

CHANG'S CONJECTURE FOR \aleph_ω

BY

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ABSTRACT

We establish, starting from some assumptions of the order of magnitude of a huge cardinal, the consistency of $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\omega_1, \omega_0)$, as well as of some other transfer properties of the type $(\kappa^+, \kappa) \rightarrow (\alpha^+, \alpha)$, where κ is singular.

0. Notations

0.1. On denotes the class of all ordinals.

$|X|$ denotes the cardinality of X .

0.2. If λ is a cardinal and α is an ordinal, $\lambda^{+\alpha}$ denotes the α th successor of λ .

0.3. If μ is a cardinal, H_μ is the set of all sets of hereditary cardinality $< \mu$.

0.4. If p is a function, $\text{dom}(p)$ is the domain of p and $\text{Rg}(p)$ is the range of p .

0.5. Let \mathcal{A} be a structure. $\text{dom}(\mathcal{A})$ is the base set of \mathcal{A} . " $X < \mathcal{A}$ " means that X is an elementary substructure of \mathcal{A} . $\text{Hull}(X, \mathcal{A})$ is the Skolem Hull of X in \mathcal{A} (with respect to a fixed well-order of $\text{dom}(\mathcal{A})$).

0.6. If j is an elementary embedding, $\text{cp}(j)$ is the critical point of j .

0.7. If G is generic over V and $a \in V$, a_G denotes the G -interpretation of a in $V[G]$.

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1. Chang's Conjectures

1.1. Let L be a first-order language. Let A^* , B^* be two classes of L -structures. Chang's Conjecture for (A^*, B^*) (over L) can be stated as follows: for every $A \in A^*$ there exists $B \in B^*$ such that $B < A$. We usually write $A^* \rightarrow B^*$.

1.2. Usually, we consider a two-cardinal form of Chang's Conjecture. Let $\kappa, \lambda, \beta, \alpha$ be four infinite cardinals. We write $(\kappa, \lambda) \rightarrow (\beta, \alpha)$ iff the following holds: for every countable first-order language L and for every L -structure $A = (\kappa, \lambda, \dots)$ there exists an $X < A$ such that $|X| = \beta$ and $|X \cap \lambda| = \alpha$.

1.3. REMARKS. (1) It is clear that we obtain the same conjecture if we look only at languages L such that $|L| < \omega$, or even only at languages L such that $|L| \leq 2$, as soon as L contains at least a binary predicate.

(2) We can also introduce a fifth cardinal μ and write $(\kappa, \lambda) \rightarrow_{<\mu} (\beta, \alpha)$ meaning that we have got to take care of all the languages L such that $|L| < \mu$.

(3) We can also define the symbol $(\kappa, \lambda) \rightarrow (\beta, <\alpha)$, which means that the elementary substructure X should be such that $|X| = \beta$ and $|X \cap \lambda| < \alpha$.

One can consult [R] for more information.

1.4. Let us go back to the case $(\kappa, \lambda) \rightarrow (\beta, \alpha)$. Several cases are "irrelevant", in the following sense: they are either provably false or provably true (in ZFC). In order to avoid these trivial cases, we must have: $\kappa > \lambda, \beta > \alpha, \kappa \geq \beta, \lambda > \alpha$.

1.5. For instance, we can consider conjectures like

$$(\omega_5, \omega_3) \rightarrow (\omega_4, \omega_2)$$

$$(\omega_3, \omega_1)$$

$$(\omega_2, \omega_0)$$

$$(\omega_3, \omega_2).$$

1.6. Note that a finite gap can never be increased. For example, $(\omega_5, \omega_4) \rightarrow (\omega_3, \omega_1)$ is inconsistent, because we can code the fact that "the universe is \leq to the successor of ω_4 ".

1.7. MORE EXAMPLES. (1) The usual Conjecture of Chang is $(\omega_2, \omega_1) \rightarrow (\omega_1, \omega_0)$.

(2) $(\kappa, \omega_1) \rightarrow (\kappa, \omega_0)$ means that κ is ω_1 -Rowbottom.

(3) $(\forall \lambda < \kappa)[(\kappa, \lambda) \twoheadrightarrow (\kappa, \omega_0)]$ means that κ is Rowbottom.

(4) Set $A^* = \{A / \text{dom}(A) = \kappa\}$ and

$$B^* = \{B / \text{dom}(B) \subseteq \kappa, \text{dom}(B) \neq \kappa, |\text{dom}(B)| = \kappa\}.$$

$A^* \twoheadrightarrow B^*$ means that κ is Jonsson.

1.8. CONSISTENCY. (1) The consistency of $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega_0)$ has been established early by Silver (unpublished), starting with an ω_1 -Erdős cardinal.

