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Strongly meager sets of size continuum

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Abstract. We will construct several models where there are no strongly meager sets of size $2^{\aleph_0}$.

1. Introduction

In this paper we work exclusively in the space 2^ω equipped with the standard product measure denoted as μ. Let $\mathcal{N}$ and $\mathcal{M}$ denote the ideal of all μ-measure zero sets, and meager subsets of 2^ω, respectively. For $x, y \in 2^\omega$, $x + y \in 2^\omega$ is defined as $(x + y)(n) = x(n) + y(n) \pmod 2$. In particular, $(2^\omega, +)$ is a group and μ is an invariant measure.

Definition 1 ([3]). *A set X of real numbers or more generally, a metric space, is strong measure zero if, for each sequence $\{\varepsilon_n : n \in \omega\}$ of positive real numbers there is a sequence $\{X_n : n \in \omega\}$ of subsets of X whose union is X, and for each n the diameter of X_n is less than ε_n.*

The family of strong measure zero subsets of 2^ω is denoted by $\mathcal{SN}$.

The following characterization of strong measure zero is the starting point for our considerations.

Theorem 2 ([7]). *The following are equivalent:*

(1) $X \in \mathcal{SN}$,
(2) *for every set $F \in \mathcal{M}$, $X + F \neq 2^\omega$.*

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This theorem indicates that the notion of strong measure zero should have its category analog. Indeed, we define after Prikry:

Definition 3. *Suppose that $X \subseteq 2^\omega$. We say that X is strongly meager if for every $H \in \mathcal{N}$, $X + H \neq 2^\omega$. Let $\mathcal{SM}$ denote the collection of strongly meager sets.*

Observe that if $z \notin X + F = \{x + f : x \in X, f \in F\}$ then $X \cap (F + z) = \emptyset$. In particular, a strong measure zero set can be covered by a translation of any dense G_δ set, and every strongly meager set can be covered by a translation of any measure one set.

Let Borel Conjecture denote the statement $\mathcal{SN} = [2^\omega]^{\aleph_0}$ and Dual Borel Conjecture the statement: $\mathcal{SM} = [2^\omega]^{\aleph_0}$.

The question whether Borel Conjecture and Dual Borel Conjecture are jointly consistent is the motivation for this paper. In particular, we are interested in what kinds of strongly meager sets exist in various models.

2. Preservation of not being strongly meager

A small modification [10] of the definition of strongly meager sets leads to a family that captures the concept of strongly meager sets while it has some additional properties.

Definition 4. *We say that $X \in \mathcal{SM}^+$ if for every $H \in \mathcal{N}$, there exists a countable set $Z \subseteq 2^\omega$ such that*

$$\forall x \in X \; Z \not\subseteq x + H.$$

Lemma 5.
(1) $\mathcal{SM} \subseteq \mathcal{SM}^+$,
(2) $\mathcal{SM}^+$ *is a σ-ideal,*
(3) *it is consistent that* $\mathcal{SM} = \mathcal{SM}^+$,
(4) *it is consistent that* $\mathcal{SM} \neq \mathcal{SM}^+$.

Proof. (1) and (2) are obvious.

(3) $\mathcal{SM} = \mathcal{SM}^+$ holds in Cohen's model, and in fact every known model where $\mathcal{SM} = [2^\omega]^{\aleph_0}$ [4].

(4) By the results of [2], assuming CH, $\mathcal{SM}$ is not a σ-ideal. $\square$

Definition 6. *Suppose that $N \prec \mathbf{H}(\chi)$ is a countable model. A G_δ set $H \in \mathcal{N}$ is big over N if $2^\omega \cap N \subseteq H$.*

We say that a proper forcing notion $\mathbb{P}$ is nice if for every $p \in \mathbb{P}$, a countable model $N \prec \mathbf{H}(\chi)$ containing $p, \mathbb{P}$ and a set H which is big over N there exists a condition $q \geq p$ such that

(1) *q is $(N, \mathbb{P})$-generic over N,*
(2) *$q \Vdash_{\mathbb{P}} H$ is big over $N[\dot{G}]$.*

Theorem 7. *Suppose that $\mathbf{V} \models X \notin \mathcal{SM}^+$ and $\mathbb{P}$ is nice. Then $\mathbf{V}^{\mathbb{P}} \models X \notin \mathcal{SM}^+$.*

Proof. Suppose that $X \notin \mathcal{SM}^+$ and let $H \in \mathcal{N}$ be a G_δ set such that

$$\forall Z \in [\mathbf{V} \cap 2^\omega]^\omega \ \exists x \in X \ Z \subseteq H + x.$$

Let $\{\dot{x}_n : n \in \omega\}$ be a $\mathbb{P}$-name for a countable set of reals, and let $p \in \mathbb{P}$. Find a countable model $N \prec \mathbf{H}(\chi)$ containing all relevant objects and let $x \in X$ be such that $N \cap 2^\omega \subseteq H + x$. Since $\mathbb{P}$ is nice, there exists $q \geq p$ such that $q \Vdash_{\mathbb{P}} \forall n \ \dot{x}_n \in H + x$. It follows that $\mathbf{V}^{\mathbb{P}} \models X \notin \mathcal{SM}^+$. □

The following theorem is a particular instance of a preservation result due to Eisworth, [5]. We give a sketch of the proof for completeness.

