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Author(s): Saharon Shelah
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HANF NUMBER OF OMITTING TYPE FOR SIMPLE FIRST-ORDER THEORIES

SAHARON SHELAH[1]

Abstract. Let T be a complete countable first-order theory such that every ultrapower of a model of T is saturated. If T has a model omitting a type p in every cardinality $< \beth_\omega$, then T has a model omitting p in every cardinality. There is also a related theorem, and an example showing the $\beth_\omega$ cannot be improved.

Let L be a first-order countable language, T a complete theory (in L), p a type (in L), that is $p = \{\varphi_n(x): n\}$ where each $\varphi_n(x)$ is a formula from L whose only free variable is x. If M is an L-model, $a \in |M|$ (= the set of elements of M) then a realizes p if for every n, $M \models \varphi_n[a]$ (= the formula $\varphi_n[a]$ is satisfied by M). If there is no element of M which realizes p, then we say M omits p. We define $m(T, p)$ as the first cardinality in which T has no model omitting p, if there is such a cardinality, and zero otherwise. We shall prove

THEOREM 1. *Suppose that for every model M of T, and every (nonprincipal) ultrafilter D over a set I, M^I/D is a saturated model. Then $m(T, p) < \beth_\omega$.*

(The assumption follows from: T is unidimensional; it is equivalent to: T is stable, and if $\bar{a} \in |M|$, $\varphi(\bar{x}, \bar{y}, \bar{a})$ is an equivalence relation in M, then the number of equivalence classes is $\|M\|$ or finite (see [6])).

THEOREM 2. *If T is stable, countable and complete, $m(T, p) > \beth_\omega$, then in every cardinality $\aleph_\alpha$, T has $\geq \min(|\alpha|^{\aleph_0}, 2^{|\alpha|-1})$ nonisomorphic models.*

THEOREM 3. *If T is superstable, countable, complete and $m(T, p) \geq \beth_\omega$, then there are formulas $\varphi_n(x, \bar{y})$ such that for every $\aleph_0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \cdots$ there is a model M of T and sequences $\bar{a}^n$ of elements of M such that $|\{c \in |M|: \models \varphi_n[c, \bar{a}^n]\}| = \lambda_n$ for every $n < \omega$.*

REMARKS. (1) We shall only sketch the proofs of Theorems 2, 3 as they are similar to that of Theorem 1.

(2) Theorem 2 is an improved version of Theorem 3 of the abstract [5], see also [7, §0].

(3) If T is not superstable, it can also be proved that it has many nonisomorphic models in every cardinality. See [6, Chapter VIII, §2].

(4) Saturated models are defined and investigated in Morley and Vaught [4] (or see [2]) where they are called universal-homogeneous models. In fact we need only the universality in Theorem 1. A model M is called universal if every model elementarily equivalent to M, of cardinality not greater than that of M, is isomorphic to an elementary submodel of M.

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(5) By Morley [3], $m(T, p) < \beth_{\omega_1}$ for any countable T. We shall generalize his proof, but use a smaller set of Skolem functions.

(6) We can replace the omitting of one type by the omitting of countably many types. Also each type may be an m-type for some $m < \omega$.

(7) The cardinality $\beth_\alpha$ is defined by induction: $\beth_0 = \aleph_0$, $\beth_\alpha = \sum_{\beta<\alpha} 2^{\beth_\beta}$.

Notations. Cardinals (usually infinite) shall be denoted by λ, μ, ordinals by i, j, k, l, α, β, ξ, ζ, natural numbers by m, n, q, r, and formulas by φ, ψ.

PROOF OF THEOREM 1. For every $\lambda \geq 0$ we define a first-order language L_λ whose nonlogical signs will be:

(1) the nonlogical signs of L,

(2) an individual constant c_k for every $k < \lambda$,

(3) for every formula $\varphi = \varphi(y_0, y_1, \ldots, y_n)$ of L, a function-symbol $f_\varphi = f_\varphi(y_1, \ldots, y_n)$.

(REMARK. In fact, this is defined not for every formula but for every pair $\langle\varphi, \langle y_0, \ldots, y_n\rangle\rangle$ where every free variable of φ belongs to $y_0, \ldots, y_n$. But we shall not distinguish strictly between φ and $\langle\varphi, \langle y_1, \ldots, y_n\rangle\rangle$, similarly for terms.)