(2) Later proofs have been given by Baumgartner (unpublished) and in [DL], without use of Martin's Axiom, and also using a Levy collapse instead of a Silver collapse.

(3) The consistency of $(\omega_{n+2}, \omega_{n+1}) \twoheadrightarrow (\omega_{n+1}, \omega_n)$ for $n \geq 1$ has been established in [K], starting with a huge cardinal.

The consistency of other forms of Chang's Conjecture can be found in [F].

1.9. CONVERSES. As to converses, the following is known.

(1) [DJK] shows that, if $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega_0)$ holds and if we set $\kappa = \omega_2$ and $\lambda = \omega_1$, then $K \vDash$ " κ is λ -Erdős" (where K denotes the Core Model).

(2) Indeed, the proof of [DJK] shows also the following. If, for some infinite cardinals κ and λ , $(\kappa^+, \kappa) \twoheadrightarrow (\lambda^+, \lambda)$ holds, then, setting $\mu = \lambda^+$, $K \vDash$ "there exists a μ -Erdős cardinal".

1.10. When we look at conjectures of the form $(\kappa^+, \kappa) \twoheadrightarrow (\lambda^+, \lambda)$, we have two quite different situations, depending on whether $\lambda = \omega_0$ or $\lambda \geq \omega_1$.

(1) We know that, if $(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega_0)$, then $K \vDash$ "there exists an ω_1 -Erdős cardinal".

(2) Conversely, if we start with an ω_1 -Erdős cardinal, we can obtain the consistency of almost any conjecture of the form $(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega_0)$, as long as we accept κ to be regular (for example, κ can be ω_7 , or the first inaccessible cardinal).

1.11. HIGHER CARDINALS. The situation changes, if we look at $(\kappa^+, \kappa) \twoheadrightarrow (\lambda^+, \lambda)$, where $\lambda \geq \omega_1$.

(1) It is proved in [DK] that, if $(\kappa^{++}, \kappa^+) \twoheadrightarrow (\kappa^+, \kappa)$ and $\kappa \geq \omega_1$, then 0^\dagger ["0 dagger"] exists.

(2) [L] derives the existence of 0^\dagger from other forms of Chang's Conjecture, such as the following:

(a) $(\kappa^+, \kappa) \twoheadrightarrow (\lambda^+, \lambda)$ and $\lambda \geq \omega_1$,

(b) $(\kappa^{+n}, \kappa) \twoheadrightarrow (\lambda^{+m}, \lambda)$ and $m \geq 2$,

(c) $(\kappa^+, \kappa) \twoheadrightarrow (\omega_1, \omega_0)$ and κ is singular.

One can also, for more on the existence of 0^\dagger following from some forms of Chang's Conjecture, look at [DKL].

1.12. Hence, by 1.11(2)(c), " $(\kappa^+, \kappa) \rightarrow (\omega_1, \omega_0)$ and κ is singular" is already a rather large cardinal axiom.

The rest of this paper is devoted to establishing the consistency of various hypotheses of the form $(\kappa^+, \kappa) \rightarrow (\lambda^+, \lambda)$, where κ is singular, in particular of $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\omega_1, \omega_0)$.

1.13. Before proceeding further, we may already note a fact, which puts some limits on the possibility of this kind of conjecture.

LEMMA 1. *Assume that κ, ρ are infinite cardinals such that κ is singular and $\rho^{\text{cof}(\kappa)} = \rho$ (in particular, $\text{cof}(\kappa) < \rho$). Then: $(\kappa^+, \kappa) \not\rightarrow (\rho^+, \rho)$.*

Proof of Lemma 1. Assume that $(\kappa^+, \kappa) \rightarrow (\rho^+, \rho)$. Set $\theta = \text{cof}(\kappa)$. Hence, $\theta < \rho$. Clearly, $\kappa^\theta \geq \kappa^+$. Let $(f_i)_{i < \kappa^+}$ be a family of κ^+ different functions, $f_i: \theta \rightarrow \kappa$. Set $F = \bigcup_{i < \kappa^+} \{i\} \times f_i$ and $\mathcal{A} = (\kappa^+, \theta, F)$. Assume that $X < \mathcal{A}$, $|X| = \rho^+$ and $|X \cap \kappa| = \rho$. By elementarity, the family $f_i \upharpoonright_X: X \cap \theta \rightarrow X \cap \kappa$ is made of $|X|$ different functions. But $|X \cap \kappa|^{|X \cap \theta|} \leq \rho^\theta \leq \rho$, a contradiction. QED Lemma 1.

2. Elementary submodels

2.1. Before dealing with singular cardinals, we shall recall a possible way of proving that certain forms of Chang's Conjecture are preserved in some generic extensions.

First of all, let us recall the following lemma.

LEMMA 2. *Assume that $\kappa, \lambda, \beta, \alpha$ are four infinite cardinals. The following are equivalent.*

(1) $(\kappa, \lambda) \rightarrow (\beta, \alpha)$.

