Rend. Circ. Mat. Palermo (2012) 61:107–115
DOI 10.1007/s12215-011-0078-7

Martin's Axiom and separated mad families

Alan Dow · Saharon Shelah

Received: 24 July 2011 / Accepted: 28 September 2011 / Published online: 12 November 2011

Abstract Two families $\mathcal{A}, \mathcal{B}$ of subsets of ω are said to be separated if there is a subset of ω which mod finite contains every member of $\mathcal{A}$ and is almost disjoint from every member of $\mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are countable disjoint subsets of an almost disjoint family, then they are separated. Luzin gaps are well-known examples of ω_1-sized subfamilies of an almost disjoint family which can not be separated. An almost disjoint family will be said to be ω_1-separated if any disjoint pair of $\leq\omega_1$-sized subsets are separated. It is known that the proper forcing axiom (PFA) implies that no maximal almost disjoint family is $\leq\omega_1$-separated. We prove that this does not follow from Martin's Axiom.

Keywords Maximal almost disjoint family · Martin's Axiom

Mathematics Subject Classification (2000) 03A35

1 Introduction

In this paper we construct a model of Martin's Axiom in which there is a maximal almost disjoint family of subsets of ω which has a strong separation property we call ω_1-separated. Combinatorial properties of almost disjoint families are fundamental and well-studied. Two sets are said to be almost disjoint, also *orthogonal* to each other, if their intersection is finite, and two families of sets are orthogonal if each member of one is orthogonal to each in the other. Two families of subsets of ω are *separated* if there is a set C which is orthogonal to

A. Dow (✉)
University of North Carolina at Charlotte, Charlotte, NC 28223, USA
e-mail: adow@uncc.edu

S. Shelah
Department of Mathematics, Rutgers University, Hill Center, Piscataway, NJ 08854-8019, USA
e-mail: shelah@math.rutgers.edu

Present address:
S. Shelah
Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem 91904, Israel

the first while $\omega \setminus C$ is orthogonal to the other (i.e. C mod finite contains each member of the first family).

One of the most famous and influential papers on orthogonal almost disjoint families is the 1947 paper of Luzin.

Proposition 1.1 [3] *There exist orthogonal $\aleph_1$-sized almost disjoint families of subsets of ω which can not be separated.*

Of course Luzin's method was based on the ideas introduced by Hausdorff in constructing (ω_1, ω_1^*)-gaps. Todorcevic [6] introduces the terminology of a Luzin gap. A pair $(\{a_\alpha\}_{\alpha\in\omega_1}, \{b_\alpha\}_{\alpha\in\omega_1})$ of families of countable sets is a Luzin gap if for all $\alpha \neq \beta$, $a_\alpha \cap b_\alpha$ is empty, while $(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$ is not empty. It is well-known that a Luzin gap can not be separated.

Abraham and Shelah [1] study the notions of Luzin sequences and Luzin* sequences. A sequence $\{a_\alpha : \alpha \in \omega_1\}$ is a Luzin sequence (respectively Luzin* sequence) if for all $\zeta \in \omega_1$ and $n \in \omega$, the set $\{\alpha \in \zeta : a_\alpha \cap a_\zeta \subset n\}$ is finite (respectively $\{\alpha \in \zeta : |a_\alpha \cap a_\zeta| < n\}$ is finite). An uncountable almost disjoint family is defined to be *inseparable* if no uncountable pair can be separated. Each Luzin* family is Luzin, and each Luzin family is inseparable. One of the results of [1] was to establish that MA(ω_1) implies that each inseparable sequence could be written as a countable union of Luzin* sequences.

We may say that an almost disjoint family is *nowhere inseparable* if it contains no inseparable subfamily. Regrettably the word *separable* in the context of almost disjoint families is already defined to mean something quite unrelated; specifically the term *completely separable almost disjoint family* is defined to mean that every set having infinite intersection with infinitely many of the members will contain a member.

There is a particularly natural example of a Luzin gap ([5, Lemma 1] and [6, p. 57]) which can best be defined as subsets of the tree $2^{<\omega}$.

Proposition 1.2 *If $\{f_\alpha : \alpha \in \omega_1\} \subset 2^\omega$, and for each $\alpha \in \omega_1$, let $a_\alpha = \{f_\alpha \restriction n : f_\alpha(n) = 0\}$ and $b_\alpha = \{f_\alpha \restriction n : f_\alpha(n) = 1\}$. Then, for all $\alpha < \beta$, $a_\alpha \cap b_\alpha = \emptyset$, and $|(a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)| = 1$.*

On the other hand, it is a very well-known result of Silver that MA(ω_1) implies that any family of ω_1 many branches themselves is nowhere inseparable [4, p. 162]. This is the genesis of the following notion.

Definition 1.3 A family $\mathcal{A} \subset [\omega]^\omega$ is *special* if

(1) $\mathcal{A}$ is an almost disjoint family and
(2) there is a function $c : [\mathcal{A}]^{<\omega} \to \omega$ and a linear ordering $<$ (or $<_c$) of $\mathcal{A}$ satisfying that, for each $n \in \omega$, if $B_1 < \cdots < B_n$ and $C_1 < \cdots < C_n$ are two sequences from $\mathcal{A}$, and if $c(\{B_1, \ldots, B_n\}) = c(\{C_1, \ldots, C_n\}) = k$, then for all $i \neq j \in \{1, \ldots, n\}$, $B_i \cap C_j$ is contained in k.