The interpretation of those functions is, of course, Skolem functions. Let $\{\tau^m(y_1^m, y_2^m, \ldots, y_{k(m)}^m): m < \omega\}$ be a list of the terms of L_0 such that $\tau^m = f(\tau_1, \ldots, \tau_q)$ implies $\tau_1, \ldots, \tau_q \in \{\tau^n: n < m\}$. Let $k_1(n) = \max_{m \leq n} k(m) < \omega$. Clearly for every m, and every permutation θ of $\{1, \ldots, k(m)\}$, there is n, such that

$$\tau^m(y_{\theta(1)}^m, \ldots, y_{\theta(k(m))}^m) = \tau^n(y_1^n, \ldots, y_{k(n)}^n).$$

It is clear that every constant term in L_λ is equal to $\tau^m(c_{l_1}, \ldots, c_{l_{k(m)}})$ for some m and $l_1 < \cdots < l_{k(m)} < \lambda$. Let B_λ be the set of constant terms of L_λ, and $B(\lambda, n)$ be those of the form $\tau^n(c_{l_1}, \ldots)$, where $l_1 < l_2 < \cdots$.

A function F from the set B_λ to a model M of T is l-proper ($l \leq \omega$) if

(1) if $k < j < \lambda$ then $F(c_k) \neq F(c_j)$;

(2) if $\tau \in B(\lambda, n)$, $n < l$, $\tau = f_\varphi(\tau_1, \ldots, \tau_m)$ then

$$M \models (\exists y_0)\, \varphi\, [y_0, F(\tau_1), \ldots, F(\tau_m)] \rightarrow \varphi[F(\tau), F(\tau_1), \ldots, F(\tau_m)].$$

F is strongly l-proper if in addition

(3) if $\tau \in B(\lambda, n)$, $n < l$, $i_1 < \cdots < i_{k(n)} < \lambda$, $j_1 < \cdots < j_{k(n)} < \lambda$ and $m < \omega$ then

$$M \models \varphi_m[F(\tau_n(c_{i_1}, \ldots, c_{i_{k(n)}}))] \equiv \varphi_m[F(\tau_n(c_{j_1}, \ldots, c_{j_{k(n)}}))].$$

Let $p = \{\varphi_m(x): m < \omega\}$.

Now let s denote a sequence of length $\leq \omega$ of natural numbers. Its length will be $l(s)$, and its nth element $s_n = s(n)$. The sequence of its first n-element ($n \leq l(s)$) will be $s|n$.

A function F from B_λ to a model M of T is s-proper if

(A) It is $l(s)$-proper,

(B) for every $i_1 < \cdots < \lambda$, and $m < l(s)$, $M \models \neg\, \varphi_{s(m)}\, [F(\tau^m(c_{i_1}, \ldots))]$.

LEMMA 4. *If $l(s) = \omega$, and for every n there is an $(s|n)$-proper function F_n from $B_{\aleph_0}$ into a model M_n of T, then $m(T, p) = 0$.*

PROOF OF LEMMA 4. By the compactness theorem, there is a model M of T and an s-proper function F from $B_{\aleph_0}$ into M. By a second use of the compactness theorem, there is a model M_1 and an s-proper function F_1 from B_λ into M_1 where λ is any cardinal. Let A be the range of F_1, and N a submodel of M_1 whose set of elements is A. Clearly $|A| \leq \lambda$ as $|B_\lambda| = \lambda$. On the other hand by condition (1) in the definition of l-proper function, $k < j < \lambda$ implies $F_1(c_k) \neq F_1(c_j)$, hence $|A| \geq | \{c_k: k < \lambda\}| = \lambda$. So the cardinality of N is λ. By the second condition in the definition of an l-proper function, and by Tarski-Vaught test (see e.g. [2]), N is an elementary submodel of M_1 and so also a model of T. By condition (B) of the definition of s-proper function, N omits p. So we clearly prove the lemma.