(2) For at least one regular cardinal $\mu \geq \kappa$, the following holds:

for every language L with $|L| \leq \omega$ and every L -structure $\mathbf{B} = (H_\mu, \in, L^{\mathbf{B}})$, there exists $Y < \mathbf{B}$ such that

(a) $|Y \cap \kappa| = \beta$,

(b) $|Y \cap \lambda| = \alpha$.

(3) For every regular cardinal $\mu \geq \kappa$, (2) holds.

PROOF OF LEMMA 2. (3) \rightarrow (2) is obvious.

(2) \rightarrow (1). Assume that L is a countable language and that $\mathcal{A} = (\kappa, \lambda, L^{\mathcal{A}})$ is

an L -structure. Set $\mathbf{B} = (H_\mu, \in, \kappa, \lambda, L^A)$ and let $Y < \mathbf{B}$ be such that $|Y \cap \kappa| = \beta$ and $|Y \cap \lambda| = \alpha$. It is clear that $Y \cap \kappa <_L A$.

(1) \rightarrow (3). Let μ be regular, such that $\mu \geq \kappa$. Let L be a countable language, and $\mathbf{B} = (H_\mu, \in, L^B, \triangleleft)$ be a structure, where \triangleleft is a well-order on H_μ . Let $(f_n)_{n < \omega}$, $f_n: [H_\mu]^n \rightarrow H_\mu$, be a complete set of Skolem functions for \mathbf{B} and define, for $n < \omega$, a function $g_n: [\kappa]^n \rightarrow \kappa$, setting, for $a \in [\kappa]^n$,

$$g_n(a) = \begin{cases} f_n(a) & \text{if } f_n(a) \in \lambda, \\ 0 & \text{if not.} \end{cases}$$

Finally, set $\mathbf{A} = (\kappa, \lambda, (g_n)_{n < \omega})$. Let, by (1), X be such that $X < \mathbf{A}$, $|X| = \beta$ and $|X \cap \lambda| = \alpha$. Set $Y = \text{Hull}(X, \mathbf{B})$. It is clear that $|Y| = |Y \cap \kappa| = |X| = \beta$. On the other hand, due to the g_n 's, we have $|Y \cap \lambda| = \alpha$. QED Lemma 2.

2.2. Let us now, as a sample, recall the following.

THEOREM 3. *Assume that $(\kappa, \alpha^+) \twoheadrightarrow (\alpha^+, \alpha)$. Let P be a set of conditions, such that P satisfies the α^+ -antichain condition. Then:*

$$V^P \models (\kappa, \alpha^+) \twoheadrightarrow (\alpha^+, |\alpha|).$$

PROOF OF THEOREM 3. Let μ be a regular cardinal, large enough so that $\mathcal{P}(P) \subseteq H_\mu$. Let G be P -generic over V . Since $\mathcal{P}(P) \subseteq H_\mu$, $(H_\mu)^{V[G]} = H_\mu[G]$. Let, in $V[G]$, $\bar{\mathbf{B}} = (H_\mu[G], \in, \bar{B})$ be a structure. Take $B \in V$, such that $B \subseteq H_\mu$ and $\bar{B} = (B)_G$. Set $\mathbf{B} = (H_\mu, \in, B, \{\alpha\}, \triangleleft)$, where \triangleleft is a well-order of H_μ . Hence, $\bar{\mathbf{B}}$ is a reduct of $\mathbf{B}[G]$.

CLAIM 1. There exists, in V , an $M < \mathbf{B}$ such that $|M \cap \kappa| = \alpha^+$, $|M \cap \alpha^+| = \alpha$, $\alpha \in M$ and $M \cap \alpha^+$ is transitive.

PROOF OF CLAIM 1. Let us work for a moment in V . Let $X < \mathbf{B}$ be such that $|X| = \alpha^+$ and $|X \cap \alpha^+| = \alpha$. Set $\xi_0 = \sup(X \cap \alpha^+)$. Clearly, $\xi_0 < \alpha^+$. Hence, $\alpha < \xi_0 < \alpha^+$. Set $M = \text{Hull}(X \cup \xi_0, \mathbf{B})$. It is clear that $|M| = \alpha^+$. Hence, it is enough to show that $M \cap \alpha^+ = \xi_0$. To see this, let $t(v_1, \dots, v_n, w)$ be a term, and let $\delta_1, \dots, \delta_n < \xi_0$ and $a \in X$ be such that $\zeta = t(\delta_1, \dots, \delta_n, a) \in \alpha^+$. We have got to show that $\zeta < \xi_0$. Take $\eta \in X \cap \xi_0$ such that $\delta_1, \dots, \delta_n < \eta$, and set $s(\eta, a) = \{t(\gamma_1, \dots, \gamma_n, a) / \gamma_1, \dots, \gamma_n < \eta\}$. Clearly, $s(\eta, a) \in X$, and $t(\vec{\delta}, a) \in s(\eta, a) \cap \alpha^+$. Hence, setting $\gamma = \sup[s(\eta, a) \cap \alpha^+]$ we see, since $\alpha^+ \in X$, that $t(\vec{\delta}, a) \leq \gamma < \xi_0$. QED Claim 1.

