

General topology meets model theory, on \mathfrak{p} and \mathfrak{t}

Maryanthe Malliaris^{a,1} and Saharon Shelah^{b,c}

^aDepartment of Mathematics, University of Chicago, Chicago, IL 60637; ^bEinstein Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem 91904, Israel; and ^cDepartment of Mathematics, Rutgers University, Piscataway, NJ 08854

Edited* by Spencer J. Bloch, University of Chicago, Chicago, IL, and approved June 4, 2013 (received for review April 2, 2013)

Cantor proved in 1874 [Cantor G (1874) *J Reine Angew Math* 77:258–262] that the continuum is uncountable, and Hilbert’s first problem asks whether it is the smallest uncountable cardinal. A program arose to study cardinal invariants of the continuum, which measure the size of the continuum in various ways. By Gödel [Gödel K (1939) *Proc Natl Acad Sci USA* 25(4):220–224] and Cohen [Cohen P (1963) *Proc Natl Acad Sci USA* 50(6):1143–1148], Hilbert’s first problem is independent of ZFC (Zermelo-Fraenkel set theory with the axiom of choice). Much work both before and since has been done on inequalities between these cardinal invariants, but some basic questions have remained open despite Cohen’s introduction of forcing. The oldest and perhaps most famous of these is whether “ $\mathfrak{p} = \mathfrak{t}$,” which was proved in a special case by Rothberger [Rothberger F (1948) *Fund Math* 35:29–46], building on Hausdorff [Hausdorff (1936) *Fund Math* 26:241–255]. In this paper we explain how our work on the structure of Keisler’s order, a large-scale classification problem in model theory, led to the solution of this problem in ZFC as well as of an a priori unrelated open question in model theory.

unstable model theory | cardinal invariants of \mathfrak{c} | maximal Keisler class

In this paper we present our solution to two long-standing, and a priori unrelated, questions: the question from set theory/general topology of whether $\mathfrak{p} = \mathfrak{t}$, the oldest problem on cardinal invariants of the continuum, and the question from model theory of whether SOP_2 is maximal in Keisler’s order. Before motivating and precisely stating these questions, we note there were two big surprises in this work: first, the connection of the two questions and, second, that the question of whether “ $\mathfrak{p} < \mathfrak{t}$ ” can be decided in ZFC (Zermelo-Fraenkel set theory with the axiom of choice).

There have been few connections between general topology and model theory, and these were exclusively in model theory: a key example is Morley’s use of ideas from general topology such as the Cantor-Bendixson derivative in proving his 1965 categoricity theorem (1), the cornerstone of modern model theory. Here we have applications in the other direction: by our proofs below, model theoretic methods solve an old problem in general topology. It seems too early to tell what further interactions will arise from these methods, but it is a good sign that *Claim 1* below will be generalized to $\mathcal{P}(\mathbb{N})/\text{fin}$; see *Discussion 21*. Full details of work presented here are available in ref. 2.

Cardinal Invariants of the Continuum

Cantor’s 1874 proof of the uncountability of the continuum (3) focuses attention on the region between \aleph_0 , the cardinality of the natural numbers, and the continuum 2^{\aleph_0} , the cardinality of the reals. A productive way to map this region is to consider properties of families of subsets of \mathbb{N} that hold for any countable family and fail for some family of size continuum. For any such property of interest, call the minimum size κ of a family for which the property may fail a cardinal invariant (or characteristic) of the continuum. Theorems about the relative sizes and interdependence of such cardinal invariants give fundamental structural information about families of subsets of \mathbb{N} and give precise ways to test the strength of countability hypotheses. As a simply stated example of such a cardinal, the bounding number \mathfrak{b} is the least size of a family $F \subseteq {}^\omega\omega$ that is not bounded, meaning that there is no single $g \in {}^\omega\omega$ that eventually dominates all $f \in F$; i.e., $f \leq^* g$ for

all $f \in F$. As an example of results, the Cichoń diagram (ref. 4, p. 424) gives implications between cardinal invariants relating primarily to measure and category.

The study of cardinal invariants of the continuum is a flourishing area, which lies at the intersection of set theory and general topology; some properties reflect ideas from measure theory, algebra, or combinatorics. The subject’s growth and development from the 1930s and 1940s, in part a response to Hilbert’s problem, to the present can be seen in the surveys of van Douwen (5), Vaughan (6), and Blass (4).

By Gödel (7) and Cohen (8), Hilbert’s first problem (whether 2^{\aleph_0} is the first uncountable cardinal) is independent of ZFC. In light of this, the study of cardinal invariants of the continuum becomes especially fertile. In most cases there are obvious ZFC results and usually independence proofs are deep and hard (using forcing, of Cohen). ZFC answers to problems that have remained open for some time are rare and, so, surprising.

The problem of “ \mathfrak{p} and \mathfrak{t} ” appears throughout the literature. In the seminal survey paper mentioned above, van Douwen cataloged and consolidated much of the work on these cardinals before 1984. He focused on six primary invariants: \mathfrak{b} , \mathfrak{p} , \mathfrak{t} [attributed to Rothberger (9, 10)], \mathfrak{d} [attributed to Katětov (11)], \mathfrak{a} [attributed to Hechler (12) and Solomon (13)] and \mathfrak{s} [attributed to Booth (14)]. As of Vaughan (6), only two inequalities about van Douwen’s cardinals remained open, that of \mathfrak{a} and \mathfrak{d} and that of \mathfrak{p} and \mathfrak{t} , and “we believe [whether $\mathfrak{p} < \mathfrak{t}$ is consistent with ZFC] to be the most interesting” (ref. 6, section 1.1). Following Shelah’s (15) proof of the independence of $\mathfrak{a} < \mathfrak{d}$, attention focused on whether $\mathfrak{p} = \mathfrak{t}$ as both the oldest and the only remaining open inequality about van Douwen’s diagram.

