq f$ (or even $g\leqq_{D'_\varnothing} f$) *then* $t$ is nice to $g$.

1.4. DEFINITION. We say that $t=(P,\underset{\sim}{D})$ is nice if it is nice to $g$ for every $g\in {}^{\aleph_1}\mathrm{Ord}$.

The following is a consequence of a theorem of Dodd and Jensen [Do J]:

1.5. THEOREM. *If $\lambda$ is a cardinal, $S\subseteq\lambda$* then:

(1) $K[S]$, *the core model, is a model of* ZFC $+(\forall\mu\geqq\lambda)2^\mu=\mu^+$.

(2) *If in $K[S]$ there is no Ramsey cardinal $\mu>\lambda$ (or much less) then $(K[S],V)$ satisfies the $\mu$-covering lemma for $\mu\geqq\lambda+\aleph_1$, i.e., if $B\in V$ is a set of ordinals of power* $\leqq\mu$ then *there is $B'\in K[S]$, $B\subseteq B'$, $V\vDash|B'|\leqq\mu$.*

(3) *If* $V\vDash(\exists\mu\geqq\lambda)(\exists\kappa)\ \mu^\kappa>\mu^+>2^\kappa$ then *in $K[S]$ there is a Ramsey cardinal $\mu>\lambda$.*

1.6. LEMMA. *Suppose* $f\in{}^{\aleph_1}\mathrm{Ord}$, $\lambda>\Pi_{i<\omega_1}|f(i)+1|$, $\lambda^{\aleph_1}>\lambda^+$ (*so* $\lambda\geqq 2^{\aleph_1}$), then *some $t$ is nice to $f$.*

PROOF. Without loss of generality $(\forall i)f(i)\geqq 2$.

Let $S\subseteq\lambda$ be such that if $g\in{}^{\aleph_1}\mathrm{Ord}$, $(\forall i<\omega_1)\ g(i)\leqq f(i)$ then $g\in L[S]$. In $K[S]$ there is a Ramsey cardinal $\mu>\lambda$ (see 1.5(3)). Let $I=\{X: X\subseteq\mu, X\cap\omega_1$ an ordinal $>0\}$. Let, for $i<\omega_1$,

$$J_i=\{X\in I: X \text{ has order type } \geqq f(i)\}.$$

Let $F$ be the minimal fine normal filter in $K[S]$ on $I$ to which each $J_i$ belongs. Now $F$ is non-trivial as $\mu$ is Ramsey.

Now for $g\in{}^{\aleph_1}\mathrm{Ord}$ such that $\bigwedge_{i<\omega_1} g(i)<f(i)$ let $\hat g$ be the function with domain $I$, $\hat g(X)=$ the $g(X\cap\omega_1)$-th member of $X$ if there is one, zero otherwise. For $\alpha<\mu$ and such $g$ let $S^\alpha_g\overset{\text{def}}{=}\{X\in I:\hat g(X)=\alpha\}$.

Let $P=\{Y: Y\subseteq I, Y\in K[S], Y\neq\varnothing \bmod F \text{ (in } K[S])\}$ ordered by inverse inclusion and we define a $P$-name

$$\underset{\sim}{D}=\{A\subseteq\omega_1:\{X\in I: X\cap\omega_1\in A\}\in\underset{\sim}{G}_P\}.$$

It is easy to check that $(P,\underset{\sim}{D})$ is nice to $f$ in $K[S]$. By the choice of $S$ this is inherited by our universe $V$.

1.7. REMARK. (1) Clearly the proof gives:

(∗) if $\lambda \to (f(i))^{<\omega}_{\aleph_1}$ for $i < \omega_1$, then there is a $t$ nice to $f$.

(2) In 1.5, instead of $\lambda^{\aleph_1} > \lambda^+$ we can use other violations of the covering lemma, e.g., $\lambda^{\mathrm{cf}\,\lambda} > \lambda^+$, $\lambda > 2^{\mathrm{cf}\,\lambda}$.

(3) In 1.1 we say $t$ is $\kappa$-pre-nice if (a) and

(b)′ $\underset{\sim}{D}$ is a $P$-name of an ultrafilter on the Boolean algebra

$$\mathscr{P}(\mathscr{P}_{<\aleph_1}(\kappa))^V = \{A : A \subseteq \mathscr{P}_{<\aleph_1}(\kappa)^V, A \in V\}$$

where

$$\mathscr{P}_{<\aleph_1}(A) = \{a : a \subseteq A, |a| < \aleph_1\}.$$

(c)′ for each $p \in \mathscr{P}$

$D^t_p \overset{\mathrm{def}}{=} \{A : A \subseteq \mathscr{P}_{<\aleph_1}(\kappa), A \in V,\ p \Vdash_P \text{“}A \in D\text{”}\}$ is a normal filter on $\mathscr{P}_{<\aleph_1}(\kappa)$.

(4) We define “$t$ is $\kappa$-nice to $g$” similarly.

(5) Suppose 1.6, we assume $g \in L[S]$ for every $g : \mathscr{P}_{<\aleph_1}(\kappa) \to \mathrm{Ord}$. Let

$$I = \{X : \varnothing \neq X \subseteq \mathscr{P}_{<\aleph_1}(\kappa)^V \cup \mu\},$$

$F$ the minimal fine normal filter to which each $J_i = \{X \in I : X$ has order type $\geqq f(i)\}$ belongs and we define $P$ similarly. We get that there is a $\kappa$-nice $t$.

(6) In 1.6 and in 1.7(5) $I \in P$ is the minimal member of $P$ and $p^t_I$ is the filter generated by the closed unbounded subsets (i.e. $D_{\omega_1}$, $D_{<\aleph_1}(\kappa)$ respectively).

(7) In $D_0$ is a normal fine filter on $\mathscr{P}_{<\aleph_1}(\kappa)$

$$D_0 = \{A_i : \kappa \leqq i < 2^{(\kappa_0^{\aleph_0})}\}$$

and $2^{\kappa_0} \leqq \kappa$, and there is a $\kappa$-nice $t$, and for some $p \in P$, $D^t_p = D_{<\aleph_1}(\kappa)$ *then* for some $\kappa_0$-nice $t_0$, for some $p \in P$, $D^{t_0} = D_0$. So e.g. if the $\lambda^\mu > \lambda^+ + 2^\mu + \beth_3(\aleph_1)^+$ every normal filter on $\omega_1$ is nice.

(8) If $(\lambda, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ we can replace $S_{<\aleph_1}(\kappa)$ by suitable $S \subseteq \{a \subseteq \kappa : |a \cap \omega_1| = \aleph_0\}$ with profit.

1.8. Theorem. *If for every $f : \aleph_1 \to (2^{2^{\aleph_1}})^+$, some $t$ is nice to $f$ then for every $f \in {}^{\aleph_1}\mathrm{Ord}$ some $t$ is nice to $f$. So, the existence of $\lambda$, $\lambda \to (\alpha)^{<\omega}_{\aleph_0}$ for every $\alpha < \beth_2(\aleph_1)^+$, is enough.*

Proof. The theorem is proved in [Sh 5] and is not really needed for our main results.

1.9. FACT. If there is $t = (P, \underset{\sim}{D})$ nice to $f$ *then* there is $t^1 = (P^1, \underset{\sim}{D}^1)$ nice to $f$ of power $\leqq \Pi_{i<\omega_1} |f(i) + 1|$.

PROOF. See [Sh 5]; it is true by the Lowenheim–Skolem argument.

## §2. Various ranks

2.1. CONVENTION.

(1) For some fixed $\sigma$, $\mathbb{E}$ : $\sigma$ is an ordinal $\geqq 1$, $\mathbb{E} \in OB_{\sigma+2}$. Usually we do not mention (in the simple version, $\sigma = 1$). Only rarely we vary them, thus adding parameters to the rank.

(2) We use A, B, C to denote the member of $OB_0$, $D$ to denote members of $OB_\sigma$, $E$ to denote members of $\mathbb{E}$.

So $\mathrm{rk}^l_E(f) = \alpha$ really means ${}^\sigma\mathrm{rk}^l_E(f, \mathbb{E}) = \alpha$ or "${}^\sigma\mathrm{rk}^l_E(f) = \alpha$ relative to $\mathbb{E}$". (Not to mention the use of $\omega_1$ rather than say $\aleph_8$ or normal filters on $\{a \subseteq \aleph_6 : |a| < \aleph_1\}$).

2.1A. REMARK. We could change the definition of $OB$, by letting, e.g.,

$$OB_1 = \left\{X : \mathscr{P}(\omega_1) - X \text{ is an } \aleph_1\text{-complete filter } D \text{ on } I, \right.$$
$$\left. I = \bigcup_{\alpha<\omega_1} I_\alpha, I_\alpha \notin X \text{ for } \alpha < \omega_1 \right\},$$

with little change in the proofs.

2.2. DEFINITION.

(1) For a $f \in {}^{\aleph_1}\mathrm{Ord}$, $E$ ($\in \mathbb{E}$, of course) and ordinal $\alpha$ we define, by induction on $\alpha$, when $\mathrm{rk}^2_E(f) \leqq \alpha$:

$\mathrm{rk}^2_E(f) \leqq \alpha$ *if* for every $D \in E$ and $g <_{E_{[D]}} f$ (equivalently, $g <_D f$)

there are $\beta < \alpha$ and $E_1 \subseteq E_{[D]}$ such that $\mathrm{rk}^2_{E_1}(g) \leqq \beta$.

(2) Let $\mathrm{rk}^2_E(f)$ be the minimal ordinal $\alpha$ such that $\mathrm{rk}^2_E(f) \leqq \alpha$, and $\infty$ if there is no such $\alpha$ (see 2.4 below).

2.2A. CONVENTION. If in $\mathrm{rk}^2_E(f)$, $E$ is illegal (mainly $E_{[D]}$ where $D \notin E$), the value will be zero or undefined, and will not be counted as appearing (e.g. 3.2); similarly for the other ranks.

2.3. FACT. If $\mathrm{rk}^2_E(f) \leqq \alpha$ holds and $\alpha \leqq \beta$ *then* $\mathrm{rk}^2_E(f) \leqq \beta$. Hence $\mathrm{rk}^2_E(f) = \alpha$ implies: $\mathrm{rk}^2_E(f) \leqq \beta$ iff $\alpha \leqq \beta$.

2.4. Definition. $\mathrm{rk}^3_E(f) = \mathrm{Min}\{\mathrm{rk}^2_{E_1}(f) : E_1 \subseteq E\}$.

2.5. Fact.
(1) $\mathrm{rk}^3_E(f) \leqq \mathrm{rk}^2_E(f)$.
(2) If $E_1 \subseteq E_2$ then $\mathrm{rk}^3_{E_1}(f) \geqq \mathrm{rk}^3_{E_2}(f)$.
(3) For every $f$, $E$ for some $E_1 \subseteq E$, $\mathrm{rk}^3_E(f) = \mathrm{rk}^2_{E_1}(f) = \mathrm{rk}^3_{E_1}(f)$.