Definition 1.4 An almost disjoint family $\mathcal{A}$ is $<\lambda$-special if each $\mathcal{A}' \in [\mathcal{A}]^{<\lambda}$ is special. We say that $\mathcal{A}$ is λ-special if it is $<\lambda^+$-special.

It is a well-known generalization of Silver's result that

Proposition 1.5 *MA_{ω_1} implies that disjoint $\leq\aleph_1$-sized subsets of an ω_1-special family are separated.*

Proof Let $\mathcal{A}$ be an ω_1-special family and let $\mathcal{A}_0$ and $\mathcal{A}_1$ be disjoint subsets of $\mathcal{A}$, each with cardinality at most ω_1. Of course if either is countable, then the fact that MA_{ω_1} implies that $\mathfrak{b} > \omega_1$, shows that they can be separated. Otherwise, let c be the function from $[\mathcal{A}_0 \cup \mathcal{A}_1]^{<\omega}$ into ω which witnesses that $\mathcal{A}_0 \cup \mathcal{A}_1$ is special. A poset Q is defined to be the family of functions q into ω such that $\text{dom}(q)$ is a finite subset of $\mathcal{A}_0 \cup \mathcal{A}_1$ and q satisfies that $(a \setminus q(a))$ is disjoint from $(b \setminus q(b))$ whenever $a \in \text{dom}(q) \cap \mathcal{A}_0$ and $b \in \text{dom}(q) \cap \mathcal{A}_1$. The poset Q is simply ordered by extension. It suffices to show that Q is ccc.

Let $\{q_\xi : \xi \in \omega_1\} \subset Q$. Towards proving that this is not an antichain, we may assume that there are integers $n, \bar{k}$ and subsets I, I_0, I_1 of n such that, for each $\xi \neq \eta \in \omega_1$,

(1) $\text{dom}(q_\xi) = \{a^\xi_\ell : \ell < n\}$ as ordered by $<_c$,
(2) $q(a^\xi_\ell) = q(a^\eta_\ell) < \bar{k}$ for all $\ell < n$,
(3) $a^\xi_\ell \in \mathcal{A}_0$ if $\ell \in I_0$,
(4) $a^\xi_\ell \in \mathcal{A}_1$ if $\ell \in I_1$,
(5) $n = I_0 \cup I_1$,
(6) $a^\xi_\ell = a^\eta_\ell$ if and only if $\ell \in I$,
(7) $c(a^\xi_0, \ldots, a^\xi_{n-1}) = c(a^\eta_0, \ldots, a^\eta_{n-1}) < \bar{k}$,
(8) $a^\xi_\ell \cap \bar{k} = a^\eta_\ell \cap \bar{k}$ for each $\ell < n$.

We check that $q_\xi \cup q_\eta \in Q$ for such ξ, η. Indeed, suppose that $a^\xi_i \in \text{dom}(q_\xi) \cap \mathcal{A}_0$ and $a^\eta_j \in \text{dom}(q_\eta) \cap \mathcal{A}_1$. It follows that $i \neq j$ and so by the hypothesis on c, we have that $a^\xi_i \cap a^\eta_j \subset \bar{k}$. In addition, $\bar{k} \cap (a^\xi_i \setminus q_\xi(a^\xi_i)) \cap (a^\eta_j \setminus q_\eta(a^\eta_j))$ is the same as $\bar{k} \cap (a^\xi_i \setminus q_\xi(a^\xi_i)) \cap (a^\xi_j \setminus q_\xi(a^\eta_j))$ and so is empty. □

For a collection $\mathcal{A} \subset [\omega]^\omega$ with the property that no finite union from $\mathcal{A}$ is cofinite, the well-known poset $\mathcal{P}_\mathcal{A}$ as defined in [4, p. 153] is ccc and forces an infinite set $a \subset \omega$ which satisfies that for all $Y \subset \omega$ (in the ground model), a is almost disjoint from Y if and only if Y is (mod finite) covered by a finite union from $\mathcal{A}$. We make a minor change so that if $\mathcal{A}$ is the empty family, then $\mathcal{P}_\mathcal{A}$ simply adds a standard Cohen real.

Definition 1.6 For a family $\mathcal{A} \subset [\omega]^\omega$, we define $\mathcal{P}_\mathcal{A}$ so that $p \in \mathcal{P}_\mathcal{A}$ if $p = (t_p, \mathcal{A}_p) \in [\omega]^{<\omega} \times [\mathcal{A}]^{<\omega}$, and $p < q$ if $t_q \subset t_p$, $\max(t_q) < \min(t_p \setminus t_q)$, $\mathcal{A}_q \subset \mathcal{A}_p$, and $t_p \cap A \subset t_q$ for all $A \in \mathcal{A}_q$.

The following result was the main motivation for this paper.