We now define \mathfrak{p} and \mathfrak{t} . Let $A \subseteq^* B$ mean $\{x : x \in A, x \notin B\}$ is finite. Let $D \subseteq [\mathbb{N}]^{\aleph_0}$ be a family of countable subsets of \mathbb{N} . Say that D has a pseudointersection if $\exists A \subseteq \mathbb{N}$, with A infinite such that $(\forall B \in D)(A \subseteq^* B)$. Say that D has the strong finite intersection property (s.f.i.p.) if every nonempty finite subfamily of D has an infinite intersection. Finally, say that D is a tower if it is linearly ordered by \supseteq^* and has no infinite pseudointersection.

Definition 1: The cardinal \mathfrak{p} is the minimum size of a family $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ such that \mathcal{F} has the s.f.i.p. but no infinite pseudointersection. The cardinal \mathfrak{t} is the minimum size of a tower.

It is easy to see that $\mathfrak{p} \leq \mathfrak{t}$, since a tower has the s.f.i.p. Hausdorff proved in 1936 (16) that $\aleph_1 \leq \mathfrak{p}$, and Rothberger proved in 1948 (in our terminology) that $\mathfrak{p} = \aleph_1 \implies \mathfrak{p} = \mathfrak{t}$. This begs the question:

Question 2. Does $\mathfrak{p} = \mathfrak{t}$?

After Rothberger (10), there has been much work on \mathfrak{p} and \mathfrak{t} , as noted in the surveys (4–6), the introduction to our paper (2), and also refs. 17–21. As noted, given the length of time the problem remained open, there was wide confidence of an independence result.

Author contributions: M.M. and S.S. designed research, performed research, and wrote the paper.

The authors declare no conflict of interest.

*This Direct Submission article had a prearranged editor.

See Commentary on page 13238.

¹To whom correspondence should be addressed. E-mail: mem@math.uchicago.edu.

Model Theory

From Morley's Theorem to Keisler's Order. Our solution to *Question 2* arose in the context of our work on a model-theoretic classification program known as Keisler's order. As a frame for these investigations, we briefly discuss model theory, stability, and ultraproducts. Model theory begins with the study of models, which are given by the data of an underlying set along with an interpretation for all function, relation, and constant symbols in a fixed background language \mathcal{L} ; (complete) theories, sets of \mathcal{L} -sentences of first-order logic true in some \mathcal{L} -model; and elementary classes, classes of \mathcal{L} -models that share the same theory, written $Th(M) = Th(N)$ or $M \equiv N$.

It is well known that any two algebraically closed fields of the same characteristic and the same uncountable size are isomorphic. Los could not find any countable first-order theory that had one model, up to isomorphism, in some but not every uncountable cardinality, and so conjectured there was none. The cornerstone of modern model theory is the following affirmative answer:

Theorem A [Morley (1)]. *If T is a countable theory, then either T is categorical [= has one model, up to isomorphism] in every uncountable cardinal or T is categorical in no uncountable cardinal.*

Proofs of Morley's theorem, due to Morley and later to Baldwin and Lachlan (22), show, in an abstract setting, that whenever a complete theory in a countable language is categorical in some uncountable cardinal, then its models admit analogs of many properties familiar from algebraically closed fields: for instance, the existence of prime models, a well-defined notion of independence generalizing algebraic independence in algebraically closed fields, and the existence of a maximal independent set whose size determines the model up to isomorphism. Often, model theoretic arguments extract local structural information from global constraints on a given elementary class (e.g., the property of having one model, up to isomorphism, in some uncountable size).

Model theory since Morley has developed in a number of ways; a central theme (and one crucial for our work) is stability, which we now describe. Let M be a model and $A \subseteq M$. The $[1]$ -types over A are the maximal consistent sets of formulas $\varphi(x)$ in one free variable with parameters from A . Informally, types describe elements that may or may not exist in a given model M but that will always exist in some extension of M , e.g., a transcendental element that will exist in field extensions of \mathbb{Q}^{alg} . A realization of a type p in a model M is an element a such that $\varphi(a)$ is true in M for all $\varphi \in p$. When λ is an infinite cardinal and the model M contains realizations of all types over all subsets of M of size $< \lambda$, we call M λ -saturated. For instance, an algebraically closed field of fixed characteristic is λ -saturated if and only if it has transcendence degree at least λ . Alternatively and informally, if a model M is considered as sitting inside a large universal domain \mathcal{M} , the 1-types over $A \subseteq M$ are the orbits under automorphisms of \mathcal{M} fixing A pointwise, and M is λ -saturated if it contains representatives of all orbits under all automorphisms of \mathcal{M} fixing some set $A \subseteq M$, $|A| < \lambda$ pointwise.

We now define "stability" in a given cardinal λ , which led to a fundamental structural dichotomy between theories that are stable in many λ and those that are never stable, *Theorem B* below.

Definition 3: T is λ -stable if there are no more than λ -types over any $M \models T$, $|M| = \lambda$.

Example 4: If M is an algebraically closed field, the type of an element over M is determined by the polynomials over M it does or does not satisfy; this gives $|M|$ types ["stable"]. If $M = (\mathbb{Q}, <)$, the types over M include all cuts; this gives $2^{|M|}$ types ["unstable"]. If M is the Rado graph, the types over M include all partitions of the vertex set of M ; this gives $2^{|M|}$ types [unstable].

The remarkable fact is that the stability or instability of models resolves into a very informative classification of theories (as in Morley's theorem above, much of the information comes from

the structural analysis that goes into the proof). In *Theorem B*, $\kappa(T) \leq |T|^+$ is an invariant of the theory.

Theorem B [Shelah (23)]. *A complete theory T is either*

- *stable, i.e., T is stable in all $\lambda = \lambda^{<\kappa(T)}$, or*
- *unstable, i.e., T is unstable in all λ .*

Under this analysis, the stable theories are "tame" or well behaved, admitting a strong structure theory, whereas the unstable theories are "wild".