Let M be as in Claim 1. Let $h: N \xrightarrow{\sim} M$ be the transitive collapse of M . Set $P' = (h^{-1})(P)$ and $H = (h^{-1})''(G)$.

CLAIM 2. H is P' -generic over N .

PROOF OF CLAIM 2. Let D be a maximal antichain in P' , such that D is first-order definable in N . By elementarity, $D \in N$ [since \mathcal{B} satisfies that every definable subset of P is a set]. By elementarity again, $N \models "|D| \leq \alpha"$, since P satisfies the α^+ -antichain condition in V . But $h \upharpoonright_{\alpha+1} = \text{Id}$, and hence $h''(D) = h(D)$. Hence, $h''(D)$ is a maximal antichain in P , and we are done.

QED Claim 2.

Since H is P' -generic over N , $N[H]$ satisfies the Truth Lemma. Since $h''(H) \subseteq G$, h has an extension to a map $h^*: N[H] \rightarrow \mathcal{B}[G]$, given by $h^*(a_H) = (h(a))_G$. Set $\bar{M} = \text{Rg}(h^*)$. Hence, $\bar{M} < \mathcal{B}[G]$, and it is clear that $|\bar{M}| = \alpha^+$. On the other hand, $N[H]$ has the same ordinals as N . Hence, $\text{On} \cap \bar{M} = (h^*)''(\text{On} \cap N[H]) = h''(\text{On} \cap N) = \text{On} \cap M$. So, $\bar{M} \cap \alpha^+ = M \cap \alpha^+$, and $|\bar{M} \cap \alpha^+| \leq \alpha$.

QED Theorem 3.

2.3. We will have to deal with situations where, keeping the notations of the proof of Theorem 3, H cannot be P' -generic over N . Hence, h will not extend to an elementary embedding from $N[H]$ into $\mathcal{B}[G]$. Hence, we have got to refine slightly the argument.

Keeping the notations of the proof of Theorem 3, we see that, due to the definition of h^* , $\bar{M} = \{a_G/a \in M\}$. We will keep this definition of \bar{M} . However, it will be false that $\bar{M} \cap \text{On} = M \cap \text{On}$ [actually, " $\bar{M} \cap \text{On} = M \cap \text{On}$ " is equivalent to " H is P' -generic over N ", as shown in [S]]. Hence, we will have to ensure that, although $\bar{M} \cap \alpha^+$ is larger than $M \cap \alpha^+$, we do not add too many ordinals.

2.4. Hence, assume that μ is a regular cardinal, and that $\mathcal{B} = (H_\mu, \in, \dots)$ is a structure, where " \dots " denote some additional predicates, including a well-order of H_μ . Let P be a set of conditions, such that $\mathcal{P}(P) \subseteq H_\mu$. Let M be an elementary substructure of \mathcal{B} . Let G be P -generic over V . Set $M[G] = \{a_G/a \in M\}$.

LEMMA 4. $M[G] < \mathcal{B}[G]$.

PROOF OF LEMMA 4. Let us first observe that the forcing relation over P restricted to \mathcal{B} is definable in \mathcal{B} , since $\mathcal{P}(P) \subseteq H_\mu$. [This point is, however, not essential in the proof, since we could always add it to \mathcal{B} as a predicate.]

To show that $M[G] < \mathcal{B}[G]$, let us take a formula $\varphi(v_1, \dots, v_n)$ and terms $a_1, \dots, a_n \in M$ and show that $M[G] \models \varphi((a_1)_G, \dots, (a_n)_G)$ iff $\mathcal{B}[G] \models \varphi((a_1)_G, \dots, (a_n)_G)$. We proceed as usual, by induction on φ , so

that we can assume that φ is of the form $(\exists x)\theta$, and that it is true in $\mathbf{B}[G]$. Let $p_0 \in G$ be such that $p_0 \Vdash_P \varphi(a_1, \dots, a_n)$. Since $\mathbf{B} \models \text{ZFC}^-$ and since $\mathcal{P}(P) \subseteq H_\mu$, the maximum principle is valid in \mathbf{B} . Hence, we can find a term $b \in H_\mu$ such that

$$\mathbf{B} \models (\forall q)[q \Vdash_P \varphi(a_1, \dots, a_n) \rightarrow q \Vdash_P \theta(b, a_1, \dots, a_n)].$$

By elementarity, such a term b must exist in M . By induction hypothesis, $M[G] \models \theta(b_G, (a_1)_G, \dots, (a_n)_G)$. QED Lemma 4.