2.6. Definition. Suppose $(P, \underset{\sim}{D})$ is pre-nice (see Definition 1.1) and let $t = (T, \underset{\sim}{D})$.
(1) We write $\varnothing$ for the minimal element of $P$.
(2) We define by induction on $\theta \geqq 1$ for $p \in P$, an object ${}^\theta D^t_p \in OB_\theta$:

$${}^1D^t_p = \{A : \neg[p \Vdash_P A \notin \underset{\sim}{D}]\},$$

$${}^{\theta+1}D^t_p = \{{}^\theta D^t_q : p \leqq q \in P\} \cup \bigcup_{p \leqq q} {}^\theta D^t_q,$$

$${}^\delta D^t_p = \cup\{{}^\theta D^t_p : \theta < \delta\}.$$

Let (in §2, §3) $D^t_p = {}^\sigma D^t_p$, $E^t_p = {}^{\sigma+1}D^t_p$, $\mathbb{E}^t_p = {}^{\sigma+2}D^t_p$.

2.6A. Observation.
(i) For any pre-nice $t = (P, \underset{\sim}{D})$, $\theta \geqq 1$, ${}^\theta D^t_p \in OB_\theta$.
(ii) For $\theta(1) \leqq \theta(2)^{\theta(1)}D^t_p \subseteq {}^{\theta(2)}D^t_p$.
(iii) For $p \leqq q$ from $P$, ${}^\theta D^t_q \subseteq {}^\theta D^t_p$.

Proof. By induction on $\theta$.

2.7. Definition.
(1) $\mathrm{rk}^4_E(f)$ is the minimal ordinal $\alpha$ such that for some pre-nice $t = (P, \underset{\sim}{D})$:
(a) $\mathbb{E}^t \subseteq \mathbb{E}$, $E^t_\varnothing = E$;
(b) $\Vdash_P$ "the order type of $\{g/\underset{\sim}{D}[G] : g \in {}^{\aleph_1}\mathrm{Ord},\ g <_{\underset{\sim}{D}[G]} f\}$ is $\leqq \alpha$".
We call $t$ a witness for $\mathrm{rk}^4_E(f)$.
(2) $\mathrm{rk}^5_E(f) = \mathrm{Min}\{\mathrm{rk}^4_{E_1}(f) : E_1 \subseteq E\}$.
We call $(t, E_1)$ a witness for $\mathrm{rk}^5_E(f)$ when $t$ is a witness for $\mathrm{rk}^4_{E_1}(f) = \alpha$, $E_1 \subseteq E$ and $\alpha = \mathrm{rk}^4_{E_1}(f)$ is $\mathrm{rk}^5_E(f)$.

2.8. Fact.
(1) $\mathrm{rk}^5_E(f) \leqq \mathrm{rk}^4_E(f)$.
(2) If $E_1 \subseteq E_2$ then $\mathrm{rk}^5_{E_1}(f) \geqq \mathrm{rk}^5_{E_2}(f)$.
(3) For every $f$, $E$ for some $E_1 \subseteq E$, $\mathrm{rk}^5_E(f) = \mathrm{rk}^4_{E_1}(f) = \mathrm{rk}^5_{E_1}(f)$.

2.9. Claim. $\mathrm{rk}^2_E(f) \leqq \mathrm{rk}^4_E(f)$.

PROOF. We prove it by induction on $\mathrm{rk}^4_E(f)$.

Let $\beta = \mathrm{rk}^4_E(f)$; if $\beta = 0$ the assertion is trivial. So there is a witness $t = (P, \underset{\sim}{D})$ for $\mathrm{rk}^4_E(f) = \beta$. We want to show $\mathrm{rk}^2_E(f) \leqq \beta$. By Definition 2.2(1) it suffices, given $D_1 \in E$ and $g <_{D_1} f$, to find $\gamma < \beta$ and $E_1 \subseteq E_{[D_1]}$ such that $\mathrm{rk}^2_{E_1}(g) \leqq \gamma$. As $t$ witnesses $\mathrm{rk}^4_E(f) = \beta$:

(i) $\Vdash_P$ "$\{h/\underset{\sim}{D}[G] : h <_{\underset{\sim}{D}[G]} f\}$ is well ordered, of order type $\leqq \beta$";

(ii) $E^t_\varnothing = E$.

As $D_1 \in E$, $E = E^t_\varnothing$, there is $p \in P$ such that $D^t_p = D_1$. Now as $g <_{D_1} f$, clearly $p \Vdash_P$ "$g/\underset{\sim}{D}[G] < f/\underset{\sim}{D}[G]$ and $\{h/\underset{\sim}{D}[G] : h/\underset{\sim}{D}[G] < f/\underset{\sim}{D}[G]\}$ has order type $\leqq \beta$". We can deduce $p \Vdash_P$ "$\{h/\underset{\sim}{D}[G] : h/\underset{\sim}{D}[G] < g/\underset{\sim}{D}[G]\}$ has order type $< \beta$" hence for some $q$, $p \leqq q \in P$ and $\gamma < \beta$

$$q \Vdash_P \text{"}\{h/\underset{\sim}{D}[G] : h/\underset{\sim}{D}[G] < g/\underset{\sim}{D}[G]\} \text{ has order type } \leqq \gamma\text{"}.$$

Let $E_1 = E^t_q$, clearly (as $p \leqq q$) $E_1 \subseteq E^t_p \subseteq E_{[D^t_p]} = E_{[D_1]}$ (see Definition 2.6) so $\mathrm{rk}^4_{E_1}(g) \leqq \gamma$ (see Definition 2.7(1); we can use for witness $t' = (P^*_1, \underset{\sim}{D} \restriction P^*_1)$ where $P^*_1 \overset{\text{def}}{=} \{r \in P : r \geqq q\}$, so $\varnothing_{P^*_1} = q$) so by the induction hypothesis (on $\beta$) $\mathrm{rk}^2_{E_1}(g) \leqq \gamma$ which is as required.

2.10. CONCLUSION. $\mathrm{rk}^3_E(f) \leqq \mathrm{rk}^5_E(f)$.

PROOF. By 2.9 (and Definitions 2.4, 2.7(2)).

2.11. CLAIM. *For $l = 3, 5$, if $g <_D f$, $D = \mathrm{Min}\, E$, then* $\mathrm{rk}^l_E(g) < \mathrm{rk}^l_E(f)$ (or both are $\infty$).

PROOF. Without loss of generality $\mathrm{rk}^l_E(f) < \infty$.

First we deal with $l = 5$.

If $E_1$ witness $\mathrm{rk}^5_E(f) = \alpha$ (i.e., $E_1 \subseteq E$, $\mathrm{rk}^4_{E_1}(f) = \alpha$) and $t = (P, \underset{\sim}{D})$ witness $\mathrm{rk}^4_{E_1}(f) = \alpha$, then $\Vdash_P$ "$\{h/\underset{\sim}{D}[G] : h/\underset{\sim}{D}[G] < f/\underset{\sim}{D}[G]\}$ has order type $\leqq \alpha$" so (as in the proof of 2.9) for some $p \in P$ and $\beta < \alpha$, $p \Vdash_P$ "$\{h/\underset{\sim}{D}[G] : h/\underset{\sim}{D}[G] < g/\underset{\sim}{D}[G]\}$ has order type $\leqq \beta$". So $E^t_p$ (which trivially is $\subseteq E^t_\varnothing = E_1 \subseteq E$) witness $\mathrm{rk}^5_E(g) \leqq \beta$ as $\mathrm{rk}^4_{E^t_p}(g) \leqq \beta$ is witnessed by $(P \restriction \{r \in P : r \geqq p\}, \underset{\sim}{D})$.

Now we prove for $l = 3$.

Let $E_0 \subseteq E$, $\alpha \overset{\text{def}}{=} \mathrm{rk}^3_E(f) = \mathrm{rk}^2_{E_0}(f)$. By Definition 2.2(1) for $\mathrm{rk}^2_{E_0}(f) \leqq \alpha$ (letting $g, D$ there be chosen here as $g$, $\mathrm{Min}\, E_0$ resp.) there are $E_1 \subseteq (E_0)_{[\mathrm{Min}\, E_0]} = E_0 \subseteq E$ and $\beta < \alpha$ such that $\mathrm{rk}^2_{E_1}(g) \leqq \beta$.

So by Definition 2.2(2), $\mathrm{rk}^3_E(g) \leqq \beta$.

2.12. CONCLUSION. *If $X = \mathrm{fil}(E)$, $l = 3, 5$ then* $\| f \|_X \leqq \mathrm{rk}^l_E(f)$.

PROOF. By the definition of $\| f \|_D$ (see §0) and 2.11.

2.13. CLAIM.
(1) For $l = 2, 4$:

$$\text{for } D \in E, \quad \mathrm{rk}_E^{l+1}(f) \leqq \mathrm{rk}_{E[D]}^{l+1}(f) \leqq \mathrm{rk}_{E[D]}^{l}(f) \leqq \mathrm{rk}_E^{l}(f).$$

(2) If $f \leqq_E g$ *then* $\mathrm{rk}_E^l(f) \leqq \mathrm{rk}_E^l(g)$ for $l = 2, 3, 4, 5$.
(3) If $f =_{\mathrm{Min}\, E} g$ *then* $\mathrm{rk}_E^l(f) = \mathrm{rk}_E^l(g)$ for $l = 2, 3, 4, 5$.

PROOF. (1) The first inequality holds by 2.5(2) [or 2.8(2)], the second by 2.5(1) [or 2.8(1)] and the third by Definition 2.2(1) [or 2.7(1), using $(p \restriction \{r : r \geqq q\}, \underset{\sim}{D})$ as in the proof of 2.11].
(2) Left to the reader.
(3) Follows from (2).

2.14. CLAIM. Suppose $l = 2, 4$, $\mathrm{rk}_E^l(f) = \mathrm{rk}_E^{l+1}(f)$. Then for every $D \in E$,

$$\mathrm{rk}_{E[D]}^{l}(f) = \mathrm{rk}_{E[D]}^{l+1}(f) = \mathrm{rk}_E^l(f) = \mathrm{rk}_E^{l+1}(f).$$

PROOF. By 2.13.

2.15. DEFINITION. (1) Let for $E \in OB_{>1}$

$$T_E(f) = \sup\{T_x(f) : x \in E \cap OB_1\} = \sup\{T_{\mathrm{fil}(E_1)}(f) : E_1 \subseteq E\}$$

*where*
(2) for $D \in OB_1$, $T_D(f) = \sup\{|F| : F \subseteq {}^{\aleph_1}\mathrm{Ord}, (\forall g \in F) g <_D f$ and for distinct $g, h$ from $F$, $g \neq_D h\}$.
(3) $T_E^*(f) = \mathrm{Min}\{T_{E_1}(f) : E_1 \subseteq E$; so $\mathrm{lev}(E_1) = \sigma + 2$, $E_1 \in \mathbb{E}\}$.