Discussion 5. *Moreover, stability is local: T is unstable if and only if it contains an unstable formula, i.e., a formula with the order property. Also, instability hides a structure/randomness phenomenon: Any unstable theory contains either a configuration close to a bipartite random (Rado) graph or one close to a definable linear order.*

Meanwhile, Keisler proposed a very different way of classifying theories (and ultrafilters), from an asymptotic (ultrapower) point of view. It is of particular interest that Keisler's order gives an outside definition of the stable theories; see *Discussion 10* below.

An Asymptotic Classification. Keisler's order, which we now explain, studies the relative complexity of theories according to the difficulty of ensuring regular ultrapowers of some $M \models T$ that are saturated.

Definition 6. The filter \mathcal{D} on I , $|I| = \lambda$ is called regular if it contains a regularizing family; i.e., $X = \{X_i : i < \lambda\} \subseteq \mathcal{D}$ such that any infinite subset of X has empty intersection.

It is consistent that all nonprincipal ultrafilters are regular. Moreover, if \mathcal{D} is regular and the background language \mathcal{L} is countable, then $M \equiv N \Rightarrow M^\lambda / \mathcal{D}$ is λ^+ -saturated if N^λ / \mathcal{D} is, so the quantification over all models in *Definition 7* below is justified.

An ultraproduct of models $\langle M_i : i \in I \rangle$ with respect to the ultrafilter \mathcal{D} on I is the structure obtained by identifying elements of the product $\prod_i M_i$ that are " \mathcal{D} -almost everywhere" equivalent (24). An ultrapower is an ultraproduct in which all of the models M_i are the same. Ultrapowers are large (and, in some sense, generic) extensions of a given model M that remain in its elementary class. Their structure reflects and depends on the choice of ultrafilter \mathcal{D} in subtle ways that are not yet well understood. However, precisely this dependence suggests it is informative to compare two theories by comparing ultrapowers of their respective models built using the same ultrafilter.

Definition 7 [Keisler's order (25)]: Let T_1, T_2 be countable theories. We say that $T_1 \trianglelefteq T_2$ if, for all infinite λ , all \mathcal{D} regular on λ , all $M_1 \models T_1$, all $M_2 \models T_2$, if $(M_2)^\lambda / \mathcal{D}$ is λ^+ -saturated, then $(M_1)^\lambda / \mathcal{D}$ is λ^+ -saturated.

Problem 1. *Determine the structure of Keisler's order.*

Keisler's order is a problem about constructing ultrafilters on the one hand and understanding theories on the other hand. We now give some intuition for this problem and some of the main developments; for further details, see, e.g., ref. 26.

The power of Keisler's order comes from the fact that we may regard types as maps from $[\lambda]^{<\aleph_0}$ into \mathcal{D} . Los's theorem associates to each type p over $A \subseteq M^\lambda / \mathcal{D}$, $|A| \leq \lambda$ a monotonic [i.e., $u \subseteq v \Rightarrow f(v) \subseteq f(u)$] map $f : [p]^{<\aleph_0} \rightarrow \mathcal{D}$ given by $\{\varphi_{i_0}(x; a_{i_0}), \dots, \varphi_{i_n}(x; a_{i_n})\} \mapsto$

$$\left\{ t \in \lambda : M \models \exists x \bigwedge_{t \leq s} \varphi_{i_t}(x; a_{i_t}[t]) \right\}$$

When \mathcal{D} is regular, such a map can be chosen to be multiplicative; i.e., $f(u \cup v) = f(u) \cap f(v)$, precisely when the type p is realized; see, e.g., ref. 26, section 1.2. Moreover, when \mathcal{D} is regular, we may choose the map so that its image is a regularizing family.

Thus, in certain cases, the problem of ensuring saturation in ultrapowers is directly reflected in a property of the underlying ultrafilter.

Definition 8 (Keisler): \mathcal{D} is λ^+ -good if every monotonic function $f: [\lambda]^{<\aleph_0} \rightarrow \mathcal{D}$ has a multiplicative refinement g ; i.e., $u, v \in [\lambda]^{<\aleph_0}$ implies $f(u) \supseteq g(u) \in \mathcal{D}$ and $g(u) \cap g(v) = g(u \cup v)$.

Good regular ultrafilters exist on all infinite λ by a theorem of Keisler assuming the generalized continuum hypothesis (27) and of Kunen in ZFC (28). If \mathcal{D} is regular and λ^+ -good, then M^λ/\mathcal{D} is therefore λ^+ -saturated for any M in a countable language. This is half of proving that Keisler's order has a maximum class. In fact, in the paper defining the order \preceq , Keisler showed the following:

Theorem C [Keisler (25)].

$$\mathbf{T}_{\min} \preceq \dots \preceq \mathbf{T}_{\max};$$

i.e., Keisler's order has a minimum class and a maximum class.

Example 9: It follows from Theorem A above that any uncountably categorical theory, e.g., the theory of algebraically closed fields of a fixed characteristic, is minimum in Keisler's order. On the other hand, one can show that any sufficiently rich theory, e.g., number theory, will code any failures of goodness into a non-realized type and will therefore be maximal.

The only Keisler-equivalence classes characterized to date were given by the following:

Theorem D [Shelah (ref. 23, chap. VI)].

$$\mathbf{T}_1 \triangleleft \mathbf{T}_2 \triangleleft \dots \triangleleft \mathbf{T}_{\max},$$

where $\mathbf{T}_1 \cup \mathbf{T}_2$ is precisely the stable theories, and

- \mathbf{T}_1 , the minimum class, consists of the theories without the finite cover property (f.c.p.).
- \mathbf{T}_2 , the next largest class, consists of stable theories with f.c.p.
- \mathbf{T}_{\max} contains all linear orders, e.g., $\text{Th}(\mathbb{Q}, <)$, although its model-theoretic identity is not known. [Indeed, the weaker order property SOP_3 suffices for maximality, by Shelah (29).]