2.5. REMARK. We could define $M[G]$ in another (equivalent) way. Let $h: N \rightarrow M$ be the transitive collapse of M . Let B denote the boolean completion of P . Assume that G is B -generic over V , and set $B' = h^{-1}(B)$ and $H = (h^{-1})''(G)$. Hence, H is an ultrafilter over the N -complete boolean algebra B' . Hence, we can define the boolean model $N^{B'}/H$. It is immediate that h extends to an elementary embedding from $N^{B'}/H$ into \mathbf{B} , and that the image of this elementary embedding is precisely $M[G]$.

3. Consistency results

3.1. We shall begin with the following theorem.

THEOREM 5. *Assume that GCH holds. Assume that there exists an elementary embedding $j: V \rightarrow N$ such that:*

(1) *N is transitive and $j \neq \text{Id}$.*

(2) *If we set $\kappa = \text{cp}(j)$ and $\lambda = j(\kappa)$, then $N^{\lambda^{+\omega+1}} \subseteq N$.*

Then there exists a generic extension of V in which $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\omega_1, \omega_0)$ holds.

PROOF OF THEOREM 5.

CLAIM 1. In V , $(\lambda^{+\omega+1}, \lambda^{+\omega}) \rightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$.

PROOF OF CLAIM 1. Let L be a countable language and $\mathcal{A} = (\lambda^{+\omega+1}, \lambda^{+\omega}, L^{\mathcal{A}})$ be a structure. Set $\pi = j \upharpoonright_{\mathcal{A}}$. π is an elementary embedding from \mathcal{A} into $j(\mathcal{A})$. Since $N^{\lambda^{+\omega+1}} \subseteq N$, $\text{Rg}(\pi) = M \in N$ and $M < j(\mathcal{A})$. It is clear that $|M| = \lambda^{+\omega+1} = j(\kappa^{+\omega+1})$ and that $M \cap j(\lambda^{+\omega}) = j''(\lambda^{+\omega})$, hence that $|M \cap j(\lambda^{+\omega})| = j(\kappa^{+\omega})$. Hence, $N \models "j(\mathcal{A})$ admits an elementary submodel of type $(j(\kappa^{+\omega+1}), j(\kappa^{+\omega}))"$. By elementarity, \mathcal{A} admits an elementary submodel M' of type $(\kappa^{+\omega+1}, \kappa^{+\omega})$. QED Claim 1.

REMARKS. (1) Since $M \cap \lambda = \kappa$ is transitive, we could also assume that $M' \cap \kappa \in \kappa$.

(2) Since $\pi \upharpoonright_\kappa = \text{Id} \upharpoonright_\kappa$, Claim 1 would still be true over languages of arbitrary cardinality $< \kappa$.

We can now forget about the elementary embedding j and use only the fact that, in V , $(\lambda^{+\omega+1}, \lambda^{+\omega}) \rightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$, and some cardinality properties, following from the GCH.

Let P_0 be the set of all finite partial functions $p: \omega \rightarrow \kappa^{+\omega}$, ordered by reverse inclusion. Hence, $|P_0| = \kappa^{+\omega}$, and hence in V^{P_0} , $\kappa^{+\omega+1} = \omega_1$. In V^{P_0} , let P_1 be the Levy set making $\lambda = \omega_2$ with countable conditions. Set $P = P_0 * P_1$. It is clear that $|P| = \lambda$, that P has the λ -antichain condition and that, in V^P , $\kappa^{+\omega+1} = \omega_1$, $\lambda = \omega_2$, $\lambda^{+\omega} = \aleph_\omega$ and $\lambda^{+\omega+1} = \aleph_{\omega+1}$. We claim that, in V^P , $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\omega_1, \omega_0)$ holds.

To see that, set $\mu = \lambda^{+\omega+1}$, and let G be P -generic over V . Since $\mathcal{P}(P) \subseteq H_\mu$, we see that $(H_\mu)^{V[G]} = H_\mu[G]$. Let $\bar{B} \in V[G]$ be such that $\bar{B} \subseteq H_\mu[G]$ and set $\bar{B} = (H_\mu[G], \in, \bar{B})$. Let $B \in V$ be such that $B \subseteq H_\mu$ and $\bar{B} = (B)_G$. Finally, set $\mathbf{B} = (H_\mu, \in, B, \{P\}, \triangleleft)$, where \triangleleft is a well-order of H_μ . Let $M \in V$ be such that $M < \mathbf{B}$, $|M \cap \lambda^{+\omega+1}| = \kappa^{+\omega+1}$ and $|M \cap \lambda^{+\omega}| = \kappa^{+\omega}$. Set $\bar{M} = M[G]$. By Lemma 4, $\bar{M} < \mathbf{B}[G]$. It is clear that $|\bar{M}| = \kappa^{+\omega+1} = (\omega_1)^{V[G]}$. Hence, it is enough to show that

CLAIM 2. $V[G] \models "|\bar{M} \cap \lambda^{+\omega}| \leq \omega_0"$.