2.16. FACT.
(1) If $E_1 \subseteq E_0$ then $T_{E_1}(f) \leqq T_{E_0}(f)$.
(2) $T_E^*(f) \leqq T_{E_D}^*(f) \leqq T_{E[D]}(f) \leqq T_E(f)$ when $D \in E$.
(3) For every $E \in \mathbb{E}$ and $f \in {}^{\aleph_1}\mathrm{Ord}$ for some $E_1 \subseteq E$:

$$T_{E_1}(f) = T_E^*(f) = T_{(E_1)[D]}(f) = T_{E[D]}^*(f) \quad \text{for every } D \in E_1.$$

PROOF. See Definition 2.15.

2.17. LEMMA. (1) $T_E(f) \leqq |\mathrm{rk}_E^l(f)| + |\mathbb{E}|$ *for* $l = 2, 4$.
(2) *If* $\mathrm{rk}_E^l(f) = \mathrm{rk}_E^{l+1}(f)$ *then* $T_E(f) \leqq |\mathrm{rk}_E^l(f)| + 2^{\aleph_1}$ *for* $l = 2, 4$.
(3) $T_E^*(f) \leqq |\mathrm{rk}_E^l(f)| + 2^{\aleph_1}$ *for* $l = 2, 3, 4, 5$.

PROOF. (1) By 2.9 without loss of generality $l = 2$. Suppose this fails, then for some $x \in E \cap OB_1$ or $x = E \in OB_1$, $T_x(f) > \lambda \overset{\text{def}}{=} |\mathrm{rk}_E^2(f)| + |\mathbb{E}|$. So there are $f_i <_x f$ for $i < \lambda^+$ such that $f_i \neq_x f_j$ for $i < j < \lambda^+$. By the definition of $\mathrm{rk}_E^2$ (see 2.2) for each $i$ for some ordinal $\alpha_i < \mathrm{rk}_E^2(f)$, and $E_i \subseteq E_{[D]}$, $\alpha_i = \mathrm{rk}_{E_i}^2(f_i) < \mathrm{rk}_E^2(f)$; without loss of generality $\mathrm{rk}_{E_i}^2(f_i) = \mathrm{rk}_{E_i}^3(f_i)$. As $\lambda \geqq |\mathrm{rk}_E^2(f)| + |\mathbb{E}|$ without loss of generality $E_i = E_0$, $\mathrm{rk}_{Ei}^2(f_i) = \gamma$. But for some $i < j$, $f_i <_D f_j$ (see 0D(1)) hence $f_i <_{\mathrm{Min}\, E_0} f_j$, contradiction to 2.11.

(2) By 2.14 (and Definition 2.15) it suffices to prove $T_x(f) \leqq |\mathrm{rk}_E^l(f)| + 2^{\aleph_1}$ for $x = \mathrm{fil}(E)$. So suppode $f_i$ $(i < \lambda^+)$ are as in the proof of (1), $\lambda \overset{\text{def}}{=} |\mathrm{rk}_E^l(f)| + 2^{\aleph_1}$. So without loss of generality $\mathrm{rk}_E^{l+1}(f_i)$ is a constant $\gamma < \mathrm{rk}_E^l(f)$, contradiction by 2.11 and 0D(1).

(3) Easy by now.

2.18. Lemma. $|\mathrm{rk}_E^l(f)| \leqq T_E(f) + |E|$ *for* $l = 2, 3, 4, 5$ *provided that* $\mathrm{rk}_E^l(f) < \infty$.

2.19. Remark. Note that $E$ has cardinality $\leqq \beth_{\sigma+1}(\aleph_1)$ and that $|\mathbb{E}|$, $|E| \geqq 2^{\aleph_1}$ and every $x \in OB_{\geqq 1}$ has cardinality $\geqq 2^{\aleph_1}$. The same applies to 2.20, 2.21.

Proof. Let $\alpha \overset{\text{def}}{=} \mathrm{rk}_E^l(f)$.

First let $l = 4$, and $t = (P, \underset{\sim}{D})$ witness $\mathrm{rk}_E^4(f) \leqq \alpha$. For every $X \in E \cap OB_1$, let $\{g_i^X : i < \lambda_X\}$ be a maximal family of functions $g \in {}^{\aleph_1}\mathrm{Ord}$, $g <_X f$, $[i \neq j \Rightarrow g_i^X \neq_X g_j^X]$. Clearly there is such a family and $\lambda_X \leqq T_X(f) \leqq T_E(f)$. Let $t = (P, \underset{\sim}{D})$ witness $\mathrm{rk}_E^4(f) \leqq \alpha$.

We can find $P^1 \subseteq P$, $|P^1| \leqq T_E(f) + |E|$ such that:

(a) $\varnothing \in P^1$;

(b) if $p \in P^1$, $D \in E_p^t$ then for some $q$, $p \leqq q \in P^1$, $D_q^t = D$;

(c) if $p \in P^1$, $g \in \{g_i^X : i < \lambda_X,\ X \in E \cap OB_1\}$, *then* for some $q$, $p \leqq q \in P^1$, and for some $\beta$ $q \Vdash_P$ "$\{h/\underset{\sim}{D}[G] : h/\underset{\sim}{D}[G] < g/\underset{\sim}{D}[G]\}$ has order type $\beta$".

It is easy to find such a $P^1$. Let $S = \{\beta$: for some $q \in P^1$ and $g \in \{g_i^X : i < \lambda_X$, $X \in E \cap OB_1\}$ we have $q \Vdash_P$ "$\{h/\underset{\sim}{D}[G] : h/\underset{\sim}{D}[G] < g/\underset{\sim}{D}[G]\}$ has order type $\beta$"$\}$. Clearly $|S| \leqq T_E(f) + |E|$, and let $\theta$ be an order prserving one-to-one function from $S$ onto some ordinal $\alpha^*$, necessarily $|\alpha^*| \leqq T_E(f) + |E|$.

Define a $P^1$-name $\underset{\sim}{D}^1 \overset{\text{def}}{=} \{A \subseteq \omega_1 : A \in V$, and for some $p \in \underset{\sim}{G}^1$, $A = \omega_1$ mod ${}^1D_p^t\}$. Easily $t^1 \overset{\text{def}}{=} (P^1, \underset{\sim}{D}^1)$ witness $\mathrm{rk}_E^4(f) \leqq \alpha^*$. The proof for $\mathrm{rk}_E^2(f)$ is similar, e.g., take a suitable elementary submodel of $(H(\lambda), \in)$, $\lambda$ large enough (or use $\mathrm{rk}_E^2(f) \leqq \mathrm{rk}_E^4(f)$).

Now for $\mathrm{rk}_E^3(f)$, $\mathrm{rk}_E^5(f)$ use their definitions (2.4, 2.7(2)) and that we have proved 2.18 for $\mathrm{rk}_E^2(f)$, $\mathrm{rk}_E^4(f)$ respectively, observing 2.16.

2.20. FACT. If $l = 2, 3, 4, 5$ $\mathrm{rk}^l_E(f) < \infty$, *then* for some $\mathbb{E}_1 \subseteq \mathbb{E}$, and $E_1 \subseteq E$ ($E_1 \in \mathbb{E}_1$) we have (for $\mathbb{E}_1$):

$$|\mathrm{rk}^l_{E_1}(f)| \leqq T_E(f) + 2^{\aleph_1} \quad \text{and} \quad |E_1| \leqq T_E(f) + 2^{\aleph_1}.$$

PROOF. Let $l = 4$.

The proof is like that of 2.17, but $P^1 \subseteq P$ has cardinality $\leqq T_E(f) + 2^{\aleph_1}$ and satisfies:

(a) $\emptyset \in P^1$;

(b) if $p \in P^1$, then $A \neq \emptyset$ mod $D^l_p$ for some $q$, $p \leqq q \in P^1$ and $A \in D^l_q$;

(c) if $p \in P^1$, $g \in \{g^X_i : i < \lambda_X, X = {}^1D^l_r$ for some $r \in P^1$ $(r \geqq p)\}$ *then* for some $q$, $p \leqq q \in P_1$ and for some $\beta$

$$q \Vdash_P \text{“}\{h/D[G] : h/D[G] < g/D[G]\} \text{ has order type } \beta\text{”}.$$

The rest should be clear, as well as the proof for $l = 2, 3, 5$.

Now by 2.17 and 2.18:

2.21. THEOREM.

(1) *For* $l = 2, 4$ *if* $\mathrm{rk}^l_E(f) < \infty$ *then*

$$|\mathrm{rk}^l_E(f)| + |\mathbb{E}| = T_E(f) + |\mathbb{E}|.$$

(2) *If* $l = 2, 4$, $\mathrm{rk}^l_E(f) = \mathrm{rk}^{l+1}_E(f) < \infty$ then

$$|\mathrm{rk}^l_E(f)| + |E| = T_E(f) + |E|.$$

(3) *For* $l = 3, 5$, *similar results hold, if* $\mathrm{rk}^l_E(f) < \infty$, *then*

$$|\mathrm{rk}^l_E(f)| + |E| = T^*_E(f) + |E|$$

[note $|E| \leqq \beth_{\sigma+1}(\aleph_1)$, $|\mathbb{E}| \leqq \beth_{\sigma+2}(\aleph_1)$].

## §3. More on ranks

3.1. CONVENTION. $\mathbb{E}$, $\sigma$ will be fixed, as in 2.1, and $A, B; D; E$ will be used similarly.

3.2. FACT. (1) If $\omega_1 = A \cup B$, $f \in {}^{\aleph_1}\mathrm{Ord}$, $l = 2, 4$, $D = \mathrm{Min}\, E$ *then*

$$\mathrm{rk}^l_E(f) = \mathrm{Max}\{\mathrm{rk}^l_{E_{[A]}}(f), \mathrm{rk}^l_{E_{[B]}}(f)\}.$$

(2) If $\omega_1 = A \cup B$, $f \in {}^{\aleph_1}\mathrm{Ord}$, $l = 3, 5$ *then*

$$\mathrm{rk}^l_E(f) = \mathrm{Min}\{r^l_{E_{[A]}}(f), \mathrm{rk}^l_{E_{[B]}}(f)\}.$$

(3) If $\omega_1 = \{i : i \in \bigcup_{j<1+i} A_j\}$ where $A_j \subseteq \omega_1$ for $j < \omega_1$ *then* for $l = 2, 4$

$$\mathrm{rk}^l_E(f) = \sup\{\mathrm{rk}^l_{E_{[D+A_i]}}(f) : i < \omega_1\}.$$

(4) If $\omega_1 = \{i : i \in \bigcup_{j<1+i} A_j\}$ *then* for $l = 3, 5$

$$\mathrm{rk}^l_E(f) = \mathrm{Min}\{\mathrm{rk}^l_{E_{[A_i]}}(f) : i < \omega_1\}.$$

PROOF. Easy, using the definitions.