Discussion 10. In Theorem D the class of stable theories, characterized in terms of counting types in Theorem B, is independently characterized in terms of saturation of ultrapowers.

Discussion 11. From Theorems C and D we see that although Keisler's order has a maximum class, we have only weak upper and lower bounds on properties for membership in this class.

In the intervening years, progress was slow, due in part to the perceived complexity of ultrafilters. Beginning in 2009, Malliaris advanced the problem of how ultrafilters and theories interact in a series of papers (30–33), motivated by Discussion 10. That is, given that Keisler's order independently detects the jump in complexity between stable and unstable theories, the goal was to leverage Keisler's order to describe gradations in complexity among the unstable theories from a more uniform point of view. See Malliaris (33) for a further discussion. For the present work, we quote the following:

Theorem E [Malliaris (30)]. For regular ultrapowers in a countable language, local saturation implies saturation. That is, Keisler's order reduces to the study of q -types.

Theorem F [Malliaris (32)]. There is a property of filters, called flexibility, which is detected by any theory that is nonlow (some formula k -divides with respect to arbitrarily large k).

Theorem G [Malliaris (32, 33)]. There is a minimum Keisler class among the theories with TP_2 , whose saturation is characterized by

existence in \mathcal{D} of internal maps between small sets and by the ultrafilter \mathcal{D} being “good for equality”.

Discussion 12. Keisler's order imposes a hierarchy on the structure/randomness phenomenon from Discussion 5: The “structured” (in some sense, rich) theories, those with linear (strict) order, are Keisler-maximum whereas the “purest” random theory, that of the Rado graph, is Keisler-minimum among the unstable theories. See ref. 33 or ref. 26, section 4. TP_2/SOP_2 is a randomness/structure phenomenon in nonsimple theories with analogies to the independence/strict order phenomenon in nonstable theories. Theorem G shows that TP_2 , the “random side” of this more complex example, also admits a Keisler-minimum class, of a particularly simple form. The very interesting idea that Keisler's order may further illuminate randomness/structure trade-offs within instability is supported by our Theorem 3 below.

Working together, Malliaris and Shelah have very recently made significant progress on the problem of Keisler's order in the 2012 papers (26, 34–36). These theorems are not prerequisites for the current proofs, so we refer the interested reader to the introduction of ref. 35 or ref. 36 for more details. However, this work provided strong motivation for addressing the long latent Question 16, which we now describe and solve in connection with $\mathfrak{p} = \mathfrak{t}$.

The Current Approach: \mathfrak{p} , \mathfrak{t} , and SOP_2

In this section we motivate and announce the main results of our present work (2), with some comments on the proof.

Orders, Trees, and the Keisler-Maximal Class. As described above, we have the following problem:

Problem 13. Give a model-theoretic characterization of the maximum class in Keisler's order.

This is an old important problem in model theory, for which we know lower bounds and upper bounds: instability on one hand, and the strict order property, or even SOP_3 , on the other (Discussion 11). The main theorems of our present work arose as part of our solution to Question 16 below, which moves the conjectured boundary of this class onto what appears to be a major dividing line for which there are strong general indications of a theory.

Before explaining this comment, let us define the following:

Definition 14: The formula φ has SOP_2 with respect to a theory T if in some sufficiently saturated $M \models T$ there are parameters $\{a_\eta : \eta \in {}^\omega 2\} \subseteq {}^{\ell(\varphi)} M$ such that

$$\text{for each } 1 \leq n < \omega \text{ and } \eta_1, \dots, \eta_n \in {}^{>2},$$

$$\{\varphi(x; a_{\eta_i}) : 1 \leq i \leq n\}$$

is consistent if and only if (iff) η_1, \dots, η_n lie along a single branch. We say the theory T has SOP_2 if some φ does, with respect to T .

Example 15: In $(\mathbb{Q}, <)$ the formula $\varphi(x; y, z) = y < x < z$ has SOP_2 ; SOP_2 also arises in the generic K_n -free graph for any $n \geq 3$, a theory well into the “independent” region of instability.

Recall from Theorem D that any theory that includes a definable linear order, e.g., $(\mathbb{Q}, <)$, or even just SOP_3 (which we leave as a black box; see ref. 29 for details), is maximal. Whereas SOP_3 retains many features of linear order, SOP_2 describes a kind of maximally inconsistent tree so did not appear amenable to the same kinds of arguments. We have the implications

$$\text{not stable} \leftarrow \text{SOP}_2 \leftarrow \text{SOP}_3 \leftarrow \text{strict order property}.$$

Following Shelah (29), the key question, which we solve below, became the following:

Question 16. *Is every theory with SOP_2 maximal in Keisler's order?*

Why is the move from SOP_3 to SOP_2 so significant? First, there is the nontrivial matter of developing a framework to compare orders and trees. Second, recent evidence suggests that SOP_2 may characterize maximality, *Conjecture 29* below.

To set the stage for the framework that led to the solution of both *Questions 16* and *2*, we present our strategy for solving *Question 16*. The first key idea was to describe the realization of SOP_2 -types in terms of a property of an ultrafilter \mathcal{D} , which could then be compared against goodness for \mathcal{D} , as by *Theorem D* above \mathcal{D} is good iff $(\mathbb{N}, <)^{\lambda}/\mathcal{D}$ has no (κ_1, κ_2) -cuts for $\kappa_1 + \kappa_2 \leq \lambda$.

