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How rigid are reduced products?☆

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Abstract

For any cardinal μ let $\mathbb{Z}^\mu$ be the additive group of all integer-valued functions $f : \mu \longrightarrow \mathbb{Z}$. The support of f is $[f] = \{i \in \mu : f(i) = f_i \neq 0\}$. Also let $\mathbb{Z}_\mu = \mathbb{Z}^\mu/\mathbb{Z}^{<\mu}$ with $\mathbb{Z}^{<\mu} = \{f \in \mathbb{Z}^\mu : |[f]| < \mu\}$. If $\mu \leqslant \chi$ are regular cardinals we analyze the question when $\mathrm{Hom}(\mathbb{Z}_\mu, \mathbb{Z}_\chi) = 0$ and obtain a complete answer under GCH and independence results in Section 8. These results and some extensions are applied to a problem on groups: Let the norm $\|G\|$ of a group G be the smallest cardinal μ with $\mathrm{Hom}(\mathbb{Z}_\mu, G) \neq 0$—this is an infinite, regular cardinal (or ∞). As a consequence we characterize those cardinals which appear as norms of groups. This allows us to analyze another problem on radicals: The norm $\|R\|$ of a radical R is the smallest cardinal μ for which there is a family $\{G_i : i \in \mu\}$ of groups such that R does not commute with the product $\prod_{i\in\mu} G_i$. Again these norms are infinite, regular cardinals and we show which cardinals appear as norms of radicals. The results extend earlier work (Arch. Math. 71 (1998) 341–348; Pacific J. Math. 118 (1985) 79–104; Colloq. Math. Soc. János Bolyai 61 (1992) 77–107) and a seminal result by Łoś on slender groups. (His elegant proof appears here in new light; Proposition 4.5.), see Fuchs [Vol. 2] (Infinite Abelian Groups, vols. I and II, Academic Press, New York, 1970 and 1973). An interesting connection to earlier (unpublished) work on model theory by (unpublished, circulated notes, 1973) is elaborated in Section 3.

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1. Introduction

A subfunctor R of the identity on the class of abelian groups is called a *radical* if $R(X/RX)=0$ for all abelian groups X. An arbitrary abelian group G gives rise to a *group radical* R_G, defined for all abelian groups X by

$$R_G X = \bigcap \{\mathrm{Ker}\,\varphi : \varphi : X \to G\}.$$

Clearly, this is a radical. In particular, $R_{\mathbb{Q}} = t$ is the torsion radical. We also mention the Chase radical v, where φ in the last display is allowed to run over homomorphisms into arbitrary $\aleph_1$-free abelian groups. It is well-known that a radical R of abelian groups commutes with direct sums. However, we know of many examples which do not commute with cartesian products. For any radical R of abelian groups which does not commute with arbitrary cartesian products we define the norm $\|R\|$ to be the least cardinal for which there exists a family, of this size, of groups G_α such that $R\prod G_\alpha \neq \prod RG_\alpha$, see [14]. This norm $\|R\|$, if it exist, is always regular, see [6] and clearly, $\|t\| = \aleph_0$. Eda [10] showed that v satisfies $\aleph_1 \leqslant \|v\| \leqslant 2^{\aleph_0}$. The value $\|v\|$ otherwise is quite arbitrary but depends on the underlying set theory. Moreover, Eda showed that each vG is generated by the countable subgroups of G with trivial dual, thus v satisfies the cardinal condition (for $\aleph_1$). But v is not a group radical, (defined also in Section 7), see [9]. If the radical R commutes with arbitrary products, then we write $\|R\| = \infty$. In fact $\|R_{\mathbb{Z}}\| = \infty$, if there are no measurable cardinals. This follows from a quite general theorem (see [9]) to the effect that every non-zero slender group G satisfies $\|R_G\| = \aleph_{\mathrm{fm}}$, where $\aleph_{\mathrm{fm}}$ denotes the first measurable cardinal, or $\|R_G\| = \infty$ if the universe admits no measurable cardinal. Assuming GCH, Corner and Göbel [6] constructed reduced products G to show that every regular cardinal κ which is not greater than any weakly compact cardinal is the norm of a suitable group radical R_G. Hence, it is very natural to study the case when GCH does not hold or if we are above the first weakly compact cardinal. An inspection of the proof in [6] shows that GCH was needed to overcome a cardinal restriction in a nice result due to Wald [27]; see also Corollary 6.2. He proved the following two theorems [27, Theorems A, B]. Let $\kappa < \aleph_{\mathrm{fm}}$ be a regular cardinal.

(i) If κ is not weakly compact, then there is a subgroup $0 \neq U \subseteq \mathbb{Z}_\kappa$ with $|U| \leqslant 2^{<\kappa}$ and trivial dual $U^*(=\mathrm{Hom}(U,\mathbb{Z})) = 0$. (If GCH holds, then $\kappa = 2^{<\kappa}$, thus $|U| \leqslant \kappa$.)
(ii) If κ is weakly compact, and A is a group of cardinality κ such that all its subgroups of cardinality $< \kappa$ are torsionless, then A is torsionless as well.

Recall that a group A is named torsionless (by Bass) if every non-trivial element is mapped to an non-trivial integer by some $\varphi \in A^*$.

Thus, we will study reduced products in general, will analyze the role of weakly compact cardinals and try to weaken the restriction $2^{<\kappa}$ above. We will also establish a link between these group theoretic questions and two set theoretic, model theoretic conditions studied intensively already in the 1970s, see Shelah [23]. Finally, we can look at our problem from two sides: as the original algebraic question as well as the one translated into model theory. Thus, we will gain a useful extension of Wald's result and, on the other hand, settle the question about radicals commuting with cartesian products pointed out above.

We will proceed as follows: In the section after the set-theoretic background we will deal with these two equivalent properties and in the next we will prove that they are equivalent to this question on reduced product of groups and obtain the desired results with contributions from either side. Finally, we apply the results to norms of groups (defined in the abstract) and then to radicals.

Suppose for the moment that the cardinals $\lambda < \aleph_{\mathrm{fm}}$ of our universe satisfy the following condition:

$$\lambda^{\aleph_0} < \lambda^{+\omega} \text{ and } 2^{\lambda} \text{ is smaller than the first weakly inaccessible cardinal above } \lambda. \tag{1.1}$$

Here $\aleph_\alpha^{+\beta} = \aleph_{\alpha+\beta}$. Condition (1.1) follows trivially from GCH.

Assuming (1.1) we will show that χ is a norm of a group if and only if $\chi \leqslant \aleph_{\mathrm{fm}}$ (if $\aleph_{\mathrm{fm}}$ exists) and it is not of the form $\chi = \lambda^+$ with λ weakly compact or $\chi = \infty$ (if $\aleph_{\mathrm{fm}}$ does not exist); see Corollary 6.3.

Moreover, χ is the norm of a radical if $\chi < \aleph_{\mathrm{fm}}$ and χ is either inaccessible or the successor of a cardinal that is not weakly compact; see Theorem 7.13 and the remark after the proof.

In the closing section we will prove and discuss various consistency results related to the above.

2. Set-theoretical preliminaries

By an ultrafilter we will mean an ultrafilter which is not principal. A cardinal κ is *weakly compact* if it satisfies the partition property $\kappa \longrightarrow (\kappa)^2$, i.e. κ is uncountable and if we write $[\kappa]^2$ for the set of all subsets of cardinal 2 in κ, then any function ('partition') $f : [\kappa]^2 \to 2$ admits a homogeneous subset of cardinal κ; recall that a subset H of κ is *homogeneous* if $f([H]^2)$ is a singleton, see [17, p. 325]. It is well-known that measurable cardinals are weakly compact and weakly compact cardinals are strong limit cardinals, see [17, p. 325, 327]. We will use the following notations for large cardinals.

Definition 2.1. Let $\aleph_{\mathrm{fm}}, \aleph_{\mathrm{fwc}}$ denote the first measurable, the first weakly compact cardinal and similarly, $\aleph_{\mathrm{fwi}}, \aleph_{\mathrm{fsi}}$ denote the first weakly inaccessible, the first strongly inaccessible cardinal, respectively (if they exist).

Recall that a cardinal κ is a *strong limit cardinal* if $2^{\lambda} < \kappa$ for all $\lambda < \kappa$. It is *weakly inaccessible* if it is a regular limit cardinal and it is *strongly inaccessible* if it is a regular cardinal which is a strong limit, see also [13,18,19].

3. Model theoretic conditions for pairs of cardinals

We will consider two properties $\mathfrak{p}_2, \mathfrak{p}_3$ for pairs (μ, λ) of infinite cardinals $\mu \leqslant \lambda$, where λ is regular, but μ need not be regular. Another property $\mathfrak{p}_1$, specially important to us in this connection, will be added in Section 4.

We will deal with two variants of the $\mathfrak{p}_i$'s, denoted by $\mathfrak{p}_i^x$. The parameter x will be either s (s for strong) or u (u for uniform). However, we are mainly interested in the uniform case. If M is a model with countable vocabulary τ_M and $\varphi(x, \overline{y})$ is a formula in the language L_{τ_M}, then let $\varphi(M, \overline{a}) = \{b \in M : M \vDash \varphi(b, \overline{a})\}$ where $\overline{a}$ is an n-tuple in M if $\overline{y}$ is an n-tuple of variables. For the property $\mathfrak{p}_2^x$ we will apply the following.

Definition 3.1. Let $\mu \leqslant \lambda$ be cardinals and $M \preccurlyeq N$ be models in a language L_{τ_M} with a unary predicate Q such that $Q^M, Q^N \subseteq \lambda$. Then we will say that Q^M is x-bounded by some $c \in Q^N$ if the following holds:

(i) $c \notin Q^M$ in case $x = s$.
(ii) $d < c$ for all $d \in Q^M$ in case $x = u$ and μ regular.
(iii) We assume for any formula $\varphi(x, \overline{y})$ in the language L_{τ_M} with $\overline{a} \in M^{\lg(\overline{y})}$ and $|\varphi(M, \overline{a})| < \mu$, that $c \notin \varphi(N, \overline{a})$ in case $x = s$ and μ is singular.

If such an element c does not exist, we will say that Q^M is x-unbounded for Q^N.

Note that, assuming that μ is regular, then (iii) is equivalent to (i).

We now express the

Property $\mathfrak{p}_2^x$. The pair of infinite cardinals (μ, λ) with $\mu \leqslant \lambda$ satisfies $\mathfrak{p}_2^x$ (we also write $(\mu, \lambda) \in \mathfrak{p}_2^x$) if there is a model M with countable vocabulary τ_M and a universe λ with two unary predicates Q_0, Q_1 and a binary predicate R representing $(\neq)$ if $x = s$ and the order relation $<$ on λ if $x = u$. Moreover, $Q_0^M = \omega,\ Q_1^M = \mu$. Then the following holds:

($\mathbb{E}$) If $M \preccurlyeq N$ is an elementary extension with universe λ and $Q_0^N = Q_0^M$, then Q_1^M is x-unbounded for Q_1^N.

If $(\mu, \mu) \in \mathfrak{p}_i^x$, we will write $\mu \in \mathfrak{p}_i^x$. We denote by $\mathfrak{P}(\mu)$ the powerset of μ with inclusion. Recall that a filter D on a boolean algebra $\mathbb{B} \subseteq \mathfrak{P}(\mu)$ is uniform if $D \cap [\mu]^{<\mu} = \emptyset$, where $[X]^{<\mu} = \{U \subseteq X : |U| < \mu\}$ for any set X. Thus, we also say that D is s-uniform if D is not principal and D is u-uniform if D is uniform in the usual sense, thus all elements in $\mathbb{B}$ are unbounded in μ.

We also have the

Property $\neg\mathfrak{p}_3^x$. The pair of infinite cardinals (μ, λ) with $\mu \leqslant \lambda$ satisfies $\neg\mathfrak{p}_3^x$ (the negation of $\mathfrak{p}_3^x$, we write $(\mu, \lambda) \notin \mathfrak{p}_3^x$) if the following holds:

For every boolean subalgebra $\mathbb{B} \subseteq \mathfrak{P}(\mu)$ with $|\mathbb{B}| \leqslant \lambda$ and any sequence

$$\langle\langle A_n^\alpha : n \in \omega\rangle : \alpha \in \lambda\rangle \quad \text{with } A^\alpha =: \bigcap_{n\in\omega} A_n^\alpha$$

of countable chains of elements $A_n^\alpha, A^\alpha \in \mathbb{B}$, there is an x-uniform ultrafilter D on $\mathbb{B}$ with the following property:

($*$) If $\alpha \in \lambda$ and $A_n^\alpha \in D$ for all $n \in \omega$, then also $A^\alpha \in D$.

We say that D is *weakly complete* if D satisfies ($*$).

We will use the connection between A_n^α and A^α at various places. Thus, $(\mu, \lambda) \in \mathfrak{p}_3^x$ provides a boolean algebra $\mathbb{B} \subseteq \mathfrak{P}(\mu)$ of size $|\mathbb{B}| \leqslant \lambda$ and a sequence $\langle\langle A_n^\alpha : n \in \omega\rangle : \alpha \in \lambda\rangle$ with $A^\alpha, A_n^\alpha \in \mathbb{B}$ such that for all x-uniform ultrafilters D on $\mathbb{B}$ there is $\alpha \in \lambda$ such that $A_n^\alpha \in D$ for all $n \in \omega$ but $A^\alpha \notin D$.

We first state some basic properties of the $\mathfrak{p}_i$s. Here it is our aim to determine the following class $\mathfrak{C}$ as good as possible in ZFC. We summarize some results from Proposition 3.3.

Remark 3.2. The class $\mathfrak{C} = \{\mu : \mu \in \mathfrak{p}_2^s,\ \mu < \aleph_{\mathrm{fm}}\}$ of cardinals is quite large. It contains all cardinals $\beth_\alpha$ $(\alpha < \aleph_{\mathrm{fm}})$ except those that are weakly compact as well as all 2-powers 2^μ for $\mu < \aleph_{\mathrm{fm}}$. Moreover, $\mathfrak{C}$ is closed under taking successor and singular limits. Assuming GCH the set $\mathfrak{C}$ is the complement of the set of all weakly compact cardinals below $\aleph_{\mathrm{fm}}$.

Proposition 3.3. *Let (μ, λ) be a pair of infinite cardinals with $\mu \leqslant \lambda$ and λ regular. Then the following holds*:

(i) $(\mu, \lambda) \in \mathfrak{p}_2^x \iff (\mu, \lambda) \in \mathfrak{p}_3^x$.
(ii) $\mathrm{cf}\ \mu < \mu \leqslant \lambda$, $(\mathrm{cf}\ \mu, \lambda) \in \mathfrak{p}_2^u \Rightarrow (\mu, \lambda) \in \mathfrak{p}_2^u$.
(iii) $(\mu, \lambda) \in \mathfrak{p}_2^s \Rightarrow (\mu, \lambda) \in \mathfrak{p}_2^u$.
(iv) *If* $\mu \leqslant \lambda_1 \leqslant \lambda_2$ *then* $((\mu, \lambda_1) \in \mathfrak{p}_2^x \Rightarrow (\mu, \lambda_2) \in \mathfrak{p}_2^x)$.
(v) *If* $2^\mu \leqslant \lambda$: $(\mu, \lambda) \notin \mathfrak{p}_2^x \iff \exists$ *an* ω_1*-complete* x*-uniform ultrafilter on* μ.

In particular, the following holds:

(a) $(\mu, 2^\mu) \notin \mathfrak{p}_2^s \iff \mu \geqslant \aleph_{\mathrm{fm}}$.
(b) $(\mu, 2^\mu) \notin \mathfrak{p}_2^u \iff \exists$ *a uniform* ω_1*-complete ultrafilter on* $\mu \iff$ *there is a uniform* $\aleph_{\mathrm{fm}}$*-complete ultrafilter on* $\aleph_{\mathrm{fm}}$.