Proposition 1.7 [2, 2.10] *PFA implies that every maximal almost disjoint family contains a Luzin sequence.*

The previous result follows easily from Lemma 1.8 which is new and illustrates some of the key ideas.

Lemma 1.8 *Let $\mathcal{U}$ be an ultrafilter on ω and suppose that $\mathcal{A}$ is an uncountable almost disjoint family satisfying that, for each $U \in \mathcal{U}$, all but countably many members of $\mathcal{A}$ meet U in an infinite set. Then there is a ccc poset which forces that $\mathcal{A}$ contains a Luzin sequence.*

Proof Clearly we may assume that $\mathcal{A}$ has cardinality ω_1 and fix an enumeration $\{a_\alpha : \alpha \in \omega_1\}$. The poset Q consists simply of conditions $q = (n_q, I_q) \in \omega \times [\omega_1]^{<\omega}$ where a condition p extends q if $I_p \supset I_q$, $n_p \geq n_q$, and for $\beta \in I_p \setminus I_q$ and all $\alpha \in I_q \cap \beta$, $a_\alpha \cap a_\beta \not\subset n_q$.

Obviously, if $G \subset Q$ is a generic filter, the desired Luzin sequence will be given by $\{a_\alpha : \alpha \in \dot{I} = \bigcup_{p\in G} I_p\}$. There will be a condition that forces $\dot{I}$ is uncountable so long as Q is ccc.

Assume now that $\{q_\xi : \xi \in \omega_1\} \subset Q$. By passing to a subcollection, we may assume there is an n so that $n = n_{q_\xi}$ for all $\xi \in \omega_1$. In addition, we may assume that the sequence $\{I_{q_\xi} : \xi \in \omega_1\}$ forms a Δ-system with root $I_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. For each $\xi \in \omega_1$, let $I_{q_\xi} \setminus I_0 = \{\beta_1^\xi, \ldots, \beta_\ell^\xi\}$ be listed in increasing order, and assume that $\alpha_m < \beta_1^\xi$. One final reduction (not really needed) is to choose a sequence of integers $\{k_1, \ldots, k_\ell\}$ so that for all $\xi \in \omega_1$ and $1 \leq i \leq \ell$, we have that $k_i \in a_{\beta_i^\xi}$.

We define a collection $T \subset ([\omega]^{<\omega})^\ell$ according to $\vec{t} = \langle t_1, \ldots, t_\ell\rangle \in T$ if $J_{\vec{t}} = \{\xi : t_i \subset a_{\beta_i^\xi} \setminus n \text{ for all } 1 \leq i \leq \ell\}$ is uncountable. Of course we have that the sequence $\langle\emptyset, \ldots, \emptyset\rangle$ is a member of T. We show that, for each $t \in T$ and $1 \leq i \leq \ell$, the set

$$U(\vec{t}, i) = \{k \in \omega : \langle t_1, \ldots, t_{i-1}, t_i \cup \{k\}, t_{i+1}, \ldots t_\ell\rangle \in T\}$$

is a member of $\mathcal{U}$. For each $k \notin U(\vec{t}, i)$, $J_{\vec{t},i,k} = \{\xi \in J_{\vec{t}} : k \in a_{\beta_i^\xi}\}$ is countable, thus we may choose $\xi \in J_{\vec{t}} \setminus \bigcup_{k\notin U(\vec{t},i)} J_{\vec{t},i,k}$ and observe that for each $k \in a_{\beta_i^\xi}$, $J_{\vec{t},i,k}$ is uncountable. This means that $\omega \setminus U(\vec{t}, i)$ is disjoint from uncountably many members of $\mathcal{A}$ and so can not be a member of $\mathcal{U}$.

Now choose $\delta \in \omega_1$ so large that for all $\vec{t} \in T$, $1 \leq i \leq \ell$ and $k \in \omega$, if $J_{\vec{t},i,k}$ is countable, then it is contained in δ and, if $U(\vec{t}, i) \in \mathcal{U}$, it meets $a_{\beta_j^\delta}$ in an infinite set. If $\vec{t} = \langle t_1, \ldots, t_\ell\rangle \in T$ and if $1 \leq i, j \leq \ell$, then there is an $k \in a_{\beta_j^\delta} \cap U(\vec{t}, i)$, which implies that $\vec{t}^* = \langle t_1, \ldots, t_{i-1}, t_i \cup \{k\}, t_{i+1}, \ldots t_\ell\rangle \in T$. Therefore a routine finite recursion shows the existence of a sequence $\vec{t} = \langle t_1, \ldots, t_\ell\rangle \in T$ satisfying that for each $1 \leq i, j \leq \ell$, $t_i \cap a_{\beta_j^\delta}$ is not empty. It is routine to verify that for any $\xi \in J_{\vec{t}}$, q_ξ and q_δ are compatible since $t_i \cap a_{\beta_j^\delta} \not\subset n$ is a witness to $a_{\beta_i^\xi} \cap a_{\beta_j^\delta} \not\subset n$ for each $1 \leq i, j \leq \ell$. □

2 An ω_1-special mad family and MA_{ω_1}

We produce a model of MA_{ω_1} in which there is an ω_1-special maximal almost disjoint family. By Proposition 1.5, this will complete the proof of the main theorem.