Definition 17: A subset Y of a partially ordered set $(X, <)$ is cofinal in X if $(\forall x \in X)(\exists y \in Y)(x \leq y)$. We use the word *coinitial* when the order is reversed. By the cofinality of a partially ordered set we mean the smallest size of a cofinal subset. Call an infinite cardinal κ regular if, considered as an ordinal, its cofinality is κ .

Note that in this paper, we have two different uses of the word “regular”: as applied to ultrafilters, *Definition 6* above, and as applied to cardinals, *Definition 17*.

Definition 18: Let κ, θ be infinite regular cardinals and $(A, <)$ be a total linear order. We say there is a (κ, θ) -cut or gap in A when we can write $A = A_1 \cup A_2$ such that (a) $A_1 \cap A_2 = \emptyset$, (b) $(\forall a \in A_1)(\forall b \in A_2)(a < b)$, (c) there is a cofinal κ -indexed sequence in $(A_1, <)$, and (d) there is a coinitial θ -indexed sequence in $(A_2, <)$. So in our terminology, cuts are not filled.

Definition 19: The regular ultrafilter \mathcal{D} on λ has κ -treetops if, whenever $M = (T, \triangleleft)$ is a tree and $N = M^{\lambda}/\mathcal{D}$ is its ultrapower, any \triangleleft -increasing sequence in N of cofinality $< \kappa$ has an upper bound.

We now have the following measure of the complexity of \mathcal{D} ,

$$\mathcal{C}(\mathcal{D}) = \{(\kappa_1, \kappa_2) : \kappa_1, \kappa_2 \text{ regular}, \kappa_1 + \kappa_2 \leq |\lambda|,$$

$$\text{and } (\mathbb{N}, <)^{\lambda}/\mathcal{D} \text{ has a } (\kappa_1, \kappa_2)\text{-cut}\},$$

recalling that $\mathcal{C}(\mathcal{D}) = \emptyset$ iff \mathcal{D} is good (“maximally complex”, that is, able to saturate ultrapowers of linear order, which belong to the Keisler-maximum class).

We have therefore translated *Question 16* to

Question 20. *Suppose \mathcal{D} is a regular ultrafilter on λ with λ^+ -treetops. Is $\mathcal{C}(\mathcal{D}) = \emptyset$?*

To illustrate some simple properties of the cofinality spectrum $\mathcal{C}(\mathcal{D})$ under λ^+ -treetops, and to motivate the later abstraction of “cofinality spectrum problems”, we now sketch the proof of a uniqueness result in this context.

Claim 1 [ref. 2, claim 3.3, in the Special Case of Ultrapowers]. *Let \mathcal{D} be a regular ultrafilter on λ , $\kappa = \text{cf}(\lambda) \leq \lambda$, and suppose \mathcal{D} has λ^+ -treetops. If $(\kappa, \theta_0) \in \mathcal{C}(\mathcal{D})$ and $(\kappa, \theta_1) \in \mathcal{C}(\mathcal{D})$, then $\theta_0 = \theta_1$.*

Proof: Let $M = (\mathbb{N}, <)$ and let $N = M^{\lambda}/\mathcal{D}$. Suppose that in N , $(\langle a_{\alpha}^0 : \alpha < \kappa \rangle, \langle b_{\epsilon}^0 : \epsilon < \theta_0 \rangle)$ represents a (κ, θ_0) -cut whereas $(\langle a_{\alpha}^1 : \alpha < \kappa \rangle, \langle b_{\epsilon}^1 : \epsilon < \theta_1 \rangle)$ represents a (κ, θ_1) -cut.

As ultrapowers commute with reducts, we may suppose, in an expanded language [e.g., expanding M to $(\mathcal{H}(\aleph_1), \epsilon)$], that M includes a definable tree T of finite sequences of pairs of natural numbers, strictly increasing in each coordinate, with \triangleleft denoting a definable partial order on T corresponding to an initial segment. Identifying N with its induced expansion to the larger language, let (T^N, \triangleleft^N) denote the \mathcal{D} -ultrapower of this tree. For ease of reading, write $c(n, i)$ for the i th coordinate of $c(n)$.

By induction on $\alpha < \kappa$, we choose $c_{\alpha} \in T^N$ and $n_{\alpha} = \max \text{dom}(c_{\alpha})$ such that $c_{\alpha}(n_{\alpha}, i) = a_{\alpha}^i$ for $i = 0, 1$. The base and successor cases are easily done. In the limit case, we will have defined a sequence $\langle c_{\beta} : \beta < \kappa \rangle$ of elements lying along a single branch in T^N . The assumption of treetops gives an upper

bound $c_* \in T^N$ for this sequence; i.e., $c_{\beta} \triangleleft c_*$ for $\beta < \kappa$. Then, in our expanded language, the set

$$\{n \leq n_* : c_*(n, 0) < a_{\alpha}^0 \wedge c_*(n, 1) < a_{\alpha}^1\}$$

is a nonempty, definable, bounded subset of \mathbb{N}^N , and so it has a maximal element m . To complete this step, set $c_{\alpha} := c_* \upharpoonright_m \langle a_{\alpha}^0, a_{\alpha}^1 \rangle$.

