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An improper arithmetically closed Borel subalgebra of $\mathcal{P}(\omega)$ mod FIN ☆

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ABSTRACT

We show the existence of a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ that satisfies the following three conditions:

- $\mathcal{A}$ is Borel (when $\mathcal{P}(\omega)$ is identified with 2^ω).
- $\mathcal{A}$ is arithmetically closed (i.e., $\mathcal{A}$ is closed under the Turing jump, and Turing reducibility).
- The forcing notion $(\mathcal{A}, \subseteq)$ modulo the ideal FIN of finite sets collapses the continuum to $\aleph_0$.

1. Introduction

For a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ let $\mathbb{P}_\mathcal{A}$ be the partial order obtained by reducing $(\mathcal{A}, \subseteq)$ modulo the ideal FIN of finite sets. Gitman [5] made an advance towards the *Scott set problem* by showing that, assuming the proper forcing axiom (PFA), if $\mathcal{A}$ is arithmetically closed and $\mathbb{P}_\mathcal{A}$ is a proper notion of forcing, then there is a model of Peano arithmetic whose standard system is $\mathcal{A}$.[1] Gitman [6] also investigated proper posets of the form $\mathbb{P}_\mathcal{A}$ and showed that the existence of proper uncountable arithmetically closed algebras $\mathcal{A} \neq \mathcal{P}(\omega)$ is consistent with ZFC. These results naturally motivate the question whether there is an arithmetically closed $\mathcal{A}$ for which $\mathbb{P}_\mathcal{A}$ is not proper. This question was answered in the affirmative by Enayat [3, Theorem D], using a highly nonconstructive reasoning that establishes the existence of an arithmetically closed $\mathcal{A}$ of power $\aleph_1$ such that $\mathbb{P}_\mathcal{A}$ collapses $\aleph_1$ (and is therefore not proper). The nonconstructive feature of the proof prompted

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[1] The Scott set problem [14, Question 1] asks whether every Scott set $\mathcal{A}$ can be realized as the standard system of a model of Peano arithmetic (a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a Scott set if $\mathcal{A}$ is closed under Turing reducibility, and every infinite subtree of ${}^{<\omega}2$ that is coded in $\mathcal{A}$ has an infinite branch that is also coded in $\mathcal{A}$). It is known that the answer to the Scott set problem is positive when $|\mathcal{A}| \leqslant \aleph_1$, and when $\mathcal{A} = \mathcal{P}(\omega)$. On the other hand, a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is arithmetically closed if $\mathcal{A}$ is closed under (1) Turing jump and (2) Turing reducibility. Note that if $\mathcal{A}$ is arithmetically closed, then $\mathcal{A}$ is a Scott set, but not vice versa.

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Question II(b) of [3], which asked whether $\mathbb{P}_{\mathcal{A}}$ is a proper poset if $\mathcal{A}$ is both arithmetically closed and *Borel* (when $\mathcal{P}(\omega)$ is identified with the Cantor set).[2]

The main result of this paper, Theorem A below, provides a strong negative answer to the above question.

Theorem A. *There is an arithmetically closed Borel subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathbb{P}_{\mathcal{A}}$ is equivalent to* $\text{Levy}(\aleph_0, 2^{\aleph_0})$.

Theorem A is established in Section 3 using a rich toolkit from set theory and model theory. For this reason, Section 2 is devoted to the description of the machinery employed in the proof of Theorem A.

Dedication. We are honored to present this paper in a special issue that celebrates Ken Kunen's far reaching achievements.

2. Preliminaries

2.1. Forcing

Given an infinite cardinal κ, $\text{Levy}(\aleph_0, \kappa)$ is the usual partial order that collapses κ to $\aleph_0$, i.e., $\text{Levy}(\aleph_0, \kappa) = ({}^{<\omega}\kappa, \subseteq)$. The following result provides a structural characterization of $\text{Levy}(\aleph_0, \kappa)$.[3]

2.1.1. Theorem *(McAloon). ([13, Theorem 14.17]) The following conditions are equivalent for a partial order $\mathbb{P}$ of infinite cardinality κ.*

(a) *$\mathbb{P}$ is equivalent*[4] *to* $\text{Levy}(\aleph_0, \kappa)$.
(b) *$\mathbb{P}$ is $(\aleph_0, \kappa)$-nowhere distributive, i.e., there is a family $\{I_n \colon n \in \omega\}$ of maximal antichains of $\mathbb{P}$ such that for every $p \in \mathbb{P}$, there is some $n < \omega$ such that there are κ elements of I_n that are compatible with p.*

2.1.2. Corollary. *([11, Lemma 26.7]) The following conditions are equivalent for a partial order $\mathbb{P}$ of cardinality $\kappa \geqslant \aleph_0$.*

(a) *$\mathbb{P}$ is equivalent to* $\text{Levy}(\aleph_0, \kappa)$.
(b) $\mathbb{V}^{\mathbb{P}} \models$ *"there is a surjection $f : \omega \to \kappa$".*

The next result shows that one can use standard techniques to build a subalgebra $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\mathbb{P}_{\mathcal{A}}$ is not proper.

2.1.3. Proposition.[5] *There is a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ of cardinality $\aleph_1$ such that $\mathbb{P}_{\mathcal{A}}$ is equivalent to* $\text{Levy}(\aleph_0, \aleph_1)$.