Theorem 8. *Let $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \delta\}$ be a countable support iteration of proper forcing notions such that every $\alpha < \delta$, $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha$ is nice. Then $\mathcal{P}_\delta$ is nice.*

Proof. The proof follows standard proof of preservation of properness. □

Lemma 9 ([8], [1] lemma 6.1.2)**.** *Let $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \delta\}$ be a countable support iteration. Assume $\alpha < \beta \leq \delta$, $p \in \mathcal{P}_\alpha$ and $\dot{\tau}$ is a $\mathcal{P}_\beta$-name for an ordinal. There is a $\mathcal{P}_\alpha$-name $\dot{\pi}$ for an ordinal and a condition $q \geq_\beta p$ such that $q{\restriction}\alpha = p$ and $q \Vdash_\beta \dot{\tau} = \dot{\pi}$.*

Let N be a countable submodel of $\mathbf{H}(\chi)$ containing $\mathbb{P}_\delta$ and let $H = \bigcap_n \bigcup_{m>n} U_m$ be big over N, where U_m's are basic open sets.

Lemma 10. *Let $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \delta\}$ be a countable support iteration and assume that for all $\alpha < \delta$, $\Vdash_\alpha$ "$\dot{\mathcal{Q}}_\alpha$ is nice." Then for all $\beta \in N \cap \delta$, for all $\alpha \in N \cap \beta$ and for all $p \in \mathcal{P}_\beta \cap N$, whenever $q \geq_\alpha p{\restriction}\alpha$ is $(N, \mathcal{P}_\alpha)$-generic, there is an $(N, \mathcal{P}_\beta)$-generic condition $r \geq_\alpha p{\restriction}\beta$ such that $r{\restriction}\alpha = q$ and $r \Vdash N[G_\beta] \cap 2^\omega \subseteq H$.*

Proof. The proof is by induction on β.

Successor step. Let $\beta = \beta' + 1$. We assume that $\beta \in N$ so $\beta' \in N$ too. Using the induction hypothesis on α, β' we can extend q to a $(N, \mathcal{P}_{\beta'})$-generic condition $q' \in \mathcal{P}_{\beta'}$ such that $q' \geq_{\beta'} p{\restriction}\beta'$ and $q'{\restriction}\alpha = q$. Hence without loss of generality we can assume $\beta = \alpha + 1$.

Since $\Vdash_\alpha N[\dot{G}_\alpha] \prec \mathbf{H}(\chi)^{\mathbf{V}[\dot{G}_\alpha]}$, q forces that there is an $(N[\dot{G}_\alpha], \dot{\mathcal{Q}}_\alpha)$-generic condition $\geq \dot{p}(\alpha)$ which forces that $N[G_\alpha] \cap 2^\omega \subseteq H$. It has a $\mathcal{P}_\alpha$-name, $r(\alpha)$ and $r = q^\frown r(\alpha)$ is $(N, \mathcal{P}_\beta)$-generic and $r \Vdash N[G_\beta] \cap 2^\omega \subseteq H$.

Limit step. Let $\beta \in N$ be a limit ordinal and $\alpha = \alpha_0 < \alpha_1 < \cdots$ be a sequence of ordinals in N such that $\sup(N \cap \beta) = \sup_{n \in \omega} \alpha_n$. Let $\langle \dot{\tau}_n : n \in \omega \rangle$ enumerate all $\mathcal{P}_\beta$-names for ordinals which are in N, let $\langle \dot{x}_n : n \in \omega \rangle$ enumerate all $\mathcal{P}_\beta$-names for reals in N with infinitely many repetitions. By induction on $n \in \omega$ using lemma 9, we define conditions $p'_n \in \mathcal{P}_\beta \cap N$, $\mathcal{P}_{\alpha_n}$-names $\dot{\tau}^\star_n \in N$ for ordinals such that for each n,

(1) $p'_0 = p$ and $p_{n+1} \geq p_n$,
(2) $p'_{n+1} \Vdash_\beta \dot{\tau}^\star_n = \dot{\tau}_n$,
(3) $p'_{n+1}{\restriction}\alpha_n = p_n{\restriction}\alpha_n$,

Up to now the proof followed standard proof of preservation of properness. Now we will make a small change relevant to the current setup.

We will refine the sequence $\{p_n : n \in \omega\}$ as follows. Let $\{p^1_m : m \in \omega\}$ and $x^0 \in 2^\omega \cap N$ be such that

(1) $p^1_0 = p'_1$,
(2) $p^1_{m+1} \geq p^1_m$, $p^1_m \in \mathcal{P}_\delta$,
(3) $p^1_m \Vdash \dot{x}_0{\restriction}m = x^0{\restriction}m$.