(vi) *If* $(\mu, 2^\mu) \in \mathfrak{p}_2^s$, *then also* $2^\mu \in \mathfrak{p}_2^s$.
(vii) *If* $(\mu, \lambda) \in \mathfrak{p}_2^s$, *then* $(\mu^+, \lambda + \mu^+) \in \mathfrak{p}_2^s$.
(viii) *Let* μ_i $(i \in \delta)$ *be an increasing, continuous chain of cardinals with* $(\mu_i, \mu) \in \mathfrak{p}_2^s$ *and* $\delta < \sup_{i \in \delta} \mu_i = \mu$. *Then also* $(\mu, \lambda) \in \mathfrak{p}_2^s$.
(ix) *If* $\mu < \aleph_{\mathrm{fm}}$ *then* $2^\mu \in \mathfrak{p}_2^s$.
(x) *If* $\mu = \beth_\alpha < \aleph_{\mathrm{fm}}$ *is not weakly compact, then* $\mu \in \mathfrak{p}_2^s$. *(Similarly, if* $\beth_\alpha \leqslant \mu$ *is smaller then the first weakly inaccessible cardinal above* $\beth_\alpha$, *then* $\mu \in \mathfrak{p}_2^s$.*)*

Remark. From (v) follows that for all

$$\lambda, \lambda' \geqslant 2^\mu : (\mu, \lambda) \in \mathfrak{p}_2^x \iff (\mu, \lambda') \in \mathfrak{p}_2^x.$$

From Proposition 3.3(i) and Corollary 4.6 follows $((\mu, 2^\mu) \notin \mathfrak{p}_2^u \iff (\mu, 2^\mu) \notin \mathfrak{p}_1)$ and by definition of $\mathfrak{p}_1$ this is equivalent to $\mathbb{Z}_\mu^* \neq 0$. In this form (v)(b)"$\Leftarrow$" is a well-known observation due to Łoś, see [12, Remark, Vol. 2, p. 161]. For Proposition 3.3(v) we also note that by the existence of an ω_1-complete ultrafilter D on μ follows that $\mu' = \min\{|E| : E \subseteq D, \bigcap E \notin D\} \leqslant \mu$ is a measurable cardinal, see [17, p. 297].

Proof. (i): Suppose that $(\mu, \lambda) \in \mathfrak{p}_3^x$. We want to show that $(\mu, \lambda) \in \mathfrak{p}_2^x$.

By property $\mathfrak{p}_3^x$ we have a boolean algebra $\mathbb{B} \subseteq \mathfrak{P}(\mu)$ of cardinality $\leqslant \lambda$ and a sequence of elements $A_n^\alpha, A^\alpha \in \mathbb{B}$ such that no x-uniform ultrafilter is weakly complete. We use this to determine a model M for $\mathfrak{p}_2^x$ and concentrate on our main case $x=u$ with μ regular. Let Q_0^M and Q_1^M be as in $\mathfrak{p}_2^u$, $\mathbb{B}=\{a_i : i \in \lambda\}$ be an enumeration and $P^M=\{(x,i) : i \in \lambda, x \in a_i\}$ which encodes $\mathbb{B}$. We also define a 2-place function $F^M : \mu \times \lambda \longrightarrow \omega \cup \{\omega\}$ by

$$F^M(x,\alpha)=\begin{cases} \min\{n \in \omega : x \notin A_n^\alpha\} & \text{if } x \notin A^\alpha, \\ \omega & \text{if } x \in A^\alpha. \end{cases}$$

Thus, we can express in M that

$$x \in A^\alpha \iff (\forall n \in \omega \ F(x,\alpha) \neq n) \iff F(x,\alpha)=\omega$$

and more: If M satisfies $\mathfrak{p}_2^x$, then the proof is complete. Otherwise, we have an elementary extension $M \preccurlyeq N$ such that $\mathfrak{p}_2^x$ does not hold for N, so N has universe λ, $Q_0^N=Q_0^M$ but there is $c \in Q_1^N$ showing that Q_1^M is x-bounded for Q_1^N. For $x=u$ and μ regular this is case (ii) when $d<c$ for all $d \in Q_1^M$.

Let $D=\{a_i : i \in \lambda, (c,i) \in P^N\}$ which is a collection of elements in $\mathbb{B}$, and we show that D is an x-uniform ultrafilter. We check two critical properties:

If $a \in D$, then $a=a_i$ for some $i \in \lambda$ and $c \in a_i$ by P^N. If $a_i, a_j \in D$, then $(c,i),(c,j) \in P^N$, hence $c \in a_i, c \in a_j$ and also $c \in a_i \cap a_j = a_k$ for some $k \in \lambda$. It follows $(c,k) \in P^N$ and therefore $a_k = a_i \cap a_j \in D$. If $a \in \mathbb{B}$, then $a=a_i, a_j=\mu \backslash a$ for some $i,j \in \lambda$ and either $c \in a_i$ or $c \in a_j$. Hence, either $a \in D$ or $\mu \backslash a \in D$ and D is an ultrafilter. Moreover, $Q_1^M=\mu$ by assumption on M as in $\mathfrak{p}_2^u$ and $x<c$ for all $x \in Q_1^M$ from above. Hence, $a \in D$ implies $a \notin [\mu]^{<\mu}$ and D is (u-)uniform.

We now show that D is weakly complete. If $\alpha \in \lambda$ and $A_n^\alpha \in D$ for all $n \in \omega$, then $c \in A^\alpha \in \mathbb{B}$ and there is $i \in \lambda$ with $a_i = A^\alpha$. Hence, $(c,i) \in P^N$ and $A^\alpha = a_i \in D$ and D is weakly complete indeed. The existence of D contradicts our hypothesis $\mathfrak{p}_3^u$, hence $\mathfrak{p}_2^u$ holds.

Conversely, suppose that $(\mu,\lambda) \in \mathfrak{p}_2^u$. Then we have a model M satisfying property $\mathfrak{p}_2^u$. This will be used to define a boolean algebra $\mathbb{B}$ and a sequence $\langle\langle A_n^\alpha : n \in \omega\rangle : \alpha \in \lambda\rangle$ which satisfy $\mathfrak{p}_3^u$. Recall that M has universe λ, $Q_0^M=\omega$ and $Q_1^M=\mu$.

If $\varphi(x,\overline{b})$ with $\overline{b} \in M^{\lg(\overline{b})}$ and $\psi=\exists x \varphi(x,\overline{b})$, then let $F_\psi(,\overline{b})$ be a Skolem function interpreting ψ in M such that $M \vDash \forall x\, Q_0(F_\psi(x,\overline{b}))$, hence $F_\psi^M(,\overline{b}) : M \longrightarrow \omega$; see [4, p. 164]. Let $\mathfrak{F}=\{F_\alpha^M(,\overline{b}_\alpha) : \alpha \in \lambda\}$ be a list of these functions. Take $\mathbb{B}'$ to be the boolean algebra of subsets of μ definable with parameters in μ, choose

$$A_n^\alpha=\{\varepsilon \in \mu : F_\alpha^M(\varepsilon,\overline{b}_\alpha) \neq n\}$$

for all $\alpha \in \lambda$ and $n \in \omega$. The sequence $\langle\langle A_n^\alpha : n \in \omega\rangle : \alpha \in \lambda\rangle$ is defined. Finally, let $\mathbb{B}$ be the weak closure of $\mathbb{B}'$, the boolean algebra generated by $\mathbb{B}'$ and $\{A^\alpha : \alpha \in \lambda\}$. If $x=u$ it remains to show that for any uniform ultrafilter D on $\mathbb{B}$ there is $\alpha \in \lambda$ such that $A_n^\alpha \in D$ for all $n \in \omega$ but $A^\alpha \notin D$.

Suppose for contradiction that there is a uniform ultrafilter D on $\mathbb{B}$ which is weakly complete, hence for all $\alpha \in \lambda$ with $A_n^\alpha \in D$ $(n \in \omega)$ follows $A^\alpha \in D$. If $\varphi(x,\overline{y})$ is a formula in the language L of M and $\overline{a} \in M^{\lg(\overline{y})}$, then $\Phi =: \{\varepsilon \in \mu : M \vDash \varphi(\varepsilon,\overline{a})\}$ is a member of $\mathbb{B}$ by definition of $\mathbb{B}'$. The following set

$$\Gamma=\{\varphi(\varepsilon,\overline{a}) : \varphi(x,\overline{y}) \in L, \overline{a} \in M^{\lg(\overline{y})}, \{\varepsilon \in \mu : M \vDash \varphi(\varepsilon,\overline{a})\} \in D\}$$

of formulas is finitely satisfiable in M. If $\varphi_i(\varepsilon, \overline{a}_i) \in \Gamma$ for $i < n$ $(n \in \omega)$, then $\Phi_i =: \{\varepsilon \in \mu : M \vDash \varphi_i(\varepsilon, \overline{a}_i)\} \in D$ and also $\Phi = \bigcap_{i<n} \Phi_i \in D$ and if $\varepsilon \in \Phi$, then $M \vDash \varphi_i(\varepsilon, \overline{a}_i)$ for all $i < n$. By the compactness theorem (see [4, p. 33]) there are elementary extensions $N \succ M$ and $c \in N$ such that $N \vDash \varphi(c, \overline{a})$ for all $\varphi(x, \overline{a}) \in \Gamma$. We may assume that M has Skolem functions, so we can choose $N = \mathfrak{H}(M \cup \{c\})$ minimal as the Skolem hull of $M \cup \{c\}$, see [4, p. 165]. Since $\{\varepsilon \in \mu : M \vDash Q_1(\varepsilon)\}$ belongs to D, we have $Q_1(x) \in \Gamma$ and $N \vDash Q_1(c)$ by the choice of c. Hence, $c \in Q_1^N$. If $\beta \in \mu$, then the interval $(\beta, \mu) = \{\varepsilon \in \mu : \beta < \varepsilon\}$ belongs to D. Otherwise, its complement $[0, \beta]$ belongs to the ultrafilter D, which is impossible because D is also uniform. Hence, $\{\varepsilon \in \mu : M \vDash \beta < \varepsilon\} \in D$ and therefore also $\beta < x \in \Gamma$, so $N \vDash \beta < c$ and $\mu \leqslant c$ is an upper bound. Next we show that $Q_0^N = Q_0^M$ (which is ω). Clearly, $Q_0^M \subseteq Q_0^N$ and if $d \in Q_0^N$, then there is a Skolem function with $d = F_\alpha^N(c, \overline{b}_\alpha)$ because N was the corresponding Skolem hull. We may assume that $F_\alpha^M(x, \overline{b}_\alpha) : M \longrightarrow Q_0^M$. If there is $n \in \omega$ such that $A_n^\alpha \notin D$, then $\{\varepsilon \in \mu : F_\alpha^M(\varepsilon, \overline{b}_\alpha) = n\} \in D$ and $(F_\alpha(x, \overline{b}_\alpha) = n) \in \Gamma$. Hence, $N \vDash F_\alpha(c, \overline{b}_\alpha) = n$ and $d = n$ because $F_\alpha(x, \overline{b}_\alpha)$ is a function. This shows $Q_0^N = Q_0^M$ if $A_n^\alpha \notin D$. Otherwise, $A_n^\alpha \in D$ for all $n \in \omega$ and $\bigcap_{n\in\omega} A_n^\alpha \in D$ by hypothesis. But from the other hypothesis $Q_0^M = \omega$ and the definition of the A_n^αs follows $\bigcap_{n\in\omega} A_n^\alpha = \emptyset$ which is also impossible in D. Hence, such a weakly closed D does not exist and $\mathfrak{p}_3^u$ follows.

The proof for $x = s$ and for singular μ is similar. □

For the proof of the remaining statements in Proposition 3.3 we first formulate some preliminary claims.

Claim 3.4. *If χ is a cardinal, then there is a model M with universe χ and a countable vocabulary such that the following holds*:

(i) *If $\mu \leqslant \lambda \leqslant \chi$ and $(\mu, \lambda) \in \mathfrak{p}_2^x$ $(x \in \{u, s\})$, then we can interpret in M by formulas with parameters a submodel $M_{\mu,\lambda}$ (depending on (μ, λ) only) that $M_{\mu,\lambda} \vDash (\mu, \lambda) \in \mathfrak{p}_2^x$.*

(ii) *If $2^\mu \leqslant \lambda$, then there are some formulas $\varphi(x, y, z)$ with parameters from M such that*

$$\{\{\alpha \in \mu : M \vDash \varphi(\alpha, \gamma, \mu) : \gamma \in 2^\mu\}\}$$

is the family $\mathfrak{P}(\mu)$ of all subsets of μ.

(iii) *There are functions $F(x, y, z)$ definable in M with parameters for every $\mu < \chi$ such that for every $\alpha \in [\mu, \mu^+)$ the sequence $\langle F^M(i, \alpha, \mu) : i < \alpha\rangle$ is μ as the set of all ordinals without repetition.*

Proof. (i) Let τ be a countable vocabulary with countably many functions and relation symbols, each in a finite number of variables. For any $(\mu, \lambda) \in \mathfrak{p}_2^x$ we have a model $M_{\mu\lambda}^x$ with language $L_{\tau_{\mu\lambda}}$ given by the property $(\mathfrak{p}_2^x)$ (for M). We may assume that $\tau_{\mu\lambda} \subseteq \tau$ and view $M_{\mu\lambda}^x$ as a τ-model, thus

$$\langle M_{\mu\lambda}^x : (\mu, \lambda) \in \mathfrak{p}_2^x, \mu \leqslant \lambda \leqslant \chi, x \in \{u, s\}\rangle$$

is a well-defined sequence of τ-models for any cardinal χ. We define a model $M = M_\chi$ with universe $\chi + 1$ and vocabulary τ^* derived from τ by replacing any n-place function F

(relation R) by an $n+3$-place function F' (relation R'). We interpret R'^M by

$$\bigcup_{x,\mu,\lambda}\{\langle x,\mu,\lambda\rangle^\wedge\overline{a}:\overline{a}\in P^{M^x_{\mu\lambda}},(\mu,\lambda)\in\mathfrak{p}_2^x,\mu\leqslant\lambda\leqslant\chi,x\in\{u,s\}\}.$$

Put $F'^M(\overline{a})=\gamma$ if $(\mu,\lambda)\in\mathfrak{p}_2^x$ for some $\mu\leqslant\lambda\leqslant\chi$, $\overline{a}=\langle x,\mu,\lambda\rangle^\wedge\overline{a}'$ and $F^{M^x_{\mu\lambda}}(\overline{a}')=\gamma$ and put $\gamma=0$ otherwise. Thus, M is a τ^*-model with universe $\chi+1$ and if $(\mu,\lambda)\in\mathfrak{p}_2^u$, then for example we can interpret $M^u_{\mu\lambda}$ by formulas in M in an obvious manner (using strings $(u,\mu,\lambda,\overline{a})$ with $\overline{a}\in(M^u_{\mu\lambda})^n$ for a suitable $n\in\omega$). Thus, in M we can express $(\mu,\lambda)\in\mathfrak{p}_2^x$ for $x=s, x=u$.

(ii) Let $\{U_\gamma:\gamma<2^\mu\}$ be an enumeration of all subsets of μ without repetition. For each $\gamma<2^\mu$ choose a map $\varphi=\varphi_{\gamma,\mu}:\mu\longrightarrow U_\gamma$ which is onto. Hence, $\varphi_{\gamma,\mu}(\alpha)=\varphi(\alpha,\gamma,\mu)$ are the formulas for all subsets of μ when γ runs through 2^μ. The subsets of μ are obtained as $\{i\in\mu:M\vDash\varphi(i,\gamma,\mu)\}$.

(iii) This is similar to (ii). If μ is as above and $\alpha\in[\mu,\mu^+)$, then $|\alpha|=\mu$ and we can choose a bijection $F^M(,\alpha,\mu):\alpha\longrightarrow\mu$ from μ onto α. For every $\gamma\in\mu$ there is exactly one $i<\alpha$ such that $\gamma=F^M(i,\alpha,\mu)$. The claim is then immediate. □

We apply Claim 3.4(ii) to derive the next claim. Claim 3.5(ii) is only needed for singular cardinals, a case which we only mention for completeness.

Claim 3.5. (i) *If* $2^\mu\leqslant\lambda$, $M\preccurlyeq N$ *and* $c\in N\setminus M$, *then*

$$(N\vDash c<2^\mu\Rightarrow\exists d\in N\setminus M\ \ and\ \ N\vDash d<\mu).$$

(ii) *Let* $\mu\leqslant\lambda\leqslant\chi$ *and* $\mu<\chi$ *and let* M *be a model that interprets a model* M' *that exemplifies* $(\mu,\lambda)\in\mathfrak{p}_2^u$ *and includes* $(\mu,<)$. *Suppose that* $Q_0^M=\omega$, $R^M=\{(\alpha,\beta):\alpha<\beta<\chi\}$ *is the order relation* $<$ *on* χ, $M\preccurlyeq N$ *and* $Q_0^M=Q_0^N$. *If* $c\in N$ *and* $N\vDash c<\mu$, *then the following holds for some formula* $\varphi(x,\overline{a})$, $(\overline{a}\in{}^{\lg(\overline{a})}M)$.

$$N\vDash\varphi(c,\overline{a})\quad and\quad|\varphi(M,\overline{a})|<\mu.$$

Proof. (i) Let $\varphi(\alpha,\gamma,\mu):\gamma\in2^\mu$ be the list of formulas given by Claim 3.4(ii) and consider the set U related to c in the elementary extension N, that is

$$U=\{i\in\mu:N\vDash\varphi(i,c,\mu)\}\subseteq\mu.$$

So there is a $\gamma\in2^\mu$ such that $U=\{i\in\mu:M\vDash\varphi(i,\gamma,\mu)\}$. Since $M\preccurlyeq N$, this also holds in N, we have $U=\{i\in\mu:N\vDash\varphi(i,\gamma,\mu)\}$. This set is a subset of μ by Claim 3.4(ii).