Proof. By a theorem of Parovičenko [18] (see also [13, Sections 5.28 and 5.29]) every Boolean algebra of cardinality $\leqslant \aleph_1$ can be embedded into $\mathcal{P}(\omega)$ mod FIN. On the other hand, $\text{Levy}(\aleph_0, \aleph_1)$ can be densely embedded into a Boolean algebra of power $\aleph_1$ since each s in $\text{Levy}(\aleph_0, \aleph_1)$ determines a basic clopen X_s set in ${}^{\omega}\omega_1$, and the Boolean algebra $\mathbb{B}$ of clopen sets generated by the family $\{X_s \colon s \in \text{Levy}(\aleph_0, \aleph_1)\}$ is of size $\aleph_1$. So by Parovičenko's theorem there is an embedding f of $\mathbb{B}$ into $\mathcal{P}(\omega)$ mod FIN. Let $\mathcal{A} := \{X \subseteq \omega \colon [X] \text{ is in the range of } f\}$. We may conclude that $\mathbb{P}_{\mathcal{A}}$ is equivalent to $\text{Levy}(\aleph_0, \aleph_1)$ since $\text{Levy}(\aleph_0, \aleph_1)$ densely embeds into $\mathbb{P}_{\mathcal{A}}$. □

2.1.4. Remark.[6] Zapletal [30, Lemma 2.3.1] used Woodin's Σ^2_1-absoluteness theorem [15, Theorem 3.2.1] to show that in the presence of the continuum hypothesis and large cardinals (more precisely: a measurable Woodin cardinal), a projective partial order $\mathbb{P}$ preserves $\aleph_1$ iff $\mathbb{P}$ is proper. Note that if $\mathcal{A}$ is Borel, then $\mathbb{P}_{\mathcal{A}}$ is projective.

2.2. Infinite combinatorics

2.2.1. Definition. Let $\mathcal{A} \subseteq [\omega]^{\omega} := \{X \subseteq \omega \colon X \text{ is infinite}\}$.

(a) $\mathcal{A}$ is *almost disjoint* (AD) if the intersection of any two distinct members of $\mathcal{A}$ is finite.
(b) An AD family $\mathcal{A}$ is *maximal almost disjoint* (MAD) if $\mathcal{A}$ has no proper extension to another AD family.

[2] Many of the other questions posed in [3] have by now been answered by Shelah; see [24] and [25].
[3] See [1, Corollary 1.15] for a generalization of Theorem 2.1.1. that characterizes other collapsing algebras.
[4] For partial orders $\mathbb{P}_1$ and $\mathbb{P}_2$, $\mathbb{P}_1$ is *equivalent* to $\mathbb{P}_2$ if they yield the same generic extensions. This can be recast algebraically as the existence of an isomorphism between $B(\overline{\mathbb{P}_1})$ and $B(\overline{\mathbb{P}_2})$, where $\overline{\mathbb{P}}$ is the separative quotient of $\mathbb{P}$, and $B(\overline{\mathbb{P}})$ is the complete Boolean algebra consisting of regular cuts of $\overline{\mathbb{P}}$ [11, Theorem 14.10].
[5] Thanks to K.P. Hart and Ken Kunen for (independently) drawing our attention to this consequence of Parovičenko's theorem.
[6] We owe this remark to Paul Larson.

(c) A MAD family $\mathcal{A}$ is *completely separable*[7] if for every set $B \in [\omega]^\omega$ either there is some $A \in \mathcal{A}$ such that $A \subseteq B$ or B is a subset of the union of a finite subfamily of $\mathcal{A}$.

Note that if $\mathcal{A}$ is a finite partition of ω, then $\mathcal{A}$ is completely separable. A routine diagonal argument, on the other hand, shows that any *infinite* MAD family $\mathcal{A} \subseteq [\omega]^\omega$ must be uncountable; and indeed it is consistent with ZFC for a MAD family to have cardinality $\aleph_1$ and $2^{\aleph_0}$ to be arbitrarily large (e.g., by adding enough Cohen reals to a model of CH). However, if $\mathcal{A}$ is an infinite completely separable MAD family, then the cardinality of $\mathcal{A}$ must be $2^{\aleph_0}$. This follows from the well-known fact that if $\mathcal{A}$ is completely separable, and $B \subseteq \omega$ is not a subset of the union of a finite subfamily of $\mathcal{A}$, then $\{A \in \mathcal{A}\colon\ A \subseteq B\}$ has cardinality continuum. *In particular, if $\mathcal{A}$ is an infinite completely separable MAD family and $B \subseteq \omega$, then* $\{A \in \mathcal{A}\colon\ A \cap B$ is infinite$\}$ is either finite or has cardinality continuum.

Hechler [9, Theorem 8.2, Lemma 9.2] showed that Martin's axiom (MA) implies the existence of a completely separable family. A similar proof yields the following result.