PROOF OF CLAIM 2. Let δ be an ordinal. Say that a is a δ -term if $a \in H_\mu$ and $\Vdash_P "a \in \delta"$. Let $T(\delta)$ denote the set of all δ -terms. It is clear that $\bar{M} \cap \lambda^{+\omega} = \{a_G / a \in T(\lambda^{+\omega}) \cap M\}$. Moreover, since $\lambda^{+\omega} = \bigcup \{\lambda^{+n} / n < \omega\}$, and since $(\forall n < \omega)[\lambda^{+n} \in M]$ (because $\omega + 1 \subseteq M$) and $\lambda^{+\omega} \in M$, we see that $\bar{M} \cap \lambda^{+\omega}$ is still equal to $\{a_G / \text{for some } \delta \in M \cap \lambda^{+\omega}, a \in T(\delta) \cap M\}$. Hence, in order to show that $|\bar{M} \cap \lambda^{+\omega}| \leq \omega$, it is enough to show that, for all $\delta \in M \cap \lambda^{+\omega}$, $V[G] \models |T(\delta) \cap M| \leq \omega$. Hence, take $\delta \in M \cap \lambda^{+\omega}$. An element $a \in T(\delta)$ can be obtained as follows. Let B denote the boolean completion of P . Define a function $f: \delta \rightarrow B$ setting, for $\alpha < \delta$, $f(\alpha) = \|a = \alpha\|^B$. It is clear that a_G is the unique $\alpha < \delta$ such that $f(\alpha) \in G$. Since P has the λ -antichain condition and $|P| = \lambda$, we see that $|B| = \lambda$ and that $B \in H_\mu$. Hence, $T(\delta) \in H_\mu$ and, moreover, $|T(\delta)| = \lambda^\delta < \lambda^{+\omega}$, since we assume that the GCH holds ($\lambda^{+\omega}$ strong limit would be enough for this point). Since $\mathbf{B} \models "|T(\delta)| \leq \lambda^{+\omega}"$, $M \models "|T(\delta)| \leq \lambda^{+\omega}"$. Hence $|T(\delta) \cap M| \leq \kappa^{+\omega}$. So, $V[G] \models |T(\delta) \cap M| \leq \omega$.

QED Theorem 5.

3.2. We can improve Theorem 5 as follows.

THEOREM 6. *Assume that the GCH holds. Assume that $1 \leq n < \omega$ and that there exists an elementary embedding $j: V \rightarrow N$ such that:*

(1) *N is transitive and $j \neq \text{Id}$.*

(2) *Setting $\kappa = \text{cp}(j)$ and $\lambda = j(\kappa)$, $N^{\lambda+\omega_n+1} \subseteq N$.*

Then there exists a generic extension of V in which $(\aleph_{\omega_n+1}, \aleph_{\omega_n}) \rightarrow (\omega_{n+1}, \omega_n)$ holds.

PROOF OF THEOREM 6. We follow the proof of Theorem 5. As first claim, we obtain the fact that $(\lambda^{+\omega_n+1}, \lambda^{+\omega_n}) \rightarrow (\kappa^{+\omega_n+1}, \kappa^{+\omega_n})$ holds in V . Moreover, by Remark (2) following the proof of Claim 1, this hypothesis holds over all languages L such that $|L| < \kappa$. In particular, this implies that we can always assume that our elementary substructure M is such that $\omega_n + 1 \subseteq M$.

Now, let P_0 be the set of all partial functions $p: \omega_n \rightarrow \kappa^{+\omega_n}$ such that $|P| < \omega_n$, ordered by reverse inclusion. The supplementary point here is that, because of the GCH, $|P_0| = \kappa^{+\omega_n}$. Hence, forcing with P_0 preserves all the cardinals γ such that $\gamma \leq \omega_n$ or $\gamma > \kappa^{+\omega_n}$, and makes $\kappa^{+\omega_n+1} = \omega_{n+1}$.

We let, in V^{P_0} , P_1 be the Levy collapse making λ the successor of $\kappa^{+\omega_n+1}$ and set $P = P_0 * P_1$. Hence $|P| = \lambda$, P has the λ -antichain condition and, in V^P , $\kappa^{+\omega_n+1} = \omega_{n+1}$, $\lambda = \omega_{n+2}$, $\lambda^{+\omega_n} = \aleph_{\omega_n}$ and $\lambda^{+\omega_n+1} = \aleph_{\omega_n+1}$.