Now suppose that (3.5) does not hold, so for every $d\in N$ with $N\vDash d\in\mu$ follows $d\in M$. Thus,

$$N\vDash\forall x\in\mu\ [(\varphi(x,\gamma,\mu)\Longleftrightarrow\varphi(x,c,\mu)],$$

but using again that $M\preccurlyeq N$, this holds in M, hence $c=\gamma$ by the unique representation of the subsets. However, $c\notin M$ and $\gamma\in M$ is a contradiction.

(ii) If M', $M \preccurlyeq N$ are as above, then let N' be the interpretation of M' in N, hence $N \preccurlyeq N'$ with $Q_0^N = Q_0^{N'}$ and $c \in N'$ satisfies $N' \vDash c < \mu$. But because of M' (which is a model for $(\mu, \lambda) \in \mathfrak{p}_2^u$) in N' the set $\{\alpha : \alpha < \mu\}$ is unbounded (for $<$). There is some $\alpha \in \mu$ such that $N' \vDash c < \alpha$, hence also for $N \vDash c < \alpha$. Now let $\varphi = \{x : x \in \alpha\}$ be the required formula. □

Proof of Proposition 3.3 (continuation).

(ii) Suppose $(\mathrm{cf}\,\mu, \lambda) \in \mathfrak{p}_2^u$. Let $\mathrm{cf}\,\mu = \mu'$ and $M_{\mu'\lambda}$ be given by property $\mathfrak{p}_2^u$. We must find a suitable model M' showing $(\mu, \lambda) \in \mathfrak{p}_2^u$ and suggest two ways. Either consider the model M given by Claim 3.4(i) and choose $M' \preccurlyeq M$ such that $|M'| = \lambda$ and $\lambda + 1 \subseteq M'$, moreover, expand Q_0, Q_1, Q_2 such that $Q_0^{M'} = \omega$, $Q_1^{M'} = \mu$, $Q_2^{M'} = \lambda$ and show that M' is as required for $(\mu, \lambda) \in \mathfrak{p}_2^u$, or construct M' directly.

Take $M_{\mu'\lambda}$ as above with $Q_0^{M_{\mu'\lambda}} = \omega$ but ${Q'_1}^{M_{\mu'\lambda}} = \mu'$ and $Q_1^{M_{\mu'\lambda}} = \mu$. Expand $M_{\mu'\lambda}$ by adding functions from Claim 3.4. (This is all what is needed from the first suggested proof.)

$$F' : \mu' \longrightarrow \mu \text{ strictly increasing, continuous and unbounded,}$$

$$F'' : \mu \longrightarrow \mu' \ (\alpha \longrightarrow F''(\alpha) = \min\{i \in \mu : F'(i) \geqslant \alpha\}).$$

Now we show that M' is a model for $(\mu, \lambda) \in \mathfrak{p}_2^u$. Clearly, $Q_0^{M'} = \omega$, $Q_1^{M'} = \mu$. If M' is not as required, then there is an elementary extension $M' \preccurlyeq N$ with universe λ, $Q_0^N = \omega$ and $Q_1^{M'} = \mu$ is u-bounded for Q_1^N, hence there is an upper bound $c \in Q_1^N$ such that $N \vDash (\alpha < c\ \forall \alpha \in \mu)$. Now apply F'', hence $M_{\mu'\lambda} \vDash (\alpha \in F''(c)\ \forall \alpha < F''(\mu) = \mu')$ which contradicts that $M_{\mu'\lambda}$ is a model exemplifying $(\mu', \lambda) \in \mathfrak{p}_2^u$.

(iii) is trivial.

(iv) By Proposition 3.3(i) we can replace $\mathfrak{p}_2^x$ by $\mathfrak{p}_3^x$. Now the proof is easy.

(v) ($\Rightarrow$) Again we can replace $\mathfrak{p}_2^x$ by $\mathfrak{p}_3^x$, choose $\mathbb{B} = \mathfrak{P}(\mu)$ and let $\langle\langle A_n^\alpha : n \in \omega\rangle : \alpha \in \lambda\rangle$ all possible sequences with $A^\alpha, A_n^\alpha \in \mathbb{B}$ because $2^\mu \leqslant \lambda$. By $\mathfrak{p}_3^x$ there is an x-uniform, ω_1-complete ultrafilter on μ, because weakly complete and complete are now the same notions.

(v) ($\Leftarrow$ $(x = s)$) Suppose that κ is measurable and $\kappa \leqslant \mu$, e.g. $\kappa = \aleph_{\mathrm{fm}}$. Then choose a κ-complete non-principal ultrafilter D on κ (see [17, p. 297]) and let M be a model with universe 2^μ, $Q_0^M = \omega$ and $Q_1^M = 2^\mu$ as in the definition of $\mathfrak{p}_2^s$. Hence, there is a canonical elementary embedding $\mathbf{j}$ of M into the model $N = M^\kappa/D$ and $\mathbf{j}$ maps Q_0^M onto Q_0^N because Q_0^M is expressed by the sentence $\psi = (\forall x\, Q_0(x) \equiv \exists n, x = n)$, but $\mathbf{j}$ maps Q_1^M properly into Q_1^N as $c = \langle \alpha : \alpha \in \kappa\rangle/D \in Q_1^N \setminus \mathbf{j}(Q_1^M)$. We may assume that $\mathbf{j}$ is the identity and N has universe 2^μ (as $|M^\kappa/D| = 2^\mu$). Hence, $M \preccurlyeq N$, $Q_0^N = \omega$ and $c \in Q_1^N \setminus Q_1^M$ violates the implication of $\mathfrak{p}_2^s$ and it follows$(\mu, 2^\mu) \notin \mathfrak{p}_2^s$.

(v) ($\Leftarrow$ $(x = u)$) By Proposition 3.3(i) we must show that $(\mu, \lambda) \notin \mathfrak{p}_3^u$. There is a uniform ω_1-complete ultrafilter D an μ. We can enumerate its elements as $D = \{A_\alpha : \alpha \in \lambda\}$ because $2^\mu \leqslant \lambda$ and let $\langle A_n^\alpha : n \in \omega\rangle$ be all countable sequences in D. Then D is also weakly complete for any boolean algebra $\mathbb{B} \subseteq \mathfrak{P}(\mu)$ and $(*)$ of Property $\mathfrak{p}_3^u$ holds.

(vi) Let $\lambda = 2^\mu$ and $\mathfrak{P}(\mu) = \{A_\alpha : \alpha \in 2^\mu\}$ without repetitions. If M is a model for $(\mu, \lambda) \in \mathfrak{p}_2^s$, then we get a new model M' with Skolem functions (w.l.o.g.) and let R be a binary relation such that $R(\beta,\alpha)$ holds if $\beta\in A_\alpha$. Moreover, $Q_0^M=Q_0^{M'}=\omega$ as usual, but we replace Q_1 by Q_1', so ${Q_1'}^{M'}=\mu$ and let $Q_1^{M'}=\lambda$. We claim that M' exemplifies $\lambda\in\mathfrak{p}_2^s$. If $M'\preccurlyeq N$, $Q_0^{M'}=Q_0^N$, then by Q_1' we have $\mu=\{\alpha: M'\vDash\alpha\in\mu\}=\{\alpha: N\vDash\alpha\in\mu\}$. Suppose M' is s-bounded by some $c\in M'\backslash N$. If $A=\{\beta\in\mu: R(\beta,c)\}\subseteq\mu$, then there is $\alpha\in 2^\mu$ with $A=A_\alpha$. Hence, $\alpha\in 2^\mu=\lambda$ (the universe of M'), but $c\in N\backslash M'$, and $N\vDash\alpha\neq c$. From $A=A_\alpha$ follows $R(\beta,\alpha)=R(\beta,c)$ for all $\beta\in\mu$, but this is a contradiction because (by enumeration of the A_αs without repetition) it follows from $\alpha\neq c$ that for at least one $z\in Q_1^{M'}=\mu$ we have $R(z,\alpha)\neq R(z,c)$.

(vii) Let M be a model for $(\mu,\lambda)\in\mathfrak{p}_2^s$ and M' a model with universe $\lambda+\mu^+$ where we can interpret M. We add functions $F_1(x,y)$ such that $\langle F_1(\beta,\alpha),\beta<\alpha\rangle$ for all $\alpha\in[\mu,\mu^+)$ lists all functions on μ without repetition, and $F_2(x)$ is the function with $F_2(F_1(\beta,\alpha))=\beta$ if $\beta<\alpha\in[\mu,\mu^+)$.

Claim. *M' is a model exemplifying $(\mu^+,\lambda+\mu^+)\in\mathfrak{p}_2^s$.*

Proof. If $M'\preccurlyeq N$ and $Q_0^{M'}=Q_0^N=\omega$ and N has universe μ^+, then $\{b\in N: M\vDash b\in\mu\}=\mu$ by $Q_1^M=\mu$. Suppose for contradiction that $c\in N\backslash M'$ is an s-upper bound and $c\in\mu^+$, then we distinguish two cases. Either for all $\alpha\in\mu^+$ follows $N\vDash\alpha<c$ or for some $\alpha>\mu$ (w.o.l.g.) $N\vDash c<\alpha$. In the first case $\langle F_1^N(\alpha,c):\alpha\in\mu^+\rangle$ is a sequence of elements in μ of length μ^+ which is impossible for cardinals. In the other case $c=F_2(F_1(c,\alpha),\alpha)$ belongs to M because $F_1(c,\alpha)\in M$ which is also a contradiction.

(viii) Let μ_i $(i\in\delta)$ be an increasing, continuous chain of cardinals with $(\mu_i,\mu)\in\mathfrak{p}_2^s$ and $\delta<\sup_{i\in\delta}=\mu$. We claim that $(\mu,\lambda)\in\mathfrak{p}_2^s$.

By Claim 3.4(i) we find a model M with universe λ such that for each $i\in\delta$ we can interpret a model M_i showing that $(\mu_i,\mu)\in\mathfrak{p}_2^s$. Let $F':\delta\longrightarrow\mu$ $(i\longrightarrow\mu_i)$ and $F:\mu\longrightarrow\delta$ be given by $F(\alpha)=\min\{i\in\delta: F'(i)\geqslant\alpha\}$ for any $\alpha\in\mu$.

If $i<\delta$, then $Q_1^{M_i}=\mu_i$ and $M\preccurlyeq N$. We have $\{c: N\vDash c<\mu_i\}=\{c: M\vDash c<\mu_i\}=\mu_i$. Suppose that N is s-bounded. Then there is $c\in N\backslash M$. We may assume that $N\vDash c\in\mu$, but then $N\vDash F(c)=i\in\delta$ as $\mathrm{cf}\,\mu\leqslant\mathrm{cf}\,\delta\leqslant\delta<\mu$ and by F' follows $c\in\mu_i\backslash M$, so M_i is s-bounded, a contradiction.

(ix) If $\mu<\aleph_{\mathrm{fm}}$, then $(\mu,2^\mu)\in\mathfrak{p}_2^s$ by Proposition 3.3(v)(a), hence $2^\mu\in\mathfrak{p}_2^s$ by Proposition 3.3(vi).

(x) The proof follows by induction on $\alpha<\aleph_{\mathrm{fm}}$. For $\alpha=0$ there is nothing to show. If $\alpha=\beta+1$, then $\beth_\alpha=2^{\beth_\beta}$ and $\beth_\alpha\in\mathfrak{p}_2^s$ by Proposition 3.3(ix). If $\mu=\beth_\alpha$ is a singular limit cardinal, then we can choose an increasing sequence μ_i $(i\in\delta=:\mathrm{cf}\,\mu<\mu)$ of cardinals which are 2-powers at successor stages, hence members $\mu_i\in\mathfrak{p}_2^s$ by (ix). Now $\beth_\alpha\in\mathfrak{p}_2^s$ follows by Proposition 3.3(viii). We may assume that $\mu=\beth_\alpha$ is a strongly inaccessible cardinal, but not weakly compact (the hypothesis in (x)). So there is a μ-tree $T=(\mu,<_*)$ (with the tree ordering $<_*$ having no μ-branches, see [17, p. 326] (on Aronszajn μ-trees). We choose a 2-place function $F: T\times T\longrightarrow T$ such that $F(c,\beta)=d$ if $c\in T$ has level $l(c)\geqslant\beta$ and d is an element of level $l(d)=\beta$. Otherwise, put $F(c,\beta)=c$ (which is an arbitrary and uninteresting choice). Moreover, let $F'=l: T\longrightarrow\mu$ $(c\longrightarrow F'(c)=l(c))$ be the map assigning the level to each tree element. Let M be a model with universe μ such that for $\alpha<\mu$ (then $\beth_\alpha<\mu$) we can interpret M_α (the model telling us that $\beth_\alpha\in\mathfrak{p}_2^s$) using only $\beth_\alpha$ as a parameter. If M does not exemplify $\mu \in \mathfrak{p}_2^s$, there is $M \preccurlyeq N$ with $Q_0^M = Q_0^N = \omega$ and if $\alpha < \mu$, then again $\{c : N \vDash c \in \beth_\alpha\} = \{c : M \vDash c \in \beth_\alpha\} = \beth_\alpha$; let $c \in N \setminus M$ be an s-bound for $\alpha < \mu$, hence $N \vDash F'(c) \geqslant \alpha$. It follows that $\langle F_1(c, \beta) : \beta < \mu \rangle$ is a μ-branch of T which is too long, a contradiction. □

Lemma 3.6. *If λ is a weakly compact cardinal, then the following holds*:

(i) $\lambda \notin \mathfrak{p}_2^s$.
(ii) *If $\lambda < \rho < \aleph_{\mathrm{fm}}$ and ρ is a not weakly compact limit cardinal, then $\rho \in \mathfrak{p}_2^s$.*

Proof. (i) If λ is weakly compact then any expansion of $M = (H(\lambda), \varepsilon)$ by countably many relations and functions has a *proper* elementary *end* extension N, i.e. there is $M \prec N$ and $(a \in H(\lambda), N \vDash b \in a) \Rightarrow b \in M$; see [7, p. 184, 185]. Consider $Q_0^M = \omega$, $Q_1^M = \lambda$ as in the definition of $\mathfrak{p}_2^s$, then Q_1^M is s-bounded because N is an end extension. Hence, (i) follows.

(ii) Choose a chain of suitable cardinals and apply Proposition 3.3(x). □

We also note

Observation 3.7. (i) *If $\lambda \in \mathfrak{p}_2^s$, then $(\mu, \lambda) \in \mathfrak{p}_2^s$.*
(ii) (ZFC+GCH) *If $\mu < \lambda < \aleph_{\mathrm{fm}}$ are cardinals and λ is not weakly compact, then $(\mu, \lambda) \in \mathfrak{p}_2^s$.*

Proof. (i) If $\lambda \in \mathfrak{p}_2^s$, then $(\lambda, \lambda) \in \mathfrak{p}_2^s$ by definition of $\mathfrak{p}_2^s$ and trivially $(\mu, \lambda) \in \mathfrak{p}_2^s$ for all $\mu \leqslant \lambda$.