2.2.2. Theorem. *The following statement* (#) *is provable within* ZFC + MA.

(#) *For every increasing sequence $\bar{n} = \langle n_i\colon\ i < \omega\rangle$ with $\lim_{i\in\omega}(n_{i+1} - n_i) = \infty$ there is a MAD family $\mathcal{A} = \mathcal{A}_{\bar{n}}$ that satisfies the following two conditions*:
(1) $\mathcal{A} \subseteq \{u \subseteq \omega\colon\ \forall i \in \omega |u \cap [n_i, n_{i+1})| = 1\}$, *and*
(2) *if $B \subseteq \omega$, then $\{A \in \mathcal{A}\colon\ A \cap B$ is infinite$\}$ is either finite or has cardinality $2^{\aleph_0}$.*

2.3. Tree indiscernibles

2.3.1. Definition. Suppose $\mathcal{M}$ is a model with signature $\tau_{\mathcal{M}}$. An indexed family $\{a_\eta\colon\ \eta \in {}^\omega 2\}$ of pairwise distinct elements of $\mathcal{M}$ is said to be a family of *tree indiscernibles in* $\mathcal{M}$ if for every $\varphi(x_0, \ldots, x_{m-1}) \in \mathsf{L}_{\omega,\omega}(\tau_{\mathcal{M}})$, there is some $n_\varphi < \omega$, such that for all natural numbers $n > n_\varphi$ and all infinite sequences $\eta_0, \ldots, \eta_{m-1} \in {}^\omega 2$, $\nu_0, \ldots, \nu_{m-1} \in {}^\omega 2$ the following implication is true

$$\Big(\bigwedge_{i<m} \eta_i \restriction n = \nu_i \restriction n\Big) \wedge \Big(\bigwedge_{i<j<m} \eta_i \restriction n \neq \eta_j \restriction n\Big)$$
$$\Rightarrow \quad \mathcal{M} \models \varphi[a_{\eta_0}, \ldots, a_{\eta_{m-1}}] \leftrightarrow \varphi[a_{\nu_0}, \ldots, a_{\nu_{m-1}}].$$

Tree indiscernibles were invented by Shelah [20,21] to prove certain 2-cardinal theorems, including $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$.[8] More recently, Shelah [23] further developed the machinery of tree indiscernibles in his work on Borel structures. In particular, he isolated a cardinal $\lambda_{\omega_1}(\aleph_0)$ that satisfies the following three properties.

- $\lambda_{\omega_1}(\aleph_0) \leqslant \beth_{\omega_1}$ [23, Definition 1.1, Conclusion 1.8].
- $\lambda_{\omega_1}(\aleph_0)$ is preserved in c.c.c. extensions [23, Claim 1.10].
- If a sentence $\psi \in \mathsf{L}_{\omega_1,\omega}$ has a model $\mathcal{M}_0$ with $|R^{\mathcal{M}_0}| \geqslant \lambda_{\omega_1}(\aleph_0)$ (where R is a distinguished unary predicate of $\mathcal{M}_0$), then ψ has a Skolemized model $\mathcal{M}$ that is generated by a family of tree indiscernibles in $R^{\mathcal{M}}$ (in particular $|R^{\mathcal{M}}| = 2^{\aleph_0}$) [23, Claim 2.1].

The above three facts immediately imply the following result. Note that $\mathsf{L}_{\omega_1,\omega}$ might gain new sentences in the passage to a generic extension.

2.3.2. Theorem. *Suppose* **V** *satisfies $\aleph_{\omega_1} = \beth_{\omega_1}$ and $\mathbb{P}$ is a c.c.c. notion of forcing. Then the following statement* (▲) *holds in* $\mathbf{V}^{\mathbb{P}}$:

(▲) *If a sentence ψ of $\mathsf{L}_{\omega_1,\omega}$ has a model $\mathcal{M}_0$ with $|R^{\mathcal{M}_0}| \geqslant \aleph_{\omega_1}$, then there is a countable first order Skolemized theory T such that the signature $\tau(T)$ of T extends the signature $\tau(\psi)$ of ψ, and $T + \psi$ has a model $\mathcal{M}$ that is generated from a family of tree indiscernibles $\{a_\eta\colon\ \eta \in {}^\omega 2\} \subseteq R^{\mathcal{M}}$.*

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[7] Completely separable families were first introduced in [9], and are referred to as "saturated families" in [7]. The question of the existence of an infinite completely separable MAD family in ZFC, posed by Erdős and Shelah [4], remains open. Shelah [26] has recently shown that (1) the existence of such families can be established within ZFC + $2^{\aleph_0} < \aleph_\omega$; and (2) the nonexistence of such families has very high large cardinal strength. See also [10] for further open questions and references.

[8] Shelah also employed tree indiscernibles in his work on classification theory [22, VII, Section 4] to show that for all $\lambda \geqslant \max\{|T|, \aleph_1\}$ T has 2^λ nonisomorphic models of cardinality λ for every complete theory T that is not superstable ([19] includes an expository account). Tree indiscernibles were also discovered by Paris and Mills ([17], [14, Theorem 3.5.3]) in the context of nonstandard models of Peano arithmetic to show, e.g., the existence of model $\mathcal{M}$ of PA with a nonstandard integer m in $\mathcal{M}$ such that the set of $\mathcal{M}$-predecessors of m is externally countable but the set of $\mathcal{M}$-predecessors of 2^m is of power $2^{\aleph_0}$ (this result is also an immediate corollary of [20, Theorem 1]).