Let us keep the notations of the proof of Theorem 5. Since $\omega_n + 1 \subseteq M$, $\{\lambda^{+i}/i < \omega_n\} \subseteq M$. Hence, $\bar{M} \cap \lambda^{+\omega_n} = \{a_G/\text{for some } \delta \in M \cap \lambda^{+\omega_n}, a \in T(\delta) \cap M\}$. Hence, it is enough to show that, for $\delta \in M \cap \lambda^{+\omega_n}$, $|T(\delta) \cap M| \leq \kappa^{+\omega_n}$. This is proven exactly as in Theorem 5. QED Theorem 6.

3.3. **REMARKS.** (1) Of course, the scheme of proof of Theorem 6 applies to a variety of situations, as soon as the right cardinals are preserved.

We did not use fully the full fact that $|P| = \lambda$ and P satisfies the λ -antichain condition. $|P| < \lambda^{+\omega}$ [resp. $|P| < \lambda^{+\omega_n}$] would have been enough.

(2) In general, we could also have argued as follows. Assume the GCH. Let ρ, κ be two regular cardinals, such that $\omega \leq \rho < \kappa$. Assume the existence of an elementary embedding $j: V \rightarrow N$ such that N is transitive, $j \neq \text{Id}$, and, if we set $\kappa = \text{cp}(j)$ and $\lambda = j(\kappa)$, then $N^{\lambda^{++}} \subseteq N$. Then, in some generic extension of V , $(\aleph_{\rho+1}, \aleph_\rho) \rightarrow (\rho^+, \rho)$ holds.

(3) As we have seen in Lemma 1, $(\aleph_{\rho+1}, \aleph_\rho) \rightarrow (\rho^{++}, \rho^+)$ will, in general, not be consistent.

(4) What about $(\aleph_{\rho+1}, \aleph_\rho) \rightarrow (\delta^+, \delta)$, with $\delta^+ \leq \rho^+$?

More specifically, what about

- (a) $(\aleph_{\omega_1+1}, \aleph_{\omega_1}) \twoheadrightarrow (\omega_1, \omega_0)$,
- (b) $(\aleph_{\omega_2+1}, \aleph_{\omega_2}) \twoheadrightarrow (\omega_2, \omega_1)$,
- (c) $(\aleph_{\omega_2+1}, \aleph_{\omega_2}) \twoheadrightarrow (\omega_1, \omega_0)$?

A glance at the proof of Theorem 6 shows that the construction does not work in this case. For, assume that $0 \leq k < n < \omega$, that κ, λ are two regular cardinals such that $\omega_n < \kappa < \lambda$ and that $(\lambda^{+\omega_n+1}, \lambda^{+\omega_n}) \twoheadrightarrow (\kappa^{+\omega_n+1}, \kappa^{+\omega_n})$ holds. Assume also that a set of conditions P has been found, such that we can establish that $(\aleph_{\omega_n+1}, \aleph_{\omega_n}) \twoheadrightarrow (\omega_{k+1}, \omega_k)$ holds in V^P , using the above hypothesis in V , and the method of proof of Theorem 6. Let G be P -generic over V . In $V[G]$, the following has to hold:

- (a) $|\kappa^{+\omega_n}| = \omega_k$,
- (b) $\lambda^{+\omega_n} = \aleph_{\omega_n}$.

Since $\omega_k < \omega_n \leq \kappa^{+\omega_n}$, (a) implies that ω_n is not a cardinal in $V[G]$, contradicting (b).

Hence, we have got to do a slight change in the proof of Theorem 6.

THEOREM 7. *Assume that the GCH holds. Assume that k, n are two integers, such that $0 < k + 1 < n$. Assume that there exists an elementary embedding $j: V \rightarrow N$ such that:*

- (a) N is transitive and $j \neq \text{Id}$.
- (b) If we set $\kappa = \text{cp}(j)$ and $\lambda = j(\kappa)$, then $N^{\lambda^{++}} \subseteq N$.

Then there exists a generic extension of V in which $(\aleph_{\omega_n+1}, \aleph_{\omega_n}) \twoheadrightarrow (\omega_{k+1}, \omega_k)$ holds.

PROOF OF THEOREM 7. As in Claim 1 of the proof of Theorem 5, we see that the relation $(\lambda^{++}, \lambda^+) \twoheadrightarrow (\kappa^{++}, \kappa^+)$ holds in V . Hence, we shall derive the result from this hypothesis, together with some cardinal assumptions.

Let P_0 be the set of all partial functions $p: \omega_k \rightarrow \kappa^{+\kappa}$, such that $|p| < \omega_k$. Using GCH, we see that $|P_0| = \kappa^{+\kappa}$. Hence, forcing with P_0 preserves all cardinals γ such that $\gamma \leq \omega_k$ or $\gamma \geq \kappa^{+\kappa+1}$. Moreover, $\Vdash_{P_0} \omega_{k+1} = \kappa^{+\kappa+1}$.