(ii) If $\mu < \lambda$, then $2^\mu = \mu^+ \leqslant \lambda$ by GCH. If $2^\mu < \lambda$ and $(\mu, \lambda) \notin \mathfrak{p}_2^s$, then $\mu \geqslant \aleph_{\mathrm{fm}}$ by Proposition 3.3(v)(a), a contradiction. So $\lambda = 2^\mu$, then $\lambda \in \mathfrak{p}_2^s$ by Proposition 3.3(ix). Hence, Observation 3.7(i) applies and the lemma follows. □

We summarize results from Lemma 3.6, Observation 3.7 and Proposition 3.3 as a

Corollary 3.8 (*ZFC+GCH*). *Let $\lambda < \aleph_{\mathrm{fm}}$ be a cardinal. Then*

$$\lambda \in \mathfrak{p}_2^s \iff \lambda \text{ is not weakly compact.}$$

Recall that $\aleph_\alpha^{+\beta} =: \aleph_{\alpha+\beta}$ for all ordinals β. Corollary 3.8 shows Remark 3.2 and we wonder if GCH can be replaced by a weaker hypothesis. Inspection of the proof shows that the following two assumption are sufficient to characterize cardinals $< \aleph_{\mathrm{fm}}$ with property $\mathfrak{p}_2^s$ (in $\mathfrak{C}$):

Remark 3.9. If all cardinals $\lambda < \aleph_{\mathrm{fm}}$ satisfy the conditions:

$\lambda^{\aleph_0} < \lambda^{+\omega}$ and 2^λ is smaller than the first weakly inaccessible cardinal above λ, then a cardinal smaller than $\aleph_{\mathrm{fm}}$ has the property $\mathfrak{p}_2^s$ (belongs to $\mathfrak{C}$) if and only if it is not weakly compact.

4. An equivalent group theoretic condition for pairs of cardinals

It is convenient to introduce at this point a notation for reduced products. Given a family of groups G_α $(\alpha \in \rho)$ indexed by a cardinal ρ, we shall write

$$\prod_{\alpha\in\rho}^{\text{red}} G_\alpha = \prod_{\alpha\in\rho} G_\alpha \Big/ \prod_{\alpha\in\rho}^{<\rho} G_\alpha,$$

where $\prod_{\alpha\in\rho}^{<\rho} G_\alpha$ is the subgroup of all elements in $\prod_{\alpha\in\rho} G_\alpha$ of support $< \rho$. In particular, the standard notation for powers and reduced powers of $\mathbb{Z}$ become

$$\mathbb{Z}^\rho = \prod_{\alpha\in\rho} \mathbb{Z}, \quad \mathbb{Z}^{<\rho} = \prod_{\alpha\in\rho}^{<\rho} \mathbb{Z} \quad \text{and} \quad \mathbb{Z}_\rho = \prod_{\alpha\in\rho}^{\text{red}} \mathbb{Z}.$$

The canonical epimorphism from product to reduced product will be denoted by π, or often simply a bar. We follow the tradition and abbreviate $\mathrm{Hom}(G, \mathbb{Z}) = G^*$.

For $I \subseteq \rho$, write $e_I \in \mathbb{Z}^\rho$ for the characteristic function

$$e_I = \sum_{i\in I} e_i.$$

Note that in $\mathbb{Z}_\rho$

$$\overline{e_I} \neq 0 \iff |I| = \rho.$$

The first part of the following lemma is obvious and the second part is the Wald–Łoś lemma, see [26] or for example [11, Proposition 3.4, p. 30] or [15]. The proof uses that for regular uncountable cardinals κ the filter $\mathfrak{F}^\kappa = \{X \subseteq \lambda : |\kappa \backslash X| < \kappa\}$ is κ-complete.

Lemma 4.1. (i) $|\mathbb{Z}_\kappa| = |\mathbb{Z}^\kappa| = 2^\kappa$.

(ii) *If κ is regular, uncountable, $G \subseteq \prod_{\alpha\in\kappa}^{\text{red}} G_\alpha$ is a reduced product for some family of groups G_α and $|G| < \kappa \Rightarrow \exists G' \subseteq \prod_{\alpha\in\kappa} G_\alpha$ and a homomorphism $\sigma : G \longrightarrow G'$ such that* $\sigma\pi = \mathrm{id}_{G'}$.

Observation 4.2. *Suppose $\mu = \mathrm{cf}\,\kappa < \kappa$. Then there is an embedding $\mathbb{Z}_\mu \hookrightarrow \mathbb{Z}_\kappa$.*

Proof. Let κ_i $(i \in \mu)$ be an increasing sequence of cardinals converging to κ, and let $\kappa_0 = 0$. We define a homomorphism

$$\varphi : \mathbb{Z}^\mu \longrightarrow \mathbb{Z}^\kappa \quad \left(\sum_{\alpha\in\mu} x_\alpha e_\alpha \longrightarrow \sum_{\beta\in\kappa} y_\beta e_\beta \right)$$

with $y_\beta = x_\alpha$ for all $\beta \in [\kappa_\alpha, \kappa_{\alpha+1})$. It follows that $\mathbb{Z}^{<\mu}\varphi \subseteq \mathbb{Z}^{<\kappa}$, and the induced homomorphism $\mathbb{Z}_\mu \longrightarrow \mathbb{Z}_\kappa$ is injective. $\square$

Note that $\mathbb{Z}_\kappa$ for cardinals κ with $\mathrm{cf}\,\kappa > \aleph_0$ is $\aleph_1$-free (by Lemma 4.1(ii)), so there are many even free subgroups $T \subseteq \mathbb{Z}_k$ with non-trivial dual. Nevertheless there are also many subgroups with trivial dual; they are related to the property $\mathfrak{p}_1$.

Property $\mathfrak{p}_1$. The pair of infinite cardinals (μ, λ) with $\mu \leqslant \lambda$ satisfied $\mathfrak{p}_1$ (we write $(\mu, \lambda) \in \mathfrak{p}_1$) if for any $J \subseteq \mu$ with $|J| = \mu$ there is a group $G = G_J$ with $\mathbb{Z}^{<\mu} \subseteq G \subseteq \mathbb{Z}^{\mu}$ and $e_J \in G$ such that $|G/\mathbb{Z}^{<\mu}| \leqslant \lambda$ and $(G/\mathbb{Z}^{<\mu})^* = 0$. Again $\lambda \in \mathfrak{p}_1$ stands for $(\lambda, \lambda) \in \mathfrak{p}_1$.

Example 4.3. $\omega \in \mathfrak{p}_1$.

Proof. Choose an infinite subset J of ω and find $G \subseteq \mathbb{Z}^{\omega}$ with $e_J \in G$ and $G/\mathbb{Z}^{(\omega)} \cong \mathbb{Q}$. $\square$

Proposition 4.4. *For (μ, λ) a pair of cardinals with $\mu \leqslant \lambda$ the following holds*:

(i) *If $(\mu, \lambda) \notin \mathfrak{p}_3^u$, then for any group G with $\mathbb{Z}^{<\mu} \subseteq G \subseteq \mathbb{Z}^{\mu}$ and $1 \neq |G/\mathbb{Z}^{<\mu}| \leqslant \lambda$ follows $(G/\mathbb{Z}^{<\mu})^* \neq 0$.*

(ii) $(\mu, \lambda) \in \mathfrak{p}_1 \Rightarrow (\mu, \lambda) \in \mathfrak{p}_3^u$.

Proof. (i) Suppose that $(\mu, \lambda) \notin \mathfrak{p}_3^u$. If G is a group satisfying the hypothesis of (i), then we want to show that $(G/\mathbb{Z}^{<\mu})^* \neq 0$.

If $z \in \mathbb{Z}$, we shall consider the z-support of any $g = \sum_{\alpha \in \mu} g_\alpha e_\alpha \in G$ to be the set

$$[g]_z = \{\alpha \in \mu : g_\alpha = z\},$$

hence $[g] = \bigcup_{0 \neq z \in \mathbb{Z}} [g]_z$ is the usual support. Fix a bijection $\iota : \mathbb{Z} \longrightarrow \omega$ with $0\iota = 0$ and rename $[g]_z = A_n^g$ if $z\iota = n$. We also consider the boolean algebra $\mathbb{B}$ generated by these A_n^g $(n \in \omega, g \in G)$ as a subalgebra of $\mathfrak{P}(\mu)$. If $g \in \mathbb{Z}^{<\mu}$, then let $[g] = \emptyset$ and using cosets we can choose $[g] = [g']$ for any $g \equiv g' \bmod \mathbb{Z}^{<\mu}$. Thus, $|\mathbb{B}| \leqslant |G/\mathbb{Z}^{<\mu}| \leqslant \lambda$. We also have sequences

$$\langle \langle A_n^g : n \in \omega \rangle : g \in G \rangle$$

(which we could label by λ because $|G/\mathbb{Z}^{<\mu}| \leqslant \lambda$). Moreover, we add the elements $A^g =: \bigcap_{n \in \omega} A_n^g$ $(g \in G)$ as generators to $\mathbb{B}$. The assumptions on $\mathfrak{p}_3^u$ are satisfied. By $(\mu, \lambda) \notin \mathfrak{p}_3^u$ there is a uniform ultrafilter D on $\mathbb{B}$ such that $(*)$ in property $\mathfrak{p}_3^x$ holds, thus D is weakly complete. Now we define a homomorphism $\varphi : G \longrightarrow \mathbb{Z}$ and let

$$g\varphi = z \iff A_{z\iota^{-1}}^g \in D.$$

The map is well-defined: If $A_m^g, A_n^g \in D$ and $m \neq n$, then $A_m^g \cap A_n^g = \emptyset$ by definition of support, but then $A_m^g \cap A_n^g = \emptyset \in D$ is a contradiction. If $g\varphi = z$ and $g'\varphi = z'$, then we must show that $(g + g')\varphi = z + z'$. However, $[g]_{z\iota^{-1}}, [g']_{z'\iota^{-1}} \in D$ and $Y = [g]_{z\iota^{-1}} \cap [g']_{z'\iota^{-1}} \in D$ as well, but then $(g + g')\restriction Y$ is the constant function with value $z + z'$, the linearity follows.

There is $g \in G \setminus \mathbb{Z}^{<\mu}$, which has support $[g]$ of size μ. We may replace G by $G^{[g]}$, the restriction of all elements of G to those of support $[g]$. This new group, an epimorphic image of the old one, is also called G. Now g has support $[g] = \mu \in D$ and $[g]_0 = \emptyset$. If $g\varphi = 0$, then $\emptyset = [g] \in D$, which is impossible. Hence, $\varphi \neq 0$. On the other hand, any $x \in \mathbb{Z}^{<\mu}$ has $[x]_0 = \mu$ up to a set of size $< \mu$. We have $\mu \setminus [x]_0 \notin D$ because D is uniform and $[x]_0 \in D$ because D is an ultrafilter, hence $x\varphi = 0$. So φ induces a non-trivial homomorphism from $G/\mathbb{Z}^{<\mu}$ and $(G/\mathbb{Z}^{<\mu})^* \neq 0$, and (i) is shown.

(ii) If $(\mu, \lambda) \notin \mathfrak{p}_3^\mu$ and $G = G_J$ is a group satisfying the hypothesis of $\mathfrak{p}_1$, then (i) applies and $(G/\mathbb{Z}^{<\mu})^* \neq 0$, hence $(\mu, \lambda) \notin \mathfrak{p}_1$. □

In Section 7 we require a stronger form of the converse of the last proposition, which we show next. Recall from the last proof that $[f]_z = \{i \in \mu : f_i = z\} \subseteq \mu$ for any $f \in \mathbb{Z}^\mu$, $z \in \mathbb{Z}$. Also recall that a pure subgroup K of the group of bounded, integer-valued functions on μ is a Specker group if with any $f \in K$ and $z \in \mathbb{Z}$ also the characteristic function $e_{[f]_z}$ belongs to K; see [12, Vol. 2, p. 172]. This notion extends naturally to subgroups of $\mathbb{Z}^\mu$, thus $K \subseteq \mathbb{Z}^\mu$ is a *Specker group* if again $e_{[f]_z} \in K$ for all $f \in K$, $z \in \mathbb{Z}$. Moreover, we say that K is closed under stretched copies of the Baer–Specker group if for any $f \in U$ follows $P_f =: \prod_{z\in\mathbb{Z}} e_{[f]_z}\mathbb{Z} \subseteq U$. Note that $f \in P_f \cong \mathbb{Z}^\alpha$ for some $\alpha \leqslant \omega$. The latter condition clearly implies Specker.

Proposition 4.5. *Let (μ, λ) be a pair of infinite cardinals with $\mu \leqslant \lambda = \lambda^{\aleph_0}$ and suppose that $(\mu, \lambda) \in \mathfrak{p}_3^\mu$ holds. For each set $J \subseteq \mu$ of size $|J| = \mu$ there is a test group $T_J = G_J/\mathbb{Z}^{<\mu} \subseteq \mathbb{Z}_\mu$ with $\mathbb{Z}^{<\mu} \subseteq G_J \subseteq \mathbb{Z}^\mu$ closed under stretched Baer–Specker groups* (*in particular T is a Specker group*) *containing the element e_J and with the following additional properties.*

(i) $1 \neq |T_J| \leqslant \lambda$.
(ii) $\mathrm{Hom}(T_J, H) = 0$ *for any slender group H* (*in particular $(\mu, \lambda) \in \mathfrak{p}_1$*).
(iii) $\mathrm{Hom}(T_J, X) = 0$ $(\forall J \subseteq \mu) \Rightarrow \mathrm{Hom}(\mathbb{Z}_\mu, X) = 0$ *for any reduced, torsion-free group X.*

Proof. (i) will be obvious by construction, and the addition in (ii) is immediate from (ii) because $\mathbb{Z}$ is slender. We fix $J \subseteq \mu$, may assume $|J| = \mu$ and suppress the index J in the proof.

(i) (The construction of T.) By $\mathfrak{p}_3^\mu$ there are a boolean algebra $\mathbb{B} \subseteq \mathfrak{P}(\mu)$ with $|\mathbb{B}| \leqslant \lambda$ and a sequence $\langle\langle A_n^\alpha : n \in \omega\rangle : \alpha \in \lambda\rangle$ with $A_n^\alpha \in \mathbb{B}$ and $A^\alpha =: \bigcap_{n\in\omega} A_n^\alpha$. As $\lambda = \lambda^{\aleph_0}$ we can assume that $\mathbb{B}$ is closed under countable intersections, in particular $A^\alpha \in \mathbb{B}$ for all $\alpha \in \lambda$. For any uniform ultrafilter D on $\mathbb{B}$ there is $\alpha \in \lambda$ with $A_n^\alpha \in D$ for all $n \in \omega$ but $A^\alpha \notin D$. (There is no weakly complete D.) We also may assume that $J \in \mathbb{B}$ and must find (from $\mathbb{B}$) $\mathbb{Z}^{<\mu} \subset G \subseteq \mathbb{Z}^\mu$ such that $e_J \in G$, $T = T_J = G/\mathbb{Z}^{<\mu}$, $|T| \leqslant \lambda$ and $\mathrm{Hom}(T, H) = 0$ for all slender groups H.

If $B_n^\alpha = \bigcap_{i\leqslant n} A_n^\alpha$, then also $B_n^\alpha \in \mathbb{B}$ and $B^\alpha =: \bigcap_{n\in\omega} B_n^\alpha = \bigcap_{n\in\omega} A_n^\alpha = A^\alpha$. Hence, if $A_n^\alpha \in D$ for all $n \in \omega$, then also $B_n^\alpha \in D$ for all $n \in \omega$ and we can assume that $A_0^\alpha = \mu$ and $\langle A_n^\alpha :\ n \in \omega\rangle$ is a descending chain converging to A^α. Now let $C_n^\alpha =: A_n^\alpha \backslash A_{n+1}^\alpha$ for all $n \in \omega$ and $C^\alpha = \bigcup_{n\in\omega} C_n^\alpha$. It follows that

$$A_n^\alpha \in D \text{ for all } n \in \omega \Longleftrightarrow C_n^\alpha \notin D \text{ for all } n \in \omega$$

using $A_{n+1}^\alpha = A_n^\alpha \cap (C_n^\alpha)^c$ and induction on n. Moreover, $A^\alpha = \mu\backslash C^\alpha$, thus

$$A^\alpha \in D \Longleftrightarrow C^\alpha \notin D.$$

The sets C_n^α $(n \in \omega)$ are pairwise disjoint and therefore the following elements $g_{\alpha h} \in \mathbb{Z}^\mu$ are well-defined. Let $h : \omega \longrightarrow \mathbb{Z}$ be any function and define $g_{\alpha h} = \sum_{i\in\mu} g_{\alpha h}(i)e_i \in \mathbb{Z}^\mu$

componentwise by

$$g_{\alpha h}(i) = \begin{cases} h(n) & \text{if } i \in C_n^\alpha \ (n \in \omega), \\ 0 & \text{if } i \in A^\alpha. \end{cases}$$

If $h : \omega \longrightarrow \mathbb{Z}$ runs over all maps, we obtain a 'stretched' copy of the Baer–Specker group $\mathbb{Z}^\omega$ inside $\mathbb{Z}^\mu$ and let

$$G = \langle g_{\alpha h}, e_A : h \in {}^\omega\mathbb{Z}, \alpha \in \lambda, A \in \mathbb{B}\rangle \subseteq \mathbb{Z}^\mu.$$

It is obvious by the definition that G is closed under stretched Baer–Specker groups; in particular G is a Specker subgroup of $\mathbb{Z}^\mu$ and trivially $2^{\aleph_0} \leqslant |G|$.