Since $x^0 \in U_k$ for infinitely many k, it follows that for some m, $p^1_m \Vdash \dot{x}_0 \in U_m$.

Let G be a $\mathcal{P}_{\alpha_0}$-generic filter over N containing $p_0{\restriction}\alpha_0$. Construct sequence $\{p^2_m : m \in \omega\}$ and $x^1 \in 2^\omega \cap N[G]$ be such that

(1) $p^2_0 = p'_1{\restriction}[\alpha_0, \delta)$,
(2) $p^2_{m+1} \geq p^2_m$, $p^2_m \in \mathcal{P}_{\alpha_0,\delta}$,
(3) $p^2_m \Vdash \dot{x}_1{\restriction}m = x^1{\restriction}m$.

As before, $N[G] \models x^1 \in U_k$ for infinitely many k, so it follows that for some $m > 0$, $N[G] \models p^2_m \Vdash_{\alpha_0,\delta} \dot{x}_1 \in U_m$. Thus there is a $\mathcal{P}_{\alpha_0}$-name $\dot{m}$ for an integer such that

$$p_2 = p_1{\restriction}\alpha_0 \star p^2_{\dot{m}} \Vdash \exists m > 0\ \dot{x}_1 \in U_m.$$

We continue in this fashion and construct a sequence $\langle p_n : n \in \omega\rangle$ such that

(1) $p_0 = p$ and $p_{n+1} \geq p_n$,
(2) $p_{n+1} \Vdash_\beta \dot{\tau}^\star_n = \dot{\tau}_n$,
(3) $p_{n+1}{\restriction}\alpha_n = p_n{\restriction}\alpha_n$,
(4) $p_{n+1} \Vdash \exists m > n\ \dot{x}_n \in U_m$.

Note that all conditions p_n are in N (while the sequence is not). Now using the induction hypothesis we define conditions $q_n \in \mathcal{P}_{\alpha_n}$ such that $q_0 = q$, $q_n \geq p_n{\restriction}\alpha_n$, $q_{n+1}{\restriction}\alpha_n = q_n$ and q_n is an $(N, \mathcal{P}_{\alpha_n})$-generic condition. We set $r = \bigcup_{n\in\omega} q_n$. Now for each n, since $\mathsf{supp}(p_{n+1}) \subseteq N$, $\mathsf{supp}(p_{n+1}) \subseteq \mathsf{dom}(r)$ and by (1) and (3), $r \geq p_{n+1}$. Hence $r \Vdash_\beta \dot{\tau}_n = \dot{\tau}^\star_n$, and $q_n = r{\restriction}\alpha_n \Vdash_{\alpha_n} \dot{\tau}^\star_n \in N$. Likewise $r \Vdash_\beta \dot{\tau}_n \in N$ for each $n \in \omega$, and r is $(N, \mathcal{P}_\beta)$-generic. Similarly, for every n, $r \Vdash \exists^\infty m\ \dot{x}_n \in U_m$. □

The following definition gives an easy to verify property of forcing notion which implies niceness.

Definition 11. *Suppose that $\mathbb{P}$ is a forcing notion satisfying Axiom A. $\mathbb{P}$ has PPP-property if for every $\mathbb{P}$-name $\dot{x}$ such that $\Vdash_{\mathbb{P}} \dot{x} \in 2^\omega$ and every $p \in \mathbb{P}$, $n \in \omega$ there exists $k \in \omega$ such that for every $\ell \in \omega$ there exists $A \in [2^\ell]^{<k}$ and $q \geq_n p$, $q \Vdash_{\mathbb{P}} \dot{x}{\restriction}\ell \in A$.*

Observe that PPP-property is a weak version of Laver property (which is equivalent to the requirement that k depends only on n but not on p). Thus Laver, Miller and Sacks forcings have property PPP (since they have the Laver property), as well as many tree forcings.

Lemma 12. *If $\mathbb{P}$ has PPP-property then $\mathbb{P}$ is nice. In particular, if $\mathbb{P}$ has Laver property then $\mathbb{P}$ is nice.*

Proof. Suppose that $N \prec \mathbf{H}(\chi)$ is a countable model containing $\mathbb{P}$ and $p \in \mathbb{P}$. Suppose that $H \in \mathcal{N}$ is big over N. Let $\langle U_j : j \in \omega\rangle$ be open sets such that $H = \bigcap_j U_j$. We want to find a condition $q \geq p$ which is $(N, \mathbb{P})$-generic such that $q \Vdash_{\mathbb{P}} 2^\omega \cap N[\dot{G}] \subseteq H$.

Basic step: Suppose that $\dot{x} \in N$ is a $\mathbb{P}$-name for a real, $n \in \omega$ and $\bar{p} \in \mathbb{P} \cap N$. Using PPP-property we construct reals $\{x_1, \ldots, x_k\} \in N \cap 2^\omega$ and a sequence of conditions $\langle \bar{p}_m : m \in \omega\rangle \in N$ such that

(1) $\bar{p} = \bar{p}_0$,
(2) $\bar{p}_{m+1} \geq_n \bar{p}_m$,
(3) $\bar{p}_m \Vdash_{\mathbb{P}} \exists j \leq k\ \dot{x}\restriction m = x_j\restriction m$.