Having completed the definition of the sequence $\langle c_{\alpha} : \alpha < \kappa \rangle$, we may again by treetops find an upper bound for this sequence, $c_{\infty} \in T^N$. Let $n_{\infty} = \max \text{dom}(c_{\infty})$. For $i \in \{0, 1\}$ and $\epsilon < \theta_i$, let $n_{\epsilon}^i = \max\{n \leq n_{\infty} : c_{\infty}(n, i) < b_{\epsilon}^i\}$. This is again a nonempty, definable, bounded subset of the nonstandard natural numbers, so each $n_{\epsilon}^0, n_{\epsilon}^1$ is well defined. However, now we have effectively “sewn together” the two original cuts. The sequences $\langle b_{\epsilon}^i : \epsilon < \theta \rangle$ and $\langle c(n_{\epsilon}^i, i) : i < \theta \rangle$ are mutually cofinal for $i = 0, 1$. Suppose for a contradiction that, in \mathbb{N}^N , the sequences $(\langle n_{\alpha} : \alpha < \kappa \rangle, \langle n_{\epsilon}^0 : \epsilon < \theta_0 \rangle)$ and $(\langle n_{\alpha} : \alpha < \kappa \rangle, \langle n_{\epsilon}^1 : \epsilon < \theta_1 \rangle)$ do not describe the same cut. Without loss of generality, $\langle n_{\epsilon}^0 : \epsilon < \theta_0 \rangle$ is not cofinal in $\langle n_{\epsilon}^1 : \epsilon < \theta_1 \rangle$. So there is some $\epsilon_* < \theta_1$ such that (recalling we have strict monotonicity in each coordinate) $c_{\infty}(n_{\epsilon_*}^1, 0)$ lies in what we assumed to be a cut described by $(\langle a_{\alpha}^0 : \alpha < \kappa \rangle, \langle b_{\epsilon}^0 : \epsilon < \theta_0 \rangle)$, a contradiction. Because we assumed θ_0, θ_1 are regular, $\theta_0 = \theta_1$, as desired. ●

Discussion 21. *In work in progress, we have an analog of Claim 1 for $\mathcal{P}(\mathbb{N})/\text{fin}$.*

Before describing our approach in full generality, we recall that the hypothesis $\mathfrak{p} < \mathfrak{t}$ also connects to questions about cuts.

Peculiar Cuts in \mathfrak{p} and \mathfrak{t} . Here we describe a further useful ingredient that connects $\mathfrak{p} < \mathfrak{t}$ to the appearance of a certain kind of cut (gap), *Definition 18*. The $*$ in \leq^* means “for all but finitely many”.

Definition 22 (peculiar cuts): For κ_1, κ_2 infinite regular cardinals, a (κ_1, κ_2) -peculiar cut in ${}^{\omega}\omega$ is a pair $(\langle f_{\alpha} : \alpha < \kappa_1 \rangle, \langle g_{\beta} : \beta < \kappa_2 \rangle)$ of sequences of functions in ${}^{\omega}\omega$ such that (a) $\alpha < \alpha' < \kappa_1$ and $\beta < \beta' < \kappa_2$ implies $f_{\alpha} \leq^* f_{\alpha'} \leq^* g_{\beta'} \leq^* g_{\beta}$, (b) $(\forall \alpha < \kappa_1)(f_{\alpha} \leq^* h)$ implies $(\exists \beta < \kappa_2)(g_{\beta} \leq^* h)$, and (c) $(\forall \beta < \kappa_2)(h \leq^* g_{\beta})$ implies $(\exists \alpha < \kappa_1)(h \leq^* f_{\alpha})$.

Theorem H [Shelah (37)]. *Assume $\mathfrak{p} < \mathfrak{t}$. Then for some regular cardinal κ there exists a (κ, \mathfrak{p}) -peculiar cut in ${}^{\omega}\omega$, where $\aleph_1 \leq \kappa < \mathfrak{p}$.*

Looking ahead, in the context of forcing, we will connect Theorem H to a general version of the “cut spectrum” $\mathcal{C}(\mathcal{D})$, as follows:

Example 23: If $G \subseteq [{}^{\omega}\omega]^{\aleph_0}$ is a generic ultrafilter, then writing $X = \prod_{G \in \mathcal{G}_0} G$, we have that $\langle f_{\alpha}/G : \alpha < \kappa_1 \rangle, \langle g_{\beta}/G : \beta < \kappa_2 \rangle$ give a (κ_1, κ_2) -cut in X .

A Common Context for Both Problems. The right framework, discovered in ref. 2, is more general than regular ultrapowers; however, in defining it we draw on intuition from the proof of *Claim 1* sketched above. The abstraction captures the fact that when studying regular ultrapowers of $(\mathbb{N}, <)$, because ultrapowers commute with reducts, we may assume that the nonstandard copy of \mathbb{N} in the ultrapower N continues to behave in a “pseudofinite” way (e.g., all nonempty definable subsets of \mathbb{N}^N have what N believes to be an $<^N$ -least element, and all such bounded subsets have a $<^N$ -greatest element). Moreover, we may expand the language to have available uniform definitions for relevant trees. Recall *Definition 18* above.

Definition 24 (informal; for full details, see ref. 2, definition 2.1, pp. 7–8): Informally, we say that the triple M, M_1, Δ has “enough set theory for trees” when $M \leq M_1$; Δ is a nonempty set of formulas defining discrete linear orders in M_1 , such that every nonempty M_1 -definable subset of any such order has a first and last element; and for each instance of a formula φ in Δ (i.e., for

each order) there is a definable tree of sequences, i.e., functions from the order to itself.

We ask that for each such tree, length, projection, concatenation, the maximal element of the domain of a given function, and the partial ordering \leq are definable; and we ask that Δ be closed under Cartesian products.

Our true context is therefore the following:

Definition 25 (ref. 2, definitions 2.5–2.11): We say that s is a cofinality spectrum problem (CSP) when

$s = (M, M_1, M^+, M_1^+, \Delta)$, where Δ is a set of formulas of $\mathcal{L}(M)$, $M \leq M_1$, and this pair can be expanded to $M^+ \leq M_1^+$ such that (M^+, M_1^+, Δ) has enough set theory for trees. When $M = M^+$ or $M_1 = M_1^+$, we may omit it.