3.3. DEFINITION. For $f \in {}^{\aleph_1}\mathrm{Ord}$ let:
(1) $A_0(f) = \{i < \omega_1 : f(i) = 0\}$;
(2) $A_1(f) = \{i < \omega_1 : f(i)$ is a successor ordinal$\}$;
(3) $A_2(f) = \{i < \omega_1 : f(i)$ is a limit ordinal$\}$.

3.4. FACT. If $f \in {}^{\aleph_1}\mathrm{Ord}$, $A_0(f) \in \mathrm{fil}\, E$, $l = 2, 3, 4, 5$ *then* $\mathrm{rk}^l_E(f) = 0$.

PROOF. Easy.

3.5. FACT. If $f, g \in {}^{\aleph_1}\mathrm{Ord}$, $\{i : f(i) = g(i) + 1\} \in \mathrm{fil}\, E$, *then*

$$\mathrm{rk}^2_E(f) = \sup\{\mathrm{rk}^3_{E_{[D]}}(g) + 1 : D \in E\}.$$

PROOF. Easy, by the definition of $\mathrm{rk}^2_E$.

3.6. FACT. (1) If $f, g \in {}^{\aleph_1}\mathrm{Ord}$, $E \in \mathbb{E}$, $l = 3, 5$, and $\{i : f(i) = g(i) + 1\} \in \mathrm{fil}\, E$ *then* $\mathrm{rk}^l_E(f) = \mathrm{rk}^l_E(g) + 1$.

(2) If $\mathrm{rk}^2_E(f) = \mathrm{rk}^3_E(f)$, $\{i : f(i) = g(i) + 1\} \in \mathrm{fil}\, E$ then $\mathrm{rk}^2_E(g) = \mathrm{rk}^3_E(g)$.

PROOF. (1) By 2.11, $\mathrm{rk}^l_E(g) + 1 \leqq \mathrm{rk}^l_E(f)$ (as $g <_{\mathrm{fil}\, E} f$).

By 2.5(3) (and 2.8(3)) for some $E_1 \subseteq E$,

$$\mathrm{rk}^l_E(g) = \mathrm{rk}^{l-1}_{E_1}(g) = \mathrm{rk}^l_{E_1}(g),$$

hence $\mathrm{rk}^l_{E_1}(g) = \mathrm{rk}^l_{(E_1)_{[D]}}(g)$ for every $D \in E_1$. So by 3.5, for $l = 3$, $\mathrm{rk}^2_{E_1}(f) = \mathrm{rk}^3_{E_1}(g) + 1$; but $\mathrm{rk}^3_E(g)$ is by the choice of $E_1$, $\mathrm{rk}^3_{E_1}(g)$ and by 2.5(2) $\mathrm{rk}^3_E(f) \leqq \mathrm{rk}^3_{E_1}(f)$, hence $\mathrm{rk}^3_E(f) \leqq \mathrm{rk}^3_E(g) + 1$. So together $\mathrm{rk}^3_E(f) = \mathrm{rk}^3_E(g) + 1$.

As for $l = 5$, use the definition directly.

(2) Let $\alpha = \mathrm{rk}^2_E(f) = \mathrm{rk}^3_E(f)$. By 3.6(1) $\mathrm{rk}^3_E(g) = \alpha - 1$ (and $\alpha - 1$ is well defined).

We can prove $\mathrm{rk}^2_E(g) \leqq \alpha - 1$, using the definition. [Let $D \in E$, $g_1 <_{E_{[D]}} g$; then by 2.14 $\mathrm{rk}^3_E(f) = \mathrm{rk}^3_{E_{[D]}}(f) = \mathrm{rk}^2_{E_{[D]}}(f) = \mathrm{rk}^2_E(f) = \alpha$ hence by 3.6(1) $\mathrm{rk}^3_{E_{[D]}}(g) = \alpha - 1$ but $\mathrm{rk}^3_{E_{[D]}}(g_1) < \mathrm{rk}^3_{E_{[D]}}(g)$ (by 2.11), hence

$$\beta \overset{\mathrm{def}}{=} \mathrm{rk}^3_{E_{[D]}}(g_1) < \alpha - 1$$

so there is $E_1 \subseteq E_{[D]}$, $\mathrm{rk}^2_{E_1}(g_1) = \beta$. So $E_1$, $\beta$ are as required.] So we proved $\mathrm{rk}^2_E(g) \leqq \alpha - 1$, but $\mathrm{rk}^2_E(g) \geqq \mathrm{rk}^3_E(g) \geqq \alpha - 1$ (see 2.5(1)) so the conclusion follows.

3.7. FACT. Suppose $A_2(f) \in \mathrm{fil}\, E$, $K \subseteq {}^{\aleph_1}\mathrm{Ord}$, $g <_{\mathrm{fil}\, E} f$ for $g \in K$, and $(\forall h \in {}^{\aleph_1}\mathrm{Ord})\, [h <_{\mathrm{fil}\, E} f \rightarrow (\exists g \in K) h \leqq_{\mathrm{fil}\, E} g]$.
Then for $l = 2$,

$$\mathrm{rk}^l_E(f) = \sup\{\mathrm{rk}^l_E(g) : g \in K\}.$$

PROOF. Let $\alpha \overset{\mathrm{def}}{=} \sup\{\mathrm{rk}^l_E(g) : g \in K\}$. Trivially [by 2.13(2)) $\mathrm{rk}^l_E(g) \leqq \mathrm{rk}^l_E(f)$ for $g \in K$, hence $\alpha \leqq \mathrm{rk}^l_E(f)$. Let us prove the other direction. Let $D \in E$, $g <_{E_{[D]}} f$. Now let us define $g_1 : g_1(i) = g(i) + 1$; clearly, as $A_2(f) \in \mathrm{fil}\, E$, $g_1 <_{\mathrm{fil}\, E} f$, hence for some $g_2 \in K$, $g_1 \leqq_{\mathrm{fil}\, E} g_2$. So $g <_{E_{[D]}} g_2$, $D \in E$ where $\mathrm{rk}^l_E(g_2) \leqq \alpha$, so there are $E_1 \subseteq E_{[D]}$, $\beta$ as required, by the definition of $\mathrm{rk}^l_E(g_2)$.

3.8. FACT. If $l = 3$, $\alpha \leqq \mathrm{rk}^l_E(f) < \infty$, *then* for some $g \leqq_E f$, and $E_1 \subseteq E$, $\mathrm{rk}^l_{E_1}(g) = \alpha$ (and $\mathrm{rk}^l_{E_1}(g) = \mathrm{rk}^{l-1}_{E_1}(g)$).

PROOF. Suppose not, then we shall prove by induction on $\beta \geqq \alpha$ that

(∗) if $g \leqq_{E_1} f$, $E_1 \subseteq E$ and $\mathrm{rk}^3_{E_1}(g) \geqq \alpha$ then $\mathrm{rk}^3_{E_1}(g) \geqq \beta$.

*For* $\beta = \alpha$: trivial, as we assume our assertion fails.
*For* $\beta = \alpha + 1$: this is the assumption (using 2.5(3)).
*For* $\beta > \alpha$ *limit*: trivial by the induction hypothesis.
*For* $\beta = \gamma + 1$, $\gamma > \alpha$: we know, by the induction hypothesis, that $\mathrm{rk}^3_{E_1}(g) \geqq \alpha + 1$ hence $\mathrm{rk}^2_{E_1}(g) \not\leqq \alpha$.
By 2.2(1):
(a) there are $D \in E_1$, and $h <_{E_{[D]}} g$ such that for no $\zeta < \alpha$ and $E_2 \subseteq (E_1)_{[D]}$ is $\mathrm{rk}^2_{E_2}(h) \leqq \zeta$.
For such $D$ and $h$, we get: for $E_2 \subseteq (E_1)_{[D]}$, $\mathrm{rk}^2_{E_2}(h) \geqq \alpha$. So by the definition of $\mathrm{rk}^3_{E_2}$, $\mathrm{rk}^3_{E_2}(h) \geqq \alpha$ for every $E_2 \subseteq (E_1)_{[D]}$. By the induction hypothesis $\mathrm{rk}^3_{E_2}(h) \geqq \gamma$ for every $E_2 \subseteq (E_1)_{[D]}$. So $D$, $h$ exemplifies $\mathrm{rk}^2_{E_2}(g) \geqq \gamma + 1 = \beta$ for every $E_2 \subseteq (E_1)_{[D]}$. Hence $\mathrm{rk}^3_{E_2}(g) \geqq \beta$ for every $E_2 \subseteq (E_1)_{[D]}$. As this holds for every $E_1$, $\mathrm{rk}^2_E(g) \geqq \beta$, hence $\mathrm{rk}^3_{E_1}(g) \geqq \beta$ for $E_1 \subseteq E$. So we have carried the induction on $\beta$, thus proved (∗). So $\mathrm{rk}^3_D(f) = \infty$, contradicting the assumption $\mathrm{rk}^2_{E_1}(f) < \infty$.

3.9. FACT. If $\alpha < \mathrm{rk}^2_E(f) < \infty$ *then* for some $E_1 \subseteq E$, $g <_{E_1} f$, $\mathrm{rk}^2_E(g) = \mathrm{rk}^3_E(g) = \alpha$.

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PROOF. By 3.8 it suffices to find $E_1 \subseteq E$, $g <_{E_1} f$, $\mathrm{rk}^3_E(g) \geqq \alpha$, which follows from 3.2–35.

3.10. LEMMA. (1) *Suppose $\kappa$ is a regular cardinal* $> |\mathbb{E}|$, $g \in {}^{\aleph_1}\mathrm{Ord}$, $E \in \mathbb{E}$ *and* $\infty > \mathrm{rk}^2_E(f) > \kappa$. *Then for some* $g_\xi \in {}^{\aleph_1}\mathrm{Ord}$ *(for* $\xi \leqq \kappa$*) and* $E_1 \subseteq E$ $(E_1 \in \mathbb{E})$ *the following holds*:

(A) $g_\kappa <_{E_1} f$;

(B) *for* $\xi < \zeta \leqq \kappa$, $g_\xi <_{E_1} g_\zeta$ *and even* $\mathrm{rk}^2_{E_1}(g_\xi) < \mathrm{rk}^3_{E_1}(g_\zeta)$;

(C) $T_{E_1}(g_\xi) < \kappa$ *for* $\xi < \kappa$;

(D) *if* $D \in E_1$, $\xi < \kappa$ *then* $T_{\mathrm{fil}(E_{[D]})}(g_\xi) < \kappa$; *and in particular* $T_{\mathrm{fil}\, E_1 + A}(g_\xi) < \kappa$ *when* $\xi < \kappa$, $A \neq \emptyset$ mod(fil $E_1$);

(E) $\xi \leqq \mathrm{rk}^2_{E_1}(g_\xi) = \mathrm{rk}^3_{E_1}(g_\xi) < \kappa$ *for* $\xi < \kappa$;

(F) $T_X(g_\kappa) = \kappa$ for $X \in \mathrm{Fil}(E_1)$;

(G) $\mathrm{rk}^2_{E_1}(g_\kappa) = \mathrm{rk}^3_{E_1}(g_\kappa) = \kappa$;

(H) *if* $g <_D g_\kappa$, $D \in \mathrm{Fil}(E)$, *then for some* $\zeta < \kappa$, $g <_{(E_1)_{[D]}} g_\zeta$.