The next result shows that for a given sentence ψ of $\mathsf{L}_{\omega_1,\omega}$ the existence of a model of ψ that is generated by tree indiscernibles is absolute.[9]

2.3.3. Theorem. *For any sentence ψ of $\mathsf{L}_{\omega_1,\omega}$ in $\mathbf{V}$ the following statement (♠) is absolute between $\mathbf{V}$ and any generic extension $\mathbf{V}^{\mathbb{P}}$:*

(♠) *There is a Skolemized model $\mathcal{M} \models \psi$ with a countable signature $\tau(\mathcal{M}) \supseteq \tau(\psi)$ such that $\mathcal{M}$ is generated from a family of tree indiscernibles $\{a_\eta\colon \eta \in {}^{\omega}2\} \subseteq R^{\mathcal{M}}$.*

Proof. It is well known[10] that for any sentence ψ of $\mathsf{L}_{\omega_1,\omega}$ with signature $\tau(\psi)$ there is a countable Skolemized first order theory T_ψ in a countable signature $\tau^+ \supseteq \tau(\psi)$ and a countable set Γ_ψ of 1-types of τ^+ such that (1) every model $\mathcal{M}$ of ψ has an expansion to a model $\mathcal{M}^+$ of T_ψ which omits the types in Γ_ψ, and (2) every model of T_ψ that omits the types in Γ_ψ satisfies ψ. Suppose ψ has a model $\mathcal{M}$ generated from a family $\{a_\eta\colon \eta \in {}^{\omega}2\}$ of tree indiscernibles in $\mathbf{V}^{\mathbb{P}}$. Then in $\mathbf{V}^{\mathbb{P}}$ we can form the multi-sorted structure $(\mathcal{M}^+, \mathcal{N}, f)$, where $\mathcal{N}$ is the standard model for second order number theory $(\omega, \mathcal{P}(\omega))$ (which is itself a two-sorted structure) and $f : \mathcal{P}(\omega) \to R^{\mathcal{M}}$ by $f(A) = a_{\chi_A}$ (where χ_A is the characteristic function of A). In particular, the signature τ^* appropriate to $(\mathcal{M}^+, \mathcal{N}, f)$ has a sort U_M for the universe of $\mathcal{M}^+$, a sort $U_{\mathcal{P}(\omega)}$ for $\mathcal{P}(\omega)$, and a sort U_ω for ω. Let θ be the conjunction of the following sentences $\theta_1, \ldots, \theta_4$ of $\mathsf{L}_{\omega_1,\omega}(\tau^*)$. Note that θ_4 is the only finitary sentence in the list.

- θ_1 expresses: ψ holds in U_M.
- θ_2 expresses: the axioms of second order arithmetic[11] (Z_2) hold in $(U_{\mathcal{P}(\omega)}, U_\omega)$.
- θ_3 expresses: U_ω is an ω-model.
- θ_4 expresses: f is an injection from $\mathcal{P}(\omega)$ into $R^{\mathcal{M}}$.

Consider the subset B of $({}^{\omega}2)^2$ that consists of elements of the form (r, s), where r codes a countable model $(\mathcal{M}_r^+, \mathcal{N}_r, f_r)$ of θ such that $\mathcal{M}_r^+$ omits the types in Γ_ψ, and s codes a function $g_s : \omega \to \omega$ that witnesses the fact that the image of f_r forms a family of tree indiscernibles in the sense of $\mathcal{N}_r$, i.e., g_s has the property that for every formula $\varphi = \varphi(x_0, \ldots, x_{m-1}) \in \mathsf{L}_{\omega,\omega}(\tau_\psi)$, if $n > g_s(\ulcorner\varphi\urcorner)$, then for all $x_0, \ldots, x_{m-1} \in U_{\mathcal{P}(\omega)}$, and for all $y_0, \ldots, y_{m-1} \in U_{\mathcal{P}(\omega)}$ the following implication is true (in what follows, φ^{U_M} is the relativization of φ to U_M)

$$\Bigl(\bigwedge_{i<m} \chi_{x_i} \restriction n = \chi_{y_i} \restriction n\Bigr) \wedge \Bigl(\bigwedge_{i<j<m} \chi_{x_i} \restriction n \neq \chi_{x_j} \restriction n\Bigr)$$
$$\to \varphi^{U_M}\bigl[f(x_0), \ldots, f(x_{m-1})\bigr] \leftrightarrow \varphi^{U_M}\bigl[f(y_0), \ldots, f(y_{m-1})\bigr].$$

It is easy to see that B is a Borel set with a Borel code c in $\mathbf{V}$. Also, by the downward Löwenheim–Skolem theorem for $\mathsf{L}_{\omega_1,\omega}$ sentences,

$$\mathbf{V}^{\mathbb{P}} \models \text{“the Borel set coded by } c \text{ is not empty”}.$$

On the other hand, the statement “the Borel set coded by c is empty” is provably equivalent (in $\mathsf{ZF} + \mathsf{DC}$) to a Π^1_1-statement [11, Lemma 25.45] and therefore by Mostowski's Π^1_1-absoluteness theorem [11, Theorem 25.4], the Borel set coded by c is nonempty in the real world $\mathbf{V}$. This shows that in $\mathbf{V}$ there is a countable model $(\mathcal{M}_0^+, \mathcal{N}_0, f_0)$ of ψ, and a function $g_0 : \omega \to \omega$ that witnesses the fact that the image of f forms a family of tree indiscernibles in the sense of $\mathcal{N}_0$ (in particular, $\mathcal{N}_0$ is an ω-model of second order arithmetic).