Since sets U_j are open it follows that for every j there is m such that $\bar{p}_m \Vdash_{\mathbb{P}} \dot{x} \in U_j$. In other words, given $\bar{p}, n \in \omega$, $j \in \omega$, and $\dot{x}$ as above we can find $\bar{q} \geq_n \bar{p}$, $\bar{q} \in N$ and $\bar{q} \Vdash_{\mathbb{P}} \dot{x} \in U_j$.

The rest of the construction is standard: we build in $\mathbf{V}$ a sequence $\langle (j_n, \dot{x}_n, A_n) : n \in \omega\rangle$ of all triples $(j, \dot{x}, A)$ where $j \in \omega$, $\dot{x} \in N$ is a $\mathbb{P}$-name for a real and $A \in N$ is a maximal antichain of $\mathbb{P}$. Using the basic step above, we build by induction a sequence $\langle p_n : n \in \omega\rangle$ such that

(1) $p_0 = p$,
(2) $p_n \in N$,
(3) $p_{n+1} \geq_n p_n$,
(4) $p_{n+1} \Vdash_{\mathbb{P}} \dot{x_n} \in U_{j_n}$,
(5) p_{n+1} is compatible with countably many elements of A_n.

Finally, let q be such that $q \geq p_n$ for all n. Clearly, q has the required properties. □

3. No small sets of size continuum

Suppose that $\mathcal{J}$ is a σ-ideal of sets of reals having Borel basis. Define $\mathcal{J}^\star = \{X \subseteq 2^\omega : \forall A \in \mathcal{J}\ X + A \neq 2^\omega\}$. In particular, $\mathcal{M}^\star = \mathcal{SN}$ and $\mathcal{N}^\star = \mathcal{SM}$.

Definition 13. *We say that $\mathcal{P}$ has property $P_{\mathcal{J}}$ if for every condition $p \in \mathcal{P}$, and a $\mathcal{P}$-name for a real $\dot{x}$ such that $p \Vdash_{\mathcal{P}} \dot{x} \notin \mathbf{V} \cap 2^\omega$ there exists a set $H \in \mathbf{V} \cap \mathcal{J}$ such that for all $z \in \mathbf{V} \cap 2^\omega$, there exists $q_z \geq p$ such that*

$$q_z \Vdash \dot{x} \in H + z.$$

Theorem 14. *Let $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2\}$ be a countable support iteration of proper forcing notions. If for every $\alpha < \omega_2$, $\mathcal{P}_{\omega_2}/\mathcal{P}_\alpha$ has property $P_{\mathcal{J}}$, then $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathcal{J}^\star \subseteq [\mathbb{R}]^{<2^{\aleph_0}}$.*

Proof. Suppose that $X \subseteq 2^\omega \cap \mathbf{V}^{\mathcal{P}_{\omega_2}}$ and $\Vdash_{\omega_2} X \in \mathcal{J}^\star$. Working in $\mathbf{V}^{\mathcal{P}_{\omega_2}}$, for every Borel set $H \in \mathcal{J}$ find a real x_H such that $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models x_H \notin H + X$. Granted that for every $\beta < \omega_2$, $\mathbf{V}^{\mathcal{P}_\beta} \models \mathsf{CH}$, we can find $\alpha < \omega_2$ such that $x_H \in \mathbf{V}^{\mathcal{P}_\alpha}$ for $H \in \mathbf{V}^{\mathcal{P}_\alpha}$. Suppose that $p \Vdash_{\omega_2} \dot{x} \notin \mathbf{V}^{\mathcal{P}_\alpha} \cap 2^\omega$. Apply property $P_{\mathcal{J}}$ to find $H \in \mathcal{J}$ with the required properties. In particular, there is $q \geq p$ such that $q \Vdash_{\omega_2} \dot{x} \in H + x_H$. It follows that $q \Vdash_{\omega_2} x_H \in H + \dot{x}$, thus $q \Vdash_{\omega_2} \dot{x} \notin X$. Since $\dot{x}$ and p were arbitrary, we conclude that $X \subseteq \mathbf{V}^{\mathcal{P}_\alpha} \cap 2^\omega$. □

Definition 15. *Suppose that $\mathbb{P}$ satisfies Axiom A. We say that $\mathbb{P}$ is strongly ω^ω-bounding if for every $p \in \mathbb{P}$, $n \in \omega$ and a $\mathbb{P}$-name $\dot{n}$, if $p \Vdash \dot{n} \in \omega$ then there exists $q \geq_n p$ and a set $A \in [\omega]^{<\omega}$ such that $q \Vdash \dot{n} \in A$.*

The following is (essentially) proved in [9], see also [1] theorem 8.2.14.