LEMMA 5. *Suppose $m(T, p) \geq \beth_\omega$ and for every model M of T, M^I/D is saturated. Suppose $l(s) = n$ and there is an s-proper function F from $B_{\aleph_0}$ into a model M of T. Let N be a model of T of cardinality $\geq \lambda = \beth_{k_1(n)+1}$ which omits p. Then we can lengthen s to a sequence s' of length $n + 1$ such that there is an s'-proper function from $L_{\aleph_0}$ into N.*

PROOF OF LEMMA 5. Let D be a (nonprincipal) ultrafilter on ω (on ultrafilters and ultraproducts e.g. [2]). Let $N_1 = N^\omega/D$. By our assumptions N_1 is saturated. By the compactness theorem, there is a model M_1 of T such that there is an s-proper function F from B_λ to M_1. By the downward Löwenheim-Skolem theorem we can assume the cardinality of M_1 is λ, and hence it is isomorphic to an elementary submodel of N_1. So there exists an s-proper function F' from B_λ into N_1. By changing F' a little, we can assume in addition that for every $\tau = f_\varphi(\tau_1, \ldots, \tau_n)$ where $\varphi = \varphi(x, y, \ldots, y)$ it holds that $N_1 \models (\exists x)\varphi(x, \tau_1, \ldots, \tau_n) \rightarrow \varphi(\tau, \tau_1, \ldots, \tau_n)$. Let $F'(\tau) = \langle a_\tau^0, a_\tau^1, \ldots a_\tau^m, \ldots\rangle_{m<\omega}/D$.

We define the function F_m from L_λ into N by $F_m(\tau) = a_\tau^m$.

Let us define an equivalence relation E among the increasing sequences of length $k_1(n)$ of ordinals $< \lambda$: $\langle i_1, \ldots, i_{k_1(n)}\rangle E \langle j_1, \ldots, j_{k_1(n)}\rangle$ if

(I) for every $r < \omega$, $F_r(c_{i_1}) = F_r(c_{i_2})$ iff $F_r(c_{j_1}) = F_r(c_{j_2})$,

(II) for every $m < n + 1, q < \omega$ and $r < \omega$,

$$N \models \varphi_q[F_r(\tau^m(c_{i_1}, \ldots, c_{i_{k(m)}}))] \Leftrightarrow N \models \varphi_q[F_r(\tau^m(c_{j_1}, \ldots, c_{j_{k(m)}}))],$$

(III) if $r < \omega$, $m < n + 1$, $\tau = \tau^m(c_{i_1}, \ldots, c_{i_{k(m)}})$, $\tau = f_\varphi(\tau_1, \ldots, \tau_p)$, and w.l.o.g. $\tau_q = \tau_q(c_{i_1}, \ldots)$, and $\tau^* = \tau^m(c_{j_1}, \ldots)$, $\tau_q^* = \tau_q(c_{j_1}, \ldots)$, then

$$N \models (\exists y)\varphi[y, F_r(\tau_1), \ldots] \rightarrow \varphi[F_r(\tau), F_r(\tau_1), \ldots] \Leftrightarrow N \models (\exists y)\,\varphi\,[y, F_r(\tau_1^*), \ldots] \rightarrow \varphi[F_r(\tau^*), \ldots].$$

It is easy to see the number of equivalence classes is $\leq 2^{\aleph_0}$. By Erdös, Hajnal and Rado [1] there is an infinite subset I of λ, such that any two increasing sequences of ordinals from I, of length $k_1(n)$ are E-equivalent. By renaming the c_i's we can assume the natural numbers belong to I. Let F_r' be F_r restricted to $B_{\aleph_0}$. Now by Los' theorem for every $m < n$

$$\{r: r < \omega, N \models \neg\, \varphi_{s(m)}[F_r(\tau^m(c_1, \ldots, c_{k(m)}))]\} \in D.$$

This implies there are r's for which F_r' satisfies condition (B) for being s-proper. By looking on the one hand at conditions (1), (2), for F to be strongly $(n + 1)$-proper, and conditions (A) and (B) for being s-proper, and on the other hand conditions (I), (II), (III) in the definition of E, it is easy to see the set of r's for which

F'_r is $(n+1)$-proper and s-proper, belongs to D. Let r_0 be such r. As F'_r is a function into N which omits p, there is m_0 such that

$$N \models \neg \varphi_{m_0}[F_{r_0}(\tau^n(c_1, \ldots, c_{k(n)}))].$$

Define $s' = \langle s_0, s_1, \ldots, s_{n-1}, m_0 \rangle$. Clearly F_{r_0} is an s'-proper function from $L_{\aleph_0}$ into N. So we prove Lemma 5.