The countable model $(\mathcal{M}_0^+, \mathcal{N}_0, f_0)$ and g_0 together provide us with a blueprint Σ for producing a model of ψ of cardinality continuum that is generated by tree indiscernibles. To construct Σ, add new constants $\{c_\eta\colon \eta \in {}^{\omega}2\}$ to the vocabulary τ^+ of $\mathcal{M}_0^+$. Then Σ is defined as follows. Given $\varphi(x_0, \ldots, x_{m-1}) \in \mathsf{L}_{\omega,\omega}(\tau_{\mathcal{M}})$, fix any $n > g_s(\ulcorner\varphi\urcorner)$, and find $\nu_0, \ldots, \nu_{m-1} \in {}^{\omega}2$ such that each ν_i is coded in $\mathcal{N}_0$ (i.e., there is some A_i in $U_{\mathcal{P}(\omega)}$ of $\mathcal{N}_0$ such that $\chi_{A_i} = \nu_i$) and $\eta_i \restriction n = \nu_i \restriction n$ for each $i < m$. Then put $\varphi[c_{\eta_0}, \ldots, c_{\eta_{m-1}}]$ or its negation in Σ depending on whether $\mathcal{M}_0^+$ respectively satisfies $\varphi[f_0(A_0), \ldots, f_{m-1}(A_{m-1})]$ or its negation. Since $\mathcal{M}_0^+$ is Skolemized, Σ uniquely determines an elementary extension $\mathcal{M}_2^+$ of $\mathcal{M}_0^+$ that is generated by tree indiscernibles. In order to arrange an elementary extension of $\mathcal{M}_0^+$ that is generated by tree indiscernibles that also satisfies ψ we need to thin $\mathcal{M}_2^+$ as follows. Since $\mathcal{M}_0^+$ omits every type in Γ_ψ and g_0 provides a witness to the tree indiscernibility of the range of f_0, we can easily construct a perfect subtree Δ of ${}^{\omega}2$ such that the submodel $\mathcal{M}_1^+$ of $\mathcal{M}_2^+$ generated by $\{c_\eta\colon \eta \in \Delta\}$ omits every type in Γ_ψ. Therefore $\mathcal{M}_1^+$ is our desired model of ψ that is generated by tree indiscernibles. □

[9] This result is stated for generic extensions, but the proof shows that this absoluteness result is true for any two ω-models $\mathbf{V}$ and $\mathbf{W}$ of $\mathsf{ZF} + \mathsf{DC}$ with $\mathcal{P}^{\mathbf{V}}(\omega) \subseteq \mathcal{P}^{\mathbf{W}}(\omega)$.

[10] Cf. [12, Chapter 11, Theorem 14] or [2, Theorem 6.18].

[11] We just need a workable theory of finite and infinite sequences, so it is more than sufficient to use RCA_0 or ACA_0 instead of Z_2 here.

2.4. Borel structures

Recall that a model $\mathcal{M}$ is said to be *totally Borel* if the universe of $\mathcal{M}$ is a Borel subset of $\mathbb{R}$, and every subset of X that is parametrically definable in $\mathcal{M}$ is a Borel set. It is known that every countable theory has an uncountable totally Borel model. This result was established by H. Friedman [28] and also (later, but independently) by Malitz, Mycielski and Reinhardt [16]. The following results are included for those readers favoring a shorter (albeit less self-contained) proof of Theorem A.

2.4.1. Theorem. *(Steinhorn [29]) If $\mathcal{M}$ is a model generated by tree indiscernibles, then $\mathcal{M}$ is isomorphic to a totally Borel model.*

2.4.2. Theorem. *(Harrington and Shelah [8]) No analytic linear order contains an uncountable well-ordered set. In particular, the cofinality of every Borel linear order with no last element is $\aleph_0$.*

3. Proof of Theorem A

Before presenting the full technical details of the proof, let us describe a high-level summary of the three stages of the argument.

- **Stage 1 Outline.** Start with the constructible universe $\mathbf{L}$ and a regular cardinal $\kappa > (\aleph_{\omega_1})^{\mathbf{L}} = (\beth_{\omega_1})^{\mathbf{L}}$. Then force $\mathsf{MA} + 2^{\aleph_0} = \kappa$ with the usual c.c.c. partial order $\mathbb{Q}$ of cardinality κ. In $\mathbf{L}^{\mathbb{Q}}$, use Theorem 2.3.2 to get hold of an ω-standard model $\mathcal{M}'$ of $\mathsf{ZFC}^- + \mathsf{MA}$ (where ZFC^- is ZFC without the powerset axiom) that is generated by tree indiscernibles.
- **Stage 2 Outline.** By Theorem 2.4.1 $\mathcal{M}'$ is a totally Borel model in $\mathbf{L}^{\mathbb{Q}}$. Combined with Theorem 2.3.3 this shows that there is also a totally Borel model $\mathcal{M}$ in $\mathbf{V}$ that shares the salient features of $\mathcal{M}'$. In particular, $\mathcal{M}$ is an ω-standard model of ZFC^- that satisfies MA and is generated by tree indiscernibles. The family $\mathcal{A}$ of Theorem A is the set of reals of $\mathcal{M}$. This family $\mathcal{A}$ is both Borel and arithmetically closed.
- **Stage 3 Outline.** By Theorem 2.4.2 every definable infinite linear order in $\mathcal{M}$ with no last element has countable cofinality. This fact, when coupled with the veracity of MA in $\mathcal{M}$, will allow us to verify that $\mathbb{P}_{\mathcal{A}}$ is $(\aleph_0, 2^{\aleph_0})$-nowhere distributive. By Theorem 2.1.1, this completes the proof of Theorem A.

We now proceed to flesh out the above outline.