Theorem 16. *If $\mathbb{P}$ is strongly ω^ω-bounding then $\mathbb{P}$ has property $P_{\mathcal{M}}$. Let $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2\}$ be a countable support iteration of proper forcing notions such that for $\alpha < \omega_2$, $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha$ is strongly ω^ω-bounding. Then $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathcal{M}^\star = \mathcal{SN} \subseteq [\mathbb{R}]^{<2^{\aleph_0}}$.*

Laver forcing (and many other forcing notions) have property $\mathbb{P}_{\mathcal{N}}$, a somewhat weaker result was proved in [11], but whether this property is preserved, particularly for posets adding unbounded reals, is unclear. Therefore we will use the following notions.

Definition 17. *Suppose that $N \prec \mathbf{H}(\chi)$. We say that a sequence of clopen sets $\mathbf{C} = \langle C_n : n \in \omega \rangle$ is big over N if*

(1) *C_n's have pairwise disjoint supports,*
(2) *$\mu(C_n) \leq 2^{-n}$ for $n \in \omega$,*
(3) *for every infinite set $X \subseteq 2^\omega$, $X \in N$, there exists infinitely many n such that $X + C_n = 2^\omega$.*

Lemma 18. *Suppose that $\langle C_n : n \in \omega \rangle$ is big over N. Then for every $x \in 2^\omega$, $\langle C_n + x : n \in \omega \rangle$ is big over N.*

Proof. For a clopen set C, if $C + X = 2^\omega$ then $C + x + X = 2^\omega + x = 2^\omega$. □

Theorem 19. *For every countable model $N \prec \mathbf{H}(\chi)$ there exists a set $H \in \mathcal{N}$ which is big over N.*

Proof. We will need the following theorem:

Theorem 20 ([6], [1] theorem 3.3.9)**.** *Suppose that $X \subseteq 2^\omega$ is an infinite set and $\varepsilon > 0$. Then there exists a clopen set $C \subseteq 2^\omega$ such that $\mu(C) < \varepsilon$ and $X + C = 2^\omega$.*

Let $N \prec \mathbf{H}(\chi)$ be a countable model. Fix an enumeration (with infinitely many repetitions), $\{X_n : n \in \omega\}$ of all infinite subsets of $N \cap 2^\omega$ that are in N. By theorem 20, for each n there exists a clopen set C_n of measure $< 2^{-n}$ such that $C_n + X_n = 2^\omega$. □

Corollary 21. *Suppose that c is a Cohen real over $\mathbf{V}$. There exists a sequence $\langle C_n : n \in \omega \rangle \in \mathbf{V}[c]$ which is big over $\mathbf{V}$.*

Definition 22. *We say that a proper forcing notion $\mathbb{P}$ is good if for every $p \in \mathbb{P}$, a countable model $N \prec \mathbf{H}(\chi)$ containing $p, \mathbb{P}$ and a sequence $\mathbf{C} = \langle C_n : n \in \omega\rangle$ which is big over N there exists a condition $q \geq p$ such that*

(1) *q is $(N, \mathbb{P})$-generic over N,*
(2) *$q \Vdash_{\mathbb{P}} \langle C_n : n \in \omega\rangle$ is big over $N[\dot{G}]$,*
(3) *$q \Vdash_{\mathbb{P}} F_{\mathbf{C}} \cap N = F_{\mathbf{C}} \cap N[\dot{G}]$, where $F_{\mathbf{C}} = \{x \in 2^\omega : \forall^\infty n\ x \notin C_n\}$.*

Theorem 23. *If $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2\}$ is a countable support iteration and for every $\alpha < \omega_2$, $\Vdash_\alpha$ "$\dot{\mathcal{Q}}_\alpha$ is good" then $\mathcal{P}_{\omega_2}$ is good.*

Proof. Similar to the proof of theorem 8. Note that the requirements are represented by G_δ sets. □

Theorem 24. *Let $\{\mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \omega_2\}$ be a countable support iteration of proper forcing notions such that every $\alpha < \omega_2$, $\Vdash_\alpha$ $\dot{\mathcal{Q}}_\alpha$ is good. Then $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models \mathcal{SM} \subseteq [\mathbb{R}]^{<2^{\aleph_0}}$.*

Proof. Suppose that $X \subseteq 2^\omega \cap \mathbf{V}^{\mathcal{P}_{\omega_2}}$ and $\mathbf{V}^{\mathcal{P}_{\omega_2}} \models X \in \mathcal{SM}$. As in the proof of theorem 14, for $H \in \mathcal{N}$ let $x_H \in 2^\omega$ be such that $x_H \notin H + X$. We can assume that $x_H \in \mathbf{V}$ if $H \in \mathbf{V}$. Suppose that $\dot{x}$ is a $\mathcal{P}_{\omega_2}$-name for a real and $p \Vdash_{\mathcal{P}_{\omega_2}} \dot{x} \notin \mathbf{V}$. Let $N \prec \mathbf{H}(\chi)$ be a countable model containing $p, \mathcal{P}_{\omega_2}$, and $\dot{x}$. Let $\langle C_n : n \in \omega\rangle \in \mathbf{V}$ be big over N. Put $H = \bigcap_m \bigcup_{n>m} C_n \in \mathcal{N}$ and let $\mathbf{C} = \langle C_n + x_H : n \in \omega\rangle$. Sequence $\mathbf{C}$ is also big over N, and $H + x_H = \bigcap_m \bigcup_{n>m} C_n + x_H$. Since $\mathcal{P}_{\omega_2}$ is good we can find $q \geq p$ such that

(1) q is $(N, \mathcal{P}_{\omega_2})$-generic over N,
(2) $q \Vdash_{\omega_2} \mathbf{C}$ is big over $N[\dot{G}]$,
(3) $q \Vdash_{\omega_2} F_{\mathbf{C}} \cap N = F_{\mathbf{C}} \cap N[\dot{G}]$.