(ii) Suppose for contradiction that there are a slender group H and a homomorphism $0 \neq \varphi \in \mathrm{Hom}(G, H)$ with $(G \cap \mathbb{Z}^{<\mu})\varphi = 0$. Consider the set

$$I = \{A \in \mathbb{B} : \forall X \subseteq A, X \in \mathbb{B} \Rightarrow e_X\varphi = 0\},$$

so φ is 'hereditarily' 0 on A and $I \subseteq \mathbb{B} \subseteq \mathfrak{P}(\mu)$. Clearly, I by definition is downwards closed. If $A_1, A_2 \in I$ and $X \subseteq A = A_1 \cup A_2$, then we partition $X = X_1 \cup X_2$ with $X_1 = X \cap A_1$ and $X_2 = X\backslash X_1$. Hence, $e_X = e_{X_1} + e_{X_2}$ and $e_X\varphi = 0$ is immediate, so I is also closed under finite unions. Next we show that I is also closed under the relevant countable unions:

$$\text{If } C_n^\alpha \in I \text{ for all } n \in \omega \text{ then also } C^\alpha \in I. \tag{4.1}$$

If $X \subseteq C^\alpha$ and $X \in \mathbb{B}$, then let $X_n = X \cap C_n^\alpha$ and define the homomorphism $\sigma : \mathbb{Z}^\omega \longrightarrow \mathbb{Z}^\mu$ for any $v = \sum_{n\in\omega} v_n e_n$ by

$$v\sigma(i) = \begin{cases} v_n & \text{if } i \in X_n \text{ for some } n \in \omega, \\ 0 & \text{if } i \in \mu\backslash X. \end{cases}$$

Note that $v\sigma = \sum_{n\in\omega} v_n e_{X_n} \in G$. If $e_\omega = \sum_{n\in\omega} e_n \in \mathbb{Z}^\omega$, then $e_n\sigma = e_{X_n}$ and $e_\omega\sigma = e_X$. Since $X \in \mathbb{B}$ and $X_n \subseteq C_n^\alpha \in I$ also $X_n \in I$ and therefore $e_n\sigma\varphi = e_{X_n}\varphi = 0$. Thus, $\sigma\varphi = 0$ since H is slender. In particular, $0 = e_\omega\sigma\varphi = e_X\varphi$ for any $X \subseteq C^\alpha$ with $X \in \mathbb{B}$, hence $C^\alpha \in I$ and (4.1) is shown.

Next we show that

$$\mathbb{B}/I \text{ is a finite boolean algebra.} \tag{4.2}$$

Otherwise, there are $C_n \in \mathbb{B}\backslash I$ $(n \in \omega)$ which are pairwise disjoint modulo I, i.e. $C_n \cap C_m \in I$ for all $n \neq m$. We can choose new representatives $C_n' = C_n\backslash\bigcup_{l\leqslant n} C_l \in \mathbb{B}$, hence these new C_n's (called C_n again) are pairwise disjoint. Let $C = \bigcup_{n\in\omega} C_n$ and choose $\alpha \in \lambda$ such that $C_n^\alpha = C_n$ for all $n \in \omega$. This is possible, because w.l.o.g. we can add all corresponding sequences $(A_n \subseteq \mu)_{n\in\omega}$ to the list given by $\mathfrak{p}_3^\mu$. Only here we use that $\lambda^{\aleph_0} = \lambda$. We have that $e_{C^\alpha}, e_{C_n^\alpha} \in G$ and $e_{C_n^\alpha}\varphi \neq 0$ for all $n \in \omega$. Next we define a homomorphism $\sigma : \mathbb{Z}^\omega \longrightarrow \mathbb{Z}^\mu$ $(v \longrightarrow v\sigma)$ as above with

$$v\sigma(i) = \begin{cases} v_n & \text{if } i \in C_n^\alpha \text{ for some } n \in \omega, \\ 0 & \text{if } i \in \mu\backslash C^\alpha. \end{cases}$$

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By definition of G it follows that $\upsilon\sigma \in G$, hence $\sigma\varphi\colon \mathbb{Z}^{\omega} \longrightarrow H$ is a well-defined homomorphism, and $e_n(\sigma\varphi) = e_{C_n^{\alpha}}\varphi \neq 0$ for all $n \in \omega$. This contradicts slenderness of H. So $\mathbb{B}/I$ is finite.

If $\mathbb{B} = I$, then $e_{C_n^{\alpha}}\varphi = 0$ for all $\alpha \in \lambda$ and $n \in \omega$, hence $g_{\alpha h}\varphi = 0$ for all $\alpha \in \lambda$ and $h \in {}^{\omega}\mathbb{Z}$. Hence, $G\varphi = 0$ contrary to our choice $\varphi \neq 0$.

If $\mathbb{B} \neq I$, then we can choose an atom $A/I \in \mathbb{B}/I$ with $A \subseteq \mu$ from the finite non-trivial boolean algebra and we also choose an ultrafilter D on $\mathbb{B}$ disjoint to I such that $A \in D$. From $(G \cap \mathbb{Z}^{<\mu})\varphi = 0$ it follows that $|X| = \mu$ for all $X \in D$, so D is uniform. Finally, we want to show that D is weakly complete, which then will contradict the assumption $\mathfrak{p}_3^{\mu}$: If $A_n^{\alpha} \in D$ for all $n \in \omega$, then $A^{\alpha} \in D$. In terms of C^{α}, C_n^{α} this is equivalent to say

(∗) If $C^{\alpha} \in D$, then $C_n^{\alpha} \in D$ for some $n \in \omega$.

If $C_n^{\alpha} \cap A \notin I$ for some $n \in \omega$, then $C_n^{\alpha} \in D$ and (∗) holds. Otherwise, $C_n^{\alpha} \cap A \in I$ for all $n \in \omega$ and w.l.o.g. we can find $\beta \in \lambda$ such that $C_n^{\beta} = C_n^{\alpha} \cap A$ for all $n \in \omega$. Hence, $C_n^{\beta} \in I$ for all $n \in \omega$. But then also $C^{\beta} = \bigcup_{n\in\omega} C_n^{\beta} \in I$ because I is countably closed by (4.1). Hence, $C^{\alpha} \cap A = C^{\beta} \in I$ and from $A \in D$ follows $C^{\alpha}\backslash A \notin D$, as D is an ultrafilter. So $C^{\alpha} = (C^{\alpha}\backslash A) \cup (C^{\alpha} \cap A)$ is a partition with $C^{\alpha} \cap A \in I$, so $C^{\alpha} \cap A \notin D$ and $C^{\alpha}\backslash A \notin D$. Hence, $C^{\alpha} \notin D$ and (∗) holds trivially. So D is weakly complete, this is a final contradiction.

(iii) We prove the contrapositive. Suppose that $\mathrm{Hom}(\mathbb{Z}_{\mu}, X) \neq 0$, where X is reduced and torsion-free. This means that there exists a non-zero homomorphism $\varphi\colon \mathbb{Z}^{\mu} \longrightarrow X$ which vanishes on $\mathbb{Z}^{<\mu}$. Consider the Nöbeling subgroup B of $\mathbb{Z}^{\mu}$ of all bounded, integer-valued functions on μ, which is generated by all e_I $(I \subseteq \mu)$; see [22]. Since $\mathbb{Z}^{\mu}/B$ is divisible, the restriction $\varphi\restriction B$ cannot vanish, so there exists a subset $J \subseteq \mu$ such that $e_J\varphi \neq 0$. This requires $e_J \notin \mathbb{Z}^{<\mu}$, in other words $|J| = \mu$. But then $e_J \in G_J$ and $\overline{e_J} \in T_J = G_J/\mathbb{Z}^{<\mu}$ is a test group. Let $\overline{\varphi}$ be the map induced by φ. Then $\overline{e_J}\,\overline{\varphi} = e_J\varphi \neq 0$, and we conclude that $0 \neq \overline{\varphi}\restriction T_J \in \mathrm{Hom}(T_J, X)$, which contradicts the assumption of (iii). □

The last two propositions give an immediate corollary.

Corollary 4.6. *Let* (μ, λ) *be a pair of infinite cardinals with* $\mu \leqslant \lambda = \lambda^{\aleph_0}$. *Then* $(\mu, \lambda) \in \mathfrak{p}_3^{\mu} \Longleftrightarrow (\mu, \lambda) \in \mathfrak{p}_1$.

Lemma 4.7. *If* $(\mu, \lambda) \in \mathfrak{p}_1$ *and* $\lambda = \lambda^{\aleph_0} < \chi$ *where* χ *is regular, then* $\mathrm{Hom}(\mathbb{Z}_{\mu}, \mathbb{Z}_{\chi}) = 0$.

Proof. Suppose for contradiction that $0 \neq \varphi \in \mathrm{Hom}(\mathbb{Z}_{\mu}, \mathbb{Z}_{\chi}) = 0$. We view φ as

$$\varphi : \mathbb{Z}^{\mu} \longrightarrow \mathbb{Z}_{\chi} \quad \text{with } \mathbb{Z}^{<\mu}\varphi = 0.$$

By the same argument as above (using the Nöbeling subgroup of $\mathbb{Z}^{\mu}$) we find $J \subseteq \mu$ such that $e_J\varphi \neq 0$. Hence, $|J| = \mu$ and choose $G =: G_J$ with $e_J \in G \subseteq \mathbb{Z}^{\mu}$ by $\mathfrak{p}_1$ and $\mathbb{Z}^{<\mu} \subseteq G$. Thus, $G\varphi = T\varphi \neq 0$ with $T = G/\mathbb{Z}^{<\mu}$, $T^* = 0$ and $|T| \leqslant \lambda < \chi$. By Lemma 4.1(ii) follows that the group $0 \neq T\varphi$ is isomorphic to a subgroup of $\mathbb{Z}^{\chi}$ and there are obvious non-trivial homomorphisms (projections from $\mathbb{Z}^{\chi}$) $\eta : T\varphi \longrightarrow \mathbb{Z}$, hence $0 \neq \varphi\eta\colon T \longrightarrow \mathbb{Z}$ contradicts $T^* = 0$. □

Remark. From the proof of Proposition 4.5 follows that condition (iii) can be (virtually) strengthened:

$$0 \neq \varphi \in \mathrm{Hom}(\mathbb{Z}_\mu, X) \Rightarrow (\forall T \subseteq \mathbb{Z}_\mu, |T| \leqslant \lambda\ \exists T \subseteq T' \subseteq \mathbb{Z}_\mu, \\ |T'| \leqslant \lambda \text{ and } \varphi\restriction T' \neq 0).$$

This applies in particular to $X = \mathbb{Z}$. If this is the case we say that $(\mu, \lambda) \in \mathfrak{p}_1^*$ which is equivalent to $(\mu, \lambda) \in \mathfrak{p}_1$ by the remark.

5. Embedding reduced products into reduced products

In this section we want to sharpen the condition $\mathrm{Hom}(\mathbb{Z}_\mu, \mathbb{Z}_\kappa) \neq 0$ from the last section replacing non-trivial homomorphisms by monomorphisms.

Recall that a group G is *torsionless* if for any $0 \neq g \in G$ there is $\varphi \in G^*$ such that $g\varphi \neq 0$. This is equivalent to say that G is isomorphic to the subgroup of some product $\mathbb{Z}^\kappa$. In a similar way we say

Definition 5.1. If G is a group and λ is a cardinal, then G is λ^+-torsionless, if all subgroups $T \subseteq G$ with $|T| \leqslant \lambda$ are torsionless.

This definition is also parallel to the notion of λ^+-free groups. We also express a new properties for pairs $\mu \leqslant \lambda$ of cardinals.

Property $\mathfrak{p}_1^+$. The pair of infinite cardinals (μ, λ) with $\mu \leqslant \lambda$ satisfies $\mathfrak{p}_1^+$ (we write $(\mu, \lambda) \in \mathfrak{p}_1^+$) if $\mathbb{Z}_\mu$ is not λ^+-torsionless. Again $\lambda \in \mathfrak{p}_1^+$ stands for $(\lambda, \lambda) \in \mathfrak{p}_1^+$.

We have an immediate

Lemma 5.2. *If λ is weakly compact, then $\lambda \notin \mathfrak{p}_1^+$.*

Proof. If λ is weakly compact, then $\lambda \notin \mathfrak{p}_3^\mu$ which is equivalent to $\lambda \notin \mathfrak{p}_1$ by Propositions 3.3 and 4.4. Hence, $\lambda \notin \mathfrak{p}_1^+$ is immediate. □

Lemma 5.2 also follows from Wald [27, Theorem B], see introduction. He argues differently and uses the weak compactness theorem for languages $L_{\mu\omega}$, see [17, Section 32]. The main theorem of this section is now the following.

Theorem 5.3. *If $\mu \leqslant \lambda$ are cardinals such that $2^\mu = \lambda^+$ and $(\mu, \lambda) \notin \mathfrak{p}_1^+$, then there is an embedding $\mathbb{Z}_\mu \hookrightarrow \mathbb{Z}_{\lambda^+}$.*

Proof. By $|\mathbb{Z}^\mu| = 2^\mu = \lambda^+$ there is a continuous, increasing chain of subgroups $T_i \subseteq \mathbb{Z}_\mu$ $(i \in 2^\mu)$ of cardinality $|T_i| < 2^\mu$. Hence, $|T_i| \leqslant \lambda$ by $2^\mu = \lambda^+$, and the T_i's constitute a λ^+-filtration of $\mathbb{Z}^\mu$. We also may assume that $|T_i| = \lambda$ for all $i \in \lambda^+$.

The set $\lambda^+ \times \lambda$ ordered lexicographically has order type λ^+. Let $\iota : \lambda^+ \longrightarrow \lambda^+ \times \lambda$ be the related order isomorphism and enumerate $T_i \setminus \{0\} = \{t_j : j \in \lambda\}$ without repetition. From

$(\mu, \lambda) \notin \mathfrak{p}_1^+$ follows that $\mathbb{Z}^\mu$ is λ^+-torsionless, hence T_i is torsionless and for any $j \in \lambda$ there is $\varphi_{ij} \in T_i^*$ with $0 \neq t_j\varphi_{ij} \in \mathbb{Z}$. Now we replace (i, j) by $k = (i, j)\iota^{-1}$ and φ_{ij} becomes φ_k. Conversely, if $k \in \lambda^+$, then $k\iota = (i, j) \in \lambda^+ \times \lambda$ and $t_i\varphi_{ij} \neq 0$.

We will use the following notation: If $f = \sum_{i\in\mu} f_i e_i \in \mathbb{Z}^\mu$ then $\overline{f} = f + \mathbb{Z}^{<\mu}$ and if $f' = \sum_{i\in\lambda^+} f_i e_i \in \mathbb{Z}^{\lambda^+}$ then $\overline{f'} = f' + \mathbb{Z}^{<\lambda^+}$. We will also replace f by g below.

If $0 \neq \overline{f} \in \mathbb{Z}_\mu$, then we want to define $\overline{f}\psi = \overline{f'} \in \mathbb{Z}_{\lambda^+}$. If $\overline{f} \in T_i$, then there is exactly one $j \in \lambda$ such that $\overline{f} = \overline{f}_j$, hence $k\iota = (i, j)$ holds for exactly one $k \in \lambda^+$ and $\overline{f}\varphi_k = \overline{f}_j\varphi_{ij} \neq 0$ is an integer. Put

$$f'_k = \begin{cases} \overline{f}\varphi_k & \text{if } \overline{f} = \overline{f}_j \in T_i \text{ for } k\iota = (i, j), \\ 0 & \text{otherwise.} \end{cases}$$

We note that for any $\overline{f} \in \mathbb{Z}_\mu$ follows $\overline{f} \in T_i$ for almost all $i \in \lambda^+$ (i.e. with possibly λ exceptions). Hence, the (second) 0-case in the displayed equation appears at most λ times and modulo $\mathbb{Z}^{<\lambda^+}$ it can be ignored. Otherwise, $f'_k \neq 0$, hence $\overline{f}\psi = \overline{f'} \neq 0$. So ψ is well-defined and injective provided it is an homomorphism.