Theorem 2.1 *It is consistent with Martin's Axiom and $\mathfrak{c} = \omega_2$ that there is a maximal almost disjoint family which is ω_1-separated, and so contains no Luzin family.*

The method to construct the model is to begin with a forcing to produce a model in which $\mathfrak{c} = \omega_2$ and there is an ω_1-special maximal almost disjoint family which remains maximal in extensions by ccc forcings of cardinality ω_1. Then the standard finite support iteration of ccc posets of cardinality ω_1 can be used to extend to a model of Martin's Axiom. Let us say that a maximal almost disjoint family is κ-indestructible if its maximality is preserved by any forcing of cardinality less than κ.

Lemma 2.2 *Suppose that κ is a regular cardinal and $\mathcal{A} \subset [\omega]^\omega$ is an almost disjoint family of cardinality κ with the property that each $Y \subset \omega$ is either (mod finite) covered by finitely*

Sh:985

many members of $\mathcal{A}$ or meets all but fewer than κ many members of $\mathcal{A}$. Then $\mathcal{A}$ is κ-indestructible.

Proof Let Q be a poset of cardinality less than κ. Assume that $\dot{Y}$ is a Q-name of a subset of ω. For each $A \in \mathcal{A}$, choose $q_A \in Q$ (if one exists) and $n_A \in \omega$ which forces that $\dot{Y} \cap A \subset n_A$. Fix any $q \in Q$ and $n \in \omega$ so that $\mathcal{A}_{q,n} = \{A \in \mathcal{A} : q_A = q, \text{ and } n_A = n\}$ has cardinality κ. It follows from the assumptions on $\mathcal{A}$ then that $\dot{Y}_q^+ = \{m \in \omega : (\exists p < q)\ p \Vdash m \in \dot{Y}\}$ is mod finite covered by finitely many members of $\mathcal{A}$. This shows that q forces that $\dot{Y}$ is also covered by finitely many members of $\dot{A}$. □

Lemma 2.3 *If $\mathcal{A}$ is an ω_2-indestructible maximal almost disjoint family, then it remains maximal in any extension by a finite support iteration of ccc posets of size at most ω_1.*

Proof Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha \in \lambda\rangle$ be a finite support iteration so that, for each α,

$$\Vdash_{P_\alpha} \dot{Q}_\alpha \text{ is ccc and } |\dot{Q}_\alpha| \le \omega_1.$$

For convenience, we may assume that for each α there is a P_α-name $\dot{<}_\alpha$ of a subset of $\omega_1 \times \omega_1$ so that $\dot{Q}_\alpha$ is forced to be the poset $(\omega_1, \dot{<}_\alpha)$. Let $\dot{Y}$ be a P_λ-name of a subset of ω. Let $\{M_\xi : \xi \in \omega_1\}$ be an increasing $\in$-chain of countable elementary submodels of $H(\theta)$ for a sufficiently large θ such that P_λ and $\dot{Y}$ are in M_0. Let $M = \bigcup_{\xi\in\omega_1} M_\xi$ and let $P' = P_\lambda \cap M$. Since P' is a ccc poset of cardinality ω_1, there is a condition $\bar{p} \in P'$ and an $A \in \mathcal{A}$ such that $\bar{p} \Vdash_{P'} \dot{Y} \cap A$ is infinite. Assume that $\bar{p} > p \in P_\lambda$ is such that there is an $n \in \omega$ with $p \Vdash_{P_\lambda} \dot{Y} \cap A \subset n$. We may assume that p satisfies that for each $\alpha \in \text{dom}(p)$, $p(\alpha)$ is an element of ω_1 (and not just forced to be). There is a $\delta \in \omega_1$ such that $p(\alpha) \in \delta$ for all $\alpha \in \text{dom}(p)$ and so that $\text{dom}(p) \cap M \in M_\delta$. It follows by elementarity that, for each $\alpha \in \text{dom}(\bar{p}) \cap \text{dom}(p) \cap M_\delta$, $p \restriction (\text{dom}(p) \cap M_\delta \cap \alpha) \Vdash p(\alpha) \dot{<}_\alpha \bar{p}(\alpha)$. Now choose $p' \in P'$ and $m \in A \setminus n$ such that $p' < p \restriction M_\delta$ and $p' \Vdash n \in \dot{Y}$. It is easily checked that p' is compatible with p (mainly since $\text{dom}(p') \cap \text{dom}(p) \subset \text{dom}(p')$). It follows that $\bar{p} \Vdash_{P_\lambda} \dot{Y} \cap A$ is infinite; and that $\mathcal{A}$ remains maximal. □

Now we define the natural poset for introducing a function to make an almost disjoint family special.

Definition 2.4 For an almost disjoint family $\mathcal{A}$ and a linear order $\prec$ of $\mathcal{A}$, the poset $Q_{\mathcal{A},\prec}$ will simply be the set of functions c such that $\text{dom}(c) = \mathcal{P}(\mathcal{A}_c)$ for some finite $\mathcal{A}_c \subset \{a_\beta : \beta < \alpha\}$ which satisfy condition (2) of Definition 1.3. The ordering on $Q_{\mathcal{A},\prec}$ is simple extension. We use $Q_\mathcal{A}$ if the choice of $\prec$ is clear from the context.