(2) *If* $l = 2, 3, 4, 5$, $\infty > \mathrm{rk}^l_E(f, \mathbb{E}) > \kappa$, $|\mathbb{E}| < \kappa$, then *there are* $g_\xi$ $(\xi \leqq \kappa)$ *and* $E_1$ *as above*.

PROOF. (1) By 3.9 there are $E_1 \subseteq E$, $g_\kappa < f$, such that $\mathrm{rk}^2_{E_1}(g_\kappa) = \mathrm{rk}^3_{E_1}(g_\kappa) = \kappa$. So it is enough to prove:

3.11. SUBFACT. If $\mathrm{rk}^2_{E_1}(g_\kappa) = \mathrm{rk}^3_{E_1}(g_\kappa) = \kappa$, $\kappa$ of cofinality $> |\mathbb{E}|$, then for some $g_\xi$ $(\xi < \kappa)$ (A)–(H) (from 3.10) are satisfied.

PROOF. Easily $A_2(g_\kappa) \in \mathrm{fil}(E_1)$ [otherwise there is $i \in \{0, 1\}$ such that $A_i(f) \neq \emptyset$ mod $\mathrm{fil}(E_1)$, hence $E_2 \stackrel{\mathrm{def}}{=} (E_1)_{[A_i(f)]} \in \mathbb{E}$ and by 2.14 (as $E_2 = (E_1)_{[(\mathrm{Min}\, E_1)_{[A_i(f)]}]}$) clearly $\mathrm{rk}^2_{E_2}(g_\kappa) = \mathrm{rk}^3_{E_2}(g_\kappa) = \kappa$ and $A_i(f) \in \mathrm{fil}(E_2)$ hence by 3.4, 3.6 $\kappa$ is zero or a successor ordinal]. By 2.13(3) without loss of generality $A_2(g_\kappa) = \omega_1$. By 3.9 for every $\xi < \kappa$ for some $E_\xi \subseteq E$, $g_\xi <_{E_\xi} g_\kappa$, $\mathrm{rk}^3_{E_\xi}(g_\xi) = \mathrm{rk}^2_{E_\xi}(g_\xi) = \xi$. As $\kappa$ has large cofinality, for some unbounded $C \subseteq \kappa$, $|C| = \kappa$, $E_\xi$ is constant for $\xi \in C$, so w.l.o.g. $E_\xi = E_1$ for $\xi \in C$.

So for every $\xi \in C$

$$(*) \qquad \xi = \mathrm{rk}^3_{E_1}(g_\xi) = \mathrm{rk}^2_{E_1}(g_\xi) < \kappa.$$

So:

$$(**) \qquad \text{if } \xi < \zeta \text{ are in } C, \quad \mathrm{rk}^2_{E_1}(g_\xi) < \zeta.$$

Now if $\xi < \zeta$ are in $C$, $A = \{i < \omega_1 : g_\xi(i) \geqq g_\zeta(i)\}$ and $A \neq \emptyset$ mod fil $E_1$ then (see 2.13(1) and 1.1(3)):

(a) $\mathrm{rk}^3_{E_1}(g_\zeta) \leqq \mathrm{rk}^2_{E_1[A]}(g_\zeta) \leqq \mathrm{rk}^2_{E_1}(g_\xi)$

hence

(b) $\mathrm{rk}^2_{E_1[A]}(g_\xi) < \zeta$.

On the other hand, applying (a) and (∗) for $\zeta$

(c) $\mathrm{rk}^2_{E_1[A]}(g_\zeta) \geqq \zeta$.

But $g_\zeta \leqq_{E_1[A]} g_\xi$, a contradiction to (b) and 2.13(3).

It follows that

$$(***) \qquad \text{for } \xi < \zeta \text{ in } C, \quad g_\xi <_{\mathrm{fil}\, E_1} g_\zeta.$$

So restricting ourselves to $\xi$, $\zeta$ in $C$, (B), (E), (F) and (G) hold. Now (C) and (A) hold by 2.17(2) (and by previous information) and (F) holds by 2.21. If (H) fails, exemplified by $g$ we can get $\mathrm{rk}^3_E(g) > \kappa$, contradiction. Lastly (D) holds by 2.16(2), 2.17.

By renaming the $g_\xi$ ($\xi \in C$) we get the desired conclusion.

(2) Left to the reader (use 3.10 for $l = 2$, 3.11 for $l = 3$, 2.9 for $l = 4$, 2.10 for $l = 5$).

3.12. Definition.

(1) $\mathbb{E}$ is $\mathrm{rk}^l$-nice to $f$ *if* for every $g \leqq f$ and $E \in \mathbb{E}$, $\mathrm{rk}^l_E(g) < \infty$ relative to $\mathbb{E}$.

(2) $\mathbb{E}$ is $\mathrm{rk}^l$-nice if for every $f$ and $E \in \mathbb{E}$, $\mathrm{rk}^l_E(f) < \infty$,

(3) $\mathbb{E}$ is nice if it is $\mathrm{rk}^4$-nice.

(4) $\mathbb{E}$ is hereditarily $\mathrm{rk}^l$-nice to $f$ *if* $\theta + 2 \leqq \sigma$ and $E_1 \in \{E\} \cup E$ such that $\mathrm{lev}(E_1) \geqq \theta + 2$ implies $E_1$ is nice to $f$; similarly for the other definitions.

3.12A. Remark. For $l = 2, 4$, $\mathrm{rk}^l$-niceness implies $\mathrm{rk}^{l+1}$-niceness. Also for $l = 2, 3$ $\mathrm{rk}^{l+2}$-niceness implies $\mathrm{rk}^l$-niceness (by 2.9, 2.10).

3.13. Fact.

(1) If $l = 4, 5$, $\mathrm{rk}^l_E(f) < \infty$ relative to $\mathbb{E}$, *then* for some $\mathbb{E}_1 \subseteq \mathbb{E}$, $E \in \mathbb{E}$, $|\mathbb{E}_1| \leqq |\mathrm{rk}^l_E(f)| + |E|$ and $\mathbb{E}_1$ is $\mathrm{rk}^l$-nice to $f$ (and $\mathrm{rk}^l_E(f, \mathbb{E}_1) = \mathrm{rk}^l_E(f, \mathbb{E})$).

(2) In fact, if $t = (P, \underset{\sim}{D})$ exemplifies $\mathrm{rk}^l_E(f) < \infty$ ($l = 4, 5$) relative to $\mathbb{E}$, then we can choose $\mathbb{E}_1 \overset{\mathrm{def}}{=} \mathbb{E}^l_\varnothing$ (see 2.6).

(3) Similar results hold for $l = 2, 3$.

Proof. Immediate.

3.14. Theorem. *The following are equivalent*:

(1) *There is a nice* $\mathbb{E} \in OB_{\sigma+2}$.

(2) *There is* $\mathbb{E}$, $\mathrm{rk}^5$*-nice for* $\beth_{\sigma+2}(\aleph_1)^+$ (*i.e., the constant function with this value*).

(3) *There is t nice to* $\beth_{\sigma+2}(\aleph_1)^+$.
(4) *For every* $f \in {}^{\aleph_1}\mathrm{Ord}$ *some t is nice to it.*

REMARK. Note that (4) does not depend on $\sigma$, so for all ordinals $\sigma \geqq 1$ the conditions are equivalent.

PROOF. (3)⇒(4): By [Sh 5].
(2)⇒(3): By the definitions.
(3)⇒(2): Easy (defining the $\mathbb{E}$ by $t$).
(1)⇒(2): By the definitions.
(4)⇒(1): By (4) for every ordinal $\alpha$ some $t^\alpha$ is nice to it (i.e., to the constant function $\alpha$). As the family of possible $\mathbb{E}^t$ is a set, and $\mathbb{E}^{t^\alpha}$ is nice to $\alpha$, and monotonicity, we are done.

3.14A. REMARK. Instead of using nice $\mathbb{E}$, another way is to use nice fine normal filters on $\mathscr{P}_{<\aleph_1}(\lambda)$. But it seems a stronger assumption.

3.15. FACT.
(1) If $\mathbb{E}$ is nice to $\beth_{\sigma+2}(\aleph_1)^+$, *then* it is nice.
(2) We can add in 3.14:

(5) $\mathrm{rk}^2_E(f, \mathbb{E}) < \infty$ for every $f: \omega_1 \to \beth_{\sigma+2}(\aleph_1)^+$.

PROOF. As in [Sh 5].

## §4. Preservative pairs

4.1. CONVENTION. $\mathbb{E} \in OB_{\sigma+2}$ will be a nice collection for this section.

4.2. DEFINITION. (1) The pair $(H_1, H_2)$ is $\mathrm{rk}^l$-preserving (i.e. ${}^\sigma\mathrm{rk}^l$- preserving) if:
(a) for $m = 1, 2$ $H_m$ is a function from the ordinals into the ordinals, $\alpha \leqq H_m(\alpha)$ and $\alpha < \beta \Rightarrow H_m(\alpha) \leqq H_m(\beta)$ (we stipulate $H_m(\infty) = \infty$, $\alpha < \infty$);
(b) for every $f \in {}^{\aleph_1}\mathrm{Ord}$, $E \in \mathbb{E}$

$$\mathrm{rk}^l_E(H_1 \circ f) \leqq H_2(\mathrm{rk}^l_E(f));$$

(Note $H \circ f \in {}^{\aleph_1}\mathrm{Ord}$, $(H \circ f)(i) = H(f(i))$.)
(2) We say $H$ is $\mathrm{rk}^l$-preserving if $(H, H)$ is.
(3) We say $(H_1, H_2)$ is $\mathrm{rk}^l$-*preserving if we restrict (b) to the case $\bigwedge_{k=2,4} [l \in \{k, k+1\} \Rightarrow \mathrm{rk}^k_E(f) = \mathrm{rk}^{k+1}_E(f)]$; this is clearly a weaker condition.

REMARK. As we shall show, proving a pair is preservative, is a bound on some powers.

4.2A. CLAIM. (1) If $l=3$, $m=5$, *or* $l=5$, $m=3$, $H_2'$ is defined by $H_2'(\alpha)=(H_2(|\alpha|^+ + \beth_{\sigma+1}(\aleph_1))^+$, and $(H_1, H_2)$ is $\mathrm{rk}^l$-*preservative *then* $(H_1, H_2')$ is $\mathrm{rk}^m$-*preservative.

(2) If we replace $\beth_{\sigma+1}(\aleph_1)$ by $\beth_{\sigma+2}(\aleph_1)$ we can omit the "*".

REMARK. For our applications an improvement in (1) will be inessential.