In V_{P_0} , let P_1 be the Levy set of conditions making $\lambda = \omega_n$. Forcing with P_1 over V_{P_0} does not destroy $\kappa^{+\kappa+1}$, precisely because $k + 1 < n$. Set $P = P_0 * P_1$. Hence $|P| = \lambda$, P has the λ -antichain condition and in V^P , $\kappa^{+\kappa+1} = \omega_{k+1}$, $\lambda = \omega_n$, $\lambda^+ = \aleph_{\omega_n}$ and $\lambda^{++} = \aleph_{\omega_n+1}$. As usual, set $\mu = \lambda^{++}$ and let $B \subseteq H_\mu$. Set $\mathbf{B} = (H_\mu, \in, B, \{P\}, \triangleleft)$, where \triangleleft is a well-order of H_μ . Let G be P -generic over V , and set $\bar{\mathbf{B}} = \mathbf{B}[G]$. We have got to find some $\bar{M} \in V[G]$ such that $\bar{M} < \bar{\mathbf{B}}$, $|\bar{M}| = \omega_{k+1}$ and $|\bar{M} \cap \lambda^+| = \omega_k$. As before, we take a set $M \in V$

such that $M < B$, $|M| = \kappa^{+\kappa+1}$ and $|M \cap \lambda^{+\lambda}| = \kappa^{+\kappa}$, and we set $\bar{M} = M[G]$. It is clear that $|\bar{M}| = \kappa^{+\kappa+1} = \omega_{k+1}$. Hence, we have got to show that, in $V[G]$, $|\bar{M} \cap \lambda^{+\lambda}| \leq \omega_k$.

To show this, we shall anew "count the $\lambda^{+\lambda}$ -terms".

Assume that $\xi \in \bar{M} \cap \lambda^{+\lambda}$. Let $a \in M$ be such that $\xi = a_G$. Applying the maximum principle in H_μ , we see that there exists a term $b \in H_\mu$ such that

- (1) $\Vdash_P b < \lambda^{+\lambda}$,
- (2) $\Vdash_P a < \lambda^{+\lambda} \rightarrow a = b$.

[In particular, for any such b , $b_G = a_G = \xi$.]

By elementarity, such a b must exist in M . This means that we can, without loss of generality, assume that

- (a) $\Vdash_P a < \lambda^{+\lambda}$.

Now, we shall use the fact that P satisfies the λ -antichain condition. This implies that

- (b) $B \models$ "for some $\delta < \lambda^{+\lambda}$, $\Vdash_P a < \delta$ ".

By elementarity, there exists some $\delta \in M \cap \lambda^{+\lambda}$ such that $\Vdash_P a < \delta$, which means that $a \in T(\delta)$ [the set of all δ -terms]. Hence, we have shown that

- (c) $\bar{M} \cap \lambda^{+\lambda} = \{a_G / \text{for some } \delta \in M \cap \lambda^{+\lambda}, a \in T(\delta) \cap M\}$.

This implies that $|\bar{M} \cap \lambda^{+\lambda}| \leq |M \cap \lambda^{+\lambda}| \cdot \sup\{|T(\delta)| / \delta \in M \cap \lambda^{+\lambda}\}$. Hence, let us verify that, in $V[G]$, this cardinal is at most ω_k .

- (i) $V[G] \models |M \cap \lambda^{+\lambda}| = |\kappa^{+\kappa}| = \omega_k$.

(ii) Assume that $\delta \in M \cap \lambda^{+\lambda}$, and let us show that $V[G] \models |T(\delta)| \leq \omega_k$. Let B denote the boolean completion of P . Since $|P| = \lambda$ and P satisfies the λ -antichain condition, we see that $|B| = \lambda$. Hence, as in the proof of Theorem 5, we see that $B \models |T(\delta)| \leq |B^\delta| = \lambda^\delta < \lambda^{+\lambda}$, since $\delta < \lambda^{+\lambda}$ and the GCH holds in V . Hence $M \models |T(\delta)| \leq \lambda^{+\lambda}$, and so $V[G] \models |T(\delta) \cap M| \leq |\kappa^{+\kappa}| \leq \omega_k$. QED Theorem 7.

3.4. REMARKS. Hence, we have solved the cases $(\aleph_{\omega_n+1}, \aleph_{\omega_n}) \rightarrow (\omega_{k+1}, \omega_k)$, except one: $(\aleph_{\omega_n+1}, \aleph_{\omega_n}) \rightarrow (\omega_n, \omega_{n-1})$, with $n \geq 1$. We shall leave this case open.

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