Since $p \Vdash \dot{x} \notin N$ it follows that $q \Vdash \dot{x} \notin F_{\mathbf{C}} = 2^\omega \setminus (H + x_H)$. In particular, $q \Vdash \dot{x} \in H + x_H$, and as before, $q \Vdash \dot{x} \notin X$. Thus $X \subseteq \mathbf{V} \cap 2^\omega$, which finishes the proof. □

4. Laver and Miller forcings

In this section we will show that Laver and Miller forcings are good, and therefore there are no strongly meager sets of size continuum in Laver's or Miller's model.

For a tree $p \subseteq 2^{<\omega}$ let $[p]$ denote the set of branches of p, and let $p_s = \{t \in p : s \subseteq t \text{ or } t \subseteq s\}$. For $s \in p$ and $m > |s|$ let $\mathsf{succ}_p^m(s) = \{t \in p : |s| + |t| = m \text{ and } s^\frown t \in p\}$. We will suppress the superscript m if its value will be clear from the context.

Let $\mathsf{split}(p) = \{s \in p : |\mathsf{succ}_p(s)| > 1\} = \bigcup_{n\in\omega} \mathsf{split}_n(p)$, where $\mathsf{split}_n(p) = \left\{s \in \mathsf{split}(p) : \left|\{t \subsetneq s : t \in \mathsf{split}(p)\}\right| = n\right\}$. The unique element of $\mathsf{split}_0(p)$ is called $\mathsf{stem}(p)$. For $s \in p$ let $A_s = \{n \in \omega : s^\frown n \in p\}$.

Definition 25. *The Laver forcing $\mathbb{L}$ is the following forcing notion:*

$$p \in \mathbb{L} \iff p \subseteq \omega^{<\omega} \text{ is a tree } \& \ \forall s \in p\ \big(|s| \geq \mathsf{stem}(p) \to |\mathsf{succ}_p(s)| = \aleph_0\big).$$

For $p, q \in \mathbb{L}$, $p \geq q$ if $p \subseteq q$.

For every $p \in \mathbb{L}$ and $s \in \omega^{<\omega}$ define a node $p(s)$ in the following way: $p(\emptyset) = \mathsf{stem}(p)$ and for $n \in \omega$ let $p(s^\frown n)$ be the n-th element of $\mathsf{succ}_p\big(p(s)\big)$.

For $p, q \in \mathbb{L}$ and $n \in \omega$ define

$$p \geq_n q \iff p \geq q \,\&\, \forall s \in n^{\leq n}\ p(s) = q(s).$$

In particular, $p \geq_0 q$ is equivalent to $p \geq q$ and $\mathsf{stem}(p) = \mathsf{stem}(q)$.

The rational perfect forcing (Miller forcing) $\mathbb{M}$ is the following forcing notion:

$$p \in \mathbb{M} \iff p \subseteq \omega^{<\omega} \text{ is a perfect tree } \& \\ \forall s \in p\ \exists t \in p\ \big(s \subseteq t \,\&\, |\mathsf{succ}_p(t)| = \aleph_0\big).$$

For $p, q \in \mathbb{M}$, $p \geq q$ if $p \subseteq q$. Without loss of generality we can assume that $|\mathsf{succ}_p(s)| = 1$ or $|\mathsf{succ}_p(s)| = \aleph_0$ for all $p \in \mathbb{M}$ and $s \in p$. Conditions of this type form a dense subset of $\mathbb{M}$.

For $p, q \in \mathbb{M}$, $n \in \omega$, we let

$$p \geq_n q \iff p \geq q \,\&\, \mathsf{split}_n(q) = \mathsf{split}_n(p).$$

Theorem 26. *Laver and Miller forcings are good.*

Proof. We will prove the theorem for the Laver forcing $\mathbb{L}$, the proof for Miller forcing is similar (and easier). □

Lemma 27. *Suppose that $p \Vdash_{\mathbb{L}} \dot{x} \in 2^\omega$. There exists $q \geq_0 p$ a set of reals $\{x_s : s \in q,\ \mathsf{stem}(q) \subseteq s\}$ such that*

(1) *if $\mathsf{stem}(q) \subseteq s \in q$ then for every n there is $r \geq_0 q_s$ such that $r \Vdash_{\mathbb{L}} \dot{x}{\restriction}n = x_s{\restriction}n$,*
(2) *for every s, mapping $n \rightsquigarrow x_{s^\frown n}$ ($n \in A_s$) is either one-to-one or constant,*
(3) $\lim_{n\in A_s} x_{s^\frown n} = x_s$.