PROOF. (1) Clearly $(H_1, H_2')$ satisfies condition (a) of 4.2(1) for being $\mathrm{rk}^m$-*preservative. As for condition (b), let $f \in {}^{\aleph_1}\mathrm{Ord}$, $\mathrm{rk}_E^m(f)=\mathrm{rk}_E^{m+1}(f)$, so:

(a) $|\mathrm{rk}_E^m(H_1 \circ f)| \leqq T_E(H_1 \circ f)+\beth_{\sigma+1}(\aleph_1)$ by 2.18, and

(b) $T_E(H_1 \circ f) \leqq |\mathrm{rk}_E^l(H_1 \circ f)| + \beth_{\sigma+1}(\aleph_1)$ by 2.21(2),

hence together

(c) $|\mathrm{rk}_E^m(H_1 \circ f)| \leqq |\mathrm{rk}_E^l(H_1 \circ f)| + \beth_{\sigma+1}(\aleph_1)$.

As $(H_1, H_2)$ is $\mathrm{rk}^l$-preservative

(d) $|\mathrm{rk}_E^l(H_1 \circ f)| \leqq H_2(\mathrm{rk}_E^l(f))$.

But similar to the proof of (c):

(e) $|\mathrm{rk}_E^l(f)| \leqq |r_E^m(f)| + \beth_{\sigma+1}(\aleph_1)$.

By (e) and monotonicity of $H_2$:

(*f*) $H_2(\mathrm{rk}_E^l(f)) \leqq H_2(|\mathrm{rk}_E^m(f)|^+ + \beth_{\sigma+1}(\aleph_1)^+)$.

But by the definition of $H_2'$:

(g) $H_2'(\mathrm{rk}_E^m(f)) = H_2(|\mathrm{rk}_E^m(f)|^+ + \beth_{\sigma+1}(\aleph_1))$.

So by (c), (d), (f) and (g) we get the conclusion (as $\beth_{\sigma+1}(\aleph_1) \leqq H_2'(\alpha)$ for every $\alpha$).

(2) Similar proof.

REMARK. So it usually doesn't matter whether we get a result for $\mathrm{rk}^3$ or $\mathrm{rk}^5$.

4.3. FACT. If $(H_1, H_2)$ satisfies (a) of 4.2, $l=3, 5$ and we are proving (b) of 4.2 by induction on $\alpha=\mathrm{rk}_E^l(f)$ (for all $f$ and $E$), we can assume

(i) $\mathrm{rk}_E^l(f)=\mathrm{rk}_E^{l-1}(f)$;

(ii) for some $l, l_1<3$, $A_l(f) \in \mathrm{Min}\, E$, $A_{l_1}(H_1 \circ f) \in \mathrm{Min}\, E$.

So without loss of generality $A_l(f)=\omega_1$, $A_{l_1}(H_1 \circ f)=\omega_1$.

PROOF. By 2.5(3), 2.8(3) for some $E_1 \subseteq E$, $\mathrm{rk}_E^l(f)=\mathrm{rk}_{E_1}^{l-1}(f)=\mathrm{rk}_{E_1}^l(f)$. So $H_2(\mathrm{rk}_E^l(f))=H_2(\mathrm{rk}_{E_1}^l(f))$ and $\mathrm{rk}_E^l(H_1 \circ f) \leqq \mathrm{rk}_{E_1}^l(H_1 \circ f)$ (by 2.5(2), 2.8(2)). So it is enough to prove that $\mathrm{rk}_{E_1}^l(H_1 \circ f) \leqq H_2(\mathrm{rk}_{E_1}^l(f))$, so (i) holds. For (ii) note that $\omega_1 = \bigcup_{l<3} A_l(f)$ and by 3.2(2) it is enough to prove for $l<3$ that if $A_l(f) \neq \emptyset$ mod $D$ *then* $\mathrm{rk}_{E_1[A_l(f)]}^l(H_1 \circ f) \leqq H_2'(\mathrm{rk}_{E_1[A_l(f)]}^l(f))$. So (ii) follows (the last phrase by 2.13(3)).

4.4. Lemma.

(1) *The function* $H = H_s = {}^{\sigma}H_s$ *defined by* $H_s(\alpha) = |\alpha|^+ + \beth_{\sigma+1}(\aleph_1)^+$ *(cardinal addition) is* $\mathrm{rk}^3$*-preserving.*

(2) *The function* $H = H'_s$ *defined by:* $H'_s(\alpha) = |\alpha|^+ + \beth_{\sigma+1}(\aleph_1)^+$ *is* $\mathrm{rk}^5$*-preserving.*

Proof. (1) In Definition 4.2 (a) is immediate, and we prove (b) by induction on $\alpha = \mathrm{rk}^3_E(f)$.

By 4.3 without loss of generality $\mathrm{rk}^2_E(f) = \mathrm{rk}^3_E(f)$ and for some $l, A_l(f) = \omega_1$.

If $\{i < \omega_1 : f(i) < (\beth_{\sigma+1}(\aleph_1)^+\} \neq \varnothing \mod \mathrm{fil}(E)$, it is enough to prove $\mathrm{rk}^3_E(\beth_{\sigma+1}(\aleph_1)^+) \leqq \beth_{\sigma+1}(\aleph_1)^+$, and for this it suffices to prove that for $f: \omega_1 \to \beth_{\sigma+1}(\aleph_1)^+$, $\mathrm{rk}^3_E(f) < \beth_{\sigma+1}(\aleph_1)^+$, which holds by 2.21(2), and cardinal arithmetic. So without loss of generality $f(i) \geqq (\beth_{\sigma+1}(\aleph_1)^+$ for every $i < \omega_1$. so clearly $\alpha \geqq (\beth_{\sigma+1}(\aleph_1)^+$. Assume that the desired conclusion fails.

Let $\mu \overset{\mathrm{def}}{=} |\alpha| + \beth_{\sigma+1}(\aleph_1) = |\alpha|$, $X = \mathrm{fil}\, E$. So $H(\mathrm{rk}^3_E(f)) = \mu^+$, $\mathrm{rk}^3_E(H \circ f) > \mu^+$. As the range of $H \circ f$ consists of limit ordinals, by 3.7 there are $g <_E H \circ f$ and $E_1 \subseteq E$ such that $\mathrm{rk}^2_{E_1}(g) = \mathrm{rk}^3_{E_1}(g) \geqq \mu^+$.

Clearly $(\forall i < \omega_1)[|g(i)| \leqq |f(i)|]$, hence $T_{E_1}(g) \leqq T_{E_1}(f)$. By 2.21(2)

$$|\mathrm{rk}^3_{E_1}(g)| \leqq T_{E_1}(g) + \beth_{\sigma+1}(\aleph_1) \leqq T_{E_1}(f) + \beth_{\sigma+1}(\aleph_1) \leqq T_E(f) + \beth_{\sigma+1}(\aleph_1)$$

$$= |\mathrm{rk}^3_E(f)| + \beth_{\sigma+1}(\aleph_1) = |\alpha| + \beth_{\sigma+1}(\aleph_1) < \mu^+$$

but $g$ was chosen such that $\mathrm{rk}^3_{E_1}(g) \geqq \mu^+$, contradiction.

(2) Same proof using 3.8 instead of 3.7.

4.5. Definition. Let $H$ be a function from the ordinals to the ordinals.

(1) $H^{\langle\alpha\rangle}$ is defined by induction on $\alpha$,

$$H^{\langle 0\rangle}(\xi) = \xi,$$

$$H^{\langle\alpha+1\rangle}(\xi) = H(H^{\langle\alpha\rangle}(\xi) + 1),$$

$$H^{\langle\alpha\rangle}(\xi) = \bigcup_{\beta<\alpha} H^{\langle\beta\rangle}(\xi) \qquad \text{for limit } \alpha;$$

(2) $H^*$ is defined by $H^*(\alpha) = H^{\langle\alpha\rangle}(0)$.

4.6. Fact. If $H$ satisfies 4.1(1)(a) then

(1) $\xi \leqq H^{\langle\alpha\rangle}(\xi) \leqq H^{\langle\alpha\rangle}(\zeta)$ for ordinals $\xi < \zeta$;

(2) $\xi \leqq H^*(\xi) < H^*(\zeta)$ for $\xi < \zeta$.

Proof. (1) Easy.

(2) $H^*(\xi) < H^*(\xi)+1 = H^{\langle\xi\rangle}(0)+1 \leqq H(H^{\langle\xi\rangle}(0)+1) = H^{\langle\xi+1\rangle}(0) = H^*(\xi+1) \leqq H^*(\zeta)$.

4.7. LEMMA. *If* $(H_1, H_2)$ *is* $\mathrm{rk}^l$*-preserving,* $l=3$ *then* $(H_1^*, H_2^*)$ *is* $\mathrm{rk}^l$*-preserving.*

REMARK. It does not matter so much that $l=5$ doesn't appear here because of 2.21, 4.2A.

PROOF. Part (a) of Definition 4.1 is easy (look carefully at $\alpha \leqq H_l^*(\alpha)$). Part (b) of Definition 4.1(1) we prove by induction on $\mathrm{rk}_E^3(f)$. By 4.3 without loss of generality $\mathrm{rk}_E^2(f) = \mathrm{rk}_E^3(f)$ and for some $m<3$, $A_m(f)=\omega_1$.

*Case* 1. $A_0(f)=\omega_1$.
So $\mathrm{rk}_E^3(f)=0$, $(H_1^*\circ f)(i) = H_1^*(0) = H_1^{\langle 0\rangle}(0) = 0$ so the assertion is $\mathrm{rk}_E^3(O_{\omega_1}) \leqq H_2^*(O_{\omega_1})$ which holds trivially.

*Case* 2. $A_1(f)=\omega_1$.
So for some $g\in {}^{\aleph_1}\mathrm{Ord}$, for every $i, f(i)=g(i)+1$. Now
(a) $\mathrm{rk}_E^3(H_1^*\circ f) = \mathrm{rk}_E^3(H_1\circ(H_1^*\circ g+1))$ [by Definition 4.5].
(b) $\mathrm{rk}_E^3(H_1\circ(H_1^*\circ g)) \leqq H_2(\mathrm{rk}_H^3(H_1^*\circ g+1))$ [by the assumption "$(H_1,H_2)$ is $\mathrm{rk}^l$-preservative"].
(c) $H_2(\mathrm{rk}_E^3(H_1^*\circ g+1)) \leqq H_2(H_2^*(\mathrm{rk}_E^3(g))+1)$ [as $g <_{\mathrm{fil}\,E} f$, by 2.11 $\mathrm{rk}_E^3(g)<\mathrm{rk}_E^3(f)$ hence by the induction hypothesis $\mathrm{rk}_E^3(H_1^*\circ g) \leqq H_2^*(\mathrm{rk}_E^3(g))$. By 3.6(1) $\mathrm{rk}_E^3(H_1^*\circ g+1) = \mathrm{rk}_E^3(H_1\circ g)+1$ so by the previous sentence $\mathrm{rk}_E^3(H_1^*\circ g+1) \leqq H_2^*(\mathrm{rk}_E^3(H_1^*\circ g))+1$; as $H_2$ is monotonically increasing we can get (c)].
(d) $H_2(H_2^*(\mathrm{rk}_E^3(g))+1) = H_2^*(\mathrm{rk}_E^3(g)+1)$ [by the definition of $H_2^*$ (i.e., 4.5)].
(e) $H_2^*(\mathrm{rk}_E^3(g)+1) \leqq H_2^*(\mathrm{rk}_E^3(f))$ [as $g<f$, by 2.11 $\mathrm{rk}_E^3(g)<\mathrm{rk}_E^3(f)$ hence $\mathrm{rk}_E^3(g)+1 \leqq \mathrm{rk}_E^3(f)$ apply $H_2^*$ is monotonic].
By (a)–(e) we finish.