Stage 1. Let $\mu = (\aleph_{\omega_1})^{\mathbf{L}} = (\beth_{\omega_1})^{\mathbf{L}}$, and fix a regular cardinal $\kappa > \mu$. By GCH in $\mathbf{L}$, $\kappa = \kappa^{<\kappa}$ holds in $\mathbf{L}$. Let $\mathbb{Q}$ be the usual c.c.c. notion of forcing $\mathsf{MA} + 2^{\aleph_0} = \kappa$ [11, Theorem 16.13]. Let $\mathcal{H}(\kappa^+)$ be the collection of sets whose transitive closure has cardinality at most κ. In the forcing extension $\mathbf{L}^{\mathbb{Q}}$ let $\mathcal{M}_0$ be an expansion of the structure $(\mathcal{H}(\kappa^+), \in)$ by Skolem functions, a well-ordering of $\mathcal{H}(\kappa^+)$, and individual constants c_n and c_ω, where $c_n^{\mathcal{M}_0} = n$, and $c_\omega^{\mathcal{M}_0} = \omega$. Let $\tau = \tau_{\mathcal{M}_0} =$ the signature of $\mathcal{M}_0$. We may assume that $\tau \in \mathbf{L}$ and τ is countable in $\mathbf{L}$, but note that $\mathsf{Th}(\mathcal{M}_0)$ need not be in $\mathbf{L}$. Of course $\mathcal{M}_0$ is a model of ZFC^-+ "$2^{\aleph_0}$ is the last cardinal" + MA.

Since $\kappa > \mu$ we may invoke Theorem 2.3.2 to obtain a model $\mathcal{M}'$ in $\mathbf{L}^{\mathbb{Q}}$ that satisfies the following five conditions:

(a′) $\mathcal{M}'$ is a model of $\mathsf{Th}(\mathcal{M}_0)$ with signature τ. In particular $\mathcal{M}'$ satisfies ZFC^-+ "$2^{\aleph_0}$ is the last cardinal" + MA.
(b′) $\mathcal{M}'$ is an ω-model, i.e., $\mathcal{M}$ omits $\{x \in c_\omega\} \cup \{x \neq c_n\colon n < \omega\}$.
(c′) There is a family $\{a'_\eta\colon \eta \in {}^{\omega}2\}$ of tree indiscernibles in $\mathcal{M}$.
(d′) For each $\eta \in {}^{\omega}2$, $\mathcal{M}' \models$ "$a'_\eta \subseteq c_\omega$" (i.e., each a'_η is a real in the sense of $\mathcal{M}'$).
(e′) $\mathcal{M}'$ is the Skolem hull of $\{a'_\eta\colon \eta \in {}^{\omega}2\}$.

Stage 2. Let $T = \{\varphi \in \mathsf{L}_{\omega,\omega}(\tau)\colon 1 \Vdash_{\mathbb{Q}} \mathcal{M}' \models \varphi\}$. Note that since $\mathcal{M}'$ is actually a $\mathbb{Q}$-name, $T \in \mathbf{L}$. By Theorem 2.3.3 there is a τ-model $\mathcal{M}$ of T in $\mathbf{V}$ and a family of tree indiscernibles $\langle a_\eta\colon \eta \in 2^\omega\rangle$ such that the following five conditions hold.

(a) $\mathcal{M}$ is a model with signature τ that satisfies ZFC^-+ "$2^{\aleph_0}$ is the last cardinal" + MA.
(b) $\mathcal{M}$ is an ω-model.
(c) There is a family $\{a_\eta\colon \eta \in {}^{\omega}2\}$ of tree indiscernibles in $\mathcal{M}$.
(d) For each $\eta \in {}^{\omega}2$, $\mathcal{M} \models$ "$a_\eta \subseteq c_\omega$".
(e) $\mathcal{M}$ is the Skolem hull of $\{a_\eta\colon \eta \in {}^{\omega}2\}$.

We may assume that the model $\mathcal{M}$ is in "reduced form", i.e., the well-founded part of $\mathcal{M}$ is transitive. In particular, $\omega^{\mathcal{M}} = \omega$, and if $\mathcal{M} \models b \subseteq c_\omega$, then $b \in \mathcal{P}(\omega)$. Let $\mathcal{A} = \{b\colon \mathcal{M} \models b \subseteq c_\omega\}$. Obviously $\mathcal{A}$ is arithmetically closed.[12] By Theorem 2.4.1 $\mathcal{A}$ is also Borel. This fact can also be established directly as follows. For any $\tau_{\mathcal{M}}$-term $\sigma = \sigma(x_0, \ldots, x_{m-1})$, $m < \omega$,

[12] Indeed $\mathcal{A}$ is even *hyperarithmetically* closed. This follows from the fact that any ω-model of $\Sigma^1_1{-}\mathsf{AC}_0$ contains all hyperarithmetical sets [27, Lemma VIII.4.15]; note that since $\mathcal{M}$ satisfies the axiom of choice, the standard model of second order arithmetic in the sense of $\mathcal{M}$ satisfies $\Sigma^1_n{-}\mathsf{AC}_0$ for all $n < \omega$.

$n^* < \omega$, and pairwise distinct $\nu_0, \ldots, \nu_{m-1} \in {}^{n^*}2$, let $\bar{\nu} = \langle \nu_i \colon i < m \rangle$, and consider the set $\mathcal{A}_{\sigma,\bar{\nu}}$ defined as follows (in the formula below $\lhd$ denotes the end extension relation among sequences):

$$\mathcal{A}_{\sigma,\bar{\nu}} := \left\{ \omega \cap \sigma^{\mathcal{M}}(a_{\eta_0}, \ldots, a_{\eta_{m-1}}) \colon \bigwedge_{i<m} \nu_i \lhd \eta_i \in {}^{\omega}2 \right\}.$$

It is sufficient to prove that $\mathcal{A}_{\sigma,\bar{\nu}}$ is Borel for any $(\sigma, \bar{\nu})$ since $\mathcal{A}$ is the union of the countable family of sets of the form $\mathcal{A}_{\sigma,\bar{\nu}}$. We can find an increasing $f : \omega \to \omega \backslash n^*$ and $\langle g_n \colon n < \omega \rangle$ for which (α), (β), and (γ) hold.