Proof. Induction on levels of p. Suppose that $|\mathsf{stem}(p)| = k$ and define trees $\{p_n : n \in \omega\}$ such that

(1) $p_0 = p$,
(2) $p_{n+1} \cap \omega^{k+n} = p_n \cap \omega^{k+n}$ for every n,
(3) reals x_s are defined for $s \in p_{n+1} \cap \omega^{k+n}$ and satisfy conditions (1) and (3).

Suppose that p_n is given, and $s \in p_n \cap \omega^{k+n}$. For each $l \in A_s$ let $r_l^s \geq_0 (p_n)_{s^\frown n}$ and $s_l \in 2^l$ be such that $r_l^s \Vdash \dot{x}{\restriction}l = s_l$. Let $A_s' \subseteq A_s$ be an infinite set such that $\bigcup_{l\in A_s'} s_l = x_s \in 2^\omega$. Finally let $p_{n+1} = \bigcup_{s\in p_n\cap\omega^{k+n}} \bigcup_{l\in A_s'} r_l^s$. Clearly, $q' = \bigcap_n p_n$ satisfies (1) and (3). By pruning it further we get $q \geq_0 q'$ that satisfies (2) as well. □

Recall that $B \subseteq p$ is a front in p, that is for every branch $x \in [p]$ there is $n \in \omega$ such that $x{\restriction}n \in B$.

Lemma 28. *Suppose that $p \in \mathbb{L}$ and $B \subseteq p$. Then either there exists $q \geq_0 p$ such that $q \cap B = \emptyset$ or there exists $q \geq_0 p$ such that B is a front in q.*

Proof. Define rank function on p as follows, for $s \in p$

1. $\mathsf{rank}_B(s) = 0$ if $\exists n \leq |s|\ s{\restriction}n \in B$,
2. If $\mathsf{rank}_B(s) \neq 0$ then

$$\mathsf{rank}_B(s) = \min\{\alpha : \exists A \in [A_s]^\omega\ \forall n \in A\ \mathsf{rank}_B(s^\frown n) < \alpha\}.$$

If $\mathsf{rank}_B(\mathsf{stem}(p))$ is defined then let $q \geq_0 p$ be such that rank_B is defined on q and $\mathsf{rank}_B(s) > \mathsf{rank}_B(t)$ whenever $t \subsetneq s$. It follows that B is a front in q.

If $\mathsf{rank}_B(\mathsf{stem}(p))$ is not defined then there is $q \geq_0 p$ such that $\mathsf{rank}_B(s)$ is undefined for $s \in q$. Thus $q \cap B = \emptyset$. $\square$

Suppose that $N \prec \mathbf{H}(\chi)$, $\langle C_n : n \in \omega\rangle$ is big over N.

We will need the following two basic constructions.

Basic Step I. Given $p \in N \cap \mathbb{L}$, $k, m \in \omega$ and an $\mathbb{L}$-name for a real $\dot{x} \in N$ such that $p \Vdash_{\mathbb{L}} \dot{x} \notin N \cap 2^\omega$ we need $q \geq_k p$ and $n > m$ such that $q \Vdash_{\mathbb{L}} \dot{x} \in C_n$.

We can assume that $k = 0$, otherwise we repeat the construction described below for each "protected" node.

Working in N we can shrink p (without changing the stem) so that it satisfies conclusion of lemma 27. Let $B = \{s \in p : \forall i, j \in A_s\ x_{s^\frown i} \neq x_{s^\frown j}\}$. Applying lemma 28 we can further shrink p (without changing the stem) so that B is a front in $[p]$. Note that if $p \cap B = \emptyset$ then $p \Vdash_{\mathbb{L}} \dot{x} = x_{\mathsf{stem}(p)}$, which contradicts the choice of $\dot{x}$.

Let $Z \subseteq B$ be a maximal antichain in p. Fix $s \in Z$ and let $X_s = \{x_{s^\frown l} : l \in A_s\}$. Since $N \models X_s \in [2^\omega]^\omega$ the set

$$B_s = \{l \in A_s : \exists n > m\ x_{s^\frown l} \in C_n\}$$

is infinite. For each $l \in B_s$ fix n_l such that $x_{s^\frown l} \in C_{n_l}$. Let $q^s_l \geq_0 p_{s^\frown l}$ be such that $q^s_l \Vdash_{\mathbb{L}} \dot{x}{\restriction}\mathsf{supp}(C_{n_l}) = x_{s^\frown l}{\restriction}\mathsf{supp}(C_{n_l})$. In particular, $q^s_l \Vdash_{\mathbb{L}} \dot{x} \in C_{n_l}$.

Let $q_s = \bigcup_{l \in B_s} q^s_l$ and let $q = \bigcup_{s \in Z} q_s$. Clearly $q \geq_0 p$ and $q \Vdash_{\mathbb{L}} \exists n > m\ \dot{x} \in C_n$.