*Case* 3. $A_2(f)=\omega_1$.
Let $K=\{H_1^*\circ g : g<_E f\}$. Easily (see 4.6(2)) for every $h\in K$, $h <_D H_1^*\circ f$. Also for every $h <_E H_1^*\circ f$ there is $g<_E f$ such that $h <_D H_1^*\circ g$ [see 4.5 and 4.6(2)]. Hence:
(a) $\mathrm{rk}_E^3(H_1^*\circ f) \leqq \mathrm{rk}_E^2(H_2^*\circ f)$ [by Definition 2.4].
(b) $\mathrm{rk}_E^2(H_1^*\circ f) = \sup\{\mathrm{rk}_{E[D]}^3(h) : h < H_1^*\circ f, D\in E\}$ [by the definition of $\mathrm{rk}_E^2$].
(c) $\sup\{\mathrm{rk}_{E[D]}^3(h) : h<H_1^*\circ g, D\in E$, for some $g<_D f\} = \sup\{\mathrm{rk}_{E[D]}^3(H_1^*\circ g)$: $g<_D f, D\in E\}$ [by what we say on $K$ above and as $\mathrm{rk}_{E[D]}^3$ is monotonic].

(d) $\sup\{\mathrm{rk}^3_{E_{[D]}}(H_1^* \circ g) : g <_D f,\ D \in E\} \leqq \sup\{H_2^*(\mathrm{rk}^3_{E_{[D]}}(g) : g <_D f,\ D \in E\}$ [apply the induction hypothesis to $g$ for each $g$, $E_{[D]}$ where $D \in E$, $g <_D f$; this is legitimate as by 2.11, $\mathrm{rk}^3_{E_{[D]}}(g) < \mathrm{rk}^3_{E_{[D]}}(f) \leqq \mathrm{rk}^2_E(f)$ and $\mathrm{rk}^3_{E_{[D]}}(f) = \mathrm{rk}^3_E(f)$ by 2.13 because we have assumed $\mathrm{rk}^3_E(f) = \mathrm{rk}^2_E(f)$].

(e) $\sup\{H_2^*(\mathrm{rk}^3_{E_{[D]}}(g)) : g <_D f,\ D \in E\} \leqq H_2^*(\sup\{\mathrm{rk}^3_{E_{[D]}}(g) : g <_D f,\ D \in E\})$ [because $H_2^*$ is monotonically increasing, see 4.6(2)].

(f) $H_2^*(\sup\{\mathrm{rk}^3_{E_{[D]}}(g) : g <_D f,\ D \in E\}) = H_2^*(\mathrm{rk}^2_E(f))$ [by definition of $\mathrm{rk}^2_E(f)$].

(g) $H_2^*(\mathrm{rk}^2_E(f)) = H_2^*(\mathrm{rk}^3_E(f))$ [as we are assuming (i) of 4.3].

By (a)–(g) we get the result.

4.8. Claim. If $(H_1^x, H_2^x)$ is $\mathrm{rk}^l$-preservative for $x = a, b$ where $l = 2, 3, 4, 5$ and $H_m = H_m^b \circ H_m^a$ for $m = 1, 2$ then $(H_1, H_2)$ is $\mathrm{rk}^l$-preservative.

Proof.

$$\mathrm{rk}^l_E(H_1 \circ f) = \mathrm{rk}^l_E((H_1^b \circ H_1^a) \circ f) = \mathrm{rk}^l_E(H_1^b \circ (H_1^a \circ f)) \leqq H_2^b(\mathrm{rk}^l_E(H_1^a \circ f))$$

$$\leqq H_2^b(H_2^a(\mathrm{rk}^l_E(f))) = (H_2^b \circ H_2^a)(\mathrm{rk}^l_E(f)) = H_2(\mathrm{rk}^l_E(f)).$$

4.9. Lemma. *Suppose $(H_1^m, H_2^m)$ is $\mathrm{rk}^l$-preservative for $m < \omega$, $l = 3$, and $H_n$ is defined by $H_n(\alpha) = \sup_{m<\omega} H_n^m(\alpha)$ then $(H_1, H_2)$ is $\mathrm{rk}^l$-preservative.*

Proof. Part (a) of Definition 4.1 is easy. Part (b) of Definition 4.1 we prove by induction on $\mathrm{rk}^l_E(f)$. By 4.3 we can assume $\mathrm{rk}^l_E(f) = \mathrm{rk}^{l-1}_E(f)$ and for some $m$, $A_m(f) = \omega_1$.

*Case* A. $A_0(f) = \omega_1$.
Easy.

*Case* B. $A_0(f) = \varnothing$, and for some $m < \omega_1$, $A = \{i < \omega_1 : (H_1^m \circ f)(i) = (H_1 \circ f)(i)\} \neq \varnothing$ mod fil $E$, then:

(a) $\mathrm{rk}^l_E(H_1 \circ f) \leqq \mathrm{rk}^l_{E[A]}(H_1 \circ f)$ [by monotonicity of $\mathrm{rk}^l$ in $E$].
(b) $\mathrm{rk}^l_{E[A]}(H_1 \circ f) = \mathrm{rk}^l_{E[A]}(H_1^m \circ f)$ [by choice of $A$];
(c) $\mathrm{rk}^l_{E[A]}(H_1^m \circ f) \leqq H_2^m(\mathrm{rk}^l_{E[A]}(f))$ [as $(H_1^m, H_2^m)$ is $\mathrm{rk}^l$-preservative];
(d) $H_2^m(\mathrm{rk}^l_{E[A]}(f)) \leqq H_2(\mathrm{rk}^l_{E[A]}(f))$ [by definition of $H_2$];
(e) $H_2(\mathrm{rk}^l_{E[A]}(f)) = H_2(\mathrm{rk}^l_E(f))$ [as $\mathrm{rk}^l_{E[A]}(f) = \mathrm{rk}^l_E(f)$ because $\mathrm{rk}^l_E(f) = \mathrm{rk}^{l-1}_E(f)$].
From these we get the conclusion.

*Case* C. $A_0(f) = \varnothing$ and for each $m$, $\{i : (H_1^m \circ f)(i) = (H_1 \circ f)(i)\} = \varnothing$ mod fil $E$.

Without loss of generality $(H_1^m \circ f)(i) < (H_1 \circ f)(i)$ for $m < \omega$, $i < \omega_1$. So for every $i$, $(H_1 \circ f)(i)$ is a limit ordinal. Note that $(\exists E_1 \subseteq E)[g <_{E_1} f] \Leftrightarrow (\exists A)[A \neq \emptyset \bmod \mathrm{fil}\, E$ and $g <_{\mathrm{fil}(E)+A} f]$.

Now if $g <_{E_{[D]}} H_1 \circ f$, $D \in E$, then necessarily for some $m = m(g, D) < \omega$

$$B = B_g = \{i < \omega_1 : g(i) < (H_1^m \circ f)(i)\} \neq \emptyset \bmod \mathrm{fil}(E)$$

hence $g <_B (H_1^m \circ f)$. Now under those circumstances

(a) $\mathrm{rk}^l_{E_{[D]}}(g) \leqq \mathrm{rk}^l_{(E[D])[B]}(H_1^m \circ f)$ (by 2.5(2)).

As $(H_1^m, H_2^m)$ is $\mathrm{rk}^l$-preservative

(b) $\mathrm{rk}^l_{(E[D])[B]}(H_1^m \circ f) \leqq H_2^m(\mathrm{rk}^l_{(E[D])[B]}(f))$.

By the definition of $H_2$

(c) $H_2^m(\mathrm{rk}^l_{(E[D])[B]}(f)) \leqq H_2(\mathrm{rk}^l_{(E[D])[B]}(f))$.

By our use of 4.3

(d) $H_2(\mathrm{rk}^l_{(E[D])[B]}(f)) \leqq H_2(\mathrm{rk}^l_E(f))$.

By (a)–(d) we finish as

$$\mathrm{rk}^3_E(H_1 \circ f) \leqq \mathrm{rk}^2_E(H_1 \circ f) = \sup\{\mathrm{rk}^3_{E_{[D]}}(g) : g <_{E[D]} H_1 \circ f, D \in E\}.$$

4.10. Conclusion. If $(H_1, H_2)$ is preservative, $\alpha < \omega_1$ *then* $(H_1^{\langle\alpha\rangle}, H_2^{\langle\alpha\rangle})$ is preservative.

Proof. By induction on $\alpha$.

$\alpha = 0$: trivial.

*$\alpha$ successor ordinal*: by 4.8.

*$\alpha$ limit*: by 4.9.

4.11. Remarks and Generalization.

(A)

(1) We can define when $(\bar{H}_1, H_2)$ is $\mathrm{rk}^l$-preserving where $\bar{H}_1 = \langle H_{1,\gamma} : \gamma < \omega_1\rangle$:

(a) $H_2$, $H_{1,\gamma}$ are functions from ordinals to ordinals, $\alpha \leqq H_{1,\gamma}(\alpha)$, $\alpha \leqq H_2(\alpha)$, and for $\alpha < \beta$, $H_{1,\gamma}(\alpha) \leqq H_{1,\gamma}(\beta)$, $H_2(\alpha) \leqq H_2(\beta)$;

(b) let for $f \in {}^{\aleph_1}\mathrm{Ord}$, $\bar{H}_1 \circ f$ be defined by $(\bar{H}_1 \circ f)(i) = H_{1,i}(f(i))$; then $\mathrm{rk}^l_E(\bar{H}_1 \circ f) \leqq H_2(\mathrm{rk}^l_E(f))$.

All the section generalizes easily, and in addition

(2) If $(\bar{H}_1^i, H_2^i)$ is $\mathrm{rk}^l$-preserving for $i < \omega_1$ and $\bar{H}_1^i = (H^i_{1,\gamma} : \gamma < \omega_1)$, $H_{1,\gamma}(\alpha) = \sup\{H^i_{1,\gamma}(\alpha) : i < 1 + \alpha\}$ and $H_2(\alpha) = \sup\{H_2^i(\alpha) : i < 1 + \alpha\}$ *then* $(\bar{H}_1, H_2)$ is $\mathrm{rk}^l$-preserving (see the proof of 4.9, use "Fodor" instead "$\aleph_1$-completeness").