We make the following observations about $Q_{\mathcal{A},\prec}$.

Lemma 2.5 *Let $\mathcal{A} \subset [\omega]^\omega$ be an almost disjoint family which is special.*

(1) *If $\prec$ is any linear ordering of $\mathcal{A}$, then the poset $Q_{\mathcal{A},\prec}$ is ccc.*
(2) *If $a \in [\omega]^\omega$ is almost disjoint from each member of $\mathcal{A}$, then $\mathcal{A} \cup \{a\}$ is also special.*

Proof Let c and $\tilde{<}$ be the witnesses (as in (2) of Definition 1.3) that $\mathcal{A}$ is special. To prove item 2, let us notice that $2c$ (doubling each value) is also a witness to $\mathcal{A}$ is special; so we may assume that c takes on only even values. Extend $\tilde{<}$ to all of $\mathcal{A} \cup \{a\}$ by declaring $b \,\tilde{<}\, a$ for all $b \in \mathcal{A}$. Also extend c by defining $c(B_1, \ldots, B_{n-1}, a)$ as follows. Choose k_0 minimal so that

$B_i \cap a \subset k_0$ for all $1 \le i < n$ and let $k_1 = c(B_1, \dots, B_{n-1})$. Now define $c(B_1, \dots, B_{n-1}, a)$ to be (the odd integer) $3^{k_0} \cdot 5^{k_1}$. Assume that $c(B_1, \dots, B_{n-1}, a) = c(C_1, \dots, C_{n-1}, a) = k$. It follows that $c(B_1, \dots, B_{n-1}) = c(C_1, \dots, C_{n-1}) < k$ and so $B_i \cap C_j \subset k$ for distinct $1 \le i, j < n$. Of course each of $B_i \cap a$ and $C_i \cap a$ are contained in k because of the choice of k_0. This proves that $\mathcal{A} \cup \{a\}$ is special.

Now we prove that $Q_{\mathcal{A},\prec}$ is ccc. Let $\{q_\xi : \xi \in \omega_1\}$ be a subset. We may assume that the family $\{\mathcal{A}_{q_\xi} : \xi \in \omega_1\}$ forms a Δ-system, each with cardinality n and with root $\mathcal{R}$. For each ξ, let $\mathcal{A}_{q_\xi}$ be enumerated in $\prec$-increasing order: $\{a_1^\xi, \dots, a_n^\xi\}$. We may assume that for each ξ, ζ and each $1 \le i \le n$, $a_i^\xi \in \mathcal{R}$ whenever $a_i^\zeta \in \mathcal{R}$. We may also assume that there is an uncountable $I_0 \subset \omega_1$ and a fixed permutation π of n such that $a^\xi_{\pi(i)} \precsim a^\xi_{\pi(i+1)}$ for each $1 \le i < n$ for all $\xi \in I_0$. Similarly, there is an uncountable $I_1 \subset I_0$, an integer k and a function $\underline{c}$ defined on the power set of n satisfying that, for each increasing sequence $1 \le \rho(1) < \cdots < \rho(\ell) \le n$,

$$c_{q_\xi}(\{a^\xi_{\rho(1)}, \dots, a^\xi_{\rho(\ell)}\}) = \underline{c}(\{\rho(1), \dots, \rho(\ell)\}) < k$$

and

$$c(\{a_1^\xi, \dots, a_n^\xi\}) < k.$$

Now choose $\xi < \zeta \in I_1$ so that for each $1 \le i \le n$, $a_i^\xi \cap k = a_i^\zeta \cap k$. By virtue of $\underline{c}$, it follows that $c_{q_\xi} \cup c_{q_\zeta}$ is a partial function on $\mathcal{P}(\mathcal{A}')$ (as in there are no disagreements), where $\mathcal{A}' = \mathcal{A}_{q_\xi} \cup \mathcal{A}_{q_\zeta}$. Extend this to a function c' with domain $\mathcal{P}(\mathcal{A}')$ so that if $c'(\{B_1, \dots, B_m\}) = c'(\{C_1, \dots, C_m\})$ then each of $\{B_1, \dots, B_m\}$ and $\{C_1, \dots, C_m\}$ are contained in one of $\mathcal{A}_{q_\xi}$ or $\mathcal{A}_{q_\zeta}$. To check that $q' = (c', \mathcal{A}')$ is a common extension of q_ξ and q_ζ we simply have to show that c' satisfies the requirement of being a specializing function. Now the only case to check is if $c'(\{B_1, \dots, B_m\}) = c'(\{C_1, \dots, C_m\}) = k' < k$ with $\{B_1, \dots, B_m\} \subset \mathcal{A}_{q_\xi}$ and $\{C_1, \dots, C_m\} \subset \mathcal{A}_{q_\zeta}$. Let ρ_0 and ρ_1 be the increasing mappings from m into n so that $\{B_1, \dots, B_m\} = \{a^\xi_{\rho_0(1)}, \dots, a^\xi_{\rho_0(m)}\}$ and $\{C_1, \dots, C_m\} = \{a^\zeta_{\rho_1(1)}, \dots, a^\zeta_{\rho_1(m)}\}$. It follows that $\underline{c}(\rho_0) = \underline{c}(\rho_1)$. Therefore $c_{q_\xi}(\{a^\xi_{\rho_0(1)}, \dots, a^\xi_{\rho_0(m)}\} = c_{q_\xi}(\{a^\xi_{\rho_1(1)}, \dots, a^\xi_{\rho_1(m)}\}) = k'$. Therefore, for $1 \le i \ne j \le m$, $a^\xi_{\rho_0(i)} \cap a^\xi_{\rho_1(j)} \subset k'$. If, for example $a^\xi_{\rho_1(j)}$ is in the root $\mathcal{R}$, then we have that $a^\xi_{\rho_0(i)} \cap a^\zeta_{\rho_1(j)} = B_i \cap C_j \subset k'$. If neither is in the root, then $\rho_0(i) \ne \rho_1(j)$ and so we know that $a^\xi_{\rho_1(i)} \cap a^\zeta_{\rho_1(j)} \subset k$ and so