CONTINUATION OF THE PROOF OF THEOREM 1. We define by induction on n: s_{n-1}, F_n, M_n such that: M_n is a model of T, F_n is an $\langle s_m : m < n \rangle$-proper function from $B_{\aleph_0}$ into M_n.

For $n = 0$, M_0 will be any infinite model of T, and F_0 any function from $B_{\aleph_0}$ into M_0, such that $r \neq q$ implies $F_0(c_r) \neq F_0(c_q)$.

If M_n, F_n and $\langle s_m : m < n \rangle$ are defined, by Lemma 5 we can define M_{n+1}, F_{n+1} and s_n such that M_{n+1} is a model of T, and F_{n+1} is an $\langle s_m : m < n+1 \rangle$-proper function from $B_{\aleph_0}$ into M_{n+1}. By Lemma 4, Theorem 1 follows.

SKETCH OF THE PROOF OF THEOREM 2. For simplicity, we shall sketch only a proof of:

(*) if $m(T, p) > \beth_\omega$, T is countable, complete and superstable, then $\lambda \geq \aleph_0$ where λ is from [6, V5.8]. The conclusion follows by [6, V5.8].

Assume T and p do not satisfy (*).

Let M be a model of T of cardinality $\geq \beth_\omega$ which omits p. Clearly $N = M^\omega/D$ is $\aleph_1$-saturated (where D is any nonprincipal ultrafilter over ω). Hence, by Shelah [6, V, §5; IV, §4], there are in N sets A, and $\{a_i^\xi : i < \alpha_\xi\}$ for $\xi < \xi_0 < \aleph_0$ such that:

(1) $|A| \leq 2^{\aleph_0}$,

(2) for each $\xi < \xi_0$, $\{a_i^\zeta : i < \alpha_\zeta\}$ is an indiscernible set over $A \cup \{a_i^\xi : i < \alpha_\xi, \xi < \xi_0, \xi \neq \zeta\}$,

(3) $\{a_i^\zeta : i < \alpha_\zeta\}$ is a maximal indiscernible set over $A \cup \{a_i^\xi : i < \alpha_\xi, \xi < \zeta\}$,

(4) N is *the* $\aleph_1$-primary model over $A \cup \{a_i^\xi : i < \alpha_\xi, \xi < \xi_0\}$.

The last demand means that the set of elements of N is $\{b_i : i < \beta_0\}$ and, for every $i < \beta_0$, b_i realizes an $\aleph_1$-isolated type over $B_i = A \cup \{a_j^\xi : j < \alpha_\xi, \xi < \xi_0\} \cup \{b_j : j < i\}$. That means that there is $B^i \subset B_i$, $|B^i| \leq \aleph_0$, such that if $\varphi(x, \bar{y})$ is a formula, $\bar{b}$ a sequence from B_i, *then* there is a formula $\psi(x, \bar{y})$ and sequence $\bar{c}$ from B^i, such that $N \models \varphi[b_i, \bar{b}]$ implies $N \models \psi[b_i, \bar{c}] \wedge (\forall x)[\psi(x, \bar{c}) \rightarrow \varphi(x, \bar{b})]$.

Now let n_1 be the first such that $\alpha_\xi < \beth_\omega$ implies $\alpha_\xi < \beth_{n_1}$ (exists as we assume $\lambda \leq \aleph_0$).

Now examining carefully the proof of Theorem 1, it clearly suffices to show: for every sequence s, $l(s) < \omega$, that:

(A) If there is an s-proper function from B_λ to M, then there is an s-proper function from B_λ to N.

(B) If there is an s-proper function F into N from B_λ, $\lambda > \beth_{n_1+2k_1(l(s))+2}$, and $\mu < \beth_\omega$, then there is an s-proper function from B_μ into N.