(α) g_n is a function from ${}^{m}({}^{f(n)}2)$ to $\{0, 1\}$.

(β) If $\eta_0, \ldots, \eta_{m-1} \in {}^{\omega}2$ and $\bigwedge_{i<m} \nu_i \lhd \eta_i \in {}^{\omega}2$ and $n < \omega$, then (using tree indiscernibility)

$$n \in \sigma^{\mathcal{M}}(a_{\eta_0}, \ldots, a_{\eta_{m-1}}) \quad \Leftrightarrow \quad g_n\big(\eta_0 \restriction f(n), \ldots, \eta_{m-1} \restriction f(n)\big) = 1.$$

(γ) If $A \subseteq \omega$, then by König's lemma, $A \in \mathcal{A}_{\sigma,\bar{\nu}}$ iff for every $n < \omega$ there are $\rho_0, \ldots, \rho_{m-1} \in {}^{f(n)}2$ such that

$$k < n \quad \Rightarrow \quad \big(k \in A \Leftrightarrow g_k\big(\rho_0 \restriction f(k), \ldots, \rho_{m-1} \restriction f(k)\big) = 1\big).$$

Note that (γ) shows that each $\mathcal{A}_{\sigma,\bar{\nu}}$ is Borel.

Stage 3. By Theorems 2.4.1 and 2.4.2 every definable linear order $(L, <_L)$ in $\mathcal{M}$ with no last element has countable cofinality. Alternatively, one can argue directly as follows. Suppose to the contrary. Then for some regular uncountable cardinal κ, there is an increasing unbounded subset $\{b_\alpha \colon \alpha < \kappa\}$ of $(L, <_L)$. Each b_α can be written in $\mathcal{M}$ as

$$b_\alpha = \sigma_\alpha(a_{\eta_0^\alpha}, \ldots, a_{\eta^\alpha_{n_\alpha - 1}}),$$

but without loss of generality, we may assume that (1) $\sigma_\alpha = \sigma$, (2) $n_\alpha = n$, (3) $\{\{\eta_0^\alpha, \ldots, \eta_{n-1}^\alpha\} \colon \alpha < \kappa\}$ forms a Δ-system [11, Theorem 9.18], and (4) $\eta_0^\alpha <_{\mathrm{lex}} \eta_1^\alpha <_{\mathrm{lex}} \cdots$ (where $<_{\mathrm{lex}}$ denotes the lexicographic relation among binary sequences). In particular, we may assume that for some $m < n$,

$$l < m \quad \Rightarrow \quad \eta_l^\alpha = \eta_l^0;$$

and

$$\eta_{l_1}^{\alpha_1} = \eta_{l_2}^{\alpha_2} \quad \Rightarrow \quad (l_1 = l_2 < m) \vee (\alpha_1, l_1) = (\alpha_2, l_2).$$

We can easily construct a *countable* $Y \subseteq \kappa$ such that if $\alpha < \kappa$ and $k < \omega$, then for some $\beta \in Y$ we have

$$\bigwedge_{l<n} \eta_l^\alpha \restriction k = \eta_l^\beta \restriction k.$$

The proof would be complete once we verify that $\{b_\beta \colon \beta \in Y\}$ is cofinal in $(L, <_L)$. Let $\alpha < \kappa$, and note that the concatenation of $\langle \eta_l^\alpha \colon l < n \rangle$ and $\langle \eta_l^{\alpha+1} \colon l \in [m, n) \rangle$ has no repetition. Choose $k < \omega$ that satisfies the following condition (∇):

(∇) **If** $\eta_l \in {}^{\omega}2$ for $l < n$, $\nu_s \in {}^{\omega}2$ for $s \in [m, n)$, and $(\eta_l \restriction k = \eta_l^\alpha \restriction k) \wedge (\nu_s \restriction k = \eta_s^{\alpha+1} \restriction k)$, **then** $\mathcal{M}$ satisfies the following biconditional:

$$\begin{aligned} &\sigma(\ldots, a_{\eta_l^\alpha}, \ldots) <_L \sigma(\ldots, a_{\eta_l^{\alpha+1}}, \ldots) \\ &\quad \leftrightarrow \quad \sigma(\ldots, a_{\eta_l}, \ldots) <_L \sigma(a_{\eta_0}, \ldots, a_{\eta_{m-1}}, a_{\nu_m}, \ldots, a_{\nu_{n-1}}). \end{aligned}$$

Lastly, choose $\beta \in Y$ such that

$$\bigwedge_{l<n} \eta_l^\beta \restriction k = \eta_l^{\alpha+1} \restriction k.$$

Hence $(b_\alpha <_L b_{\alpha+1}) \Leftrightarrow (b_\alpha <_L b_\beta)$, which shows that $\{b_\beta \colon \beta \in Y\}$ is cofinal in $(L, <_L)$ and concludes the proof. $\square$

We now complete the proof of Theorem A by showing that $\mathbb{P}_{\mathcal{A}}$ is equivalent to $\text{Levy}(\aleph_0, 2^{\aleph_0})$. By Theorem 2.1.1, it suffices to establish the following claim.