$$B_i \cap C_j = a^\xi_{\rho_0(i)} \cap a^\zeta_{\rho_1(j)} \cap k = a^\xi_{\rho_1(i)} \cap (a^\xi_{\rho_1(j)} \cap k) \subset k'.$$

□

Theorem 2.6 *If $2^{\omega_1} = \omega_2$, then there is a ccc poset of cardinality ω_2 which forces that there is an ω_2-indestructible maximal almost disjoint family which is ω_1-special.*

Proof We define a finite support iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha \in \omega_2\rangle$ and a sequence $\langle \dot{c}_\alpha, \dot{a}_\alpha : \alpha \in \omega_2\rangle$ so that, for each $\alpha < \omega_2$,

(1) for each $\beta < \alpha$, $\dot{a}_\beta$ is a P_α-name of a subset of ω,
(2) $\dot{c}_\alpha$ is a $P_{\alpha+1}$-name such that $P_{\alpha+1}$ forces that $\dot{c}_\alpha$ witnesses that $\dot{A}_\alpha = \{\dot{a}_\beta : \beta < \alpha\}$ is special,
(3) $\dot{Q}_\alpha$ is a P_α-name of a product, $\dot{Q}^0_\alpha \times \dot{Q}^1_\alpha$, of cardinality at most ω_1,

(4) $\dot{Q}^0_\alpha$ is the P_α-name of $P_{\mathcal{A}_\alpha}$ (as in Definition 1.6),
(5) $\dot{Q}^1_\alpha$ is the P_α-name of the poset $Q_{\mathcal{A}_\alpha}$ as in Definition 2.4 which adds $\dot{c}_\alpha$.

To prove that the poset P_{ω_2} is ccc it is sufficient to prove by induction on α, that each $\dot{Q}^1_\alpha$ is forced to be ccc. Before doing so, let us assume that P_{ω_2} satisfies the above conditions and check that this will prove the statement of the theorem. In particular, we check that the collection $\{\dot{a}_\alpha : \alpha \in \omega_2\}$ of P_{ω_2}-names is forced to be a maximal almost disjoint family which is ω_1-special and ω_2-indestructible. Clearly condition 2 implies that it is forced to be ω_1-special (which implies almost disjoint). To show that it will be ω_2-indestructible (and therefore maximal) we check that it will satisfy the hypothesis of Lemma 2.2. If $\dot{Y}$ is any P_{ω_2}-name of a subset of ω, there is an $\alpha < \omega_2$ such that $\dot{Y}$ is a P_α-name. By condition 2 and the remarks preceding Definition 1.6, we have that it is forced that $\dot{Y}$ meets $\dot{a}_\beta$ for all $\beta \geq \alpha$.

Now assume, by induction on α, that P_α is ccc and let G_α be a P_α-generic filter. If α is a successor then Lemma 2.5 shows that $Q_{\mathcal{A}_\alpha}$ is ccc, so we assume that α is a limit ordinal. Working in $V[G_\alpha]$, consider an uncountable collection $\{q_\xi : \xi \in \omega_1\} \subset Q_{\mathcal{A}_\alpha}$. If α had countable cofinality, then there would be a $\mu < \alpha$ such that $J = \{\xi : q_\xi \in Q_{\mathcal{A}_\mu}\}$ is uncountable. Again, by Lemma 2.5, $Q_{\mathcal{A}_\mu}$ is (still) ccc and so two members of $\{q_\xi : \xi \in J\}$ would be compatible in $Q_{\mathcal{A}_\mu}$ and also in $Q_{\mathcal{A}_\alpha}$. Therefore we may assume that α has cofinality ω_1.

By passing to a subcollection, we may assume that there is some $n \in \omega$ such that $\mathcal{A}_{q_\xi}$ has cardinality n for all ξ. For each ξ, fix $\{\beta^\xi_i : i < n\} \subset \alpha$ so that $\mathcal{A}_{q_\xi} = \{a_{\beta^\xi_i} : i < n\}$. Fix a function $\underset{\sim}{c}$ from $\mathcal{P}(n)$ into ω so that for some uncountable set $\Gamma \subset \omega_1$, and all increasing sequences $1 \leq \rho(1) < \cdots < \rho(\ell) \leq n$, $\bar{c}(\{\rho(1), \ldots, \rho(\ell)\}) = c_{q_\xi}(\{a_{\beta^\xi_{\rho(1)}}, \ldots, a_{\beta^\xi_{\rho(\ell)}}\})$. For simplicity of notation, we again just assume that Γ is all of ω_1.