Basic Step II. Given $p \in N \cap \mathbb{L}$, $k, l \in \omega$ and an $\mathbb{L}$-name $\dot{X} = \{\dot{x}_j : j \in \omega\}$ for a countable set of reals in $N^{\mathbb{L}}$ we need $q \geq_k p$ and $n > l$ such that $q \Vdash_{\mathbb{L}} \dot{X} + C_n = 2^\omega$.

As before we can assume that $k = 0$. Define sequences $\langle q_j : j \in \omega\rangle$, $\{s^j_i : i \leq j, j \in \omega\}$ such that

(1) $q_0 = p$,
(2) $q_{j+1} \geq_0 q_j$,
(3) $|s^j_i| > j$ for $i \leq j$,
(4) $s^j_u \neq s^j_v$ if $u, v \leq j, u \neq v$,
(5) $s^j_u \subseteq s^i_u$ if $u \leq j \leq i$.
(6) $q_{j+1} \Vdash_{\mathbb{L}} \forall i \leq n\ s^j_i \subseteq \dot{x}_i$,

Let $x_i = \bigcup_{j>i} s^j_i$. Reals x_i are pairwise different and $X = \{x_i : i \in \omega\} \in N$. By the assumption, there exists $n > l$ such that $X + C_n = 2^\omega$. Find $l \in \omega$ such that

$C_n \subseteq 2^l$ and let j be so large that $q_j \Vdash_{\mathbb{L}} \{x{\restriction}l : x \in X\} = \{s_i^j{\restriction}l : i \leq j\}$. Clearly, $q_j \Vdash_{\mathbb{L}} \dot{X} + C_n = 2^\omega$.

The rest of the proof is standard induction. We proceed as in the proof of lemma 12. Let $\{(\dot{X}_n, \dot{x}_n, A_n) : n \in \omega\}$ be an enumeration of all sets $(\dot{X}, \dot{x}, A)$ in N where $\dot{X}$ is an $\mathbb{L}$-name for a countable sets of reals in N, $\dot{x}$ is a $\mathbb{L}$-name for a real such that $\Vdash \dot{x} \notin N \cap 2^\omega$, A is a maximal antichain in $\mathbb{L}$. Using the basic steps above, we build by induction a sequence $\langle p_n : n \in \omega\rangle$ such that

(1) $p_0 = p$,
(2) $p_{n+1} \geq_n p_n$,
(3) $p_{n+1} \Vdash \exists m > n\ \dot{x}_n \in C_m$,
(4) $p_{n+1} \Vdash \exists m > n\ \dot{X}_n + C_m = 2^\omega$,
(5) p_{n+1} is compatible with countably many elements of A_n.

Note that conditions produced in step I are not necessarily in N, but they are of form $\bigcup_{s \in Z} q_s$ where $Z \in N$ and $q_{s^\frown l} \in N$ for $s \in Z$, $l \in \omega$. This suffices to carry out the construction. □

It is not clear whether every forcing notion having Laver property (or PPP property) is nice, but many of them are.

5. Strongly meager sets of size continuum

The simplest case when we have large strongly meager sets is when there are many random reals around.

Lemma 29. *If* $\mathsf{cov}(\mathcal{N}) = 2^{\aleph_0}$ *then* $[2^\omega]^{<2^{\aleph_0}} \subsetneq \mathcal{SM}$.

Proof. Suppose that $|X| < 2^{\aleph_0}$ and $H \in \mathcal{N}$. It follows that $H + X = \bigcup_{x \in X} H + x \neq 2^\omega$. Therefore $X \in \mathcal{SM}$.

To construct a strongly meager set of size continuum fix an enumeration $\{H_\alpha : \alpha < 2^{\aleph_0}\}$ of Borel measure zero sets and construct inductively reals $\{x_\alpha, z_\alpha : \alpha < 2^{\aleph_0}\}$ such that

(1) $\{x_\beta : \beta < \alpha\} \cap (H_\alpha + z_\alpha) = \emptyset$,
(2) $x_\alpha \notin \bigcup_{\beta < \alpha} H_\beta + z_\beta$.

Note that $X = \{x_\alpha : \alpha < 2^{\aleph_0}\}$ is the required set since $z_\beta \notin H_\beta + X$. □

Theorem 30. *It is consistent with* ZFC *that* $\mathsf{cov}(\mathcal{N}) < 2^{\aleph_0}$ *and there exists a strongly meager set of size continuum.*

Proof. Let $\mathbf{V}'$ be a model obtained by adding $\aleph_2$ Cohen reals to a model $\mathbf{V}_0 \models \mathsf{CH}$, and let $\mathbf{V}$ be obtained by adding $\aleph_1$ random reals (side-by-side) to $\mathbf{V}'$. It is easy to see that $\mathbf{V} \models \mathbf{V}' \cap 2^\omega \in \mathcal{SM}$ (as witnessed by random reals), and it is well known ([12], [1] theorem 3.3.24) that $\mathbf{V} \models \mathsf{cov}(\mathcal{N}) = \aleph_1$. □

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