(B)

(1) Let $\lambda \geqq \aleph_1$, $I \subseteq \{a : a \subseteq \lambda, \aleph_0 = |a \cap \omega_1|\}$.

We can replace the "normal filters on $\omega_1$" in the definition of $OB_1$ by filters over $I$ which are fine (i.e., for $\gamma<\lambda$, $\{t\in I:\gamma\in t\}\in D$) and normal (i.e., if $A_\gamma\in D$ for $\gamma<\lambda$ then $\{t\in I:\bigwedge_{\gamma\in t}t\in A_\gamma\}\in D$) (hence $\aleph_1$-complete). We can then use consistently $I$ instead of $\omega_1$. In (1) of (A) above we have $\bar{H}_1=\langle H_{1,t}:t\in I\rangle$, so (2) of (A) above becomes stronger. Using this we may need (C)(2) below.

(C)

(1) Of course if $H_1'\leqq H_1$ [i.e., $(\forall\alpha)H_1'(\alpha)\leqq H_1(\alpha)$] and $H_2\leqq H_2'$, and $H_1'H_2'$ satisfies (a) of 3.1 and $(H_1,H_2)$ is $\mathrm{rk}^l$-preservative *then* $(H_1',H_2')$ is $\mathrm{rk}^l$-preservative.

## §5. Conclusion

By 3.14, 1.6, (1.2) for our purpose we can assume

5.1. Hypothesis. $\mathbb{E}$ is a nice collection.

5.2. Theorem. *Suppose* $(H_1,H_2)$ *is* ${}^{\sigma}\mathrm{rk}^l$*-preservative for* $\mathbb{E}$, $l=3,5$ *and* $\mathrm{rk}^l_E(f,\mathbb{E})\geqq\beth_{\sigma+1}(\aleph_1)$ *(and* $\mathbb{E}$ *is nice).*

(1) $T^*_E(H_1\circ f,\mathbb{E})\leqq H_2(\mathrm{rk}^l_E(f,\mathbb{E}))$.

(2) *If* $\mathrm{cf}(\delta)=\aleph_1$, $(\forall\mu<\aleph_\delta)[\mu^{\aleph_0}<\aleph_\delta]$, $\aleph_\delta>\beth_{\sigma+1}(\aleph_1)$, $f\in{}^{\aleph_1}\mathrm{Ord}$ *is constant such that for every* $i<\omega_1$, $(H_1\circ f)(i)=\aleph_\delta$, then $\aleph_\delta^{\aleph_1}\leqq H_2(\mathrm{rk}^l_E(f))$.

(3) *If* $\mathrm{cf}(\delta)=\aleph_1$, $(\forall\mu<\aleph_\delta)[\mu^{\aleph_0}<\aleph_\delta]$, $\aleph_\delta>\beth_{\sigma+1}(\aleph_1)$, $f\in{}^{\aleph_1}\mathrm{Ord}$, $f(i)=\omega_1$, $\aleph_\delta=H_1(\omega_1)$, then $\aleph_\delta^{\aleph_1}\leqq H_2((\mathrm{rk}^l_E(f))\leqq H_2((\beth_{\sigma+1}(\aleph_1)^+)$ *(when* $H_2$ *is strictly increasing the last inequality is strict).*

(4) *If* $\mathrm{rk}^l_E(H_1\circ f,\mathbb{E})\geqq\beth_{\sigma+2}(\aleph_1)$ *then* $T_E(H_1\circ f,\mathbb{E}_2)\leqq H_2(\mathrm{rk}^l_E(f,\mathbb{E}))$.

Proof. Easy.

(1), (4) By 2.21 and Definition 4.2.

(2) By Galvin–Hajnal [GH] (see e.g., [Sh 5, 2.8]) $T_E(f)=\aleph_\delta^{\aleph_1}$ for $E\in\mathbb{E}$; now use (1).

(3) Use (2) and remember that, by 2.18, $\mathrm{rk}^l_E(f,\mathbb{E})<\beth_{\sigma+1}(\aleph_1)^+$.

5.3. Definition. Let $C_0=\{\lambda:\lambda$ an infinite cardinal$\}$, $C_{i+1}=\{\lambda\in C_i:C_i\cap\lambda$ has order type $\lambda\}$, $C_\delta=\bigcap_{i<\delta}C_i$.

5.4. Definition. (1) Let us define $\aleph^i_\alpha(\lambda)$ by induction on $i$:

*Case* (i). $\aleph^0_\alpha(\lambda)=\lambda^{+\alpha}$.

*Case* (ii). $\aleph^{i+1}_\alpha(\lambda)$ is defined by induction on $\alpha$:

$$\aleph_0^{i+1}(\lambda) = \lambda,$$

$$\aleph_{\alpha+1}^{i+1}(\lambda) = \aleph_\gamma^i(\aleph_0) \quad \text{where } \gamma = \aleph_\alpha^{i+1}(\lambda) + 1,$$

$$\aleph_\delta^{i+1}(\lambda) = \bigcup_{\alpha<\delta} \aleph_\alpha^{i+1}(\lambda).$$

*Case* (iii). $\aleph_\alpha^\xi(\lambda) = \bigcup_{\zeta<\xi} \aleph_\alpha^\zeta(\lambda)$ (for $\xi$ a limit ordinal).

(2) Let $\aleph_\alpha^i(\zeta) = \aleph_\alpha^i(|\zeta| + \aleph_0)$ for any ordinal $\zeta$.

5.5. Fact. (1) $\aleph_\alpha^i(\lambda)$ is a monotonically increasing function of $i$, $\alpha$, $\lambda$ (but not necessarily strictly).

(2) $\aleph_\alpha^i(\lambda) \geqq \lambda, \alpha, i$.

(3) $\aleph_\alpha^i(\lambda)$ is strictly increasing in $\alpha$ when $i$ is a successor.

(4) $\{\aleph_\delta^{i+1}(\lambda) : \delta \text{ a limit ordinal}\}$ is equal to $\{\mu : \aleph_\mu^i(\lambda) = \mu\}$ (i.e., a set of fixed points of $\aleph_x^i(\lambda)$ (as a function in $x$).

(5) For $\xi$ limit $\{\aleph_\delta^\xi(\lambda) : \delta \text{ an ordinal}\}$ is equal to $\bigcap_{i<\xi}\{\mu : \aleph_\mu^i(\lambda) = \mu\}$.

(6) For $i > 0$ $\{\aleph_\delta^i(\aleph_0) : \delta \text{ or } i \text{ is a limit ordinal}\}$ is equal to $C_i$.

(7) $\aleph_{\alpha+\beta}^i(\lambda) = \aleph_\beta^i(\aleph_\alpha^i(\lambda))$.

(8) If $H(\alpha) \overset{\text{def}}{=} \aleph_\alpha^i(\aleph_0)$ then $H^*(\alpha) = \aleph_\alpha^{i+1}(\aleph_0)$.

(9) $\aleph_\alpha^\zeta(\beth_{\sigma+1}(\aleph_1)) = {}^\sigma H_s^{\langle 1+\zeta\rangle}(\alpha)$ (see 4.4, 4.5).

5.6. Conclusion. (1) For $\zeta < \omega_1$, if $\lambda \overset{\text{def}}{=} \aleph_{\omega_1}^\zeta(\beth_2(\aleph_1))$, $(\forall \mu < \lambda)[\mu^{\aleph_0} < \lambda]$ then $(\aleph_{\omega_1}^\zeta(\beth_2(\aleph_1)))^{\aleph_1} < \aleph_{(\beth_2(\aleph_1))^+}^\zeta(\beth_2(\aleph_1))$.

(2) If $\zeta < \omega_1$, $\lambda$ is the $\omega_1$-th member of $C_\zeta$, $\lambda > \beth_2(\aleph_1)$, $(\forall \mu < \lambda)(\mu^{\aleph_0} < \lambda)$ *then* $\lambda^{\aleph_1}$ is smaller than the $(\beth_2(\aleph_1))^+$-th member of $C_\zeta$.

Proof. (1) Let $\sigma = 0$. Use 4.4, 4.10 and 5.5, 5.4(5).

(2) Use 5.6(1) and 5.5(6) (and definition of $C_\zeta$).

5.7. Lemma. *The function* $H = H^{ia}$ *is* ${}^\sigma\text{rk}^3$*-preservative, where*

$$H^{ia}(\alpha) \overset{\text{def}}{=} \text{Min}\{\lambda : \lambda \textit{ is weakly inaccessible}, \lambda > \beth_{\sigma+1}(\aleph_1), \lambda \geqq \alpha\}.$$

Proof. Part (a) of Definition 3.1 is easy. Suppose for $f$ and $D$ part (b) of Definition 3.1 fails, so

$$\text{rk}_E^3(H^{ia}, f) > \lambda \overset{\text{def}}{=} H^{ia}(\text{rk}_E^3(f)).$$

As in 4.3 w.l.o.g. $\text{rk}_E^3(f) = \text{rk}_E^2(f)$. So by 3.10 there are $g_\zeta \in {}^{\aleph_1}\text{Ord}$ for $\zeta \leqq \lambda$, $g_\lambda <_E H^{ia} \circ f$, $[\zeta < \xi \leqq \lambda \Rightarrow g_\zeta <_E g_\xi]$, and $\text{rk}_E^2(g_\zeta) = \text{rk}_E^3(g_\zeta) < \lambda$ for $\zeta < \lambda$.

As in [Sh 5, 5.x] we can prove that $A_i = \{i < \omega_1 : g_\lambda(i) \text{ is weakly inaccessible}\} \in \text{fil}\, E$.

Also $A_2 = \{i < \omega_1 : g_\lambda(i) < H^{ia}(f(i))\} \in \text{fil}\, E$, hence $A_0 \stackrel{\text{def}}{=} A_1 \cap A_2 \in \text{fil}\, E$ but for $i < \omega_1$, as $g_\lambda(i)$ is $< H^{ia}(f(i))$ and is weakly inaccessible it follows that $g_\lambda(i) < f(i)$ (see definition of $H^{ia}$!). So $g_\lambda <_{\text{fil}\, E} f$; so $\lambda = \text{rk}^3_E(g) < \text{rk}^3_E(f) \leqq H^{ia}(\text{rk}^3_E(f)) = \lambda$, contradiction.

5.8. Lemma. *$H^{\alpha-m}$ is ${}^\sigma\text{rk}^3$-preservative where*

$$H^{\alpha-m}(\alpha) = \text{Min}\{\lambda : \lambda \text{ is weakly } \alpha\text{-Mahlo},\ \lambda > \beth_{\sigma+1}(\aleph_1)^+ + |\alpha|\}$$

*when $\alpha < \omega_1$.*

Proof. E.g., like the proofs in [Sh 5, §7]; by 2.21 we can deal with $\text{rk}^5$-preservation, and using the ultrapower by a generic filter (chosen as in 3.10) we have no problem.

Remark. See [Sh 7] for more.

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