If $\overline{f}, \overline{g} \in \mathbb{Z}_\mu$, then there is $j < \lambda^+$ such that $\overline{f}, \overline{g} \in T'_i$ for all $i \geqslant j$ and consider the kth coordinates f'_k, g'_k of the corresponding images $\overline{f}\psi, \overline{g}\psi$ for large enough k. We have

$$f'_k + g'_k = \overline{f}\varphi_k + \overline{g}\varphi_k = (\overline{f} + \overline{g})\varphi_k$$

because φ_k is a homomorphism. The right-hand side, however, is the kth coordinate of the image of $\overline{f} + \overline{g}$, hence ψ is also additive. □

The properties $\mathfrak{p}_1$ and $\mathfrak{p}_1^+$ are in most cases the same. To see this we repeat some natural notations used in Section 4:

If $f = \sum_{i\in\mu} f_i e_i \in \mathbb{Z}^\mu$, then as above we write $\overline{f} = f + \mathbb{Z}^{<\mu}$ and will consider the z-support $[f]_z = \{i \in \mu : f_i = z\} \subseteq \mu$. Let $f^z = e_{[f]_z} z$ be the z-component of f based on the characteristic function $e_{[f]_z}$. Hence, $f \in P_f = \prod_{z\in\mathbb{Z}} e_{[f]_z}\mathbb{Z} \subseteq \mathbb{Z}^\mu$ is isomorphic to $\mathbb{Z}^\alpha$ for some ordinal $\alpha \leqslant \omega$. Recall that $U \subseteq \mathbb{Z}^\mu$ is a Specker groups if it is closed under those characteristic function. Moreover, recall the stronger condition when U is closed under stretched copies of the Baer–Specker groups: If $f \in U$, then also $P_f \subseteq U$.

Lemma 5.4. *Let $\mu \leqslant \lambda$ be infinite cardinals.*

(i) *If $2^{\aleph_0} \leqslant \lambda$ and $\aleph_0 < \mathrm{cf}\,\mu$, then $((\mu, \lambda) \in \mathfrak{p}_1^+ \Rightarrow (\mu, \lambda) \in \mathfrak{p}_1)$.*
(ii) *$(\mu, \lambda) \in \mathfrak{p}_1 \Rightarrow (\mu, \lambda) \in \mathfrak{p}_1^+$.*

Proof. (ii) is obvious. It remains to show $((\mu, \lambda) \in \mathfrak{p}_1^+ \Rightarrow (\mu, \lambda) \in \mathfrak{p}_1)$.

(i) Assume for contradiction $(\mu, \lambda) \in \mathfrak{p}_1^+$, $(\mu, \lambda) \notin \mathfrak{p}_1$.

By $(\mu, \lambda) \in \mathfrak{p}_1^+$ the group $\mathbb{Z}_\mu$ is not λ^+-torsionless and there is a subgroups $T \subseteq \mathbb{Z}_\mu$ of size λ which is not torsionless. Hence, there is $0 \neq \overline{f} \in T$ such that $\overline{f}\varphi = 0$ for all $\varphi \in T^*$. For a pair of sets $J, K \subseteq \mu$ with $|J| = |K| = \mu$ we also fix a bijection $\gamma = \gamma_{JK} : J \longrightarrow K$.

This map induces an isomorphism

$$\gamma^* = \gamma^*_{JK} : \mathbb{Z}^K \longrightarrow \mathbb{Z}^J \left(h = \sum_{i \in K} e_i h_i \longrightarrow h\gamma^* = \sum_{i \in J} e_i h_{i\gamma} \right).$$

Also consider the canonical projection

$$\pi_K : \mathbb{Z}^\mu \longrightarrow \mathbb{Z}^K \left(g = \sum_{i \in \mu} g_i e_i \longrightarrow g_K = \sum_{i \in K} g_i e_i \right).$$

From $(\mu, \lambda) \notin \mathfrak{p}_1$ follows $(\mu, \lambda) \notin \mathfrak{p}_3^\mu$ (Proposition 4.4(ii)) and by Proposition 4.4(i) follows that any group $0 \neq G \subseteq \mathbb{Z}_\mu$ with $|G| \leqslant \lambda$ has a non-trivial dual $G^* \neq 0$. We want to use a non-trivial homomorphism from G^* to map $\overline{f}$ to a non-trivial integer which will be a contradiction. Thus, we choose a very homogeneous extension G of T which allows enough endomorphisms. We can choose G arbitrarily only taking care of its size. Thus, pick $G \subseteq \mathbb{Z}_\mu$ subject to the following conditions:

(i) $|G| = \lambda$, then (automatically) there is $0 \neq \psi \in G^*$.
(ii) $T \subseteq G$.
(iii) $\overline{f} \neq 0 = \overline{f}\varphi$ for all $\varphi \in G^*$ (because $\varphi \restriction T \in T^*$ and $\overline{f} \in T$).
(iv) $\mathbb{Z}^{<\mu} \subseteq G' \subseteq \mathbb{Z}^\mu$ and $G'/\mathbb{Z}^{<\mu} = G$.
(v) G' is closed under stretched Baer–Specker groups.
(vi) If $e_J \in G'$, $n \in \mathbb{Z}$, then clearly $\pi_{[f]_n}\gamma^*_{J[f]_n} : \mathbb{Z}^\mu \longrightarrow \mathbb{Z}^J$ and require that $(\pi_{[f]_n} \restriction G')\gamma^*_{J[f]_n} \in \operatorname{End} G'$, i.e. if $g = \sum_{i \in \mu} g_i e_i \in G'$, then $g_{[f]_n}\gamma^*_{J[f]_n} \in G'$.

We note that this choice is possible but needs $2^{\aleph_0} \leqslant \lambda$. Let $\overline{h} \in G'$ with $\overline{h}\psi \neq 0$ from (i). By the above notations we have $h = \sum_{n \in \mathbb{Z}} e_{[h]_n} n$ and $e_{[h]_n} \in G'$ by (v). Moreover, let $F = \{n \in \mathbb{Z} \setminus \{0\} : \overline{e_{[h]_n}}\psi \neq 0\}$.

We now distinguish two cases. If F is infinite, then define $\Theta : \mathbb{Z}^{\mathbb{Z}} \longrightarrow \prod_{n \in \mathbb{Z}} e_{[h]_n}\mathbb{Z}$ by $e_n\Theta = h^n$ for all $n \in \mathbb{Z}$ (which extends naturally as required). But then $(\prod_{n \in \mathbb{Z}} e_{[h]_n}\mathbb{Z})\Theta \subseteq G'$, $\Theta\psi \in \operatorname{Hom}(\mathbb{Z}^{\mathbb{Z}}, \mathbb{Z})$ and $e_n\Theta\psi \neq 0$ for infinitely many $n \in \mathbb{Z}$, contradicts that $\mathbb{Z}$ is slender.

Thus, F is finite. If $F = \emptyset$, then $0 \neq \overline{h}\psi = \sum_{n \in F} e_{[h]_n}\psi = 0$ is a contradiction. Hence, there is $0 \neq n \in F$ with $|[h]_n| = \mu$. Let $J = [h]_n$. From (v) follows $e_J \in G'$. We can assume that $h = e_J$, hence $e_J\psi \neq 0$. Now we compose $(\pi_{[f]_n} \restriction G')\gamma^*_{J[f]_n} \in \operatorname{End} G'$ and $\pi_{[f]_n}\gamma^*_{J[f]_n}\psi \in G'^*$. It follows that

$$f\pi_{[f]_n}\gamma^*_{J[f]_n}\psi = e_{[f]_n} n\gamma^*_{J[f]_n}\psi = e_J n\psi \neq 0$$

and this contradicts (iii). □

It follows the

Corollary 5.5 (*ZFC+GCH*). *Let* $\mu \leqslant \chi$ *be infinite cardinals such that* χ *is a successor cardinal. Then* $\mathrm{Hom}(\mathbb{Z}_\mu, \mathbb{Z}_\chi) \neq 0 \iff \mathbb{Z}_\mu \hookrightarrow \mathbb{Z}_\chi$.

Proof. $\Leftarrow$ is trivial. Conversely, from $\mathrm{Hom}(\mathbb{Z}_\mu, \mathbb{Z}_\lambda) \neq 0$ follows $\mu^+ = \chi$ and $(\mu, \chi) \notin \mathfrak{p}_1^+$ or the trivial case $\mu = \chi$, thus Theorem 5.3 applies. □

6. The norm of a group

The *norms* of a radical and of a group as defined in Göbel [14] provide a tool for investigating commutation of radicals with cartesian products. Recall that the *norm* $\|G\|$ of a non-zero group G is defined to be

$$\|G\| = \min\{\kappa : \kappa = \aleph_{\mathrm{fm}} \text{ or } \kappa \text{ is a cardinal with } \mathrm{Hom}(\mathbb{Z}_\kappa, G) \neq 0\}$$

if this cardinal (equivalently if $\aleph_{\mathrm{fm}}$) exists, or $\|G\| = \infty$ otherwise; here again $\aleph_{\mathrm{fm}}$ is the least measurable cardinal. It is an easy exercise, using a result of Balcerzyk and Hulanicki (see [12, Vol. 1, p. 176, 177]), to prove that

$$G \text{ is cotorsion-free} \iff \|G\| > \aleph_0. \tag{6.1}$$

In this context we may remark that G is *strongly cotorsion-free* if and only if $\|G\| \in \{\aleph_{\mathrm{fm}}, \infty\}$ (see [9]).

Observation 6.1. *The norm* $\|G\|$ *of a group* G *is always a regular cardinal* (*where* $\aleph_0$ *is allowed*).

Proof. If $0 \neq \sigma \in \mathrm{Hom}(\mathbb{Z}_\kappa, G)$, then by Observation 4.2 we can find an embedding $\mathbb{Z}_{\mathrm{cf}\,\kappa} \hookrightarrow \mathbb{Z}_\kappa$ such that the composite of the two maps is not 0 as well, hence $\|G\|$ must be regular. □

Recall from Remark 3.2 that the class $\mathfrak{C}$ is large, and if $\mu = \mu^{\aleph_0} \in \mathfrak{C}$, then $\mu \in \mathfrak{p}_2^u$. This is reflected in our next main result of this section, which will follow from previous considerations, mainly from Section 4.

Corollary 6.2. *Let* χ *be a regular cardinal* $< \aleph_{\mathrm{fm}}$. *Then* $\|\mathbb{Z}_\chi\| = \chi$ (*or equivalently* $\mathrm{Hom}(\mathbb{Z}_\mu, \mathbb{Z}_\chi) = 0$ *for all infinite cardinals* $\mu < \chi$) *if one of the following conditions holds*:

(i) $\mathrm{Hom}(\mathbb{Z}_\mu, \mathbb{Z}_\chi) = 0$ *for all* regular *cardinals* $\mu < \chi$.
(ii) *For each regular cardinal* μ *there is a cardinal* λ *such that* $\mu \leqslant \lambda = \lambda^{\aleph_0} < \chi$ *and* $(\mu, \lambda) \in \mathfrak{p}_2^u$.
(iii) $\chi = \lambda^+$ *is a successor and for all regular* $\mu \leqslant \lambda$ *follows* $(\mu, \lambda) \in \mathfrak{p}_2^u$.
(iv) $\chi = \lambda^+$ *and* $\lambda \in \mathfrak{p}_2^s$.
(v) $\chi = \beth_\delta$ *with* δ *a limit ordinal*.
(vi) $\chi = \beth_\delta^+$ *for some ordinal* δ *such that* $\beth_\delta$ *is not weakly compact*.
(vii) (*Assuming GCH*) χ *is not a successor of a weakly compact cardinal*.

Remarks. Cardinals μ with cf $\mu = \aleph_0$ (including $\mu = \aleph_0$) also belong to the list of cardinals as in the corollary, but for more trivial reasons: The group $\mathbb{Z}_{\aleph_0}$ is algebraically compact (see [12, Vol. 1, p. 176, 177]). Similarly, $\mathbb{Z}_\mu$ is algebraically compact for cf $\mu = \aleph_0$ by Balcerzyk [2, Theorem 3]. Moreover, epimorphic images of these groups are cotorsion (see [12, Proposition 54.1, Vol. 1, p. 234]). Since $\mathbb{Z}_\chi$ is cotorsion-free by Lemma 4.1(ii) (χ is regular), in these cases it follows automatically that $\mathrm{Hom}(\mathbb{Z}_\mu, \mathbb{Z}_\chi) = 0$. The same argument but using that algebraically compact groups are pure injective also shows that $\mathrm{Hom}(\mathbb{Z}_\mu, \mathbb{Z}_\chi) \neq 0$ if cf $\chi = \aleph_0$. The assumption that χ in the corollary is regular could be removed, but the assumption cf $\mu > \aleph_0$ is necessary as just seen.

While Corollary 6.2 deals with the problem $\mathrm{Hom}(\mathbb{Z}_\mu, \mathbb{Z}_\chi) = 0$ for cardinals $\mu < \chi$, also the question for $\mu > \chi$ is interesting (but not needed in our context): Assuming $V = L$, Donder [8] showed that for any uncountable cardinals μ the only ω_1-complete, μ^+-saturated ideal on μ containing all subsets of cardinality $< \mu$ is $\mathfrak{P}(\mu)$. (An ideal $\mathbb{E}$ on μ is ρ-saturated for some cardinal ρ if and only if any set $S \subseteq \mathfrak{P}(\mu) \setminus \mathbb{E}$ with $X \cap Y \in \mathbb{E}$ for all $X \neq Y \in S$ has size $< \rho$.) Again assuming $V = L$, Wald [26] applied this and a result from Göbel et al. [16] to show for cotorsion-free groups G with $|G| \leqslant \mu$ follows $\mathrm{Hom}(\mathbb{Z}_\mu, G) = 0$. Finally, observe that any $\mathbb{Z}_\chi$ (χ regular uncountable) is cotorsion-free by Lemma 4.1(ii) and $|\mathbb{Z}_\chi| = 2^\chi$. Hence, $\mathrm{Hom}(\mathbb{Z}_\mu, \mathbb{Z}_\chi) = 0$ if $\mu \geqslant 2^\chi$.

Proof. (i) Follows from Observation 6.1 which shows that norms of groups are always regular.

(ii) Follows from Lemma 4.7.

(iii) We want to apply Proposition 4.5 directly and replace $(\mu, \lambda) \in \mathfrak{p}_2^u$ by $(\mu, \lambda) \in \mathfrak{p}_3^u$ (Proposition 3.3(i)). If $0 \neq \varphi : \mathbb{Z}_\mu \longrightarrow \mathbb{Z}_\chi$ for $\chi = \lambda^+$, then by Proposition 4.5(iii) there is $T \subseteq \mathbb{Z}_\mu$ with $|T| \leqslant \lambda$ and $\varphi \restriction T \neq 0$. Hence, $0 \neq T\varphi \subseteq \mathbb{Z}_{\lambda^+}$ and also $|T\varphi| \leqslant \lambda$. The non-trivial group $T\varphi$ is isomorphic to a subgroup of $\mathbb{Z}^{\lambda^+}$ by Lemma 4.1 and $(T\varphi)^* \neq 0$, thus $T^* \neq 0$ contradicts Proposition 4.5(ii).

(iv) From $\lambda \in \mathfrak{p}_2^s$ follows $(\mu, \lambda) \in \mathfrak{p}_2^s$ for all regular cardinals $\mu \leqslant \lambda$ (Observation 3.7(i)). Thus, $(\mu, \lambda) \in \mathfrak{p}_2^u$ by Proposition 3.3(iii) and (iv) follows from (iii).

(v) If $\mu < \chi$, then put $\lambda = 2^\mu$ which satisfies $\mu < \lambda = \lambda^{\aleph_0} < \chi$ and $(\mu, \lambda) \in \mathfrak{p}_2^u$ by Proposition 3.3(v). Hence, (ii) applies.

(vi) If $\kappa = \beth_\delta$ is not weakly compact, then $\kappa \in \mathfrak{p}_2^s$ by Proposition 3.3(x) and the corollary follows for κ^+ from (iv).