(C) If there is an s-proper function from B_λ into N, $\lambda \geq \beth_{n+k_1(l(s))+1}$, then there is an s-proper, strongly $(l(s)+1)$-proper function from $B_{\beth_n}$ into M (for any $n < \omega$). By (A), (B), (C) we can easily define s_m by induction, such that for every $n < \omega$, there is an $\langle s_m : m < n \rangle$-proper function from $B_{\beth_{n_1+2k_1(n)+1}}$ into M. Now (A) is self-evident, and (C) follows from the proof of Lemma 5.

Let us prove (B). In the description of N as an $\aleph_1$-primary model in (4) above, we can assume $b_j \in B_i \Rightarrow B^j \subseteq B^i$ and $b_i \notin B_i$. We expand N by the following relation, functions and constants:

(i) an individual constant a for each $a \in A \cup \{a_j^\xi : j < \alpha_\xi, \xi < \xi_0 \text{ and } \alpha_\xi < \beth_\omega\}$,

(ii) a one-place relation $P_\xi = \{a_j^\xi : j < \alpha_\xi\}$ for each $\xi < \xi_0$ and a one-place relation $Q = \{b_i : i < \beta_0\}$, $Q_0 = A$,

(iii) a two-place relation $< \; = \{\langle b_j, b_i\rangle : j < i < \beta_0\}$,

(iv) one-place function symbols G_n, such that $B_i = \{G_n(b_i) : n < \omega\}$,

(v) the functions f_φ defined naturally (i.e. to "commute" with F).

We get a model N^*. By [1], the sequence $\langle F(c_i) : i < \lambda\rangle$ has a subsequence of length ω_1, all of whose increasing subsequences of length $\leq 2k_1(l(s)) + 1$ realizes the same type in N^* (notice w.l.o.g. $k(n) \leq 1$). So we assume $\langle F(c_i) : i < \omega_1\rangle$ is such a subsequence and now we lengthen it naturally.

PROOF OF THEOREM 3. Left to the reader who should use [6, IX, 2.3].

LEMMA 6. *In Theorem* 1, *we cannot replace* $\beth_\omega$ *by a smaller cardinal, even if* T *is also superstable.*

PROOF. On $V = \{\eta : \eta$ a sequence of ones and zeros of length $\omega\}$ we define additions $\eta_1 + \eta_2 = \eta_3$ iff for each n, $\eta_1(n) + \eta_2(n) = \eta_3(n)$ mod 2. Note that V is an abelian group, and in it $\eta = -\eta$ so subtraction is addition. Let k be a fixed natural number. Let us define a model M:

$|M| = V \cup V^{k+1}$,

$P = V$ (a one-place relation),

$P_n = \{\eta \in V : \eta(n) = 0\} \cup \{\langle \eta_0, \ldots, \eta_k\rangle : \eta_k(n) = 0\}$,

0 the sequence of zeros of length ω,

$+$ a two-place function: on V as above,

$$\langle \eta_0, \ldots, \eta_k\rangle + \eta = \eta + \langle \eta_0, \ldots, \eta_k\rangle = \langle \eta_0, \ldots, \eta_{k-1}, \eta_k + \eta\rangle$$

and 0 otherwise.

F_l a one-place function, for $l < k$, $F_l(\eta) = \eta$, $F_l(\langle \eta_0, \ldots, \eta_k\rangle) = \eta_l$,

E an equivalence relation, $\langle \eta_0, \ldots, \eta_k\rangle E \langle \eta^0, \ldots, \eta^k\rangle$ iff $\eta_l = \eta^l$ for each $l < k$.

Now $T = \mathrm{Th}(M)$ is countable, complete, has elimination of quantifiers and is superstable and unidimensional. Let us define a type

$$p = \{P(x_l), F_m(y_l) = x_{l+i(m,l)}, \neg P(y_l) : l \leq k, m < k, i(m, l) \text{ is } 0 \text{ if } m < l \text{ and } 1 \text{ otherwise}\} \cup \{P_n(y_l) \equiv P_n(y_{l+1}) : n < \omega, l < k\} \cup \{\textstyle\bigwedge_{l<m\leq k} x_l \neq x_m\}.$$

Now T has a model of cardinality λ omitting p iff $\lambda \nrightarrow (k+1)^k_{2^{\aleph_0}}$, iff $\lambda \leq \beth_{k+1}$.

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HEBREW UNIVERSITY
JERUSALEM, ISRAEL