Appendix

Remarks on some cardinal invariants of the continuum

by

S. Shelah
Department of Mathematics
Hebrew University
Jerusalem, Israel

A.1. THEOREM. **ZFC** $\vdash \mathfrak{d} \leq \mathfrak{i}$.

A.2. NOTATION. Let $\mathcal{A} \subseteq [\omega]^\omega$ and $\mathrm{Fn}(\mathcal{A}) = \{\, f : f$ finite, $\mathrm{dom}(f) \subseteq \mathcal{A}$, $\mathrm{rng}(f) \subseteq \{0,1\}\,\}$. For the following f, g and h will always range over $\mathrm{Fn}(\mathcal{A})$.

For $f \in \mathrm{Fn}(\mathcal{A})$, let

$$X_f = \bigcap_{a \in \mathrm{dom}(f)} a^{f(a)},$$

where $a^1 = a$, $a^0 = \omega - a$.

From now on let $\mathcal{A}$ be independent i.e., X_f is infinite for all f.

Let $I = I_\mathcal{A} = \{\, A \subseteq \omega : \forall f \exists g \supseteq f \quad (X_g \cap A \text{ is finite})\,\}$.

Clearly, I is an ideal containing all finite sets and $X_f \notin I_\mathcal{A}$ for all f.

A.3. LEMMA (Assuming $|\mathcal{A}| < \mathfrak{d}$). *Let $E \in I_\mathcal{A}$ and assume that $f \in \mathrm{Fn}(\mathcal{A})$ and $A_0, A_1, \ldots, A_n, \ldots \in \mathcal{A}$ $(n \in \omega)$ are such that $\mathrm{dom}(f) \subseteq \mathcal{A}' =^{def} \mathcal{A} - \{A_0, \ldots\}$, then there is a set E' such that:*

(α) $E' \in I_\mathcal{A}$.
(β) $E' \cap E = \emptyset$
(γ) $\forall g \in \mathrm{Fn}(\mathcal{A}')$: If $g \supseteq f$ then $X_g \cap E'$ is infinite.

PROOF. For any $H\colon \omega \to \omega$ let

$$E'_H = \bigcup_n \left[(A_n - \bigcup_{i<n} A_i) \cap H(n) \right] - E.$$

Then clearly $E'_H \in I_\mathcal{A}$ and $E'_H \cap E = \emptyset$, so any E'_H satisfies (α) and (β). We have to find a suitable H such that (γ) is satisfied.

Note that if $g \supseteq f$ and $\mathrm{dom}(g) \subset \mathcal{A}'$ then $X_g \cap (A_n - \bigcup_{i<n} A_i) - E \notin I_\mathcal{A}$, so in particular it is infinite. (Since $(A_n - \bigcup_{i<n} A_i)$ is of the form X_h for some h with $\mathrm{dom}(h) \cap \mathrm{dom}(g) = \emptyset$, it is not in $I_\mathcal{A}$.)

For each $g \in \mathrm{Fn}(\mathcal{A}')$ extending f, let

$$H_g(n) = \min(X_g \cap (A_n - \bigcup_{i<n} A_i) - E).$$

Clearly H_g is a 1-to-1 function, and if $H_g(n) < H(n)$, then $H_g(n) \in X_g \cap E'_H$. Hence if for infinitely many n,

$$H_g(n) < H(n)$$

then

$$X_g \cap E'_H \text{ is infinite.}$$

Since $|\mathcal{A}| < \mathfrak{d}$, we can find H such that for all $g \in \mathrm{Fn}(\mathcal{A})$ with $g \supseteq f$ there are infinitely many n for which $H_g(n) < H(n)$.

Then $E' = E'_H$ satisfies the requirements of the lemma. □

PROOF OF THE THEOREM: Assume $\mathcal{A}$ is an independent family of size $< \mathfrak{d}$. We will show that $\mathcal{A}$ is not maximal.

Let $N \prec \langle H(\lambda), \in \rangle$ for sufficiently large λ with N countable and $\mathcal{A} \in N$. Let $\{f_n : n \in \omega\}$ list $\mathrm{Fn}(\mathcal{A}) \cap N$, such that each element of $\mathrm{Fn}(\mathcal{A}) \cap N$ appears with even and with odd index. By induction choose $E_n \in N$ such that:

(A) $E_n \in I_{\mathcal{A}}$

(B) $E_n \cap (\bigcup_{l<n} E_l) = \emptyset$

(C) If $f_n \subseteq g \in \mathrm{Fn}(\mathcal{A})$ and $\mathrm{dom}(g) \cap N = \mathrm{dom}(f_n)$ then $X_g \cap E_n$ is infinite

We can do this by the previous lemma, letting $E = \bigcup_{l<n} E_l$, $f = f_n$ and $\{A_0, A_1, \ldots\} \in N$ be some family disjoint from $\mathrm{dom}(f_n)$. (we can have $E_n \in N$ by elementarity of N).

Now let $Y = \bigcup_n E_{2n}$. Then $\mathcal{A} \cup \{Y\}$ is independent: Let $g \in \mathrm{Fn}(\mathcal{A})$. Find n such that $f_{2n} = g \cap N$. Then $X_g \cap Y$ contains $X_g \cap E_{2n}$ which is infinite. If $g \cap N = f_{2k+1}$ for some k then $X_g \cap (\omega - Y)$ contains $X_g \cap E_{2k+1}$ which is also infinite. This finishes the proof of the theorem. □

A.4. REMARK. $\mathfrak{d} < \mathfrak{i}$ is consistent: e.g., take a model of **CH** and add $\aleph_2$ many random reals with countable support. Then the old reals still form a dominating family. But an independent family of size ω_1 must be in an intermediate model, so it cannot be maximal, since the next random real will be independent from it. We can understand this argument more generally: if the set of reals is not the union of fewer than λ sets of measure zero, then any independent family of subsets of ω has cardinality at least λ. So if P is the forcing of the measure algebra of dimension $\lambda > \aleph_0$ then in V^P one has $\mathfrak{i} \geq \lambda$, whereas $\mathfrak{d}$ is not changed by forcing with P.