Choose $m \leq n$ maximal (possibly $m = 0$) so that there is a $\mu < \alpha$ such that $\beta^\xi_m < \mu$ for uncountably many (and, by another reduction, all) ξ. Further, choose a sufficiently large $\underline{k}$ and arrange that for each $\xi \leq \zeta$ and increasing sequence $1 \leq \rho(1) < \cdots < \rho(\ell) \leq m$

$$c_\mu(\{a_{\beta^\xi_\rho(1)}, \ldots, a_{\beta^\xi_\rho(\ell)}\}) = c_\mu(\{a_{\beta^\zeta_\rho(1)}, \ldots, a_{\beta^\zeta_\rho(\ell)}\}) < \underline{k}.$$

Further arrange that $\underline{k}$ is sufficiently large (and by a further reduction) that there is a fixed sequence $\{t_1, \ldots, t_n\} \subset \mathcal{P}(\underline{k})$ so that for all ξ and for distinct $1 \leq i, j \leq n$, $a_{\beta^\xi_i} \cap a_{\beta^\xi_j} \subset \underline{k}$ and $a_{\beta^\xi_i} \cap \underline{k} = t_i$. Finally for $\xi < \zeta$, we may assume that $\beta^\xi_n < \beta^\zeta_{m+1}$.

Now we return to the extension $V[G_\alpha \cap P_\mu]$ and fix elements $p_\xi \in P_\alpha$ for $\xi \in \omega_1$ so that $p_\xi \restriction \mu \in G_\alpha$ and p_ξ forces that q_ξ has the desired properties developed above. We may assume several things about p_ξ:

(1) p_ξ has determined the sequence $\{\beta^\xi_i : 1 \leq i \leq n\}$ and the value of $\underset{\sim}{c}$,
(2) $\beta^\xi_i \in \mathrm{dom}(p_\xi)$ for each $m < i \leq n$,
(3) for all $\gamma \in \mathrm{dom}(p_\xi)$,
    (a) $p_\xi \restriction \gamma$ forces that $p_\xi(\gamma) = ((t^\xi_\gamma, \mathcal{A}^\xi_\gamma), c^\xi_\gamma) \in \dot{Q}^0_\gamma \times \dot{Q}^1_\gamma$,
    (b) t^ξ_γ is a member of $[\omega]^{<\omega}$ and $t^\xi_\gamma \not\subset \underline{k}$,
    (c) there is a finite set $I^\xi_\gamma \subset \gamma \cap \mathrm{dom}(p_\xi)$ such that $\mathcal{A}^\xi_\gamma = \{\dot{a}_\delta : \delta \in I^\xi_\gamma\}$,
    (d) the domain of c^ξ_γ is $\mathcal{P}(A^\xi_\gamma)$ and the integer values have been determined.

Now we may choose an uncountable $J \subset \omega_1$ so that the sequence $\{\mathrm{dom}(p_\xi) : \xi \in \omega_1\}$ forms a Δ-system (and by possibly increasing μ) with root contained in μ.

Since P_α is ccc, it is routine to find $\xi < \zeta$ such that

(1) p_ξ and p_ζ are compatible,

(2) $\max(\mathrm{dom}(p_\xi)) < \min(\mathrm{dom}(p_\zeta) \setminus \mu)$,
(3) $t^\xi_{\beta^\xi_i} = t^\zeta_{\beta^\zeta_i}$ end-extends t_i for each $m < i \le n$.

Now we show that p_ξ and p_ζ have a common extension which forces that q_ξ and q_ζ are compatible. This will complete the proof that $\Vdash_{P_\alpha} \dot{Q}^1_\alpha$ is ccc.

Define $\bar{p}$ so that $\mathrm{dom}(\bar{p}) = \mathrm{dom}(p_\xi) \cup \mathrm{dom}(p_\zeta)$. The definition of $\bar{p} \restriction \mu$ is any member of G_μ which is below each of $p_\xi \restriction \mu$ and $p_\zeta \restriction \mu$. For each $m < i \le n$, and $\gamma = \beta^\xi_i$,

$$\bar{p}(\gamma) = ((t^\xi_\gamma, \mathcal{A}^\xi_\gamma \cup \{\dot{a}_{\beta^\zeta_j} : 1 \le j \le m\}), c^\xi_\gamma)$$

and for $\gamma = \beta^\zeta_i$

$$\bar{p}(\gamma) = ((t^\zeta_\gamma, \mathcal{A}^\zeta_\gamma \cup \{\dot{a}_{\beta^\xi_j} : 1 \le j \le n\}), c^\zeta_\gamma).$$

For other $\gamma \in \mathrm{dom}(p_\xi) \setminus \mu$, define $\bar{p}(\gamma) = p_\xi(\gamma)$, and similarly, for other $\gamma \in \mathrm{dom}(p_\zeta) \setminus \mu$, $\bar{p}(\gamma) = p_\zeta(\gamma)$.