3.1. Claim. *There is a family $\{I_n \colon n \in \omega\}$ of maximal antichains in $\mathbb{P}_{\mathcal{A}}$ such that for every $p \in \mathbb{P}_{\mathcal{A}}$ there is some $n < \omega$ such that $\{q \in I_n \colon p$ and q are compatible$\}$ has cardinality $2^{\aleph_0}$.*

Proof. Recall that MA holds in $\mathcal{M}$ (see condition (a) of Stage 2). Hence $\mathcal{M}$ satisfies "there is a $2^{\aleph_0}$-scale $\{f_\alpha \colon \alpha < 2^{\aleph_0}\}$ in $({}^{\omega}\omega, <_*)$" [11, Corollary 16.25]. In other words, $\mathcal{M}$ satisfies

$$\forall g : \omega \to \omega\ \exists \alpha < 2^{\aleph_0} \text{ such that } g <_* f_\alpha \ \big(\text{i.e., } g(n) < f_\alpha(n) \text{ for sufficiently large } n\big),$$
$$\text{and } f_\alpha <_* f_\beta \text{ whenever } \alpha < \beta < 2^{\aleph_0}.$$

Therefore, using Claim 3.1 we may fix a *countable* family of functions $F = \{f_n\colon n \in \omega\} \subseteq {}^\omega\omega \cap \mathcal{M}$ such that for every $g \in {}^\omega\omega \cap \mathcal{M}$ there is some $f_n \in F$ such that $g <_* f_n$. Of course we may assume that f_n is an increasing function for each $n \in \omega$. For each $f_n \in F$, let $\overline{f_n} \in \mathcal{M}$ be an auxiliary function defined by $\overline{f_n}(0) = f_n(0)$ and

$$\forall i \in \omega \quad \overline{f_n}(i+1) = i + f_n\big(\overline{f_n}(i) + 2\big).$$

Since $\mathcal{M}$ satisfies (#), and $\lim_{n\in\omega} \overline{f_n}(i+1) - \overline{f_n}(i) = \infty$, there is some family $I_n \in \mathcal{M}$ with $I_n \subseteq [\omega]^\omega$ such that for all $A \in I_n$ and for all $i < \omega$

$$\big|A \cap \big[\overline{f_n}(i), \overline{f_n}(i+1)\big)\big| = 1,$$

and for each $B \in \mathcal{P}(\omega) \cap \mathcal{M}$, $\mathcal{M}$ satisfies

$$\{A \in \mathcal{A}\colon\ A \cap B \text{ is infinite}\} \text{ is either finite or has cardinality continuum.}$$

We now verify that $\langle I_n\colon n < \omega\rangle$ exemplifies condition (b) of Theorem 2.1.1. Given a condition $p = [B] \in \mathbb{P}_{\mathcal{A}}$, we may assume that B is infinite. It is routine to construct a strictly increasing function $g \in {}^\omega\omega \cap \mathcal{M}$ by recursion such that $g(0) = 0$, and

$$\forall k \big|B \cap \big[g(k), g(k+1)\big)\big| \geqslant g(k). \tag{1}$$

Choose $f_n \in F$ and $i_0 \in \omega$ such that $g(i) < f_n(i)$ for all $i \geqslant i_0$, and let

$$Y := \big\{i\colon\ \exists k\big(\overline{f_n}(i) < g(k) < g(k+1) < \overline{f_n}(i+1)\big)\big\}.$$

We wish to show that $i \in Y$ for all $i \geqslant i_0$ (the fact that Y is infinite will come handy below). To this end, suppose $i \geqslant i_0$. Since $\langle g(s)\colon\ s < \omega\rangle$ is a strictly increasing sequence, we can find $k < \omega$ such that k is the first s such that $\overline{f_n}(i) < g(s)$. Hence $g(k-1) \leqslant \overline{f_n}(i)$, which in turn implies that $k - 1 \leqslant \overline{f_n}(i)$ (since g is strictly increasing), and therefore

$$k + 1 \leqslant \overline{f_n}(i) + 2. \tag{2}$$

Using the strictly increasing feature of g one more time, (2) yields

$$g(k+1) \leqslant g\big(\overline{f_n}(i) + 2\big). \tag{3}$$

On the other hand, since $\overline{f_n}(i) + 2 \geqslant i \geqslant i_0$, and f_n dominates g for $i \geqslant i_0$

$$g\big(\overline{f_n}(i) + 2\big) < f_n\big(\overline{f_n}(i) + 2\big) < i + f_n\big(\overline{f_n}(i) + 2\big) = \overline{f_n}(i+1). \tag{4}$$

By putting (3) and (4) together, we obtain $g(k+1) < \overline{f_n}(i+1)$. This shows that Y includes every $i \geqslant i_0$.

Now let $\mathcal{F}_B := \{A \in I_n\colon\ A \cap B \text{ is infinite}\}$, and note that $\mathcal{F}_B \in \mathcal{M}$. Thanks to (#) $\mathcal{M}$ satisfies "$\mathcal{F}_B$ is finite or has cardinality continuum". But $\mathcal{F}_B$ cannot be finite, since each $A \in \mathcal{F}_B$ has only one element in each interval $[\overline{f_n}(i), \overline{f_n}(i+1))$, whereas B has more than $\overline{f_n}(i)$ members for infinitely many values of i, thanks to (1) and the fact that Y is infinite. Hence $\mathcal{F}_B$ has cardinality $2^{\aleph_0}$ in the sense of $\mathcal{M}$, and therefore in the real world as well, since $\mathcal{M}$ has continuum-many reals. □

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