(vii) If χ is a limit cardinal then (vii) follows from (v). If $\chi = \kappa^+$ is a successor, then $\kappa \neq \aleph_0$ and κ is not weakly compact, so (vii) follows from (vi). □

Corollary 6.2 strengthens a theorem by Wald [27]. He uses that cardinals ρ that have partitions $f : [\rho]^2 \to 2$ which admit homogeneous subset of cardinality ρ must be weakly compact. If ρ is not weakly compact, there is a family f_j $(j < \rho)$ of functions $f_j : j \to 2$ where $f_j(i) = f(\{i, j\})$ and if $\alpha \in {}^{\rho>}2$ set $N_\alpha = \{j < \rho | f_j \supseteq \alpha\}$. For any $\alpha \in {}^{\rho}2$ the set $N_{\alpha\restriction\nu}$ has cardinality $< \rho$ for all large enough $\nu < \rho$. This sequence of sets is then used to construct test subgroups of $\mathbb{Z}_\rho$ that correspond to our smaller test groups T_J in Proposition 4.5. Wald's method was also used in [6].

Assuming GCH we can characterize those cardinals which are group norms.

Corollary 6.3 (*ZFC+GCH*). *The following conditions for a cardinal χ or $\chi=\infty$ are equivalent*:

(i) *There is a group G with $\|G\| = \chi$.*

(ii) $\begin{cases} \aleph_{\mathrm{fm}} \text{ does not exist}: & \chi = \infty \text{ or } \chi = \lambda^+, \lambda \text{ is not a weakly compact.} \\ \aleph_{\mathrm{fm}} \text{ exists}: & \chi = \aleph_{\mathrm{fm}} \text{ or } \chi = \lambda^+ < \aleph_{\mathrm{fm}}, \lambda \text{ is not a weakly compact.} \end{cases}$

(iii) $\chi = \|\mathbb{Z}_\chi\| \neq \infty$ *or* $\chi = \|\mathbb{Z}\| = \infty$.

Proof. If $\aleph_{\mathrm{fm}}$ does not exist and $\chi=\infty$, then $\|\mathbb{Z}\|=\infty$ follows by definition of the norm and Łoś's theorem on slender groups, see [12, Theorem 94.1, Vol. 2, p. 161] and the corollary holds in this case. If $\aleph_{\mathrm{fm}}$ exists, then $\|\mathbb{Z}\| = \aleph_{\mathrm{fm}}$ follows by the same theorem and Łoś's observation that $(\mathbb{Z}_{\aleph_{\mathrm{fm}}})^* \neq 0$ (see [12, Remark, Vol. 2, p. 161] or Proposition 3.3. Now suppose that $\chi \neq \aleph_{\mathrm{fm}}$ is a cardinal (and not ∞).

Then (ii) $\Rightarrow$ (iii) follows from Corollary 6.2(vii) and (iii) $\Rightarrow$ (i) is trivial.

We want to derive (ii) from (i). We may assume that $\chi > \aleph_0$. Then G in (i) must be cotorsion-free (see beginning of this section) and in particular G is torsion-free. From $(\mathbb{Z}_{\aleph_{\mathrm{fm}}})^* \neq 0$ also follows $\mathrm{Hom}(\mathbb{Z}_{\aleph_{\mathrm{fm}}}, G) \neq 0$ and $\|G\| \leqslant \aleph_{\mathrm{fm}}$. If $\chi = \lambda^+$ is a successor of a weakly compact cardinal $\lambda < \aleph_{\mathrm{fm}}$, then $\lambda \notin \mathfrak{p}_2^s$ by Corollary 3.8 and also $\lambda \notin \mathfrak{p}_1^+$ by Lemma 5.4, Proposition 4.4 and Proposition 3.3. From $\|G\| = \chi$ follows that there is a non-trivial homomorphism $\varphi: \mathbb{Z}_{\lambda^+} \longrightarrow G$ and by Theorem 5.3 there is an embedding $\mathbb{Z}_\lambda \hookrightarrow \mathbb{Z}_{\lambda^+}$. The automorphism group Aut $\mathbb{Z}_{\lambda^+}$ acts transitive on the pure elements of $\mathbb{Z}_{\lambda^+}$. We compose the embedding by a suitable automorphism and obtain a new embedding $\psi : \mathbb{Z}_\lambda \hookrightarrow \mathbb{Z}_{\lambda^+}$ which does not map into the kernel of φ. Thus, $\psi\varphi \neq 0$ and $\|G\| \leqslant \lambda < \chi$ is a contradiction. Hence, λ cannot be a weakly compact cardinal. □

7. Radicals of groups and their norms

The *norm* of a group is defined in Section 6. Now we add the notion of the norm of a radical from [14], and we will relate the two notions. The norm $\|R\|$ of a radical R is the least cardinal κ for which there exists a cartesian product $X = \prod_{\alpha\in\kappa} X_\alpha$ such that $RX \neq \prod_{\alpha\in\kappa} RX_\alpha$, or $\|R\| = \infty$ if no such cardinal κ exists. Clearly, $RX \subseteq \prod_{\alpha<\kappa} RX_\alpha$ and always $\kappa \geqslant \aleph_0$. Here is an obvious

Example 7.1. The torsion radical is $t = R_{\mathbb{Q}}$ and $\|t\| = \aleph_0$.

We shall make crucial use of the following elementary result.

Lemma 7.2. *For a regular uncountable cardinal κ, let $G := \prod^{\mathrm{red}}_{\alpha\in\kappa} G_\alpha$ be the reduced product of a family of κ groups G_α, and let H be any group of cardinality $< \kappa$. Then*

$$\mathrm{Hom}(H, G) = 0 \iff \mathrm{Hom}(H, G_\alpha) \neq 0 \quad \textit{for fewer than } \kappa \textit{ values of } \alpha \in \kappa.$$

Proof. By Lemma 4.1(ii), any homomorphism $\theta : H \to G$ lifts to a homomorphism $\theta' : H \to \prod_{\alpha\in\kappa} G_\alpha$; and every such θ' will map H into $\prod^{<\kappa}_{\alpha\in\kappa} G_\alpha$ if, and only if, fewer than κ of the $\mathrm{Hom}(H, G_\alpha)$ are non-zero. □

In order to relate the group norm to the radical norm it becomes important that the factors of these reduced products are semi-rigid. This is an extra condition added to the next examples. In order to find lower bounds for norms of suitable radicals $\|R\|$ we calculate the norms of radicals defined by certain reduced products G. This is possible in two cases, which suffice for our needs.

The *radical* R_G *associated with a group* G is defined by

$$R_G X = \bigcap\{\mathrm{Ker}\,\varphi | \varphi \in \mathrm{Hom}(X, G)\} \text{ i.e. } R_G X \text{ is the } \textit{reject} \text{ of } G \text{ in } X.$$

Definition 7.3. A family $\{G_\alpha : \alpha < \rho\}$ of groups is semi-rigid if $\mathrm{Hom}(G_\alpha, G_\beta) = 0$ for all $\alpha < \beta < \rho$.

Corollary 7.4. *For a regular uncountable cardinal* κ, *let* $G := \prod^{\mathrm{red}}_{\alpha\in\kappa} G_\alpha$, *where* G_α $(\alpha \in \kappa)$ *is a family of non-zero groups of cardinal* $|G_\alpha| < \kappa$ *which is semi-rigid. Then*

$$R_G \prod_{\alpha\in\kappa} G_\alpha \subseteq \prod_{\alpha<\kappa}^{<\kappa} G_\alpha \subsetneq \prod_{\alpha\in\kappa} G_\alpha = \prod_{\alpha\in\kappa} R_G G_\alpha.$$

Proof. By Lemma 7.2 each $\mathrm{Hom}(G_\alpha, G)=0$, in other words $R_G G_\alpha = G_\alpha$. But $R_G \prod_{\alpha\in\kappa} G_\alpha$ is contained in the kernel of the natural epimorphism $\prod_{\alpha\in\kappa} G_\alpha \longrightarrow G$, and this kernel $\prod^{<\kappa}_{\beta<\kappa} G_\beta$ is already a proper subgroup of $\prod_{\alpha\in\kappa} G_\alpha$. □

Observation 7.5. *The norm* $\|R\|$ *of a radical* R *is always a regular cardinal* (*where* $\aleph_0$ *is allowed*).

Proof. If κ is a singular cardinal then $\kappa = \dot\bigcup_{\alpha<\rho} I_\alpha$ where $\rho = \mathrm{cf}\,\kappa$ is the cofinality of κ; here $\rho < \kappa$ and $|I_\alpha| < \kappa$ for all $\alpha < \rho$. Any product over κ can be written as a product over ρ of products over the I_α, hence $\|R\|$ must be regular. □

Our next Proposition 7.6 extends part of Corollary 6.2 to groups different from $\mathbb{Z}$; see also Lemma 4.7.

Proposition 7.6. *Let* $\mu \leqslant \lambda < \rho$ *be cardinals with* ρ *regular. Moreover, let* $(\mu, \lambda) \in \mathfrak{p}_2^u$ *and* $G = \prod^{\mathrm{red}}_{\alpha\in\rho} G_\alpha$ *be a reduced product of a semi-rigid family* $\{G_\alpha : \alpha \in \rho\}$ *of slender groups of cardinality* $< \rho$. *Then* $\mathrm{Hom}(\mathbb{Z}_\mu, G) = 0$.

Proof. Suppose for contradiction that there is a homomorphism $0 \neq \varphi : \mathbb{Z}_\mu \longrightarrow G$. Condition $\mathfrak{p}_2^u$ is equivalent $\mathfrak{p}_3^u$ (Proposition 3.3(i)). By $(\mu, \lambda) \in \mathfrak{p}_3^u$ and Proposition 4.5 there is a subgroup $T \subseteq \mathbb{Z}_\mu$ such that $|T| \leqslant \lambda < \rho$ and $\varphi\restriction T \neq 0$. The homomorphism $\varphi\restriction T$ lifts to a non-trivial homomorphism $\varphi' : T \longrightarrow \prod_{\alpha\in\rho} G_\alpha$ by Lemma 4.1. It also follows $\mathrm{Hom}(T, G_\alpha) = 0$ from Proposition 4.5(ii) and slenderness, thus $\varphi' = 0$ is a contradiction. □

Lemma 7.7. *Let $\lambda < \rho$ be cardinals, ρ regular and suppose that $(\mu, \lambda) \in \mathfrak{p}_2^\mu$ for all $\mu \leqslant \lambda$. If $G = \prod_{\alpha\in\rho}^{\mathrm{red}} G_\alpha$ is a reduced product of a semi-rigid family $\{G_\alpha : \alpha \in \rho\}$ of non-trivial slender groups of cardinality $< \rho$, then the following holds*:

(i) $\lambda^+ \leqslant \|G\| \leqslant \rho$.
(ii) *If $\rho = \lambda^+$, then $\|G\| = \rho$.*

Proof. Clearly, (ii) follows from (i) and it remains to show (i).

By Observation 6.1 we can restrict ourself to regular cardinals μ and apply Proposition 7.6. Hence, $\mathrm{Hom}(\mathbb{Z}_\mu, G) = 0$ for all regular $\mu \leqslant \lambda$. By definition of norm follows $\lambda^+ \leqslant \|G\|$. There is an obvious embedding $\mathbb{Z}_\rho \longrightarrow G$, thus also $\|G\| \leqslant \rho$. $\square$

The case of strongly inaccessible cardinals is particularly easy. Here we determine the group, its inaccessible norm and the additional algebraic properties explicitly.

Proposition 7.8. *If $\kappa < \aleph_{\mathrm{fm}}$ is strongly inaccessible, then there is a reduced product $G := \prod_{\alpha\in\kappa}^{\mathrm{red}} \mathbb{Z}_{\kappa_\alpha}$ of norm $\|G\| = \kappa$.*

Proof. Since κ is a regular strong limit cardinal, there is an increasing sequence $\kappa_\alpha \in \mathfrak{p}_2^s$ $(\alpha \in \kappa)$ of regular cardinals with $\sup_{\alpha\in\rho} \kappa_\alpha = \kappa$, see Proposition 3.3(ix). The corresponding family of cotorsion-free groups $\mathbb{Z}_{\kappa_\alpha}$ of reduced products is semi-rigid by Corollary 6.2. We take the reduced product $G = \prod_{\alpha\in\kappa}^{\mathrm{red}} \mathbb{Z}_{\kappa_\alpha}$. Again $\mathbb{Z}_\kappa \hookrightarrow G$, so $\mu := \|G\| \leqslant \kappa$ and consider $\mu < \kappa$ which is regular and let $\psi : \mathbb{Z}_\mu \longrightarrow G$ be any homomorphism. Then $2^\mu < \kappa$ and by Lemma 4.1(ii) we also have $\varphi : \mathbb{Z}_\mu \longrightarrow \prod_{\alpha\in\kappa} \mathbb{Z}_{\kappa_\alpha}$ which induces ψ (modulo $\prod_{\alpha\in\kappa}^{<\kappa} \mathbb{Z}_{\kappa_\alpha}$). There is also $\alpha_0 \in \kappa$ such that $2^\mu < \kappa_\alpha$ for all $\alpha \geqslant \alpha_0$, hence $\mathrm{Hom}(\mathbb{Z}_\mu, \mathbb{Z}_{\kappa_\alpha}) = 0$ for all $\alpha \geqslant \alpha_0$ and $\psi = 0$ follows. This shows that $\|G\| = \kappa$. $\square$

The other case comes from our work in the last sections on the norm of a group.

Proposition 7.9. *Let $\rho = \lambda^+$ be cardinals. Then there is a reduced product of a semi-rigid family $\{G_\alpha : \alpha \in \rho\}$ of slender groups of cardinality λ such that $G = \prod_{\alpha\in\rho}^{\mathrm{red}} G_\alpha$ and the group G has norm $\|G\| = \rho$ if one the following holds*:

(a) $\lambda \in \mathfrak{p}_2^s$.
(b) *$\lambda = 2^\mu < \aleph_{\mathrm{fm}}$ for some cardinal μ and λ is not weakly compact.*
(c) *$\lambda < \aleph_{\mathrm{fm}}$ is not weakly compact and $\lambda = \beth_\alpha$ for some ordinal α.*
(d) ZFC+GCH *$\lambda < \aleph_{\mathrm{fm}}$ is not weakly compact.*

Proof. Here we apply an old result about the existence of rigid sets of slender abelian groups from Corner and Göbel [5], which utilizes Shelah's black box. There is a rigid system $\{G_\alpha : \alpha \in 2^\lambda\}$ of slender abelian groups G_α of cardinality λ, see [5, Theorem 7.4 (b), p. 466]. We choose a subsystem of size $\rho = \lambda^+$.

(a) From $\lambda \in \mathfrak{p}_2^s$ follows $(\mu, \lambda) \in \mathfrak{p}_2^s$ (Observation 3.7(i)). Hence, Lemma 7.7(ii) applies.
(b) By Proposition 3.3(ix) and (x) follows $\lambda \in \mathfrak{p}_2^s$, thus (a) applies.

(c) Apply (b).
(d) Apply (c). □

Assuming GCH we have a characterization of those cardinals $< \aleph_{\mathrm{fm}}$ which are norms of reduced products of semi-rigid families of groups; this stronger form of Corollary 6.3 is needed for norms of radicals immediately.

Corollary 7.10 (*ZFC+GCH*). *An infinite cardinal ρ is the norm of a group if and only if it is the norm of a reduced product of a semi-rigid family of groups. This is the case if and only if ρ is inaccessible or $\rho \leqslant \aleph_{\mathrm{fm}}$ is a successor cardinal but not the successor of a weakly compact cardinal.*

Proof. The existence of groups for these two kinds of norms follows from Proposition 7.9(d) and Proposition 7.8. Conversely, from Corollary 6.3(i) ⇒ (ii) follows that successors of weakly compact cardinals are not norms of groups. □

If the radical R_G is associated with a group G, then the group norm and the radical norm are related by

Lemma 7.11. *If G is an abelian group, then $\|G\| \leqslant \|R_G\|$.*

Proof (*See also Göbel [14]*). We write $\kappa = \|R_G\|$. If $\kappa = \infty$, we have nothing to do. Assuming that $\kappa < \infty$, by definition we may consider a cartesian product $X = \prod_{\alpha\in\kappa} X_\alpha$ such that $R_G X \neq \prod_{\alpha\in\kappa} R_G X_\alpha$, thus $R_G X \subsetneq \prod_{\alpha\in\kappa} R_G X_\alpha$. This means that there exist a homomorphism $\varphi : X \to G$ and an element $x = \sum_{\alpha\in\kappa} x_\alpha \in \prod_{\alpha\in\kappa} R_G X_\alpha$ such that $x\varphi \neq 0$. By the minimality of κ, R_G commutes with $\prod^{<\kappa}_{\alpha\in\kappa}$,so φ vanishes on the subgroup $\prod^{<\kappa}_{\alpha\in\kappa} R_G X_\alpha$, and therefore also on $\prod^{<\kappa}_{\alpha\in\kappa} x_\alpha\mathbb{Z}$. Hence, φ induces a non-trivial homomorphism $\mathbb{Z}_\kappa \to G$, and this implies that $\|G\| \leqslant \kappa$. □

Observation 7.12. *For any regular, uncountable cardinal κ, let G_α $(\alpha < \kappa)$ be a semi-rigid family of non-zero groups of cardinal $< \kappa$, and let $G = \prod^{\mathrm{red}}_{\alpha<\kappa} G_\alpha$. Then $\|R_G\| \leqslant \kappa$.*

Proof. This is an immediate consequence of Corollary 7.4. □

Theorem 7.13. (i) *If H is a cotorsion-free group and $|H| < \aleph_{\mathrm{fm}}$ then $\|R_H\| \leqslant \aleph_{\mathrm{fm}}$ is a regular cardinal.*

(ii) (a) *If $\aleph_{\mathrm{fm}}$ exists, then* $\|R_{\mathbb{Z}}\| = \aleph_{\mathrm{fm}}$.
(b) *If $\aleph_{\mathrm{fm}}$ does not exist, then* $\|R_{\mathbb{Z}}\| = \infty$.
(c) $\|R_{\mathbb{Q}}\| = \|t\| = \aleph_0$.