We show that $\bar{p}$ forces that $c_{q_\xi} \cup c_{q_\zeta}$, as a partial function on $\mathcal{P}(\mathcal{A}_{q_\xi} \cup \mathcal{A}_{q_\zeta})$, does not have any violations of the requirements on a specializing function. As was shown in Lemma 2.5 this ensures that there is a suitable extension, thus showing that q_ξ and q_ζ are compatible.

Let us first observe that $\bar{p}$ forces that $a_{\beta^\xi_i} \cap a_{\beta^\zeta_j}$ is contained in $\underline{k}$ for all distinct $1 \le i, j \le n$. If $m < j \le n$ then it follows that this intersection is contained in $a_{\beta^\xi_i} \cap t^\zeta_{\beta^\zeta_j}$ because of the fact that β^ξ_i was added appropriately in the definition of $\bar{p}(\beta^\zeta_j)$. Then, since $t^\xi_{\beta^\xi_j} = t^\zeta_{\beta^\xi_j}$, we have that $a_{\beta^\xi_i} \cap a_{\beta^\zeta_j} = a_{\beta^\xi_i} \cap a_{\beta^\xi_j}$, which was forced by p_ξ to be contained in $\underline{k}$. This is also the case if $j \le m < i$, since β^ζ_j was appropriately added in the definition of $\bar{p}(\beta^\xi_i)$. If $1 \le i, j \le m$, then the fact that

$$c_\mu(\{a_{\beta^\xi_i}, a_{\beta^\xi_j}\}) = c_\mu(\{a_{\beta^\zeta_i}, a_{\beta^\zeta_j}\}) < \underline{k}$$

ensures that $a_{\beta^\xi_i} \cap a_{\beta^\zeta_j}$ is contained in $\underline{k}$. Since $\bar{p}$ also forces that $a_{\beta^\xi_j} \cap \underline{k} = a_{\beta^\zeta_j} \cap \underline{k}$, it also follows that for distinct $1 \le i, j \le n$,

$$a_{\beta^\xi_i} \cap a_{\beta^\zeta_j} = a_{\beta^\xi_i} \cap (a_{\beta^\zeta_j} \cap \underline{k}) = a_{\beta^\xi_i} \cap (a_{\beta^\xi_j} \cap \underline{k}) = a_{\beta^\xi_i} \cap a_{\beta^\xi_j}. \tag{2.1}$$

Suppose that ρ_0 and ρ_1 are increasing functions from $\{1, \ldots, \ell\}$ into $\{1, \ldots, n\}$. Assume that $\underline{c}(\{\rho_0(1), \ldots, \rho_0(\ell)\}) = \underline{c}(\{\rho_1(1), \ldots, \rho_1(\ell)\} = k$. By the choice of $\underline{c}$, not only is ρ_0 and ρ_1 coding an arbitrary instance where c_{q_ξ} will equal c_{q_ζ} but also, if $\rho_0 \ne \rho_1$, where c_{q_ξ} will agree with itself. From this latter fact, we can see that if $i \ne j$, then $\imath' = \rho_0(i) \ne \rho_1(j) = \jmath'$, since $a_{\beta^\xi_{\imath'}}$ is required to be almost disjoint from $a_{\beta^\xi_{\jmath}}$. Now suppose that i is in the range of ρ_0 and $j \ne i$ is in the range of ρ_1. We must show that $a_{\beta^\xi_i} \cap a_{\beta^\zeta_j}$ is contained in k. We know already, by virtue of c_{q_ξ}, that $a_{\beta^\xi_i} \cap a_{\beta^\xi_j}$ is contained in k; and so, by equation 2.1, $a_{\beta^\xi_i} \cap a_{\beta^\zeta_j}$ is contained in k as required. □

Acknowledgements Research of the first author was supported by NSF grant No. NSF-DMS 20060114. The research of the second author was supported by the United States-Israel Binational Science Foundation (BSF Grant No. 2002323), and by the NSF grant No. NSF-DMS 0600940. This is paper number F1090 in the second author's personal listing.

References

1. Abraham, U., Shelah, S.: Lusin sequences under CH and under Martin's axiom. Fundam. Math. **169**(2), 97–103 (2001). MR1852375 (2002k:03069)
2. Dow, A.: Sequential order under PFA. Canad. Math. Bull. **54**, 270–276 (2011)
3. Luzin, N.N.: On subsets of the series of natural numbers. Izv. Akad. Nauk SSSR, Ser. Mat. **11**, 403–410 (1947). MR0021576 (9,82c)
4. Martin, D.A., Solovay, R.M.: Internal Cohen extensions. Ann. Math. Log. **2**(2), 143–178 (1970). MR0270904 (42 #5787)
5. Steprāns, J.: Strong Q-sequences and variations on Martin's axiom. Can. J. Math. **37**(4), 730–746 (1985). MR801424 (87c:03106)
6. Todorčević, S.: Analytic gaps. Fundam. Math. **150**(1), 55–66 (1996). MR1387957 (98j:03070)