In this context we define the following:

- 1) $\text{Or}(s) = \text{Or}(\Delta^s, M_1^s)$ is the set of orders defined in M_1 by instances of formulas from Δ .
 - 2) $\mathcal{C}^{\text{ct}}(s) = \{(\kappa_1, \kappa_2) : \text{There is } \mathbf{a} \in \text{Or}(s, M_1) \text{ such that the linear order } \leq_{\mathbf{a}} \text{ on } X_{\mathbf{a}} \text{ has a } (\kappa_1, \kappa_2)\text{-cut}\}$.
- Note that by definition of cut in *Definition 18* above, κ_1, κ_2 are regular.
- 3) $\text{Tr}(s) = \{\mathcal{T}_{\mathbf{a}} : \mathbf{a} \in \text{Or}(s)\}$ is the set of definable trees associated to orders in $\text{Or}(s)$.
 - 4) $\mathcal{C}^{\text{ttp}}(s) = \{\kappa : \kappa \geq \aleph_0, \mathbf{a} \in \text{Or}(s) \text{ and there is in the tree } \mathcal{T}_{\mathbf{a}} \text{ a strictly increasing sequence of cofinality } \text{cf}(\kappa) \text{ with no upper bound}\}$.
 - 5) Let t_s be $\min \mathcal{C}^{\text{ttp}}(s)$ and let p_s be $\min\{\kappa : (\kappa_1, \kappa_2) \in \mathcal{C}^{\text{ct}}(s) \text{ and } \kappa = \kappa_1 + \kappa_2\}$.

Note that by definition of $\mathcal{C}^{\text{ct}}(s)$ and $\mathcal{C}^{\text{ttp}}(s)$, both t_s and p_s are regular.

We focus on $\mathcal{C}(s, t_s)$, where this means

- 6) $\mathcal{C}(s, t_s) = \{(\kappa_1, \kappa_2) : \kappa_1 + \kappa_2 < t_s, (\kappa_1, \kappa_2) \in \mathcal{C}^{\text{ct}}(s)\}$.

First Reduction. If for every CSP s , $\mathcal{C}(s, t_s) = \emptyset$, then SOP_2 is maximal in Keisler's order. Why? This simply translates *Question 20* to this more general context: The issue is whether knowing there are always paths through trees is sufficient to guarantee that there are no cuts of small cofinality.

However, our new context also encompasses other problems.

Second Reduction. To deal with p and t in this context we look for a cofinality spectrum problem and for a cut, recalling *Theorem H* above.

Suppose we are given $M = (\mathcal{H}(\aleph_1), \epsilon)$, a forcing notion $\mathbf{Q} = ([\omega]^{\aleph_0}, \supseteq^*)$, \mathbf{V} a transitive model of ZFC, and G a fixed generic subset of \mathbf{Q} , forced to be an ultrafilter on the Boolean algebra $\mathcal{P}(\omega)^{\mathbf{V}}$.

Consider $N = M^{\omega}/G$ in the forcing extension $\mathbf{V}[G]$. Recall that first, the diagonal embedding is elementary, so $M \leq N$; and second, \mathbf{Q} is t -complete so forcing with \mathbf{Q} adds no new bounded subsets of t , and $p^{\mathbf{V}} < t^{\mathbf{V}}$ iff $p^{\mathbf{V}[G]} < t^{\mathbf{V}[G]}$.

We focus attention on the cofinality spectrum problem $s_* = (M, N, \Delta_{\text{psf}})$, where Δ_{psf} is the set of definable linear orders in N that are covered by G -ultraproducts of finite linear orders. We assume $p < t$ and prove several claims.

Claim 2 (ref. 2, claim 5.8, Assuming $p < t$). $t \leq t_{s_*}$.

Proof sketch: Without loss of generality, work in the tree $T^N = (\omega > \omega, \leq)^N$. Let $(\tilde{f}_\alpha/G : \alpha < \theta)$ be a proposed path in N , $\theta < t$. Let $B \Vdash \text{“}(\tilde{f}_\alpha/G : \alpha < \theta) \text{ is } \leq^N\text{-increasing in } T^N\text{”}$, and without loss of generality $B \Vdash \text{“}\tilde{f}_\alpha = f_\alpha\text{”}$ as there are no new sequences of length $< t$. For each $n \in B$, look at the cone Y_α^n above $f_\alpha(n)$ in the index model T^M . Let $Y_\alpha = \bigcup_{n \in B} Y_\alpha^n$. Then $\alpha < \beta \Rightarrow Y_\beta \subseteq^* Y_\alpha$, as otherwise we contradict the choice of B . Thus, $\{Y_\alpha : \alpha < \theta\}$ is a tower of length

$\theta < t$, and by definition of t as the tower number, there is an infinite Y with $Y \subseteq^* Y_\alpha$ for all $\alpha < \theta$. From any such Y it is easy to build an upper bound for our given path.

Claim 3 (ref. 2, claim 5.15, Assuming $p < t$). $p_{s_*} \leq p$.

The proof shows, using *Theorem H* and the definition of G , that if $p < t$, then for some nonconstant $g_0 \in {}^\omega \omega$, $\prod_G g_0(n)$ contains a (κ, p) -cut, where κ is regular, $\aleph_1 \leq \kappa < p$.

Conclusion 26 (ref. 2, conclusion 5.16). If $p < t$ then $\mathcal{C}(s_*, t_{s_*}) \neq \emptyset$.

We have shown that if for every CSP s , $\mathcal{C}(s, t_s) = \emptyset$, then $p = t$. This completes the second reduction.

Our Main Theorems. In Malliaris and Shelah (2) we prove the following. The first, main structure theorem is *Theorem 2*, proved by model-theoretic means; some key lemmas are described in the next section.

Theorem 2 (ref. 2, theorem 3.66). For any CSP s , $\mathcal{C}(s, t_s) = \emptyset$.

By the first reduction described above, *Theorem 2* gives the following:

Theorem 3 (ref. 2, theorem 4.48). SOP_2 is maximal in Keisler's order.