(iii) ZFC+GCH. *If $\rho = \aleph_0$ or $\rho < \aleph_{\mathrm{fm}}$ and ρ is either inaccessible or the successor of a cardinal that is not weakly compact, then there is a group G such that $|G| < \aleph_{\mathrm{fm}}$ and $\|R_G\| = \|G\| = \rho$.*

Proof. (i) If $\overline{H} = H^{\aleph_1}/H^{(\aleph_1)}$ then $\mathrm{Hom}(\overline{H}, H) = 0$ H because H is cotorsion-free (and $\aleph_1 < \aleph_{\mathrm{fm}}$). It follows that $R_H\overline{H} = \overline{H}$, hence also $(R_H\overline{H})^{\aleph_{\mathrm{fm}}} = \overline{H}^{\aleph_{\mathrm{fm}}}$. On the other hand, there is an epimorphism $\sigma: H^{\aleph_{\rm fm}} \longrightarrow H$ using the measure on $\aleph_{\rm fm}$, see Łoś's observation [12, Remark, Vol. 2, p. 161]. Write $H^{\aleph_{\rm fm}}$ as $(H^{\aleph_1})^{\aleph_{\rm fm}}$ and note that $H^{(\aleph_{\rm fm})}\sigma = 0$. Thus, σ induces an epimorphism $\overline{H}^{\aleph_{\rm fm}} \longrightarrow H$ and clearly $R_H(\overline{H}^{\aleph_{\rm fm}}) \neq \overline{H}^{\aleph_{\rm fm}}$,thus $\|R_H\| \leqslant \aleph_{\rm fm}$. Moreover, $\|R_H\|$ is regular by Observation 6.1.

(ii) From Lemma 7.11 follows $\|\mathbb{Z}\| \leqslant \|R_{\mathbb{Z}}\|$. Moreover, $\|\mathbb{Z}\| = \aleph_{\rm fm}$ or $\|\mathbb{Z}\| = \infty$ (see the proof of Corollary 6.3), hence (ii) holds if $\aleph_{\rm fm}$ does not exist. Otherwise, apply (i) for $H = \mathbb{Z}$. (c) is well-known.

(iii) Choose G from Proposition 7.6 or Proposition 7.8, thus $\|G\| = \rho \leqslant \aleph_{\rm fm}$. Moreover, $\|G\| \leqslant \|R_G\| \leqslant \rho$ from Lemma 7.11 and Observation 7.12, hence equality holds. □

We note that GCH in Theorem 7.13 can be replaced by the weaker condition on cardinals in Remark 3.9 as is immediate by inspection of the proof.

7.1. Rigid families of with prescribed norms

Propositions 7.8, 7.9, and thus Corollary 7.10 can be extended to fully rigid families of groups. If λ is a cardinal, then the family $\{G_X : X \subseteq \lambda\}$ of groups G_X of cardinality λ is fully rigid if the following holds.

$$\mathrm{Hom}_R(G_X, G_{X'}) \cong \begin{cases} \mathbb{Z} & \text{if } X \subseteq X', \\ 0 & \text{if } X \not\subseteq X'. \end{cases}$$

We consider the case corresponding to Proposition 7.9 and let $\rho = \lambda^+$. Replace the fully rigid family of slender groups of cardinality λ in [5, Theorem 7.4 (b), p. 466] by a family $\{G_{X_\alpha} : X \subseteq \lambda, \alpha \in \rho\}$ such that the following holds:

$$\mathrm{Hom}_R(G_{X_\alpha}, G_{X'_\beta}) \cong \begin{cases} \mathbb{Z} & \text{if } \alpha = \beta, X \subseteq X', \\ 0 & \text{if } \alpha \neq \beta \text{ or } X \not\subseteq X'. \end{cases}$$

The put $G_X = \prod^{\rm red}_{\alpha\in\rho} G_{X_\alpha}$. Then by the above and Proposition 7.9 follows the

Observation 7.14. *If $\lambda \in \mathfrak{p}^s_2$ and $\rho = \lambda^+$, then there is a fully rigid family of groups G_X with $\|G_X\| = \rho$ for all $X \subseteq \lambda$.*

The case corresponding to Proposition 7.8 is very similar. Observation 7.14 can be transferred to radicals. We obtain a rigid family of radicals $R^X = R_{G_X}$ $(X \subseteq \lambda)$: If $X \subseteq X'$, then there is an injective map $G_X \hookrightarrow G_{X'}$ corresponding to $1 \in \mathbb{Z}$ for example. Thus, $R^X G_{X'} = 0$. If $X \not\subseteq X'$, then $\mathrm{Hom}(G_X, G_{X'}) = 0$ and therefore $R^X G_{X'} = G_{X'}$ follows. Thus, the family R^X of radicals has norm $\|R^X\| = \rho$ (by Theorem 7.13) and is rigid in the sense that for all $X, X' \subseteq \lambda$ the following holds.

$$R^X G_{X'} = \begin{cases} 0 & \text{if } X \subseteq X', \\ G_{X'} & \text{if } X \not\subseteq X'. \end{cases}$$

8. Consistency results

We will use Fodor's lemma [17, p. 59], but in a form often more useful (but also known). For convenience we include the short proof.

Lemma 8.1. *Let κ be a regular, uncountable cardinal and $\{T_i : i \in \kappa\}$ be an increasing, continuous chain of sets of cardinality $< \kappa$. If $S \subseteq \kappa$ is stationary and $f(i) \in T_i$ for all $i \in S$, then there is $j \in S$ such that*

$$\{i \in S : f(i) \in T_j\} \subseteq \kappa$$

is stationary.

Proof. Inductively choose bijections $h_i : T_i \longrightarrow \alpha_i$ for ordinals α_i such that the following holds:

(i) If $i < j$, then $h_i \subseteq h_j$.
(ii) If j is a limit ordinal, then $h_j = \bigcup_{i<j} h_i$.
(iii) The sequence of ordinals $\alpha_i \in \kappa$ $(i \in \kappa)$ is increasing continuously.

If $j = 0$, then choose any bijection $h_0 : T_0 \longrightarrow \alpha_0 = |T_0|$. If $j = i + 1$, then let $\alpha_j = \alpha_i + |T_j \backslash T_i|$ be the ordinal sum and choose $h_j = h_i \cup h'_j$ with a bijection $h'_j : T_j \backslash T_i \longrightarrow [\alpha_i, \alpha_j)$. Thus, $\alpha_j \in \kappa$ and $h_j(f(j)) < \alpha_j$ from $f(j) \in T_j$, $j \in S$. Finally, let $h = \bigcup_{i\in\kappa} h_i$ and $f' = h \circ f : S' \longrightarrow \kappa$; again $h(f(j)) < \alpha_j$ for all $j \in S$. The set $C = \{\delta \in \kappa : \delta \text{ a limit}, \ \alpha_\delta = \delta\}$ is a cub in κ, thus $S' = S \cap C$ is stationary in κ. If $i \in S'$, then $\alpha_i = i$ and $h(f(j)) < j$ for all $j \in S'$. The map $S' \longrightarrow \kappa$ $(i \longrightarrow f'(i))$ is regressive. By the usual Lemma of Fodor [17, p. 59] this map is constant on a stationary subset $S'' \subseteq S'$. There is $j \in S''$ such that $f'(i) = j$ for all $i \in S''$. Then $f(i) \in T_j$ for all $i \in S''$. □

We first note the following.

Lemma 8.2. *If $V = L[D]$ and D is a normal measure on κ, then κ is the only measurable cardinal.*

See [17, p. 361] for a proof. Moreover, we apply several known consistency results. Recall the following.

Definition 8.3. A *transversal* of a set $\mathbb{A}$ of sets is a one to one map $T : \mathbb{A} \longrightarrow \bigcup \mathbb{A}$ with $T(A) \in A$ for all $A \in \mathbb{A}$.

Theorem 8.4. (i) *It is consistent with GCH and the existence of $\aleph_{\mathrm{fm}}$ that $\aleph_{\mathrm{fm}}$ is (strongly) compact.*

(ii) *Assume* (GCH+$\aleph_{\mathrm{fm}}$ *is compact*).

(a) *For all regular cardinals μ the following holds:*

$$\mu \in \mathfrak{p}_2^u \Longleftrightarrow \mu \in \mathfrak{p}_2^s \Longleftrightarrow (\mu < \aleph_{\mathrm{fm}} \text{ and } \mu \text{ is not weakly compact}).$$

(b) *For all regular cardinals $\mu \geqslant \aleph_{\mathrm{fm}}$ follows $\mu \notin \mathfrak{p}_2^u$ (thus $\mu \notin \mathfrak{p}_2^s$).*

(iii) *Let V be a model of set theory and* $\mu \in V$ *be a regular cardinal* $\geqslant \aleph_{\mathrm{fm}}$. *If* $\mu^+ \leqslant \lambda < 2^\mu$ *and* $A \subseteq \lambda$, *then there is a model* $V' \subseteq V$ *with GCH such that* $A \in V'$ *and* $V' \vDash \mu$ *is measurable*.

(iv) *If* $\aleph_{\mathrm{fm}} \leqslant \mu < \mu^+ \leqslant \lambda < 2^\mu$, *then there is a model of set theory with* μ *measurable and* $V \vDash (\mu, \lambda) \notin \mathfrak{p}_2^s$ *and* $(\mu, \lambda) \in \mathfrak{p}_2^u$.

Proof. (i) is due to Magidor [21] which shows that the first measurable cardinal can be compact. If one forces over a ground model containing a compact cardinal that satisfies GCH, the generic extension satisfies GCH as well. A different proof was given recently by Apter and Cummings in [1].

(ii)(a) If $\mu < \aleph_{\mathrm{fm}}$ is not weakly compact and GCH holds, then $\mu \in \mathfrak{p}_2^s$, thus $\mu \in \mathfrak{p}_2^u$ follows by Proposition 3.3(i), (iii) and (ix), (x), see also Observation 3.7 and Lemma 3.7. Conversely, if $\mu \in \mathfrak{p}_2^u$ then $\mu \in \mathfrak{p}_2^s$ by GCH (Remark 3.2) and $\mu < \aleph_{\mathrm{fm}}$ from case (b).

(ii)(b) If $\mu \geqslant \aleph_{\mathrm{fm}}$, then there is a uniform ω_1-complete ultrafilter D on $\mathfrak{P}(\mu)$ because $\aleph_{\mathrm{fm}}$ is compact, see [17]. Thus, $\mu \notin \mathfrak{p}_3^u$ and by Proposition 3.3(i), (iii) follows (b). □

(iii) is well-known, see [17].

(iv) We apply (iii), hence μ above is measurable. We have $\aleph_{\mathrm{fm}} \leqslant \mu$ regular, then there is an ω_1-complete ultrafilter on μ and $\mu \notin \mathfrak{p}_2^s$ follows from Proposition 3.3(v).

By GCH and $\mu > \aleph_{\mathrm{fm}}$ there is a stationary non-reflecting set $S \subseteq \mu^o =: \{\delta \in \mu, \operatorname{cf} \delta = \omega\}$, see [25]. Let A_δ be a ladder on δ for each $\delta \in S$ and $\mathbb{A}_S = \{A_\delta : \delta \in S\}$. The set $\mathbb{A}_S$ has no transversal as follows immediately from the version of Fodor's Lemma 8.1 above. On the other hand, for every regular cardinal μ the set $\mathbb{A}_\mu = \{X \subseteq \mu : |X| < \mu\}$ has a transversal as follows from the proof in [24, p. 1271]. (The proof is an induction on $\alpha < \mu$ for all sets $\mathbb{A}_\mu^\alpha = \{A_\beta \in \mathbb{A}_\mu : \beta \in \alpha \cap S\}$ for any fixed stationary subset S of μ and enumeration of $\mathbb{A}$.)

Now choose a model $M = (\mu, P, Q_0, Q_1, R, F, G)$ for property $\mathfrak{p}_2^u$ with $P^M = S$ the stationary set above, $Q_0^M = \omega$, $Q_1^M = \mu$, $R^M = \{(\alpha, \beta) : \alpha \in A_\beta, \beta \in S\}$ coding $\mathbb{A}_S$, $F^M : S \times S \longrightarrow \bigcup \mathbb{A}$ a 2-adic function such that $F(\alpha, \beta) \in A_\alpha$ for all $\alpha < \beta \in S$ which is one to one in the first coordinate, i.e. if $\alpha < \beta < \gamma$ and $\alpha, \beta \in S$, then $F(\alpha, \gamma) \neq F(\beta, \gamma)$ and $G^M : \omega \times S \longrightarrow \kappa$ another 2-adic function such that for each $\alpha \in S$ the sequence $\langle G(n, \alpha) : n \in \omega \rangle = A_\alpha$ lists a ladder at α (without repetition). Thus, $R(G(n, \alpha), \alpha)$ for all $n \in \omega$. These functions exist because $\mathbb{A}_\mu$ has a transversal. Suppose for contradiction that $\mu \notin \mathfrak{p}_2^u$. Then M has an elementary extension $M \preccurlyeq N$ with $Q_0^N = Q_0^M = \omega$ and there is $c \in N \setminus M$ which is an upper bound, i.e. if $\alpha \in \mu$, then $M \vDash \alpha < c$.

Note that $A_\alpha = F(\alpha, c)$ is a ladder at $\alpha \in S$. If $\alpha \neq \beta \in S$, then $F(\alpha, c) \neq F(\beta, c)$ and from $R(F(\alpha, c), \alpha)$ follows $\alpha \in F(\alpha, c)$. We have $M \vDash (\forall x, y(P(x), x < y \to R(F(x, y), x)))$ from above, thus also $N \vDash (\forall x, y(P(x), x < y \to R(F(x, y), x)))$. In particular, from $M \vDash \alpha < c$ follows $N \vDash R(F(\alpha, c), \alpha)$ for all $\alpha \in S$. From G and $(N \vDash R(b, \alpha) \to b \in A_\alpha)$ follows that $\langle F(\alpha, c) : \alpha \in S \rangle$ is a transversal, a contradiction. Hence, $\mu \in \mathfrak{p}_2^u$ and (iv) holds. □

Finally, we add without proof the following claim which is similar to Theorem 8.4(iv) but even stronger. The remark uses partially ordered Cohen sets (reals) and well-known properties on Cohen forcing; here are references for pedestrians [3,20].

Remark 8.5. (i) There are models of V_0 of set theory such that $V_0 \vDash \kappa < \mu < \chi$ are cardinals with μ super compact $\kappa = \kappa^{<\kappa}$.

(ii) Let $P = Cohen_{\kappa,\chi}$ and $G \subseteq P$ be generic over $V_1 = V_0[G]$. Then the following holds in V_1.

(a) No cardinals collapse.
(b) The cofinalities remain unchanged.
(c) $\chi \leqslant 2^\kappa$.
(d) If $\mu \leqslant \lambda < \chi$ and $A \subseteq \lambda$, then in V_1 there is a κ-complete ultrafilter D on $\mathfrak{P}(\mu)$ such that for all $X \subseteq \lambda$, $X \in V_0[A]$ either $X \in D$ or $\mu \backslash X \in D$.

(iii) In V_1 follows from (ii)(d) that $(\mu, \lambda) \notin \mathfrak{p}_2^u$.

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