By some further arguments in the case of ultrafilters, our methods and *Theorem 2* also give a new characterization of Keisler's notion of goodness:

Theorem 4 (ref. 2, theorem 4.49). For a regular ultrafilter \mathcal{D} on λ , (a) $\kappa \leq \lambda \Rightarrow (\kappa, \kappa) \notin \mathcal{C}(\mathcal{D})$ iff (b) $\mathcal{C}(\mathcal{D}) = \emptyset$ iff (c) \mathcal{D} is λ^+ -good.

By *Theorem 3* and *Theorem G* above, we obtain the following:

Theorem 5 (ref. 2, theorem 4.51). The minimum TP_2 class is the minimum nonsimple class in Keisler's order.

Finally, by the *Second Reduction* described above, *Theorem 2* gives the following:

Theorem 6 (ref. 2, theorem 5.17). $p = t$

These results give insight into other structures, as well. For instance, it is a corollary of these methods that in any model of Peano arithmetic, if for all $\kappa < \lambda$ the underlying order has no (κ, κ) -cuts (gaps), then the model itself is λ -saturated.

Key Steps in the Proof. Here we state the intermediate claims in the proof that $\mathcal{C}(s, t_s) = \emptyset$, *Theorem 2* above. [For the connection to p, t , and SOP_2 , see *First Reduction* and *Second Reduction* above.]

Claim 4 (Uniqueness) (ref. 2, claim 3.3). Let s be a cofinality spectrum problem. Then for each regular $\kappa \leq p_s$, $\kappa < t_s$:

- 1) $(\kappa, \lambda) \in \mathcal{C}^{\text{ct}}(s)$ for precisely one λ .
- 2) $(\kappa, \lambda) \in \mathcal{C}^{\text{ct}}(s)$ iff $(\lambda, \kappa) \in \mathcal{C}^{\text{ct}}(s)$.

This proof amounts to a generalization of *Claim 1* above to the general context of cofinality spectrum problems.

Claim 5 (Antisymmetry) (ref. 2, claim 3.45). Let s be a cofinality spectrum problem. Then for all regular κ such that $\kappa \leq p_s$, $\kappa < t_s$, we have that $(\kappa, \kappa) \notin \mathcal{C}^{\text{ct}}(s)$.

The proof involves constructing a path through a tree of pairs of natural numbers, in the spirit of *Claims 1* and *4*.

Corollary 27 (cf. ref. 2, corollary 3.9). Let s be a cofinality spectrum problem. In light of the above we may, without loss of generality, study $\mathcal{C}(s, t_s)$ by looking at

$$\{(\kappa_1, \kappa_2) : (\kappa_1, \kappa_2) \in \mathcal{C}(s, t_s), \kappa_1 < \kappa_2\}.$$

Corollary 27 summarizes the two previous claims.

Claim 6 [on $\text{lcf}(\aleph_0)$] (ref. 2, claim 3.58). Let s be a cofinality spectrum problem and suppose $\text{lcf}(\aleph_1, s) \geq \aleph_2$. Then $(\aleph_0, \aleph_1) \notin \mathcal{C}(s, \aleph_s) \rightarrow (\aleph_0, \aleph) \notin \mathcal{C}(s, \aleph_s)$ for all regular $\aleph \leq \aleph_s, \aleph < \aleph_s$.

The proof of Claim 6 is an order of magnitude more complicated and requires using (or developing) more general trees; to always have these available, we prove from the definition that cofinality spectrum problems always have available a certain amount of Peano arithmetic. Claim 6 may be read as saying that the only case of concern for \aleph_0 is a “close” asymmetric cut, (\aleph_0, \aleph_1) . Indeed, in the general result, Lemma 1 below, the a priori most difficult case to rule out was a (\aleph, \aleph) -cut when $\aleph^+ = \aleph$ and $\aleph^+ = \aleph_s$.

Lemma 1 [Main Lemma (ref. 2, lemma 3.65)]. Let s be a cofinality spectrum problem. Suppose that \aleph, λ are regular and $\aleph < \lambda = \aleph_s < \aleph_s$. Then $(\aleph, \lambda) \notin \mathcal{C}(s, \aleph_s)$.

This lemma, which supercedes and substantially generalizes Claim 6, is the core of the argument. Its proof requires formalizing a robust picture of how sets of large size may be carried along one side of a cut to overspill into the other when the cofinality on either side is small relative to the size of “tree-tops”. We also show that cofinality spectrum problems have available an internal notion of cardinality. So for suitably chosen representations of cuts such overspill is enough for

a contradiction, which rules out the existence of cuts of the given cofinality (2). Ref. 2, discussion 3.61 is an informal description of the proof.

With these results in hand we prove the paper’s fundamental result:

Proof (of Theorem 2): If $\aleph_s \leq \aleph_s$, then by definition of $\aleph_s, \mathcal{C}(s, \aleph_s) = \emptyset$. So we may assume $\aleph_s < \aleph_s$. Let \aleph, λ be such that $\aleph + \lambda = \aleph_s$ and $(\aleph, \lambda) \in \mathcal{C}(s, \aleph_s)$. By Claims 4 and 5 and Corollary 27, $\aleph \neq \lambda$ and we may assume $\aleph < \lambda = \aleph_s$. Then the hypotheses of Lemma 1 are satisfied, so $(\aleph, \lambda) \notin \mathcal{C}(s, \aleph_s)$. This completes the proof. ●

Discussion 28. Regarding Keisler’s order, the work presented here is related to the nonstructure side, giving sufficient conditions for maximality.

Conjecture 29. SOP_2 characterizes maximality in Keisler’s order.

ACKNOWLEDGMENTS. M.M. was partially supported by National Science Foundation (NSF) Grant DMS-1001666, by NSF Grant DMS-1101597 (to S.S.), and by a Gödel research fellowship. S.S. was partially supported by the Israel Science Foundation Grants 710/07 and 1053/11. This is paper E74 in S.S.’s list of publications.

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